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Mathematical modeling of oval arches. A study of the George V and Neuilly Bridges

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ARTICLE INFO

ABSTRACT

In this work we approach the mathematical modeling of oval arches of n centers and we present an analytical study of their geometry given the expressions of the elements that define them and the tangency points as a function of span, sagita and the radius of the circumferences of which they are formed, using *Mathematica* software to perform interactive graphs and calculations. This allows to mathematically model an existing arch or to design the construction of a new one. We apply the results obtained in the modeling of the Orléans Bridge over the River Loire (1751–1760) whose construction — initiated by Hupeau and completed by Perronet — made of three-centered oval arches and in the Neuilly Bridge over the River Seine, close to Paris, which is formed of eleven-centered oval arches.

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1. Historic introduction

"The Oval is a closed circle which
Has no beginning, end or centre.
With four points it is always formed.
Which, for the most part lie within.
In other words, they are intersected
With lines that meet at a point.
This is how these oval bodies are formed,
Vessels and other main things."[1]
Ioan de Arphe i Villafañe;
De varia conmensuración para la escultura y arquitectura.
Book I, Chapter III. Trata de ovalos y cómo se forman.

The oval arch is an oval-shaped arch that has been used extensively throughout history. Made up of an uneven number of circumference arches tangent to each other and to the jambs and at the spring, we can find three, five, six, nine, eleven centers. . .

Located in the Czech Republic in Paleolithic times, there was already evidence of an oval-plan room [1]: "In Dolni-Vestonice, in Moravia (Czech Republic), remains of an important settlement were found, near a river, at the foot of the Pavlov mountains, with clear signs of having been occupied for many years (−29000 to −19000). One of its most important rooms was oval shaped; 15 by 9 metres which was paved with cobblestones and surrounded by posts." (Own translation)

In the 3rd century B.C., towards 250−220 B.C., the Greek mathematician Apollonius of Pergamus, — a disciple of Euclid — in dealing with conics showed that the ellipse, the parabola, the circumference and the hyperbola are achieved by cutting a straight cone through a plane in different positions. From that point on, many have dedicated to the study of the ellipse and the flat figure that comes closest to it, the oval. Many were those who gathered the oval/elliptic forms: Serlio [2], Hernán Ruiz el Joven [3], Alonso de Vandelvira [4], Juan de Torija [5] and Simonin [6], to name a few. Regarding the use of this type of form in Greek architecture, Choisy [7] (vol. 1, p. 352) hints to Ionic fluting that had the semi-oval profile, often in semi-circle.

Concerning the use of the oval shape in Roman architecture Choisy [7] (vol. 1, p. 575) refers to the structural difficulties in the amphitheatres that were as a result of the oval plan with which they were drawn. With reference to the elliptic form, these were present in the remaining arches of the intersection of edge vaults with equal barrels used by the Romans. Choisy [7] (vol. 1, p. 518–519) also states that the oldest edge vault could be the one found in the tomb of Pergamus, that dates back to the time of Attales.

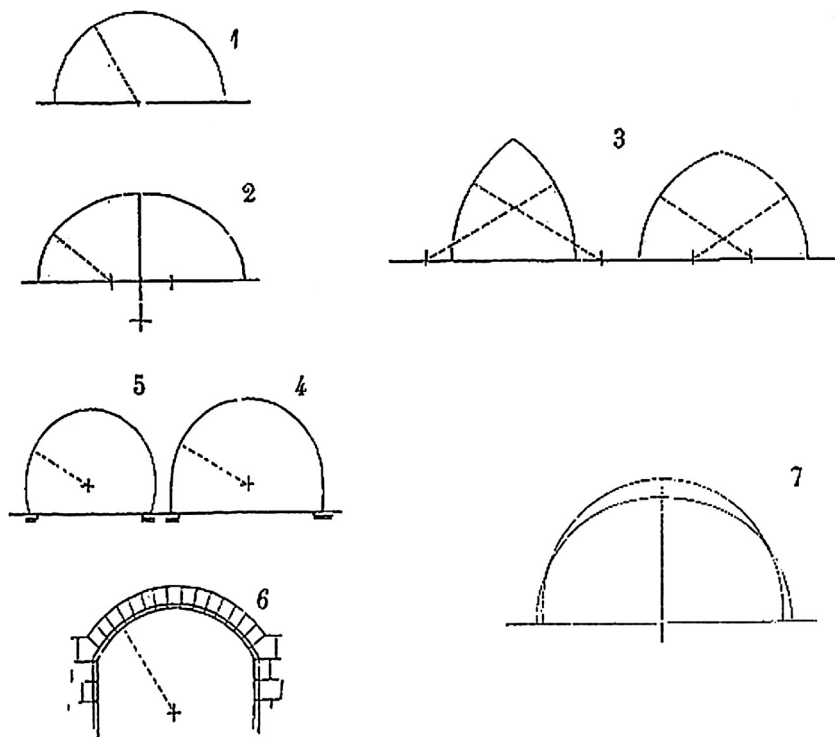* Corresponding author.
[1] Own translation.

Fig. 1. Types of arches used in the Middle Ages according to Viollet-le-Duc.

