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A NOVEL APPROACH FOR ESTIMATING THE ARL-BIAS SEVERITY OF SHEWHART P-CHARTS

Abstract: *The Shewhart p-Chart is a widely used and taught control chart that monitors the process fraction of nonconforming, p. The control limits of this chart are computed from an equation derived from the central limit theorem that assumes the parameters p and sample size, n, provide a good approximation of the binomial to the normal distribution. It is known that if the approximation is poor the chart becomes ARL-biased with a consequent deterioration of its process improvement detection capability. Hence, if one is to use this chart its ARL-biased severity ought to be established in advance. Three parameters are used to estimate a chart´s ARL-bias severity, namely the in-control ARL, the relative ARL-bias and the maximum out of control ARL. However, calculating these parameters is a task that requires of lengthy calculations that often entails the use of specialised software. In this paper we present a new approach that facilitates the estimation of the bias severity, this approach consists of a closed-form equation that eases the calculations of the ARL-bias parameters and of a graphical method that permits assessing a p-Chart´s ARL-bias severity level according to the severity zone in which its n and p values fall into*.

Keywords: *attribute chart, Average run length, ARL-bias, control chart, fraction non-conforming, p-Chart.*

1. Introduction

A control chart is a statistical tool that is extensively used for process monitoring in manufacturing and non-manufacturing industries. The Shewhart charts are among the most extensively used and are included in all statistical quality control textbooks. Depending on the data used to monitor the process, these charts can be divided into two groups: i. *Variable charts* (when continuous numerical data is used) and ii. *Attribute charts* (when count-type data is used). Montgomery (2009, pág. 226) provides a good insight into the theory and application of these charts.

Among the most popular single-attribute Shewhart charts are the so called *p* and *np* charts that monitor the process fraction of nonconforming and the number of nonconforming respectively. These charts are used in cases when it is more convenient to classify process-produced items as

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conforming or nonconforming than to measure their quality characteristics, or when only purely qualitative count data exist to monitor the process.

Let's suppose that X is the number of nonconforming products found when a random sample of n products are inspected, if the process is stable and the inspected products are independent, then X has a binomial distribution with parameters n and p. Let $x_i(i = 1,2,3,...)$ denote the observations of X when the inspection is implemented. If p is known, or if it has been estimated, and if n is sufficiently large; then the process can be monitored by plotting the number of nonconforming products x_i on an *np-Chart* or by plotting the fraction of nonconforming products $y_i = x_i/n$ on a *p-Chart* with limits determined by equations (1a) and (1b). These equations are based on the central limit theorem that presumes the binomial distribution to be approximately normal. In this paper for reasons of conciseness we analyse exclusively the *p-Chart* but the results also apply to the *np-Chart*.

When monitoring a process with a p-Chart, there will be points y_i that fall inside and outside the chart limits. The monitored process is deemed to be in-control *(or IC)* if the y_i that fall outside the chart limits do so according to the false alarm rate at which the chart was designed, a typical alarm rate is 1/370. If y_i fall outside the limits at a higher rate, then the process is deemed to be out of control *(or OC)*. The chart could be OC because of: 1) p has increased (i.e. the process has deteriorated) or 2) p has decreased (i.e. the process has improved).

The capability of the chart to detect an OC state is commonly measured by means of the Average Run Length (ARL), that being the average number of y_i points plotted within the limits before a y_i point appears outside them when p has shifted away from the in control target value (Mitra, 2008, pág. 277). To graphically observe this capability, one could plot the p-shift vs ARL, obtaining in that way the chart´s ARL curve. Figure 1 shows an ideal ARL curve, we present the x-axis as the percentual change $(p_1/p) - 1$, where p is the process fraction of nonconforming and p_1 denotes the actual p-shift. In this case $p_1 = \delta p$, with δ being the relative shift coefficient that can take values above or below one. For example: if p=0.1 and δ=1.1 the resulting p_1 is 0.11 and hence $(p_1/p) - 1$=0.1, denoting that the process fraction of nonconforming has suffered a 10% positive shift.
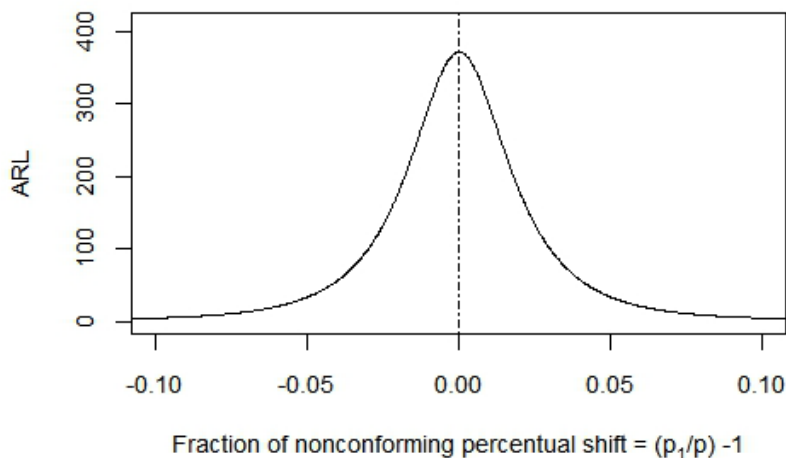

Figure 1. Example of an unbiased ARL curve

In Figure 1, when $(p_1/p) - 1 = 0$, there is no p-shift and its corresponding ARL value is the IC-ARL (or ARL_0), whilst when $(p_1/p) - 1 \neq 0$ there is a p-shift and its corresponding ARL is the OC-ARL (or ARL_1). It can be seen that always $ARL_1 < ARL_0$ and that the curve is symmetric with respect to the y-axis. A p-Chart with an ARL curve like the one shown in Figure 1 will have equal capability to detect process deterioration or improvement.

The ARL curve shown in Figure 1 is known as an unbiased ARL curve. Acosta-Mejia (1999) defined a control chart as being ARL-unbiased if its ARL curve decreased as the process parameter moved away from the in-control value, for a p-Chart the process parameter is p. However, Ryan (2011, pág. 195) points out that when $p < 0.5$ the binomial distribution becomes skewed instead of normal and that in this case the ARL required by the p-Chart to detect a decrease in p will be higher than the ARL needed to detect an increase. When this happens we effectively have an ARL-biased chart with a diminished process improvement detection capability.

