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Space time geometry in the atomic hydrogenoid system. Approach to a dust relativistic model from causal quantum mechanics

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We intend to use the description of the electron orbital trajectory in the de Broglie-Bohm (dBB) theory to assimilate to a geodesic corresponding to the General Relativity (GR) and get from it physical conclusions. The dBB approach indicates us the existence of a non-local quantum field (corresponding with the quantum potential), an electromagnetic field and a comparatively very weak gravitatory field, together with a translation kinetic energy of electron. If we admit that those fields and kinetic energy can deform the space time, according to Einstein's field equations (and to avoid the violation of the equivalence principle as well), we can made the hypothesis that the geodesics of this space-time deformation coincide with the orbits belonging to the dBB approach (hypothesis that is coherent with the stability of matter). From it, we deduce a general equation that relates the components of the metric tensor. Then we find an appropriate metric for it, by modification of an exact solution of Einstein's field equations, which corresponds to dust in cylindrical symmetry. The found model proofs to be in agreement with the basic physical features of the hydrogen quantum system, particularly with the independence of the electron kinetic momentum in relation with the orbit radius. Moreover, the model can be done Minkowski-like for a macroscopic short distance with a convenient election of a constant. According to this approach, the guiding function of the wave on the particle could be identified with the deformations of the space-time and the stability of matter would be easily justified by the null acceleration corresponding to a geodesic orbit.

Keywords: De Broglie Bohm; curvature of space time; metric tensor; general relativity; hydrogen-like atoms; electron trajectory; quantum potential; wave function; numerical methods; geodesics; Lorenz geometry.

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1. Introduction

The idea that the curvature of space time can explain the movement of the atomic particles is not new. Indeed, we could trace it to the early times of the quantum mechanics.

The starting point of our subject is the work of de Broglie that extended the duality wave particle established by Einstein for the photons, to all particles, and strived to assimilate the tracks of the particle from mechanics and the rays of waves from optics.

The method used by de Broglie was to identify the trajectory of a light ray described by the Fermat principle with a particle trajectory described by the Maupertuis principle, to connect the undulatoy and corpuscular theories. In his famous thesis [1], de Broglie considered the undulatory and the corpuscular theories that each represented a part of the phenomena and he looked for a kind of synthesis, equating the forecasts of both theories. It suggest us a methodological principle, which could be called the "coincidence principle": when a physical phenomenon can be described by two theories with opposite principles (*i.e.* corpuscular and undulatory approaches), by equating their results in a common physical feature (*i.e.* the trajectories) in a syntesis effort one can get significant physical advances. We can obtain a kind of "fusion of opposites". So could happen with the dBB and GR approach, regarding electron movement in systems like the hydrogen atom.

In the deduction of its well-known equation, Schrödinger considered a non-Euclidean metrics, based in the kinetic energy of the particle. As it is well known, Schröinger deducted its evolution equation of quantum systems from the classical equation of Hamilton Jacobi and from the already mentioned work on wave - corpuscle duality of de Broglie. The idea that the dynamics of quantum systems could be determined by space-time deformations could be already originated in the Schrödinger thoughts [2]. Indeed, in his deduction he postulated the expression (in the space coordinates q_k of the particles, being T the kinetic energy in function of the velocities):

$$ds^2 = 2T(q_k, \dot{q}_k)dt^2 \tag{1}$$

that brings us to a quadratic form, equivalent to a three dimensional metrics, that helped him to follow from the Hamilton Jacobi equation towards its well-known equation. But this metrics did not serve him to characterize the space-time performance of the particle entourage, because Schröinger considered the particle a mere "image point" without physical meaning; the physical reality was assigned by him to the wavelike being (wave parcel). The dual theory of de Broglie became in Schöinger an only-undulatory theory. Anyway, the relationship of Eq. (1) with our approach will be treated by us in a separate paper.

Later on and in the frame of orthodox quantum mechanics, Wheeler developed a particular conception of geometric-dynamics (term who Einstein used for the first time), where the geometry of space-time indeed plays a role in the performance of the quantum system. He assigned a geometrical picture to the quantum processes, based on the standard version of quantum mechanics. He treated to reduce all physical entities to geometrical beings. But, as Fiscaletti says, [3] it is impossible a geometric-dynamical description of quantum process from the orthodox quantum interpretation, because this interpretation does not permit to consider any event description in space-time. Wheeler remained in the field of orthodox quantum mechanics where no place exists for the particles trajectory. So his model is far from our approach that, according to the de Broglie original ideas, considers simultaneously real both the particle and the matter-wave, unified in a non-classical way.

In some opposite way runs the approach of Dürr, Goldstein, Tumulka and Zanghi [4] from the 1990 years regarding the quantum model of de Broglie-Bohm. They made the first consistent geometric-dynamical conception, as long as we know, giving the limelight to the trajectories. The wave function the effective wave function for systems, isolated, in relation with an entourage or even with the complete universe-has there a nomological character, indicating the evolutive rules. Their approach misleads the role of the quantum potential. We can mention the interesting attempts of these authors to develop the geometrodinamical approach by invoking a "time foliation". In this approach, a leaf of this foliation would be a tridimensional hypersurface where all the points have a status of simultaneity. It has been studied in connection with the Bohm model [5,6] and also for curved spaces [7]. This interesting concept emerges from the empirical fact of entanglement, but is difficult to conciliate with relativity.

It must be mentioned that M. Atiq, M. Karamian and M. Golshani [8] developed a quasi-Newtonian approach. They deduced the quantum potential independently of the Schrödinger equation, so without need of the wave function. By using the quantum potential in the frame of classical mechanics they argued to describe the non-classical effects. But they need the S and R functions that configure the wave function in the dBB approach.

Some years earlier and more related with our proposal we find approaches in the frame of a geometric-dynamical dual theory and in a curved space-time: A. Shojai and F. Shojai [9] developed an approach where the motion of a spinless particle is equivalent to a movement in a curved space time, of conformal character over a flat metric, the conformal factor being the Bohm quantum potential. They formulated a conformal metrics in a Weyl geometry that is a generalization of the Riemann geometry. They melted there the gravitation and the quantum potential coming to the conclusion that the quantum phenomena can derivate from space-time deformations. In this interesting approach the authors start from the Bohm version of the Klein Gordon equation; their conclusion is in this case dual. But their approach is quite different to ours, and moreover no concrete metrics results for microphysical systems are calculated by them, in our knowledge.