Viollet-le-Duc in [8] (vol. 1 *Arc*, term, p. 45–46) classifies the arches used in the Middle Ages in three main categories (Fig. 1): in 1, arch *plein centre* (semi-circular) and its variations: 4, stilted arch; 5, en fer à cheval or horseshoe and 6, with the centre below the spring; in 2, *arcs surbaissés ou en anse de panier* (surbased arches or oval arches) and in 3, *arcs en ogive ou en tiers-point* (ogive arches)

Viollet sketched the oval arches as such but stated that "they are formed by a semi-ellipse" although "with a larger diameter at the base", alluding, therefore, to the oval shape of the oval arch. In this description Viollet states the confusion between the oval and ellipse terminologies that have developed throughout history. Numerous authors have studied the oval and elliptical geometry. To name a few: Gentil Baldrich [9,10], Fernández Gómez [11], Rosin [12], Huerta Fernández [13], [14], Garcia Jara [15,16], López Mozo [17], Barrallo [18], Mazzotti [19–21], and Capilla and Calvo [22].

Viollet in [8] (vol. 1, term, p. 45) shows that until the 11th century the rounded arch and its variations was the only one used in constructions, apart from some rare exceptions. With regards to *surbaissés ou en anse de panier* (surbased or basket-handle arches), these are often found in vaults from the Romanic period. It is said that they are nothing more than the result of a deformation produced by the separation of the walls (Fig. 1, 7), having been originally built with a semi-circular arch.

Stated by Viollet-le-Duc himself [8] (vol. 1, term *Arc*, p. 46) that towards the middle of the 16th century, renaissance architects wanted to definitely exclude the use of the pointed arches that were being progressively adopted by the medieval architects in the province of France and the Occident. The substitution of those pointed arches for the oval arches towards the end of the 16th century was exemplified by Viollet in Saint-Eustache de Paris. Although he speaks of arches in the shape of ellipse, he refers to oval arches but with a small diameter in the base considering the curve as "unpleasant, difficult to sketch, more difficult to prepare and less resistant than the pointed arch" (own translation)

According to what we have seen, Viollet states that the oval arches were used in the middle of the 16th century by renaissance architects, frequently used in the Renaissance period instead of the pointed arches. This period of time to which Viollet refers to, in some regions coincided with the final Gothic period, as is the case in Spain, where the arches were very much used during the Renaissance and Baroque periods. These types of arches were used in religious constructions as well as civil and military; in doorways, vault arches and bridges, etc. Müller (1971) and Gentil Baldrich [10] carried out an in-depth study of the oval geometry in Renaissance and Baroque architecture.[2]

As an example we want to mention the oval arches used in La Lonja of Valencia, declared a World Heritage Site in 1996 and built between 1483 and 1498. In the chapel next to the columned hall, we find a three-centered oval arch in formeret arches (wall ribs) of the starred vault that covers it, as well as in the diagonals and larger tiercerons. In Capilla [23], a study is made of these arches and their geometry as well as the spatial geometry of the starred vault.

Oval arches continued to be used in later centuries. The oval shapes, as stated by Huerta [13] (p. 462), were used to "design oval arches for bridges: the oval solves the problem of designing with circumference arches, a surbased arch of vertical springs. Oval domes also existed that allowed for the adjustment of the dimensions of the plan and in the elevation". The bridges built by Perronet in the 18th century are a clear example in the use of these types of arches, those of which we analyze in this article.

2. Geometric and mathematical analysis of oval arches

Jean Rodolphe Perronet (1708–1794) was a French engineer who, among other things, "revolutionized the design and construction of bridges, favored the use of oval arches of several centers, reduced the thickness of the columns and improved the design of
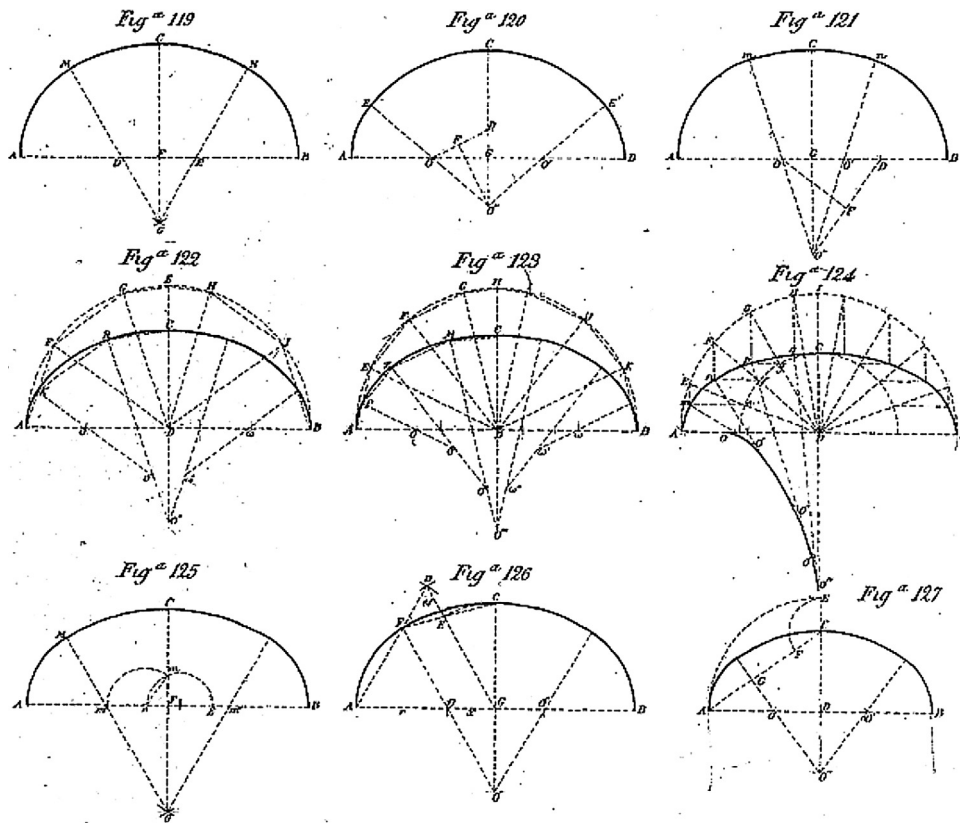
Fig. 2. Representation of oval arches according to Rovira y Rabassa ([25]: 1st part, print 12).