A rule of thumb often given in statistical text books is that the binomial distribution will be approximately normal so long as np exceeds 5 or 10. This rule, however, has been shown that does not provide satisfactory results when applied to p-Charts (Ryan T. P., 2011, pág. 183), mainly because the fractions of nonconforming in real-world production processes are very low, that is to say $p \ll 0.5$. This situation effectively means that Shewhart p-Charts, even if they are designed following commonly given guidelines, could often be ARL-biased; an important fact hardly ever mentioned in Quality Control textbooks. Brown et al., (2001), comments on the deficiencies of several rules of thumb given in textbooks regarding binomial approximation to the normal distribution.

A very effective way of finding out if a p-Chart is ARL-biased is to compute the bias parameters and then use these to determine the bias severity. Three parameters are used

to determine a chart´s ARL-bias severity, namely 1.-The in control ARL (or ARL_0) 2.-The maximum peak value of the ARL curve (or ARL_M) and 3.-The relative ARL-bias (or ARL-bias(%)). However obtaining these parameters, particularly the ARL_M and the ARL-bias(%), is a task that often requires of lengthy calculations and the use of computer programs like Matlab or R, especially if one wishes to examine several combinations of n and p.

The fact that the p-Chart is a widely taught and used chart, led us to the conclusion that simpler methods for estimating its ARL-bias severity had to be developed. To this end, a new approach that facilitates the estimation of the ARL-bias severity was devised; this approach consists of a closed-form equation that eases the estimation of the ARL-bias parameters and of a graphical method whereby a p-Chart´s ARL-bias severity level can be deduced with only its n and p values.

The research carried out along with the results and methods developed are presented in the following sections.

2. Materials and method

2.1. The three sigma p-Chart limits and the IC alpha tails

Equations (1a) and (1b) are used to compute the control limits of the Shewhart p-Chart, these equations are derived using the central limit theorem under the assumption that the binomial distribution is approximately normal. A common textbook rule of thumb says that a good approximation is obtained when $np \geq 5$ (or 10).

Upper Control Limit:

$$UCL = p + Z_{(1-\alpha/2)}\sqrt{\frac{p(1-p)}{n}} \qquad (1a)$$

Lower Control Limit:

$$LCL = p - Z_{(1-\alpha/2)}\sqrt{\frac{p(1-p)}{n}} \qquad (1b)$$

Where:

α = Type I error probability (Alpha error)

n= sample size

p= Known process fraction of nonconforming

Note: If the process fraction of nonconforming is not known but has been estimated, then $\bar{p}$ or p' are typically used in equations (1a) and (1b) to differentiate this case from the known p case.

In equations (1a) and (1b), $Z_{(1-\alpha/2)}$ is the $(1-\alpha/2)th$ quantile of the $N(0,1)$ distribution. To obtain the widely used three sigma p-Chart an alpha error probability value of α= 0.0027 is used. This means that when $Z_{(1-\alpha/2)} = 3$, and the process is under IC conditions, the lower and upper alpha tail error probabilities should be equal to α/2 or 0.00135. In this paper we call these tail values *IC alpha tails*.

Below a certain sample size, *n*, equation (1b) will compute negative LCLs, in that case is customarily to set the LCL=0. Given that the ARL-bias happens only when LCL $\neq$ 0, our calculations and analysis where done for $n \geq 1 + 9(1-p)/p$ which ensures the existence of the lower control limit.

2.2. Relation between the In-Control ARL and the IC alpha tails

As mentioned in the introduction a chart´s ARL can be classified into two types, ARL_0 and ARL_1. Let's recall that the IC-ARL, or ARL_0, is the average number of points that fall within the chart limits before a point fall outside them when the process is IC. In this case, giving that the process is IC, the points that fall outside the limits are considered to be false alarms.

The ARL_0 is a function of the IC alpha tails and can be calculated by means of equations (2), (3) and (4).

$$ARL_0 = \frac{1}{\alpha_o} \qquad (2)$$

Where:

$$\alpha_o = \alpha_L + \alpha_U =$$
$$Total\ IC\ alpha\ error\ probability$$

$$\alpha_U = IC\ upper\ alpha\ tail$$
$$\alpha_U = P\{x \geq nUCL|_p\}$$
$$\alpha_U = 1 - \sum_0^{nUCL} \binom{n}{x} p^x(1-p)^{n-x}$$
$$x = 0,1,2 \dots nUCL \qquad (3)$$

$$\alpha_L = IC\ lower\ alpha\ tail$$
$$\alpha_L = P\{x \leq nLCL|_p\}$$
$$\alpha_L = \sum_0^{nLCL} \binom{n}{x} p^x(1-p)^{n-x}$$
$$x = 0,1,2 \dots nLCL \qquad (4)$$

As previously shown a three sigma p-Chart should have, in theory, IC alpha tail values equal to 0.00135. Applying equation (2) we obtain that in this case the expected ARL_0 is: $1/(\alpha_L + \alpha_U)= 1/(0.00135 + 0.00135) = 1/(0.0027) \approx 370$. This means that when the process is working under IC conditions, in average there would be one false alarm, or one point erroneously plotted outside the chart limits, every 370 samples. However, the expected value of ARL_0=370 is rarely achieved, we show why this happens in the following section.

### 2.2.1. IC alpha tails oscillation and its consequence on the ARL_0

Ryan (2011, pág. 190) shows a table that includes actual IC alpha tails values for several *n* and *p* combinations and uses this to conclude that the alpha tails for the three sigma p-Chart, deviate from the expected 0.00135 value quite substantially. To supplement Ryan´s work we show graphically the actual oscillating behaviour of the IC alpha tails and its consequence on the ARL_0.

Figure 2 shows an example of the oscillation of the IC alpha tails in function of *n* for p=0.02. At low *n* values the tails get further apart with the lower tail getting closer to zero, whilst at higher *n* values the tails get closer

converging towards the expected 0.00135 value. We analysed a range of p between 0.01- 0.2 and found this behaviour to be similar for all those p values.