More recently, Novello, Salim and Falciano [10] suggested that the quantum phenomena can be interpreted as the manifestation of a non-Euclidean geometry in the three-dimensional space, in the picture of Weyl geometry, in an interesting paper. They came to the conclusion to identify the quantum potential with the scalar curvature of this space. In our approach we do not leave the pseudo Riemann or Lorenz geometry that is characteristic of relativity.

Licata and Fiscaletti propose another geometric-dynamical approach related to a very different consideration, [11]: the entropic effect derived from the microstates that characterize the quantum system. The trajectories emerge in this approach by a thermal bath, characteristic of the Quantum Theory of Fields.

A common feature of most of those lasts papers [9,10,11] is to postulate a microphysical action and derive from it, by variation of it, the equations of the movement and the related physical features including the metrics of the space, in a Weyl geometry.

Other approaches related with the quantum entropy have been worked out by Sbitnev [12,13]. In these approaches, the quantum entropy, related to the number of possible microstates, plays a fundamental role. The vacuum is assimilated to a fluid, which can be treated by the Navier-Stokes equation.

The already mentioned two authors, Licata and Fiscaletti, summarized different concepts about the quantum potential in an interesting book [14].

1.1. Goal and structure of this work

We focus in this paper the relationship between the de Broglie-Bohm (dBB) approach, also known as Causal Quantum Mechanics, and the General Relativity (GR), in order to go deeper in the physical concepts of stationary and entangled states like the electron in hydrogen-like atom.

Our purpose is to investigate the metric structure of the space-time in the case of stationary quantum systems, where the geodesic hypothesis seems reasonable, as well as to try to explain the guidance of the particle by the matter-wave by means of a deformation of the space-time produced by the microscopic atomic system itself. It is a first approach, which excludes the spin and gravitation.

The main purpose of this paper is to find a metric of the space-time according to the General Relativity where the expected trajectory in the Bohm model for hydrogen-like atoms would be a geodesic. This metric must be also in accordance with the main features of the hydrogen atom. It is also an objective to go deeper into the role of the wave function on the spacetime deformation and on the particle guidance. We remain in the Lorenz pseudo Riemann geometry.

Of course we do not pretend any theoretical synthesis between the two theories, and, furthermore, our approach is only an approximation to detect profitable features for a deeper theoretical progress. We must state that we do not make any attempt to "unify" both dBB and GR theories. Indeed, dBB and GR are very dissimilar theories. dBB is not Lorentz covariant and furthermore it works with an non local field, the so called quantum field and the associated quantum potential and quantum energy. GR is a local theory. In addition, dBB is a theory that fusions the undulatory and the corpuscular approaches, while the GR is a corpuscular theory.

We assume that the space-time can be curved at the microscopic scale, under the effect of the mass and energy elements that forms the atomic system. We start from the trajectory that follows electrons in a hydrogen atom, according to the dBB model. We assimilate these trajectories with geodesics of space-time, in the frame to the GR. This is coherent with stability of matter, because no acceleration implies no energy emission. Indeed, while in dBB the electron trajectory is well determined if we suppose an initial position and velocity, in the GR we must replace, in the trajectory equation, the Lorenz force for another one, coherent with the quantum phenomena. In this paper we choice to make null this force, making the covariant acceleration of the electron equal to zero and therefore establishing the energy constancy of the electron movement, briefly, the well-known Bohr postulate.

On this subjet, it is interessant to take into account the considerations of F. Goded Echevarria [15]. According to him, most force fields can influence the geometry of space time, not only the gravitatory field. The presence of two or more of such fields can interact with the geometry and we can expect some fuzzy impress on it. It happens so with the gravitatory and electromagnetic fields. We can therefore expect something similar with the electromagnetic and quantum fields. The same author also get us inspired in another aspect: he compares (a) the geodesics of the sun system according the GR and (b) the classical equation of Binet for central fields, applicable to the same system [15]. It results that both have the same structure. Then he can derive relevant physical conclusions, *i.e.* regarding the Mercury perihelion. Our treatment of the electron orbit in both GR and dBB can be considered a similar methodological approach.

In this sense, we must remark that the gravitational mass is not the only agent that can deform the space time: any energy can do it, according to the Einstein's field equations. Moreover, any potential energy must contribute to the inertial mass, according to $E = mc^2$. Consequently, its contribution must also be done to the gravitational mass or the equivalence principle would be violated.

The structure of the paper is the following:

- In Sec. 2.1 we present the general features of the dBB model.
- In Sec. 2.2 we describe the electron trajectory equation in a hydrogen-like atom in the dBB approach.
- In Sec. 2.3 we progress from the GR general geodesics to an equation inter the metric tensor components.
- In Sec. 2.4 we adopt a dust metrics from an exact solution of the Einstein's field equations, with cylindrical symmetry and gather the corresponding metrics for our case. We test the solution's coherence and we conclude that the metrics must be reformulated.
- In Sec. 2.5 we define the correction of the metrics and solve the related equations to get the metrics finally adopted. We justify an approximation made by comparing numerical and analytical solutions. We find a metrics of space-time that is successfully tested to assure the coherence with the basic physical features, like the independence of kinetic moment regarding the orbit radius and that the space-time must be considered plane for a large value of nucleus-electron distance.
- Finally in Sec. 3 we explain some conclusions about our results.

2. Development

As previously said, we make the hypothesis that on stationary quantum systems the curvature of the space-time must be of a shape that the movement of particles tracks geodesics, and that this fact explains the stability of systems like atoms and molecules. We refer to the atom of hydrogen in a certain orbital. The fact that the velocity of the particles in such systems is low enough compared with the velocity of the light allows us to use the non-relativistic dBB approach and the GR in the limit for low velocities. We use the Riemannian geometry, where the general relativity is formulated.