the centering and un-centering techniques. In addition, he devised a new rule to calculate the thickness of the bridge vault in the keystone which led to notably less thickness than the rule set by Gautier" [24] (p. 233–244).

Perronet "built thirteen bridges and designed another eight. His works inspired admiration for its audacity and lightness: The Bridge of Neuilly (1774), Saint-Maxence (1772–86) and the Concorde (1787–91) are probably the most representative". (Huerta, [13] (p. 356)). He used diminished arches on some bridges and oval arches on others. And so, on the Neuilly Bridge over the River Seine, close to Paris, on the bridge of Saint-Edme in Nogent-sur-Seine or the bridge over the River Neva in Saint-Pétersbourg he used eleven-centered oval arches.

The design of an oval arch from a graphic-mathematical point of view was already addressed by Rovira y Rabassa [25] in his treatise *Estereotomia de la piedra* (*The Stereotomy of stone*). It contains the description and design of various oval arches of which he says, "it takes the name of oval arch, when it has the shape similar to that of an ellipse, its different parts are drawn with a series of circumference arches, whose centers are of uneven numbers." (Rovira y Rabassa [25] (p. 39))

Like Perronet, Rovira y Rabassa [25] (p. 309) also refers to the predilection of using oval arches instead of elliptical arches. He textually states that they are preferred "since it is easier to design the normal arches to the same curve and these curves are more uniform in the different panels". In the treatise mentioned, Rovira y Rabassa [25] (print 12) gathers the design of various three, five and seven centered oval arches as well as an oval arch of any number of centers (Fig. 2). It gives recommendations in choosing the number of centers of an oval arch in relation to the span and sagita.

But Rovira y Rabassa not only showed the graphic construction of the different types of oval arches mentioned but also incorporated mathematical calculations. In this latter case, the design of various-centered oval arches was planned from the span and sagita through the construction of an ellipse and affinity. Designing the *evolute* of said ellipse and the tangent lines to it, successively cutting it, achieve the centers of the oval arch that will substitute the curves of the ellipse, as can be observed in Figure 124 of print 12 shown in the *Estereotoma de la piedra (The Stereotomy of stone)* (Fig. 2, Fig. 124).

In 2014, Mazzotti [19] studied the eggs and the polycentric curves — open and closed — through a Euclidean approach. He develops these concepts in a second part [20] by applying them to ovals, drawing them using a given number of certain parameters.

In 2017, Mazzotti [21] presents an exhaustive treatment of ovals and, moreover, he derives the analytic formulas which establish the relationship between the different parameters involved in the construction of four-centre ovals. In addition, he shows the application to two representative buildings of the Italian architecture located in Rome: the dome of San Carlo alle Quattro Fontane of Borromini in the chapter 7, co-written with Margherita Caputo, and the Colosseum in the chapter 8.

In this paper, the mathematical study that we present will allow us to obtain in table from, the expressions of elements that define oval arches as well as modeling and designing oval arches with symbolic calculation programs (*Mathematica*®, *Geo-geb*ra) regardless of the number of centers and the initial information (span, sagita, radius. . .). As a consequence of our study we analyse the geometry of two Perronet's bridges: the Jorge V Bridge, which crosses the Loire River that flows through the city of Orléans, built with
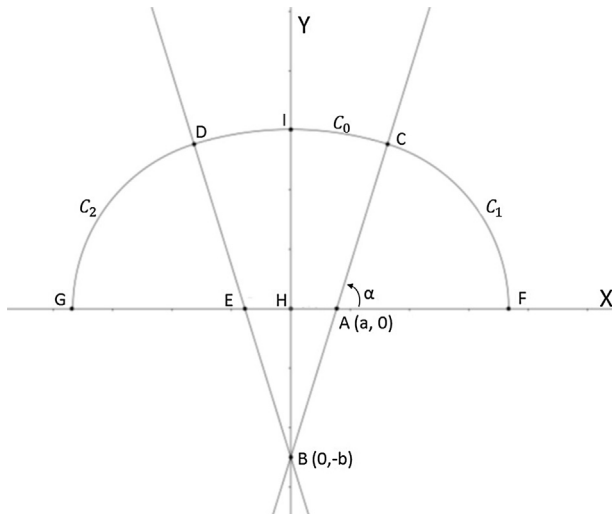
Fig. 3. Three centered oval arch.

a three-centered oval arch, and the Neuilly Bridge, over the Seine River close to Paris with eleven centers.