Since the ARL_0 is a function of the IC alpha tails then this value also oscillates, Figure 3 shows the corresponding ARL_0 for the alpha tails of Figure 2. Notice that often the ARL_0 is below 370, in those cases there will be more false alarms than expected, giving the erroneous impression that the process is out of control.
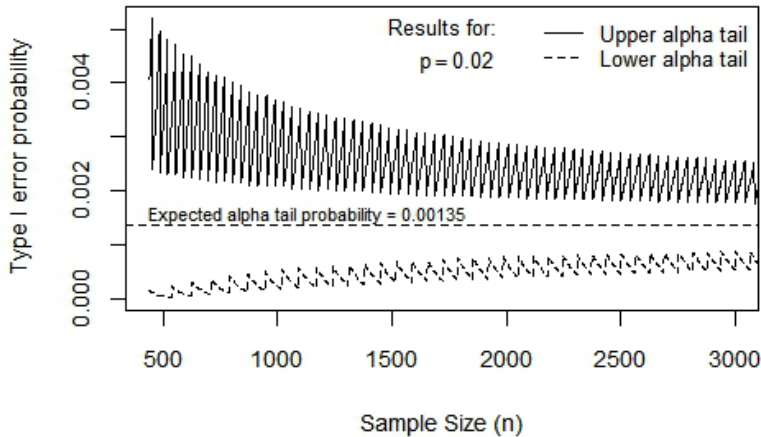
Figure 2. Example of the oscillation of the IC alpha tails according to sample size
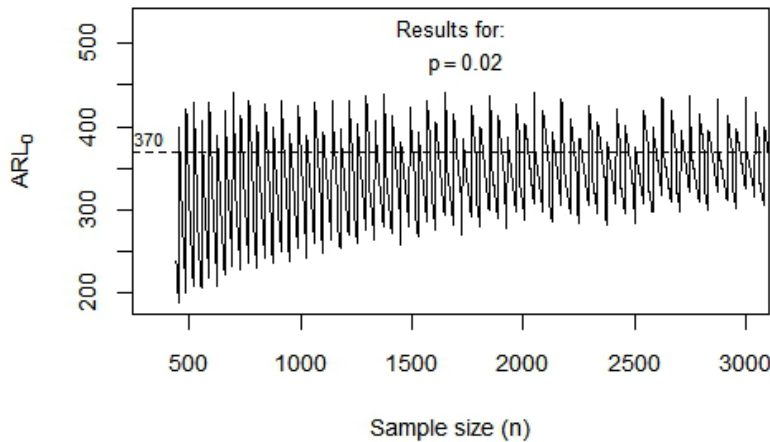
Figure 3. Example of the ARL_0 oscillation according to sample size

2.2.2. The out of control ARL

The OC-ARL, or ARL_1, can be defined as the average number of points that fall within the chart limits before a point fall outside them when the process is OC. As show in Figure 1, the ARL_1 should ideally be lower than the ARL_0; this means that when the process is OC, the points that fall outside the limits should do so at a higher rate that when the process is IC.

The ARL_1 is calculated by means of equations (5) and (6).

$$ARL_1 = \frac{1}{1-\beta} \qquad (5)$$

$$\beta = P\{x \le nUCL|_{p=p_1}\} - P\{x \le nLCL|_{p=p_1}\} \tag{6}$$

Where:

p_1= p-shift *p_1 could be: p> p_1 > p

x= number of nonconforming units

β= Beta error probability (calculated with the cumulative binomial distribution)

2.3. The ARL-bias severity level

As explained on the introduction section, a p-Chart is considered to be ARL-biased if its ARL curve increases as p_1 moves away from the in-control value p. Figure 4 and Figure 5 show several examples of ARL biased curves for p=0.02 and p=0.1. Notice that in all cases the ARL curve peak value is higher than the ARL_0 and that the x-axis location this peak is different to that of the ARL_0, which is located at the x-axis zero point. These variations of the ARL curve peak value relative to the ARL_0 can be used to quantify and determine the bias severity level of an ARL curve.

In this paper we use the following parameters to quantify the ARL-bias severity level:

1) The ARL-ratio = ARL_M /ARL_0
   Where:
   ARL_M = ARL curve peak value
   ARL_0= In-Control ARL
   Ideally (ARL_M /ARL_0) =1, otherwise the higher the ratio the greater the severity level. In this paper most of the time we present the ARL_M and ARL_0 values instead of their ratio, we choose to do this because they are more informative and the ratio can be easily calculated with those values in hand.

2) The relative ARL-bias, denoted in this paper as ARL-bias(%). It is the percentage difference between the location of the ARL_M and the ARL_0 in the x-axis. Ideally the ARL-bias(%) should be equal to zero, otherwise the further apart from zero the greater the severity level.

In a later section we present a closed-form equation by which this parameter can be estimated and also provide criteria that serves to establish the severity level according to the estimated ARL-bias(%) value.

Let's now focus on the curve for p=0.02 and n=600 of Figure 4. This curve has ARL-ratio= 3.92, ARL_0=354, ARL_M=1389 and ARL-bias(%)= -16%. This curve is clearly biased since for a wide range of p_1<p the ARL_1 is higher than the ARL_0. The p_1 at which the ARL_M occurs, or p_{1-M}, can be calculated using p_{1-M}= [p + p(ARL-bias(%)/100)], in the example the p_{1-M} = 0.0168. Also the p_1 at which the bias region starts, or p_{1-S}, can be calculated using p_{1-S}= [p + 2p(ARL-bias(%)/100)], in the example the p_{1-S} = 0.0136. It is easy to deduce that a p-Chart with these ARL characteristics would have a severely diminished process improvement detection capability.

Notice that if p_{1-M} is known then, by means of equations (5) and (6), the ARL_M could be calculated. Hence, the ARL_M can be estimated if the value of ARL-bias(%) is known.