Let us introduce the basic ideas of the dBB approach:

2.1. Basic postulates of de Broglie-Bohm interpretation

As mentioned, de Broglie-Bohm interpretation of Quantum Mechanics, [16,17], also called Causal Quantum Mechanics, proceeds logically from the classical analytical mechanics. De Broglie assimilated the corpuscular trajectory and the undulatory ray [1] and from there Schröinger derived his equation. Bohm analysed the implications of this equation for a generic wave/mechanical action form (Eq. (2)). The dBB is a dualistic theory that synthetizes the corpuscular and wave opposite approaches, with a very deep physical meaning. A minimalist conception of this can be summarized as follows [18]:

1) A microscopic physical system comprises a wave propagating in the space-time together with a point particle, which moves under the guidance of the wave.

2) The wave is mathematically described by $\Psi(\mathbf{x}, t) \in C$, a solution of the Schrödinger equation for the specific system, expressible as:

$$\Psi(\mathbf{x}, t) = Re^{iS/\hbar} \tag{2}$$

R and S being real functions of x and t.

3) The momentum of the particle is given by the gradient of S:

$$\vec{p} = \nabla S(\vec{x}, t) \tag{3}$$

4) From those 3 axioms and a statistical consideration, one can establish that the probability density is given by the function R expressed in (2).The probability that a particle could be found between $\mathbf{x}$ i $\mathbf{x} + d\mathbf{x}$ is given by

$$P(\vec{\mathbf{x}}, \vec{\mathbf{x}} + d\vec{\mathbf{x}}) = \mid \Psi \mid^2 d\vec{\mathbf{x}} = R^2(\vec{\mathbf{x}}, t)d\vec{\mathbf{x}} \tag{4}$$

Based on these assumptions one is able to calculate the trajectories of the particles, assuming initial conditions of position and velocity, which act as parameters of the bundles of trajectories. An example of such calculations is the trajectories of an electron beam at the double slit experiment [19].

Replacing (2) in the Schröinger equation we get what is called the quantum Hamilton Jacoby equation:

$$\frac{\partial S}{\partial t} + \frac{(\nabla^2 S)}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V(\vec{\mathbf{x}}, t) = 0 \tag{5}$$

while in the second and forth terms we recognise the kinetic and potential energies respectively, we note the third term

$$V_q \equiv -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} \tag{6}$$

which is called the quantum potential, for it appears as addition to the classical Hamilton-Jacobi equation. Its non-local character is shown in the quotient $\nabla^2 R/R$ that depends on the shape of R, not only on its value. This quantum potential plays an important role in the dBB dynamics.

2.2. Equations of the trajectories of the electron in a hydrogen atom

The peculiarity of our approach lies in trying to link the perspective of (a) the de Broglie Bohmian mechanics, derived from considerations about the quantification of the classical Hamilton-Jacobi equation (Schrödinger equation for stationary systems) and (b) Einstein field equations exactly solved, and that for almost classic systems, leading to specific models of space-time metrics.

For our purposes, we consider a hydrogen atom as a physical system particularly simple and well-studied. We are thus in the frame of an elementary theory of electron motion without considering the spin. The orbits described by the electron correspond to stationary states; therefore we hypothesize that the electron in the atom describes a geodesic of the space-time with constant total energy and null (covariant) acceleration.

Since the trajectories of an electron in a hydrogen atom are calculable by non-relativistic dBB approach -considering that the electron speed in the hydrogen atom is, as generally admitted, small compared with the light speed- and that the state of the potential at the atomic system is known, we can think in deriving a metric tensor that describes the geometry of space-time in the electron environment.

Let us consider then a hydrogen atom. Assume a spherical coordinate system centred in the nucleus. Let us suppose the function of the whole system factorable between the function of the nucleus and the electron; moreover, approaching the equivalent mass of the electron to its proper mass, regardless of corrections. The wave equation for the electron in a steady state must be solution of the time-independent Schrödinger equation and its solutions must be Eigen functions of the squared kinetic moment operator, together, with the projection of the kinetic moment on the OZ axis. We get the following expressions for the squared kinetic momentum operator and the OZ component of that [20]:

$$L_z = -i\hbar\partial_{\{\varphi\}} \tag{7}$$

$$L^2 = -i\hbar^2\left[\frac{1}{\sin\theta}\partial_{\{\theta\}}\left(\sin\theta\partial_{\{\theta\}} + \frac{1}{\sin^2}\partial^2_{\{\varphi^2\}}\right)\right] \tag{8}$$

Their eigenfunctions are spherical harmonics, which admit a representation of separable variables such as

$$\Psi_{Elu}(r, \theta, \phi) = g_{Elu}(r)f_{lu}(\theta)e^{i(u\varphi-(Et/\hbar))} \tag{9}$$

where f and g are real functions; E is the energy of the stationary state and l and u the orbital and magnetic -azimuth- quantum numbers l $\in \{0, 1, 2 \ldots n-1\}$; u $\in \{-l, -l+1 \ldots l-1, l\}$) and where the exponent of e is $iS/\hbar$:

$$e^{i(u\varphi-(Et/\hbar))} = e^{(i/\hbar)S(r,\theta,\varphi,t)} \tag{10}$$

so the phase S reads:

$$S(r, \theta, \varphi t) = h\hbar\varphi - Et \tag{11}$$

According to dBB theory, we can compute such velocities, always in spherical coordinates (m electron mass):

$$\begin{aligned} v_r = \dot{r} &= \frac{1}{m}\partial_r S \\ &= \frac{1}{m}\partial_r(u\hbar\varphi - Et) = 0 \quad \text{so we see:} \quad \dot{r} = 0 \end{aligned} \tag{12}$$

$$v_\varphi = r\sin\theta\dot{\varphi} = \frac{1}{mr\sin\theta}\partial_\varphi S = \frac{1}{mr\sin\theta}(u\hbar\varphi - Et)$$
$$= \frac{u\hbar}{mr\sin\theta}, \quad \text{so:} \quad \dot{\varphi} = \frac{u\hbar}{mr^2\sin^2\theta} \tag{13}$$
$$v_\theta r\dot{\theta} = \frac{1}{mr}\partial_\theta S = \frac{1}{mr}\partial_\theta(n\hbar\varphi - Et)$$
$$= 0, \quad \text{so:} \quad \dot{\theta} = 0 \tag{14}$$