2.1. Mathematical analysis of a three-centered oval arch

Let us first consider the simplest case, the three-centered oval arch. The design of an arch going through three points, the two at the spring and the keystone, is a problem that offers many solutions depending on the data and existing conditions. Fig. 3 shows an arch of this type where the sagita is $f = IH$ and span $2\ell = GF$. We place the origin of the coordinates in the center of the line that meet the two spring points and in symmetry with the arch, we will only consider the positive half-plane of the abscissa. The central circumference C_0 has its center in the point $B = (0, -b)$ and radius $R = BI = f + b$ and the lateral circumference C_1, of center $A = (a, 0)$ and radius $r = AF = \ell - a$, $\left(a < \ell \right)$.

We look for point C where they meet and are tangent the two circumferences. If we refer to α as the angle of the circular sector of the exterior circumference, it is evident that the coordinates (x, y) of point C are defined as the following

$$x = a + r\cos\alpha = a + \frac{(\ell - a)\,a}{\sqrt{a^2 + b^2}}, \quad y = r\sin\alpha = \frac{(\ell - a)\,b}{\sqrt{a^2 + b^2}}$$

where a and b are related by the equation

$$b = a\tan\alpha$$

Case 1: Supposing in the first instance a three-centered oval arch knowing the semi-span, ℓ, and the radius, r, of the circumference C_2, (equivalently known ℓ and a).

With this data,

$$a^2 + b^2 = a^2 + a^2\tan^2\alpha = \frac{a^2}{(\cos\alpha)^2}$$

And so, the radius of the central circumference C_1 and the sagita of the arch are determined by

$$R = \ell - a + +\sqrt{a^2 + b^2} = \ell - a + \frac{a}{\cos\alpha} = \ell + a\left(\frac{1}{\cos\alpha} - 1\right)$$

$$f = R - b = \ell + a\left(1/\cos\alpha - 1\right) - a\tan\alpha = \ell - a\left(\frac{\sin\alpha + \cos\alpha - 1}{\cos\alpha}\right)$$

The Table 1 has the different elements that determine a three-centered oval arch, in this case for different angles.

Case 2: Let us suppose that the sagita f and the radius R of the central circumference C_1 are known. In this case, the lesser radius and the sagita depend on the angle α

$$b = R - f$$

$$a = \frac{b}{\tan\alpha} = \frac{R - f}{\tan\alpha}$$

$$a^2 + b^2 = \frac{(R - f)^2}{(\sin\alpha)^2}$$

$$r = R - \sqrt{a^2 + b^2} = R - \frac{R - f}{\sin\alpha}$$

$$\ell = a + r = \frac{R - f}{\tan\alpha} + R - \frac{R - f}{\sin\alpha}$$

$$= R + (R - f)\left(\frac{1}{\tan\alpha} - \frac{1}{\sin\alpha}\right)$$

$$= R + \frac{R - f}{\sin\alpha}(\cos\alpha - 1).$$

The coordinates of the tangent point in this case are given by

$$x = a + r\cos\alpha = R\cos\alpha$$

$$y = r\sin\alpha = \left(R - \frac{R - f}{\sin\alpha}\right)\sin\alpha = f + R(\sin\alpha - 1)$$

Table 2 is the table with data that would be obtained for $30°$, $45°$ and $60°$ angles.

Case 3: We fix the sagita and semi-span. In this case from the equation:

$$R = r + \sqrt{a^2 + b^2} = b + f$$

we obtain:

$$\sqrt{a^2 + b^2} - b = f - r$$

Given that the right term is always positive, the equation that must comply with the radius r is:

$$f - r > 0 \Leftrightarrow a > \ell - f$$

In the Table 3, we present the values that would be obtained for the different elements that form the arch in function of the possible values of a and the existing proportion between a and b (or equivalently in function of $\tan\alpha$).

2.2. Orléans bridge over the Loire River

The Orléans Bridge crosses the Loire River that flows through the city of Orléans. It is also known the Royal or National Bridge and when WW1 ended, it was also known George V bridge, in honor of the King of England with the same name.

The engineer Robert Pitrou $-\dagger$ 1750 $-$ designed in 1749 the first project that was never carried out. After his death, the King replaced him with the first engineer of the Ponts et Chaussées, Jean Hupeau. The final location of Hupeau's bridge was just a few feet away from the original. It was opened to the public at the end of 1760 but the final reception for the work did not take place until the 1763. The work was directed by the engineer of Ponts et Chaussées, Robert Soyer, until 1763, and was supervised by Hupeau. On the 17th of October that same year after the death of Hupeau, the work was passed onto Jean Rodolphe Perronet for its completion.