Notice also that the ARL-bias severity for the curves of Figures 4 and 5, reduces or increases in direct relation to the sample size n, with the severity for the combination p= 0.1 and n=3200 being almost negligible. This indicates that for low p values it is possible to obtain p-Charts with quasi ARL-unbiased curves so long as n is very large.
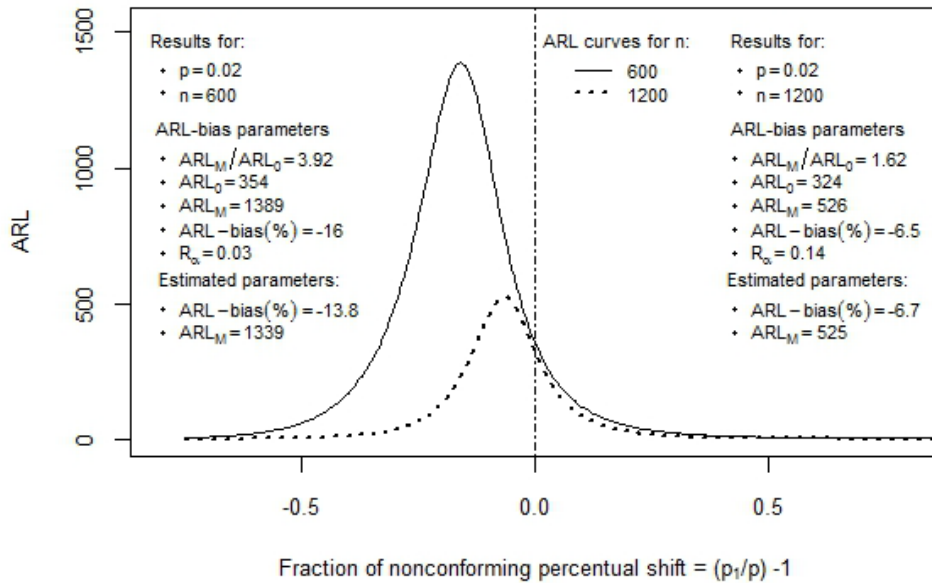
Figure 4. Examples of biased ARL curves for p= 0.02, n=600 and n=1200
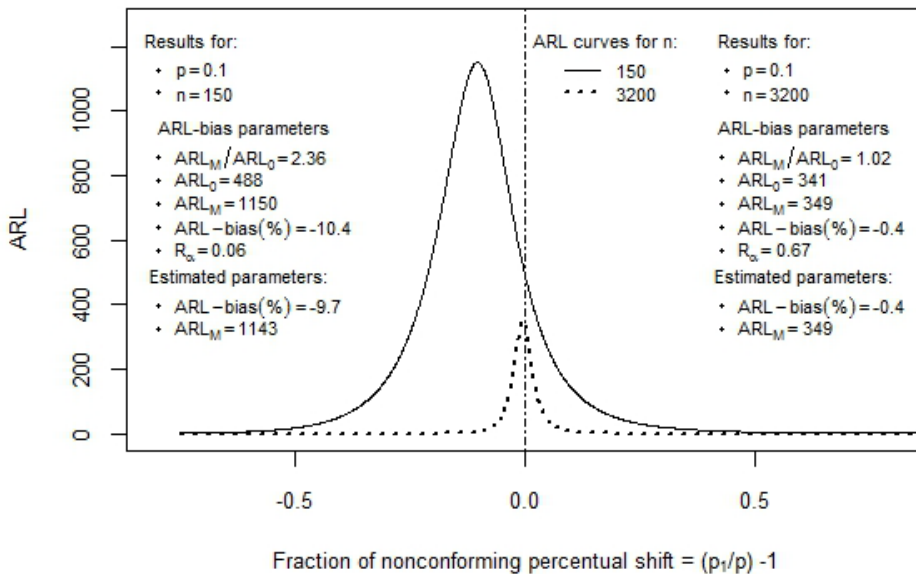

Figure 5. Example of a biased ARL curve and of a quasi-unbiased ARL curve

2.4. The IC alpha tails ratio - R_α

First of all lets defined the IC alpha tails ratio as $R_\alpha = \alpha_L/\alpha_U$. Figure 6 shows the R_α variation for the IC alpha tails for p=0.02 between n=442 and n=10000, it can be seen

that the R_α oscillates in function of *n* and that its value gets closer to zero at low sample sizes.

We plotted ARL curves for p=0.02 for different *n* values computing the ARL-bias severity and the R_α for each case, as an

example see the R_α values of Figure 4. The results we obtained showed that for p=0.02 the ARL-bias severity appeared to increase as the R_α decreased.

By means of plotting individual ARL curves we managed to study the relation between the R_α and the ARL-bias severity for a range of $0.01 \leq p \leq 0.20$; it was found that the severity tended to increase as the R_α decreased for all cases. From this preliminary study we concluded that a nonlinear relationship

appeared to exist between the R_α and the ARL-bias(%).

In order to ascertain the mathematical relationship that existed between the R_α and the ARL-bias(%), we carry out an in depth analysis that resulted on the development of a new approach for determining a p-Chart´s ARL-bias severity. The results and findings of this analysis are summarised in the following section.
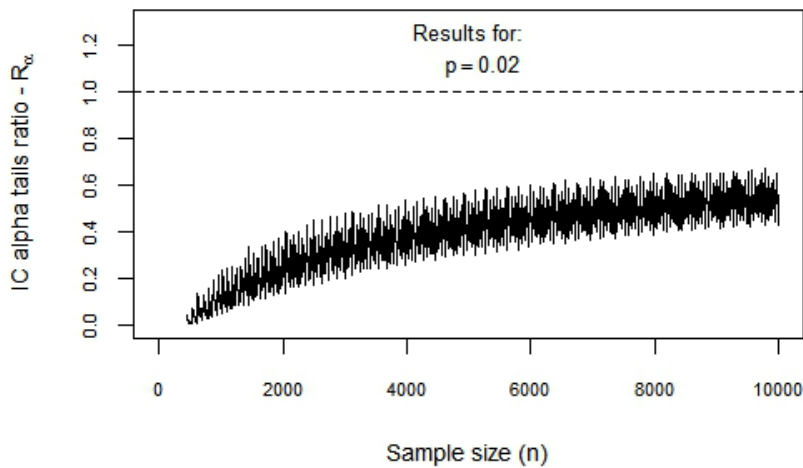
Figure 6. Example of the oscillation of the IC alpha tails ratio according to sample size

3. Results

3.1. A closed-form equation for estimating the relative ARL-bias

To explore the relationship between R_α and ARL-bias(%) we created an algorithm capable of computing R_α and ARL-bias(%) for any combination of *p* and *n*. We carried out extensive computer analysis for fractions of nonconforming between $0.001 \leq p \leq 0.2$ with samples sizes of up to 10000. We computed the R_α and its corresponding ARL-bias(%) for each sample size.