Integrating with respect to time we have the geodesics equation:

$$r = r_0\varphi = \varphi_0 + \frac{u\hbar t}{mr^2\sin^2\theta}\theta = \theta_0 \tag{15}$$

corresponding, for $u > 0$, to circumferences at constant distance from the origin, initial phase and azimuthal angle centred on the axis azimuth. It is to note the cylindrical symmetry of the orbits, with u as multiplicative factor on the linear and angular velocity. Therefore, the symmetry of the problem leads to use cylindrical coordinates to simplify them to the maximum. In cylindrical coordinates centred in the nucleus (p, φ, z) the hourly equations of the trajectory of an electron in the dBB approach will therefore read:

$$\rho = \rho_0; \quad \varphi = \varphi_0 + \frac{u\hbar t}{m\varphi^2}; \quad z = z_0 \tag{16}$$

2.3. Geodesics equation

Now consider that the space-time at a microscopic level can be described by a Riemannian model. Then their geodesics will be given by

$$\frac{d^2x^j}{ds^2} + \Gamma^j_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 0 \tag{17}$$

Equations (17) have the physical meaning to impose on the particle zero acceleration, calculated as covariant derivative of the equations of motion.

We will consider as the parameter s the proper time of the electron that, taking into account that its velocity is of the order of 0.02c, (see Eq. 26) we assimilate to the time of the observer. Therefore $X^4 = -ct$.

According to Eq. (15) the equations of the orbit of the electron from Bohmian quantum mechanics are

$$X^1 = \rho = \rho_0 \tag{18}$$
$$X^2 = \varphi = \frac{u\hbar t}{m\rho^2}(\text{with} \quad \varphi_0 = 0) \tag{19}$$
$$X^3 = Z = Z_0 \tag{20}$$
$$X^4 = -ct \tag{21}$$

But in fact we will use from now as coordinate $X^4 = t$ instead of ct, for easier comparisons. We will change to $X^4 = ct$ at the end of this paper.

Concerning the four-velocity, we note that the only nonzero derivatives with respect to time are regarding the X^2 and X^4 and that all second derivatives are zero.

$$V^1 = 0 \tag{22}$$
$$V^2 = \omega = \frac{u\hbar}{m\rho^2} \tag{23}$$
$$V^3 = 0; \tag{24}$$
$$V^4 = -c \tag{25}$$

We remark that the kinetic momentum, as derived from Eq. (23) is quantized and independent of the radius

$$L_z = mv\rho = u\hbar \tag{26}$$

Thus, replacing the velocities on Eqs. (17) of geodesics we obtain

$$\omega^2\Gamma^j_{22} + 2\omega c\Gamma^j_{24} + c^2\Gamma^j_{44} = 0 \tag{27}$$

where $j = 1, 2, 3, 4$. Let us now replace the affine connectors according to the related components of the metric tensor as

$$\Gamma^j_{ik} = \frac{1}{2}g^{jk}(\partial_k g_{ih} + \partial_i g_{hk} - \partial_h g_{ki}) \tag{28}$$

We now calculate the affine connectors that are needed for the four equations of the paths. Taking into account that the elements of the metric tensor do not depend on t for representing a stationary situation neither of φ neither of z due to the cylindrical symmetry of the trajectory one can write

$$\Gamma^j_{22} = \frac{1}{2}g^{jh}(\partial_2 g_{h2} + \partial_2 g_{h2} - \partial_h g_{22})$$
$$= -\frac{1}{2}g^{jh}\partial_h g_{22} = -\frac{1}{2}g^{j1}\partial_1 g_{22} \tag{29}$$

and in the same way one obtains

$$\Gamma^j_{24} = \frac{1}{2}g^{jh}(\partial_2 g_{4h} + \partial_4 g_{h2} - \partial_h g_{24})$$
$$= -\frac{1}{2}g^{jh}\partial_h g_{24} = -\frac{1}{2}g^{j1}\partial_1 g_{24} \tag{30}$$

and

$$\Gamma^j_{44} = \frac{1}{2}g^{jh}(\partial_4 g_{4h} + \partial_4 g_{h4} - \partial_h g_{44})$$
$$= -\frac{1}{2}g^{jh}\partial_h g_{24} = -\frac{1}{2}g^{j1}\partial_1 g_{44} \tag{31}$$

We substitute now the affine connectors calculated in the 4 equations of the geodesics (27):

$$\omega^2 g^{j1}\partial_1 g_{22} + 2\omega c g^{j1}\partial_1 g_{24} + c^2 g^{j1}\partial_1 g_{44} = 0 \tag{32}$$

that is a contract form of 4 equations with $j = 1, 2, 3, 4$. To develop these equations one needs to compute the contravariant metric tensor g^{ij}. We know that, if α_{ij} is the attached of

g_{ij} to the determinant of its matrix, that has the value $g > 0$ for the metric condition (Gram array), one has

$$g^{ij} = \frac{\alpha_{ij}}{g} \tag{33}$$

We can assure that at least g_{11} is not null and therefore we can simplify (32) as:

$$\omega^2 \partial_1 g_{22} + 2\omega c \partial_1 g_{24} + c^2 \partial_1 g_{44} = 0 \tag{34}$$

which, by substituting ω for their value in Eq. (19) reads

$$\left(\frac{u\hbar}{m\rho^2}\right)^2 \partial_1 g_{22} + 2c \frac{u\hbar}{m\rho^2} \partial_1 g_{44} + c^2 \partial_1 g_{44} = 0 \tag{35}$$

and simplifying

$$\left(\frac{u\hbar}{m}\right)^2 \partial_1 g_{22} + 2c\rho^2 \frac{u\hbar}{m} \partial_1 g_{24} + c^2 \rho^4 \partial_1 g_{44} = 0 \tag{36}$$

We introduce the b constant to lighten expressions

$$b = \frac{u\hbar}{m} \tag{37}$$

It interests us to take as a reference the electron of level $2p$: $n = 2$ and $u = 1$. For $u = 1$ and the mass of the electron, we obtain $b = 1.158 \times 10 - 4$ J.s/kg.

Let us write Eq. (36) this way

$$b^2 \partial_1 g_{22} + 2bc\rho^2 \partial_1 g_{24} + c^2 \rho^4 \partial_1 g_{44} = 0 \tag{38}$$

This equation applies to cylindrical and axial metrics, as has been said. This will be the starting equation, representing the path of the particle with axial kinetic moment quantized; the concerned elements of the metric tensor must satisfy this equation.