In Fig. 4, the plan, the elevation, and section of the bridge built in 1751 can be seen.

Table 1
Elements that determine a three-centered oval arch for different angles, fixed span and the radius of the lateral circumference.

	Centers of the circumferences	Relationship between a, b and r	Radius	Tangent points	Sagita
$\alpha = 30°$	$A = (\ell - r, 0)$ $B = \left(0, \dfrac{r-\ell}{\sqrt{3}}\right)$	$b = \dfrac{a}{\sqrt{3}}$ $a = \ell - r$	$R = \ell + a\left(\dfrac{2-\sqrt{3}}{\sqrt{3}}\right)$ r fixed	$x = ar\dfrac{\sqrt{3}}{2}$ $y = \dfrac{r}{2}$	$f = \dfrac{\ell}{\sqrt{3}} + r\left(1 - \dfrac{1}{\sqrt{3}}\right)$
$\alpha = 45°$	$A = (\ell - r, 0)$ $B = (0, r-\ell)$	$b = a$ $a = \ell - r$	$R = \ell + a\left(\sqrt{2} - 1\right)$ r fixed	$x = a + \dfrac{r}{\sqrt{2}}$ $y = \dfrac{r}{\sqrt{2}}$	$f = \left(\ell + r\sqrt{2}\right)\left(\sqrt{2} - 1\right)$
$\alpha = 60°$	$A = (\ell - r, 0)$ $B = \left(0, \sqrt{3}\left(r - \ell\right)\right)$	$b = a\sqrt{3}$ $a = \ell - r$	$R = 2\ell - r$ r fixed	$x = a + \dfrac{r}{2}$ $y = \dfrac{r\sqrt{3}}{2}$	$f = \ell\left(2 - \sqrt{3}\right) + r\left(\sqrt{3} - 1\right)$

Table 2
Value of elements that determine a three-centered oval arch, for different angles, fixed the sagita and the radius of the central circumference.

	Centers of the circumferences	Relationship between a, b and r	Radius	Tangent points	Span $\left(= 2\ell\right)$
$\alpha = 30°$	$A = (R - f)\sqrt{3}, \ 0$ $B = (0, \ f - R)$	$b = R - f$ $a = b\sqrt{3}$	R fixed $r = 2f - R$	$x = R\dfrac{\sqrt{3}}{2}$ $y = f - \dfrac{R}{2}$	$\ell = R - 2(R - f)\left(\sqrt{3}\right)$
$\alpha = 45°$	$A = (R - f, \ 0)$ $B = (0, \ f - R)$	$b = R - f a = b$	R fixed $r = R$	$x = \dfrac{R}{\sqrt{2}}$ $y = R\left(\dfrac{1}{\sqrt{2}} - 1\right) + f$	$\ell = R - (R - f)\left(\sqrt{2} - 1\right)$
$\alpha = 60°$	$A = \left(\dfrac{R - f}{\sqrt{3}}, \ 0\right)$ $B = (0, \ f - R)$	$b = R - f$ $a = \dfrac{b}{\sqrt{3}}$	R fixed $r = R - \dfrac{2}{\sqrt{3}}(R - f)$	$x = \dfrac{R}{2}$ $y = f + R\left(\dfrac{\sqrt{3}}{2} - 1\right)$	$\ell = \left(R - \dfrac{R-f}{3}\right)$
α	$A = \left(\dfrac{R - f}{\tan\alpha}, \ 0\right)$ $B = (0, \ f - R)$	$b = R - f$ $a = \dfrac{b}{\tan\alpha}$	R fixed $r = R - \dfrac{R - f}{\sin\alpha}$	$x = r\cos\alpha$ $y = f + R(\sin\alpha - 1)$	$\ell = R - \dfrac{R-f}{\sin\alpha}(\cos\alpha - 1)$

Table 3
Determination of the centers, radius and tangent points of a three-centered oval arch knowing a related proportion between *a* and *b*.