Figure 7 shows an example of the results, the width of the x-axis corresponds to a very small increment of R_α, from 0.2 to 0.23; it can be seen that the ARL-bias(%) oscillates in a random like manner.

The results also revealed that, although almost imperceptible for very small R_α increments, the ARL-bias(%) gradually increases as R_α increases. Figure 8 shows the full extent of the ARL-bias(%) variation for p=0.04 for R_α values between 0.0048 and 0.77. It can be seen that the overall variation is logarithmic and that the ARL-bias(%) rapidly approaches zero for $R_\alpha > 0.5$. The results for all other *p* were similar with only the curve slope varying for each case.
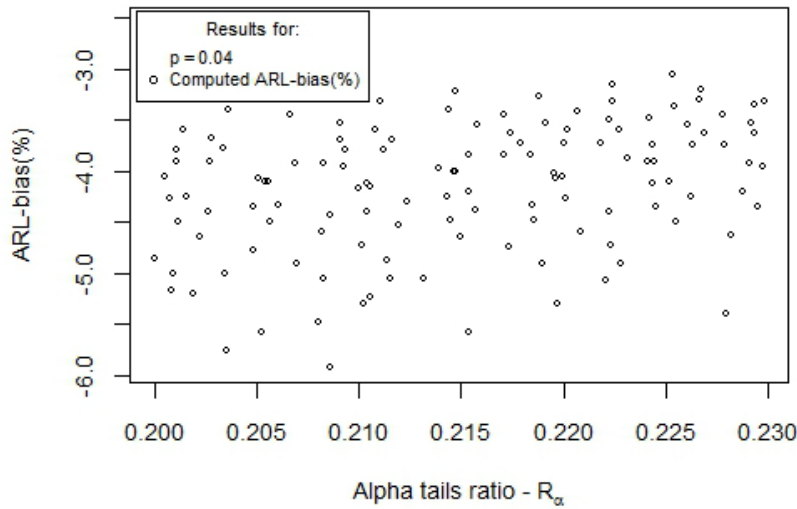
Figure 7. Example of the ARL-bias(%) oscillation for very small R_α increments
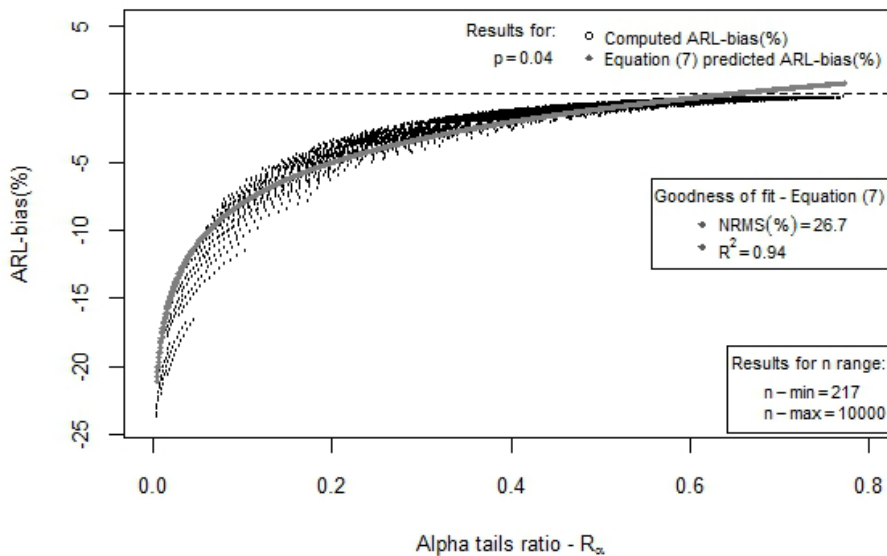
Figure 8: Example of the logarithmic relationship between ARL-bias(%) & R_α and of the predicted ARL-bias(%) values using formula 7

By applying multiple regression analysis to the results, we managed to developed equation (7). This closed-form equation gives reasonable estimates of the ARL-bias (%) for p between $0.006 \leq p \leq 0.2$.

$$ARL\text{-}bias(\%) \approx a + b \qquad (7)$$

Where:

$$a = 0.89 - 0.87 log_e(p)$$

$$b = log_e(R_\alpha)\,[2.63 - 0.52\,log_e(p)]$$

p = Process fraction of nonconforming

R_α= Alpha tails ratio (as defined in section 2.4)

An example of the goodness of fit of the results obtained with equation (7) is shown in

Figure 8. As can be seen, the formula gives reasonably accurate estimations of the ARL-bias(%). However, above certain R_α threshold values, that are specific for each *p,* equation (7) will compute ARL-bias(%) above zero; for example for p=0.04 this threshold is 0.64 whilst for p=0.08 it is 0.72. Given that our results showed that the ARL-bias(%) for R_α above those thresholds are always very close to zero, we recommend the use of the following guidelines when equation (7) computes values above zero:

1) Give a value of -0.5% if estimated relative ARL-bias is: $0 <$ ARL-bias(%)≤ 1.

2) Give a value of 0% if estimated relative ARL-bias is: ARL-bias(%) > 1.

Examples of ARL-bias(%) and ARL_M estimations obtained by means of formula (7) are presented in Figures 4 and 5 along with the true values for comparison.

3.1.1. ARL-bias severity level criteria

We also carried an study of the relationship between the ARL-bias severity level parameters ARL-ratio and ARL-bias(%). This analysis was done for $0.006 \leq p \leq 0.2$ with samples sizes of up to 10000, we computed the severity level parameters for each sample size.