Let us make an early advance of the physical meaning of the previous equations. As we will see later, the derivative of g_{44} will be considered null and g_{22} , g_{24} will be only functions of the radius. If we considered these facts in Eq. (34) we obtain the approximate relation:

$$\omega g'_{22} + 2c g'_{24} = 0 \tag{35 bis}$$

where "the comma" indicates derivation with respect to the radius. So it shows dependence between g_{22} and g_{24}, that we will verify this fact later on (94).

2.4. "Ansatz" for a cylindrical ds^2

The exact analytical resolution of the Einstein field equations led to a comprehensive set of metrics corresponding to different situations. In general, for every kind of situation we may consider three possibilities, depending on the energy momentum tensor: empty space, fluid and field. Within fluid type, when its pressure is zero, we have the type of "dust" particles dissociated comparable to each other. This is the situation that we assume as first approximation in this paper.

Taking into account the cylindrical symmetry expressed by the trajectory, we start from the cylindrical solution proposed to this case by Stephani and Kramer [21]. This is the result of several contributions, including Winicour (1975), Wishweshwara-Winicour (1977) and King (1974). Specializing the van Stockung class (1937) we get:

$$ds^2 = e^{-a^2\rho^2}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 - (cdt + a\rho^2 d\varphi)^2 \tag{39}$$

where a (*in cursive*) is a parameter that is determined from the equations of the geodesic. We remark that if $a = 0$ the Minkowski metric is obtained. For $\rho \to \infty$ we do not obtain a Minkowskian flat space; but for our purposes it suffices to be so at a certain ρ that limits the validity of the proposed model that, we advance, is of the order of 10^{-6} m.

Anyway, this metric is not totally satisfactory for our case since, as it will be shown, the parameter a results to be a function of ρ and the development that followed assuming that a is constant is not correct, but we consider it as a first approach. The elements of the metric tensor will be

$$g_{11} = g_{33} = e^{-a^2\rho^2} \tag{40}$$

$$g_{22} = \rho^2 - \rho^4 a^2 = \rho^2(1 - a^2\rho^2) \tag{41}$$

$$g_{24} = -ca\rho^2 \tag{42}$$

$$g_{44} = -c^2 \tag{43}$$

$$g_{12} = g_{13} = g_{23} = g_{14} = g_{34} = 0 \tag{44}$$

The derivatives of the components relieving for the equations of the paths are

$$g'_{22} = 2\rho - 4\rho^3 a^2 \tag{45}$$

$$g'_{24} = -2c\rho a \tag{46}$$

$$g'_{44} = 0 \tag{47}$$

We substitute in Eq. (38) the partial derivatives for the total with regard to the radius

$$b^2 g'_{22} + 2bc\rho^2 g'_{24} + c^2\rho^4 g'_{44} = 0 \tag{48}$$

and obtain

$$b^2(2\rho - 4\rho^3 a^2) + 2bc\rho^2(-2c\rho a) = 0 \tag{49}$$

$$b^2(2\rho - 4\rho^3 a^2) - 4bc^2\rho^3 a = 0 \tag{50}$$

and simplifying

$$b^2(1 - 2\rho^2 a^2) - 2bc^2\rho^2 a = 0 \tag{51}$$

or better

$$2b\rho^2 a^2 + 2c^2\rho^2 a - b = 0 \tag{52}$$

and finally we obtain

$$a^2 + \frac{c^2}{b} a - \frac{1}{2\rho^2} = 0 \tag{53}$$

The resolution of this equation provides the parameter a as function of ρ, contrary to our hypothesis that this parameter was a constant, independent of the radius. This fact has physical meaning, as we refer later on. We see the values that acquire the metric tensor

$$g_{ij} = \begin{matrix} e^{-a^2\rho^2} & 0 & 0 & 0 \\ 0 & \rho^2 - \rho^4 a^2 & 0 & -c\rho^2 a \\ 0 & 0 & e^{-a^2\rho^2} & 0 \\ 0 & -c\rho^2 a & 0 & -c^2 \end{matrix} \tag{54}$$

If we compare the results obtained with the Minkowski metric tensor gmij in polar coordinates

$$\begin{aligned} gm_{11} = gm_{33} = 1gm_{22} = \rho^2 gm_{44} = -cgm_{12} = gm_{13} \\ = gm_{14} = gm_{23} = gm_{24} = gm_{44} = 0 \end{aligned} \tag{55}$$

$$gm_{ij} = \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & \rho^2 & & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -c^2 \end{matrix} \tag{56}$$

we obtain the difference tensor

$$\epsilon_{ij} = \begin{matrix} 1 - e^{-a^2\rho^2} & 0 & 0 & 0 \\ 0 & \rho^4 a^2 & 0 & c\rho^2 a \\ 0 & 0 & 1 - e^{-a^2\rho^2} & 0 \\ 0 & c\rho^2 a & 0 & 0 \end{matrix} \tag{57}$$

which for any value different from 0 cannot match with a flat space (Minkowski-like situation) as a limit.

Regarding velocity and kinetic moment we find the following. The derivatives of the elements of the metric tensor that will affect us for the path are

$$g'_{22} = 2\rho - 4\rho^3 a^2 \tag{58}$$

$$g'_{24} = -2c\rho a \tag{59}$$

The affine connectors are

$$\begin{aligned} \Gamma^j_{22} &= \frac{1}{2} g^{jh}(\partial_2 g_{2h} + \partial_2 g_{h2} - \partial_h g_{22}) \\ &= \frac{-1}{2} g^{jh}\partial_h g_{22} = \frac{-1}{2} g^{11}\partial_1 g_{24} \end{aligned} \tag{60}$$

$$\begin{aligned} \Gamma^j_{24} &= \frac{1}{2} g^{jh}(\partial_2 g_{4h} + \partial_4 g_{h2} - \partial_h g_{24}) \\ &= \frac{-1}{2} g^{jh}\partial_h g_{24} = \frac{-1}{2} g^{11}\partial_1 g_{24} \end{aligned} \tag{61}$$

$$\begin{aligned} \Gamma^j_{44} &= \frac{1}{2} g^{jh}(\partial_4 g_{4h} + \partial_4 g_{h4} - \partial_h g_{44}) \\ &= \frac{-1}{2} g^{jh}\partial_h g_{24} = \frac{-1}{2} g^{11}\partial_1 g_{44} \end{aligned} \tag{62}$$