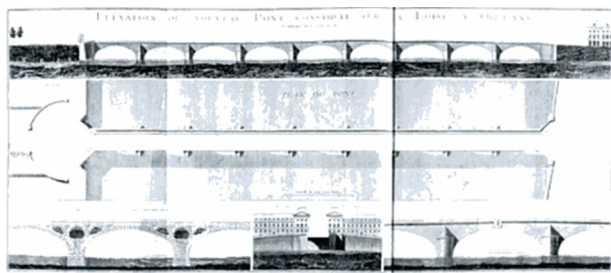|	$\alpha = \dfrac{3\ell}{5}$	$\alpha = \dfrac{4\ell}{5}$	$\alpha = \dfrac{2\ell}{5}$	$\alpha = k\ell, \ \ k > \dfrac{\ell - f}{\ell}$
$\dfrac{b}{a} = \beta,$ $\beta \in \mathbb{R}$	Radius $R = f + \beta\dfrac{3\ell}{5}$ $r = \dfrac{2\ell}{5}$ Centers $A = \left(\dfrac{3\ell}{5}, \ 0\right)$ $B = \left(0, \ -\dfrac{\beta 3\ell}{5}\right)$ Tangent point $\dfrac{\ell}{5}\left(3 + \dfrac{2}{\sqrt{1+\beta^2}}, \ \dfrac{2\beta}{\sqrt{1+\beta^2}}\right)$	Radius $R = f + \beta\dfrac{4\ell}{5}$ $r = \dfrac{\ell}{5}$ Centers $A = \left(\dfrac{4\ell}{5}, \ 0\right)$ $B = \left(0, \ -\dfrac{\beta 4\ell}{5}\right)$ Tangent point $\dfrac{\ell}{5}\left(4 + \dfrac{1}{\sqrt{1+\beta^2}}, \ \dfrac{\beta}{\sqrt{1+\beta^2}}\right)$	Radius $R = f + \beta\dfrac{2\ell}{3}$ $r = \dfrac{\ell}{3}$ Centers $A = \left(\dfrac{2\ell}{3}, \ 0\right)$ $B = \left(0, \ -\dfrac{\beta 2\ell}{3}\right)$ Tangent point $\dfrac{\ell}{3}\left(2 + \dfrac{1}{\sqrt{1+\beta^2}}, \ \dfrac{\beta}{\sqrt{1+\beta^2}}\right)$	Radius $R = \ell + \beta k\ell$ $r = \ell(1 - k)$ Centers $A = \left(k\ell, 0\right)$ $B = \left(0, -\beta k\ell\right)$ Tangent point $\ell\left(k + \dfrac{1-k}{\sqrt{1+\beta^2}}, \ \dfrac{(1-k)\beta}{\sqrt{1+\beta^2}}\right)$

Fig. 4. Plan, elevation and sections of the Orléans Bridge, built in 1751. (Perronet, *La construcción de puentes en el siglo XVIII*, p. 110).

As stated by Perronet [26] (p. 107–109) himself, the new bridge was built in stone, made up of nine vaults in contrast to the nineteen that made up the old one. The length of the bridge between buttresses is 166 toise (old French measurement) and 4 feet (325 m) and its width 46 feet (15 m). The central vault is slightly higher in dimension to the rest; 100 feet (32.5 m) of span and 28 feet (9.1 m) of sagita, while the others have 92 feet (30 m) of span and 25 feet (8.1 m) of sagita. The four central columns are 18 feet (5.8 m) thick and the other four 17 (5.5 m). The two buttresses measure 22 feet (7.15 m thick).

Following Perronet [26] (p. 108–109), the vaults have an oval shape, made up of three-centered oval arches. The central arch is thicker than the remaining ones: 6 feet 6 inches (2.1 m) in the keystone of the central arch, whereas the other arches are 5 feet 6 inches (1.8 m) (both in the buttresses as well as the others) (Fig. 4).

Applying the mathematical study previously carried out, we represented the oval arch of the central section of the Orléans Bridge with the *Mathematica*® program. From the sagita, 9.1 m and the span 32.5 m with the command "Manipulate", drawing the arch for the different values of *a* and *b*. Among several possibilities obtained, we venture two constructions hypotheses (Fig. 5)

Hypothesis 1 (Fig. 5.1):

$\alpha = 60°$ is obtained as follows: $a = 9.77$; $b = 16.92$ and the tangent point $C = (13, 5.61)$.
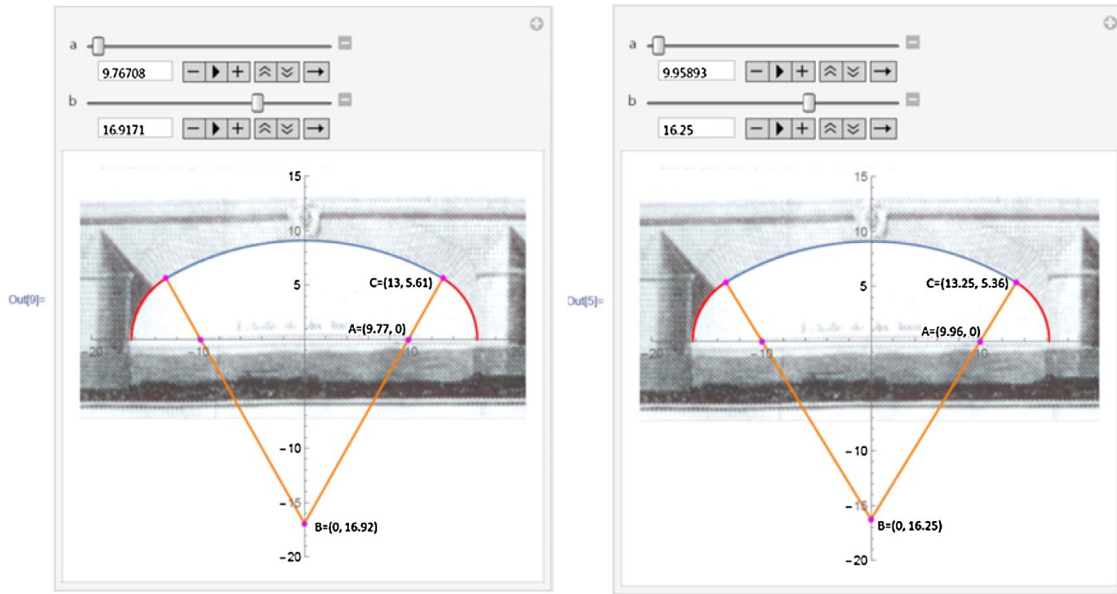
Hypothesis 2 (Fig. 5.2):

Fig. 5. Representation of the oval arch in the Orléans Bridge over the Loire River made the Mathematica program.