Figure 9 shows an example of the results for p=0.04, as can be seen the ARL-ratio tends to increase as the ARL-bias(%) decreases. This behaviour is similar for all other *p* values.
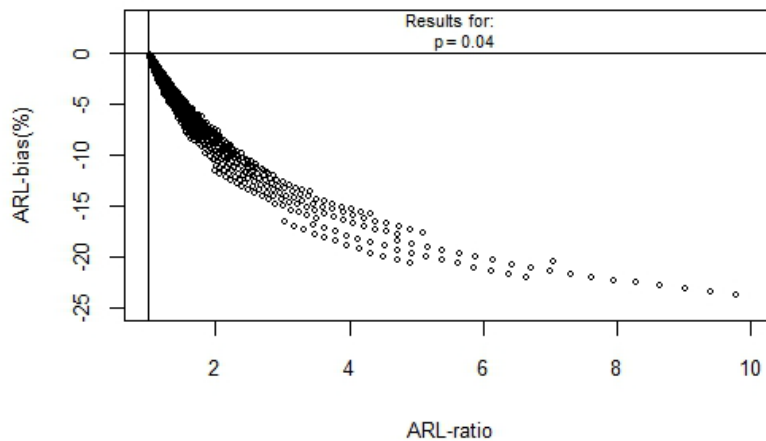

Figure 9. Example of the relationship between the parameters ARL-ratio and ARL-bias(%)

Based on the results of the ARL-bias(%) study summarised in this section, we conceived the severity level criteria shown in Table 1. This table complements equation (7)

as it permits to establish a p-Chart´s ARL-bias severity level by means of the estimated ARL-bias(%).

Table 1. ARL-bias severity level criteria in function of the relative ARL-bias

Severity level	ARL-bias(%)
Considerable	ARL-bias(%) < -10%
Significant	-10% ≤ ARL-bias(%) < -3%
Moderate	-3% ≤ ARL-bias(%) < -1%
Slight	-1% ≤ ARL-bias(%) < -0.5%
Negligible	-0.5% ≤ ARL-bias(%) ≤ 0%

Analysis of the results also led us to conclude that a p-Chart can be considered quasi ARL-unbiased if its severity level parameters meet the following criteria: a) ARL-bias (%) ≤ -1% and b) ARL-ratio ≤ 1.1. In general these criteria are met by *p* and *n* combinations that produce slight or negligible severity levels.

3.1.2. A procedure for estimating the ARL-bias severity level parameters

The following procedure can be followed to estimate a p-Chart´s ARL-bias severity level using equation (7).

1) Calculate the control limits

2) Calculate the IC alpha tails α_L and α_U (these can be easily computed using the cumulative binomial distribution available in commonly used software like Excel)

3) Use the IC alpha tails to determine the R_α and ARL_0

4) Use formula (7) to estimate the ARL-bias(%)

5) Use Table 1 to establish the severity level

6) Calculate p_{1-M} and use equation (5) to estimate the ARL_M (The β value can be easily computed using the cumulative binomial distribution available in commonly used software like Excel)

For example, a chart with p=0.03 and n=400 will have an ARL-bias (%) ≈ -13.5, an ARL_0=373, an ARL_M ≈1395 and ARL-ratio ≈ 3.74. The ARL-biased severity level of this chart is considerable and it should not be used if process improvement detection is required.

Various authors have proposed diverse methods to obtain attribute Shewhart Chart´s with reduced ARL-biased severity, among others: Ryan et al. (1997), Acosta (1999), Ryan (2011), Oliveira et al (2012), Morais (2016), and Paulino et al (2016).

3.2. ARL-bias zones for p-Charts

In the introduction section we mentioned that when p≪0.5 the common text-book rule of thumb of np>5 or 10 does not seem to provide satisfactory approximations of the binomial to the normal distribution and that in consequence for low fractions on nonconforming, say 0.01≤p≤0.14, a p-Chart could often be ARL-biased. With this in mind and in order to establish the conditions required to obtain quasi ARL-unbiased p-Charts, we decided to investigate the effect of *p* and *n* on the ARL-bias(%).

We carried out an study for *p* between 0.01≤ p ≤0.14 and *n* up to 10000, computing the ARL-bias(%) of each *n* and *p* combination. Figure 10 shows an example of the results for p=0.04. We classified the results according to the following categories: a) sample sizes with an ARL-bias(%) < -1% ; b) sample sizes with an ARL-bias(%) = -1% and c) sample sizes with an ARL-bias(%) > -1%.

With the obtained results we constructed several "*ARL-biased zones*" that are in function of *n* and *p*. These zones are very useful as they can be used to quickly deduce a p-Chart´s ARL-bias severity level by means of its *n* and *p* values. These zones are: the no LCL zone, the ARL-biased zone, the ARL-biased transition zone and the quasi ARL-unbiased zones.
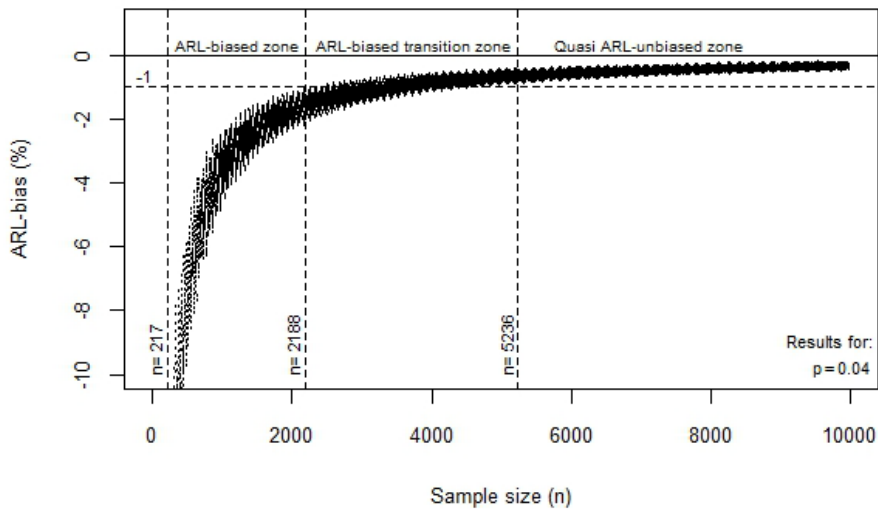
Figure 10. Example of the ARL-bias(%) variation in function of sample size and of the ARL-bias zones for p=0.04

3.2.1. The no LCL zone

Figure 11 shows the "No LCL zone", this zone contains combinations of *p* and *n* that produce p-Charts without lower control limits. When a p-chart has no LCL it has null process improvement detection capability.