We consider now the value of the component z of the kinetic moment. As previously said, from the dBB model we must have in our case

$$L_z = mv\rho = u\hbar \tag{63}$$

$$v = \frac{u\hbar}{m\rho} = \frac{b}{\rho} \tag{64}$$

$$L_z = mb \tag{65}$$

independent of the radius. By other side, the equations of the trajectory in this case read

$$\Gamma^j_{ik}\frac{dx^i}{ds}\frac{dx^k}{ds} = 0 \tag{66}$$

and so

$$\Gamma^1_{22}\omega^2 - \Gamma^1_{24}c\omega = 0 \tag{67}$$

from there we get

$$\omega = \frac{c\Gamma^1_{24}}{\Gamma^1_{22}} = \frac{cg'_{24}}{g'_{22}} = \frac{-2c^2\rho a}{2\rho - 4\rho^3 a^2} = \frac{c^2 a}{2\rho^2 a^2 - 1} \tag{68}$$

The electron must have, besides a quantized energy, a quantified value of the kinetic moment, regardless of the radius. In other words, for any value of the radius we must have the same value of kinetic moment. This feature serves as a contrast to the adequacy of the model. But we have

$$v = \omega\rho = \frac{c^2 a\rho}{2\rho^2 a^2 - 1} \tag{69}$$

and the z component of the kinetic moment would be:

$$L_z = mv\rho = m\frac{c^2 a}{2a^2 - \frac{1}{\rho^2}} = \tag{70}$$

that, as we can see, is a function of the radius, against (65). That lack and the previously mentioned one will disappear with the model that we consider later, with the parameter a function of ρ.

Conversely, the shape of the path is consistent with the hypothesis. Integrating the components of the four-velocity (the only non-zero component is $v^2 = \omega$) with respect to time we have:

$$\begin{aligned} X^1 &= \rho_0 \\ X^2 &= \omega t \ \text{ (with } \varphi_0 = 0) \\ X^3 &= z_0 \\ X^4 &= -ct \end{aligned} \tag{71}$$

So it would be a uniform circular motion in a plane perpendicular to OZ, consistently with the dBB model.

We advance one conclusion: from the performed calculations it result that the cylindrical metric with the parameter a indicated, allows us to interpret, with the indicated lacks, the movement of the electron in a curved space time; the curvature of this being caused by the global quantum interaction, including the electrostatic field and quantum potential, so that the deformation of the space-time would produce the guidance of the particle. This guidance is the "role" of the wave function .

Anyway, we remark the approximated character of this determination, as a is not constant with regard to ρ, as we have already been stated.

2.5. Consideration of a as a function of ρ.

The development of the previous point has shown us the following inconvenient of the metric mentioned there:

a) We reach the mathematical conclusion that a is not any constant parameter but a function of ρ.

b) It is not possible to find a solution, even by adjusting the equations that conserve the kinetical moment by assigning to ρ an arbitrary value.

c) Finally, we should also mention that it is not possible to adjust the equations (selection of constant k) in order to make a flat space-time for a macroscopically reduced ρ.

A possible solution seems to improve the model supposing that a is not a constant but a function of ρ that will be called y.

Therefore, the proposed metric, with $y = y(\rho)$ will be:

$$ds^2 = e^{-y^2(\rho)\rho^2}(d\rho^2 + dz^2) + \rho^2 d\varphi^2 - (cdt + y(\rho)\rho^2 d\varphi)^2, \tag{72}$$

and the components of the metric tensor will be

$$g_{11} = g_{33} = e^{-y^2(\rho)\rho^2} \tag{73}$$

$$g_{22} = \rho^2 - \rho^4 y^2 \tag{74}$$

$$g_{24} = -c\rho^2 y \tag{75}$$

$$g_{44} = -c^2 \tag{76}$$

$$g_{12} = g_{12} = g_{23} = g_{14} = g_{34} = 0 \tag{77}$$

The (total) derivatives of the components remaining from the equations for the paths read as

$$g'_{22} = 2\rho - 4\rho^3 y^2 - 2\rho^4 yy' \tag{78}$$

$$g'_{24} = -2c\rho y - c\rho^2 y' \tag{79}$$

$$g'_{44} = 0 \tag{80}$$

Substituting in Eq. (48), we obtain:

$$b^2 g'_{22} + 2bc\rho^2 g'_{24} + c^2\rho^4 g'_{44} = 0 \tag{81}$$

and so

$$b^2(2\rho - 4\rho^3 y^2 - 2\rho^4 yy') + 2bc\rho^2(-2c\rho y - c\rho^2 y') = 0 \tag{82}$$

Dividing by $-2b\rho$ and grouping

$$(b\rho^3 y + c^2\rho^3)y' + 2b\rho^2 y^2 + 2c^2\rho^2 y = b \tag{83}$$

Performing the algebra to solve for y':

$$y' + \frac{2b\rho^2 y^2 + 2c^2\rho^2 y}{b\rho^3 y + c^2\rho^3} = \frac{b}{b\rho^3 y + c^2\rho^3} \tag{84}$$

which can be written

$$y' + \frac{2y}{\rho} = \frac{b}{b\rho^3 y + c^2\rho^3} \tag{85}$$

To solve analytically this equation we will do an approximation: the denominator of the second member the first term is clearly of lesser order that the second one, (by $\ll c^2$), as it is easy to see by solving numerically that the differential equation that gives the following Fig. 1: so, we see that the maximum of y (5.11×10^6) multiplied by b (1.158×10^{-4}) is on the order of 5.913×10^2, that is very reduced against c^2. With this approximation the equation remains

$$y' + \frac{2y}{\rho} = \frac{b}{c^2\rho^3} \tag{86}$$

differential equation of first linear order that has the following solution

$$y = \frac{b}{c^2}\rho^{-2}\ln k\rho \tag{87}$$

with k as a constant of the bundle of solutions. Comparing Eq. (87) with the numerical results the agreement between both are within the 0.11%.