The upper limit of this zone was calculated using $n = 1 + 9(1 - p)/p$. ARL-biased charts will start to appear above this limit.
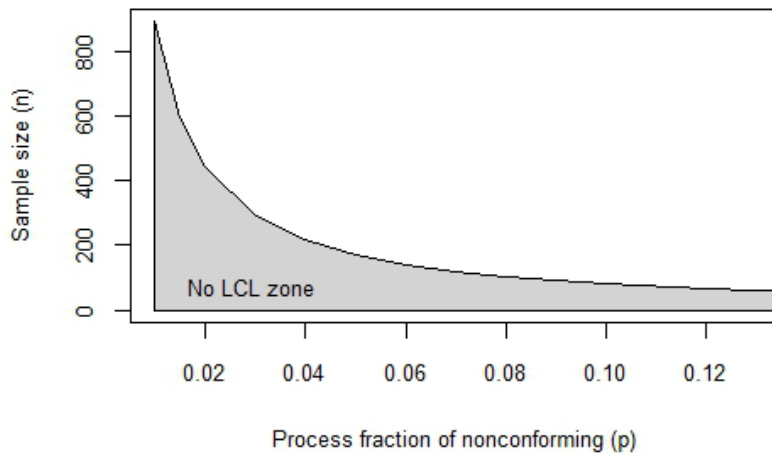
Figure 11. The no LCL zone for Shewhart p-Charts

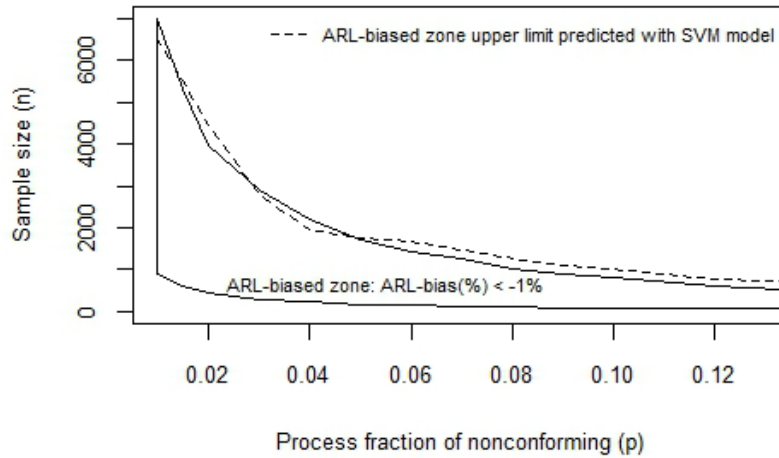### 3.2.2. ARL-biased zone

Figure 12 show the "ARL-biased zone", this zone contains *n* and *p* combinations that produce ARL-bias(%) values below -1%. In this zone the closer the sample size gets to the upper limit the closer the ARL-bias(%) will get to -1%, resulting in mainly moderate ARL-bias severity levels. In the other hand, the closer the sample size gets to the lower

limit the greater the severity, obtaining levels that could vary between significant and considerable.

Examples of the bias severity in this zone are presented in Table 2 for p=0.04 and increasing sample size n.

Figure 12. The ARL-biased zone

Table 2. Examples of bias severity levels according to sample size in the ARL-biased zone

p	n	R_α	ARL_0	ARL_M	ARL-bias(%)	Severity
0.04	220	0.04	276	958	-17.8	Considerable
0.04	292	0.02	262	1068	-16.8	Considerable
0.04	400	0.1	268	522	-9.5	Significant
0.04	1100	0.27	352	447	-3.3	Significant
0.04	2000	0.38	388	443	-1.7	Moderate

Since the upper limit of the ARL-biased zone marks the threshold where the n and p combinations start to produce charts with an ARL-bias(%) $\geq$ -1%, then this limit could be thought of as the threshold where p and n starts to provide a good approximation of the binomial to the normal distribution. In other words, above this threshold the binomial distribution skewness becomes so small to the point of being almost symmetric and hence the central limit theorem normal approximation is greatly improved.

As can be seen in the np values required to obtain a good approximation are considerably above the np > 5 rule of thumb mentioned in the introduction section. Notice also that the np value is case specific and that it varies logarithmically.

Table 3 has selected values that would suffice to draw the upper boundary of the ARL-biased zone. Figure 12 also shows the upper limit values predicted with a Support Vector Machine (SVM) regression model built using the values given in Table 3.

Table 3. Selected n values for the upper limit of the ARL-bias zone

p	0.01	0.02	0.04	0.06	0.08	0.1	0.12	0.14
n	7003	3976	2188	1434	1026	806	609	468

3.2.3. The ARL-bias transition zone
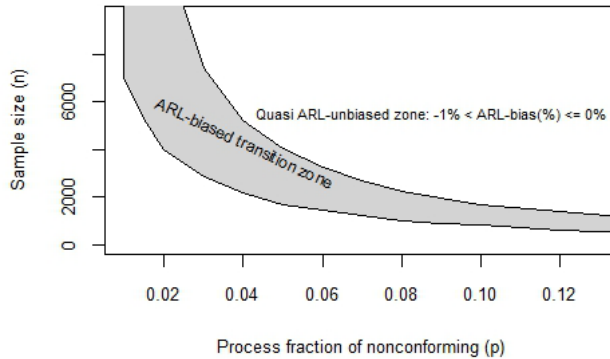
Figure 13 shows the "ARL-biased transition zone". This zone contains n and p combinations that produce ARL-bias(%) values in the vicinity of -1%, giving rise to

ARL-bias severity levels that could vary between moderate, slight and negligible. Moderate levels happen mostly for sample sizes close to the lower boundary, whilst negligible levels start to appear for sample

sizes close to the upper boundary.