Replacing y in the metric tensor elements one gets

$$g_{11} = g_{33} = e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} = (k\rho)^{\frac{-2b^2}{c^4\rho^2}} \tag{88}$$

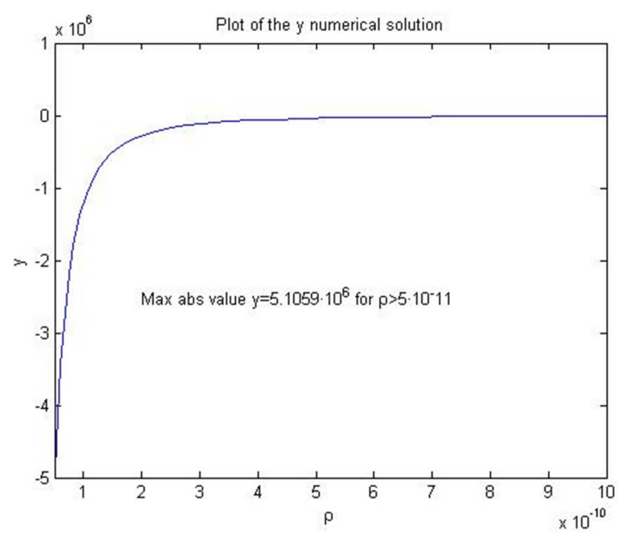

FIGURE 1. The differential equation of y solved numerically

$$g_{22} = \rho^2 - \frac{b^2}{c^4}(\ln k\rho)^2 \tag{89}$$

$$g_{24} = -\frac{b}{c}\ln k\rho \tag{90}$$

In the matrix form the metric tensor is

$$g_{ij} = \begin{matrix} e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & \rho^2 - \frac{b^2}{c^4}(\ln k\rho)^2 & 0 & \frac{-b}{c}\ln k\rho \\ 0 & 0 & e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 \\ 0 & \frac{-b}{c}\ln k\rho & 0 & -c^2 \end{matrix} \tag{91}$$

The difference tensor compared to the Minkowski metric tensor in cylindrical coordinates reads

$$\epsilon_{ij} = \begin{matrix} 1 - e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & \frac{b^2}{c^4}(\ln k\rho)^2 & 0 & \frac{b}{c}\ln k\rho \\ 0 & 0 & 1 - e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 \\ 0 & \frac{b}{c}\ln k\rho & 0 & 0 \end{matrix} \tag{92}$$

The obtained metric has a term out of diagonal (g_{24}) that binds angular coordinate with time. This fact for stationary systems can surprise somebody but is accepted for such situations [22].

Consider now the conservation of kinetic moment of the particle to which we calculate its velocity based on the equations of the trajectory. Returning to Eq. (34) with partial derivatives transformed in totals respect to the radius, the trajectory equation reads

$$\omega^2 g'_{22} + 2c\omega g'_{24} + c^2 g'_{44} = 0 \tag{93}$$

and as g'_{44} vanishes, one has

$$\omega^2 g'_{22} + 2c g'_{24} = 0 \tag{94}$$

so we come to (35 bis). From there the linear velocity can be expressed as

$$v = \rho\omega = \frac{-2c\rho g'_{24}}{g'_{22}} \tag{95}$$

and substituting the derivative of the tensors for their values in function of y and y'

$$v = \frac{-2c\rho g'_{24}}{g'_{22}} = 2c\rho\frac{-2c\rho y - c\rho^2 y'}{2\rho - 4\rho^3 y^2 - 2\rho^4 yy'}$$
$$= c^2\rho\frac{2y + \rho y'}{1 - 2\rho^2 y^2 - \rho^3 yy'} \tag{96}$$

We observe there that the velocity of the particle is independent of the constant of integration k. We will show that it faithfully reproduces the initial hypothesis on the kinetical moment regarding independence of ρ. Indeed, substituting y and y' and taking into account that the denominator is practically the unity, the previous expression remains

$$v = \frac{b}{\rho} \tag{97}$$

So the condition (94) is fulfilled. Furthermore, we can easily prove approximately (94) in a more direct way. For it, let us write (94), taking into account (23) in the following way:

$$\frac{g'_{22}}{g'_{24}} = \frac{-2c\rho^2}{b} \tag{98}$$

Now we substitute in (98) the values of g'_{22} and g'_{24} from (89) and (90) and we obtain:

$$\frac{g'_{22}}{g'_{24}} = \frac{-2c\rho^2}{b} + \frac{2b\ln k\rho}{c^3} \tag{99}$$

The discrepancy between (98) and (99) is the second term of second member of (99), that is of some 20 orders of magnitude lower in absolute value compared with the first term, for the range of values of ρ and $k = 10^6$ as is fixed later on

$$\left|\frac{2c\rho^2}{b}\right| \gg \left|\frac{2bln k\rho}{c^3}\right| \tag{100}$$

The equations of the trajectory are reproduced in our approach; particularly for φ we get, integrating (97) against time and replacing b, the Eq. (19). Furthermore, then we can ensure consistency of the model towards linear momentum. With this, the kinetic moment reads

$$L_z = mv\rho = m\frac{b}{\rho}\rho = mb = m\frac{u\hbar}{m} = u\hbar \tag{101}$$

as it is required, independent of ρ. The metrics expressed by (92) is not Minkowskian for large radius because the terms g_{22} and $g_{22} = g_{42}$ do not tend to 0 when ρ tends to infinity. All other terms tend to the flat space situation when ρ tends to infinity. But for the coherence of our approximate model we only need that the metrics shows a flat space performance

TABLE I. Exponent and tensor values in center of mass coordinates.

$\rho(m)$	2.5×10^{-11}	2×10^{-10}	7.5×10^{-10}	0.000001
$y(\times10^{6})$	-21.845	-0.27435	-0.0164817	0
$g_{11}=g_{33}=$	1	1	1	1
g_{22}	6.25×10^{-22}	4.00×10^{-20}	5.625×10^{-19}	1×10^{-12}
$g_{24}(\times10^{-9})$	4.0931	3.2899	2.7794	0

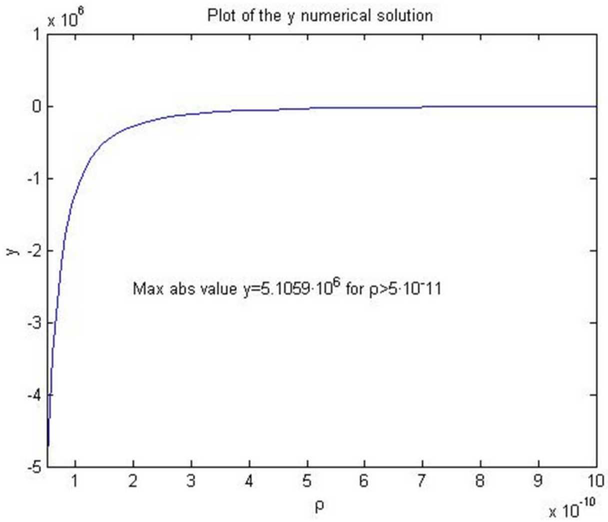

FIGURE 2. g_{24} respect to the radius.

at the limit of the model, for a distance large enough of, let's say, 10^{-6} m. With this condition, we can provisionally fix the value of the constant k as 10^6 m^{-1} , so for $\rho = 10^{-6}$ m, $\ln k\rho = 0$. We show that in the following Fig. 2.

With this choice of k, the values of the metric tensor are, for various values of the radius:

The difference compared to the corresponding tensor in cylindrical coordinates flat space is limited virtually to component g_{24} and its symmetric one g_{42}.

2.6. Metric tensor using $X^4 = ct$

So far, for easiness we used t as coordinate. For further use it is convenient to express the metric tensor using $X^4 = ct$, always in cylindrical coordinates. So it is equivalent to apply the transformation:

$$X_1' = X_1;\;\; X_2' = X_2;\;\; X_3' = X_3;\;\; X_4' = cX_4; \tag{102}$$

We use the transformation:

$$g_{ij}^T = \frac{\partial x_k}{\partial x_i'}\frac{\partial x_l}{\partial x_j'} g_{ij} \tag{103}$$

and we come easily to:

$$g_{ij} = \begin{matrix} e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & \rho^2 - \frac{b^2}{c^4}(\ln k\rho)^2 & 0 & -b\ln k\rho \\ 0 & 0 & e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 \\ 0 & -b\ln k\rho & 0 & -1 \end{matrix} \tag{104}$$

The discrepancy with respect to the Minkowskian metrics in cylindrical coordinates reads again like (92):

$$\epsilon_{ij} = \begin{matrix} 1-e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 & 0 & 0 \\ 0 & \frac{b^2}{c^4}(\ln k\rho)^2 & 0 & b\ln k\rho \\ 0 & 0 & 1-e^{\frac{-b^2}{c^4}\left(\frac{\ln k\rho}{\rho}\right)^2} & 0 \\ 0 & b\ln k\rho & 0 & 0 \end{matrix} \tag{105}$$

that can be schematically approximated as:

$$\epsilon_{ij} = \begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b\ln k\rho \\ 0 & 0 & 0 & 0 \\ 0 & b\ln k\rho & 0 & 0 \end{matrix} \tag{106}$$

3. Conclusions

The following conclusions are framed in the simplicity of the model, *i.e.* excluding spin or fully relativistic quantum model.

We made the hypothesis of assimilate the electron's trajectory in the dBB for the Hydrogen-like atoms with geodesics of space-time in the GR, corresponding to the fact that in the stationary states there is no loose of energy and that in a geodesic the acceleration is null. We made our exploration in the orbital 2p of the hydrogen atom.

We made this exploration in pseudo-Riemanian Lorenz geometry. We find a metric of this geometry by working out an exact solution metric of Einstein's field equations corresponding to dust cylindrical model, dually modified. To establish the analytic form of this metric, we must make an approximation that proved to be coherent with the corresponding numerical solution to a high degree. This modification could be related to the fact that the relation between electron and nucleus cannot be merely assimilated to "dust´´ (the electron) turning around an axle, but deeper holistic considerations should be taken into account, in relation with quantum potential.

The found metrics proved to be coherent respect the linear momentum and the constancy of the kinetic momentum, independently of the radius at which the electron turns around, that is a variable in the dBB model. Anyway, at a convenient radius, large enough, the deformation of space time vanishes, at the radius limit of validity of the model, by election of (only) a constant. We must highlight that in our approach a) emerges a relationship between the components g_{22} and g_{24} of the metrics and b) the major discrepancy with the flat metrics is a non-null g_{24} component that binds angle and time.

So we concluded that it is possible to explain the mentioned electron trajectories by a deformation of the space time. This deformation is in relationship with the quantum potential of the atom system, through the wave function. The role of the wave function appears therefore as defining the deformation of space time where the particle moves. Both, wave and the particles (electron and nucleus, the last considered at rest), are in a non-separable relationship: the wave guides the particle and the particle determines the wave, by the process of limitation that introduces the quantization (Schrödinger equation), that in fact derives from the first assumption of de Broglie [1]: to identify the particle trajectory and the corresponding light ray by stationary principles.

This feature permits us to recognise the same ontological character to the particle and to the wave function. The relation between the particle guiding and the quantum system connects in some extension with the relation between the metric structure and the energy stress impulsion of the Einstein's field equation: in both cases the system "tells" the space time how to deform, and the space time structure "tells" the body how to move. This relationship is reinforced by the fact that the GR recognises a dynamical dependence between matter/energy and space-time.

The coherence of the model here presented allow us to consider the stability of atomic matter under the insight that the electron follows a geodesic of the space time, and therefore with null covariant acceleration so the electron does not emit energy because is not accelerating, despite its rotation around the nucleus.

The capability of the Lorenz geometry to be used in this approach shall be also highlighted as it distinguished sharply the time and the spatial coordinates. Of course, it is not excluded the use of the Weyl geometry, as a generalization of Riemann geometry, for more general quantum systems.

As an extension of this approach, in a separate paper we will develop the calculations regarding Ricci tensors, scalar curvature and further relativistic elements. Specially, we will calculate the stress-energy tensor and interpret it in function of the physical features of the system.

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