Examples of the bias severity in this zone can be seen in Table 4 for p=0.04 and increasing sample size.

Figure 13. The ARL-biased transition and the quasi ARL-unbiased zones

Table 4. Examples of severity levels according to sample size in the ARL-biased transition zone

p=0.04					
n	R_α	ARL_0	ARL_M	bias(%)	Severity
2190	0.54	372	394	-1.2	Moderate
2686	0.33	350	411	-1.7	Moderate
3150	0.49	315	339	-1	Slight
5150	0.55	394	414	-0.8	Slight
5236	0.46	355	384	-1	Slight

3.2.4. The quasi ARL-unbiased zone

The quasi ARL-unbiased zone, shown in Figure 13, contains n and p combinations that produce relative ARL-bias values between -1% and 0% and hence the severity levels in this zone vary from slight to negligible. Slight levels occurs mostly for sample sizes closer to the boundary limit whilst negligible levels occur mostly for sample sizes that are far

away from that limit.

Since the severity levels in this zone could only be slight or negligible, then the p and n combinations that fall in this zone will always produce quasi ARL-unbiased charts. Example of bias severity levels in this zone can be seen in Table 5 for p=0.04 and increasing sample size n.

Table 5. Examples of severity levels according to sample size in the ARL-biased transition zone

p=0.04					
n	R_α	ARL_0	ARL_M	bias(%)	Severity
5237	0.56	407	427	-0.8	Slight
5250	0.57	359	375	-0.8	Slight
6000	0.59	364	379	-0.5	Negligible
8000	0.63	362	373	-0.5	Negligible
10000	0.67	351	359	-0.2	Negligible

Since in this zone the ARL-bias severity varies between slight to negligible, the lower boundary of this zone, or equivalently the upper boundary of the ARL-biased transition zone, could be thought of as the threshold where *p* and *n* start to provide a very good approximation of the binomial to the normal distribution.
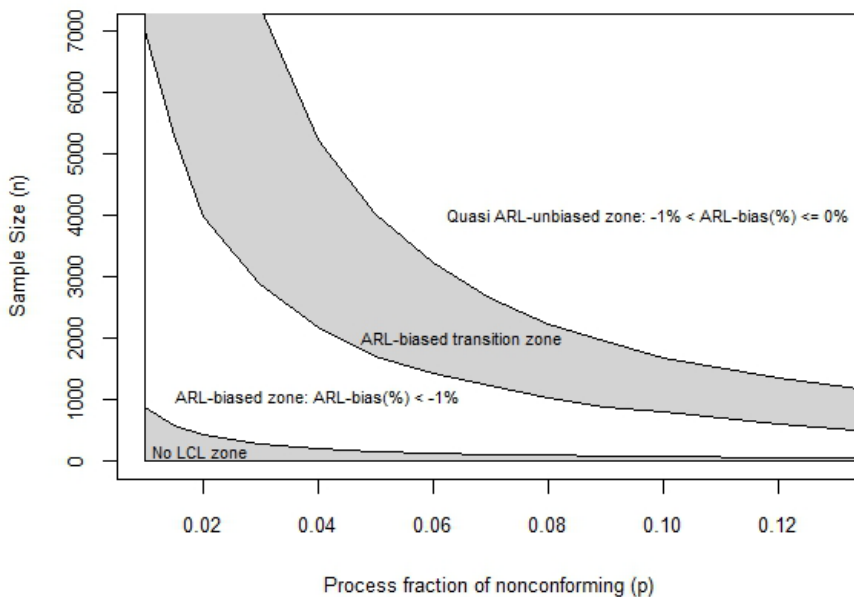
Table 6 has selected values that would suffice to draw the lower boundary of the quasi ARL-biased zone.

Table 6. Selected n values for the upper limit of the ARL-bias zone

p	0.01	0.02	0.04	0.06	0.08	0.1	0.12	0.14
n	38447	12547	5236	3224	2243	1689	1103	956

Figure 14 shows all the ARL-biased zones in one graph.

Figure 14. ARL-bias zones for Shewhart p-Charts

4. Conclusions

The objective of the research summarised in this paper, was to develop methods that would facilitate estimating a p-Chart´s ARL-bias severity. To reach this objective a whole new approach was devised for *p* values commonly found in real-world production processes, this approach is composed of:

1) A closed-formed equation through which the relative bias, ARL-bias(%), can be easily estimated, and
2) A graphical method whereby a chart´s ARL-bias severity level can

be inferred depending on the severity zone where its *n* and *p* values fall into.

To complement the equation for the relative bias, a severity level criteria table was constructed. Five severity levels, varying from considerable to negligible, were established according to specific ARL-bias(%) ranges.

The ARL-bias severity zone method offers a quick and simple way for deducing a p-Chart´s severity level. Four zones were defined, namely: the no LCL zone, the ARL-biased zone, the ARL-biased transition zone

and the quasi ARL-unbiased zone. The advantage of using the bias zones is that a rough estimation of the bias severity level of several n and p combinations can be obtained without having to do any calculations. Naturally, if a more precise estimation is required then the formula method should be used.

As a general rule we recommend that only p-Charts with severity levels slight or negligible, in other words only quasi ARL-unbiased p-Charts, be deemed as optimum for process monitoring. In any case, a p-Chart´s suitability to monitor a process must be ascertain analysing its ARL-bias severity parameters in conjunction with the specific process monitoring requirements, for example it could happen that process improvement detection may not be required, in such case charts with moderate or other severity levels could be considered as acceptable.

A finding that we would like to highlight is that the upper boundaries of the ARL-biased zone and of the ARL-bias transition zone provide empirical threshold limits, above which, p and n start to provide a good and a very good approximation of the binomial to the normal distribution respectively and that these thresholds are considerably above the $np \geq 5$ (or 10) rule of thumb commonly given in text books.

The approach for estimating the ARL-bias severity presented herein makes the task of determining a p-Chart´s ARL-bias severity a lot simpler. We are confident this approach will be useful for quality control practitioners, students or anyone interested on applying a p-Chart for process monitoring.

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