$b = \ell = 16.25$. Now is $a = 9.96$ and $C = (13.25, 5.36)$. The angle α value, in this case, $58.5°$.

Superimposing the graph obtained with *Mathematica*® with the detail drawing of the bottom right of Fig. 7, we observe the plausibility of the hypotheses that we are proposing. Both are geometrically very similar.

A third theory is possible built through a graphic procedure (Fig. 6). In this case: $\alpha = 60.75°$, $a = 9.67$ y $b = 17.28$.

2.3. Mathematical analysis of an oval arch of 2n + 1 centers

The process previously described in section 2.1 can be generalized to mathematically express the geometry of an oval arch of $m = 2n + 1$ centers. In this case we realize is more operative to place the origin of the coordinates in the center of the circumference of the largest radius, the center of the arch, as shown on Fig. 7.

Let $C_0, C_1, \ldots, C_n$ be the circumferences that make up the right part of the arch, with radius $r_0, r_1, \ldots, r_n$, and centers

$$B_0 = (a_0, \quad b_0) = (0, \quad 0), \quad B_1 = (a_1 b_1), \quad B_2 = (a_1 + a_2, \quad b_1 + b_2), \ldots,$$

$$B_n = (a_1 + a_2 + \ldots + a_n, \quad b_1 + b_2 + \ldots + b_n)$$

Respectively, the equations of these circumferences are:

$$C_j : \left(x - \sum_{i=0}^{j} a_i\right)^2 + \left(y - \sum_{i=0}^{j} b_i\right)^2 = r_j^2, \quad j = 0, \quad 1, \quad 2, \quad \ldots, \quad n$$

For each circumference,

$$r_j + d\left(B_j, \quad B_{j-1}\right) = r_{j-1}, \quad j = 1, \quad 2, \quad \ldots, \quad n$$

Therefore, by recurrence we obtain

$$r_j = r_0 - \sum_{i=1}^{j} d\left(B_i, \quad B_{i-1}\right), \quad j = 1, \quad 2, \quad \ldots, \quad n$$

If we denote $A = a_1 + a_2 + \ldots + a_n$ and $B = b_1 + b_2 + \ldots + b_n$, the conditions that must be fulfilled are:

$$A + r_n = \ell$$

$$r_0 = r_n + \sum_{i=1}^{n-1} d(B_i, \quad B_{i-1}) \tag{1}$$

$$= r_n + \sum_{i=1}^{n-1} \sqrt{a_i^2 + b_i^2}$$

$$= f + \sum_{i=1}^{n} b_i = f + B$$

where ℓ is the semi-span and f is the sagita.

We look for $A_1, A_2, \ldots, A_n$ where they meet and are tangent two and two the circumferences $C_0, C_1, \ldots, C_n$.

By deriving implicitly in the equations, we obtain the condition that the tangent y' must fulfill in each one of the circumferences:

$$C_j : 2\left(x - \sum_{i=0}^{j} a_i\right) + 2\left(y - \sum_{i=0}^{j} b_i\right) y' = 0, \quad j = 0, \quad 1, \quad 2, \quad \ldots, \quad n$$

Since at each point A_j the two adjacent circumferences must be tangent C_{j-1} and C_j, the derivative y' at that point must be the same in both. Isolating and equalizing in each of the circumferences we obtain:

$$\frac{x - \sum_{i=1}^{j-1} a_i}{y - \sum_{i=1}^{j-1} b_i} = \frac{x - \sum_{i=1}^{j} a_i}{y - \sum_{i=1}^{j} b_i}$$

And substituting in the equation of C_{j-1}, we get the coordinates (x, y) of the point A_j:

$$x = \frac{r_{j-1} a_j}{\sqrt{a_j^2 + b_j^2}} + a_1 + a_2 + \ldots + a_{j-1}$$
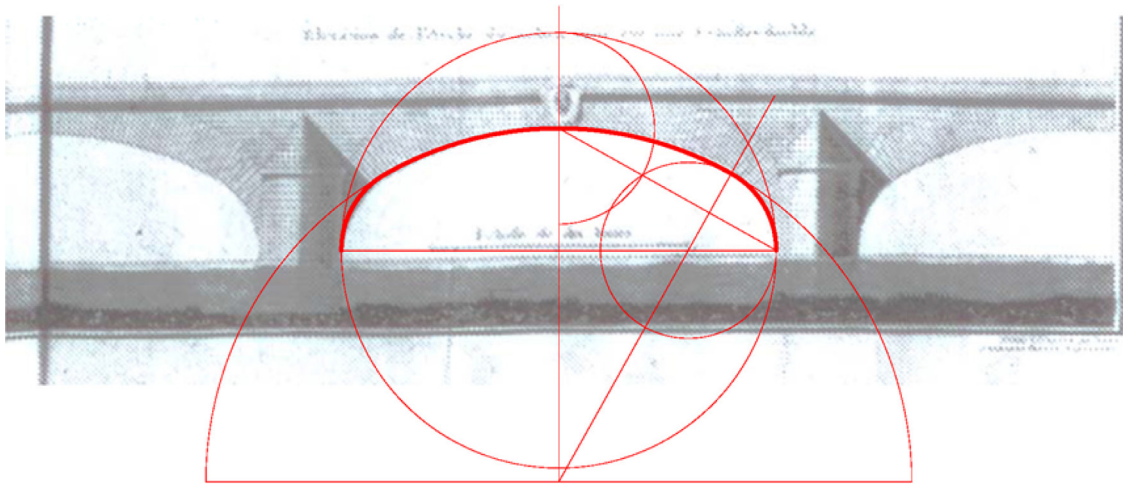
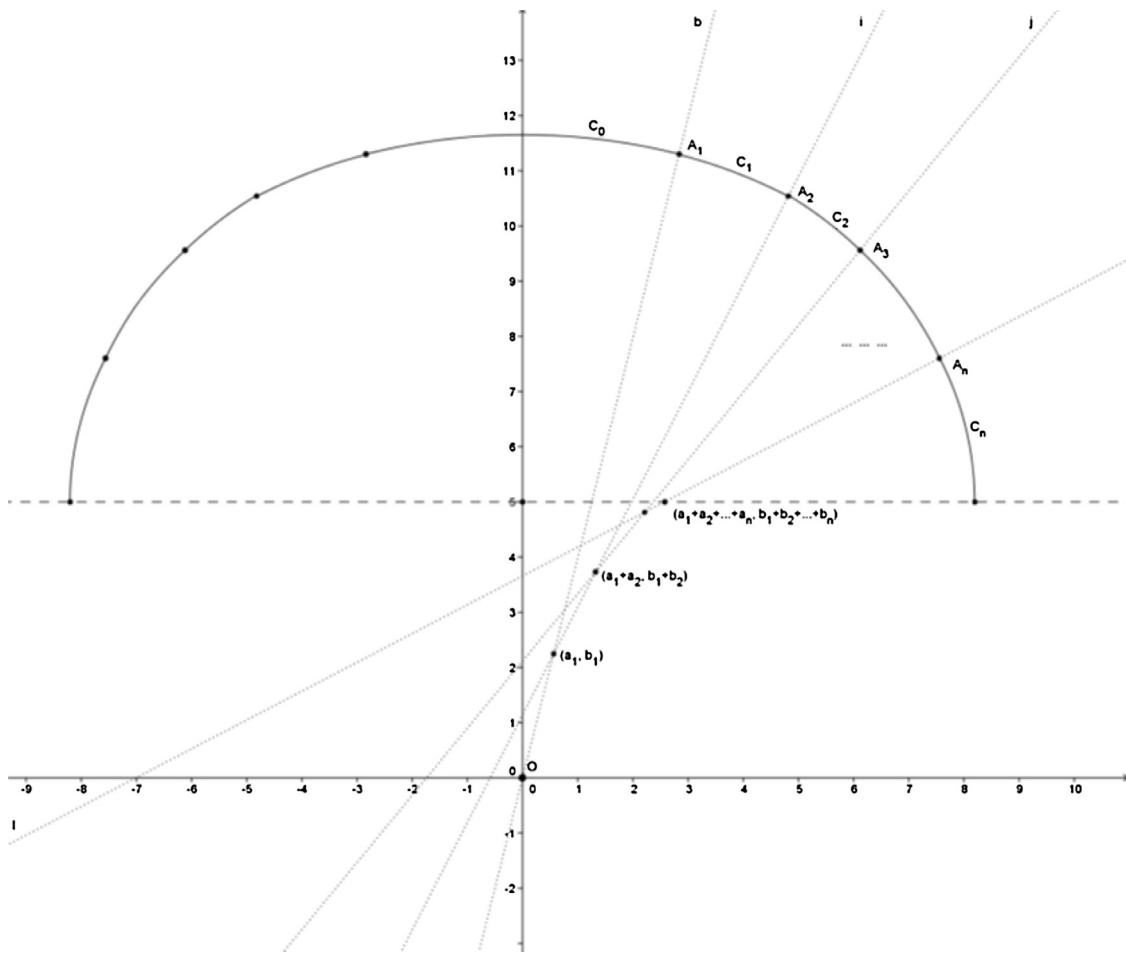**Fig. 6.** Representation of the geometry of the arches through geometric procedure.

Fig. 7. Construction scheme of the oval arch of $2n+1$ centers.

$$y = \frac{r_{j-1}b_j}{\sqrt{a_j^2 + b_j^2}} + b_1 + b_2 + \dots + b_{j-1}$$

By symmetry, we would obtain the intersections of the left side of the arc.

In addition, the slopes of the lines that join the centers with the point of the circumference that corresponds to them must gradually decrease, that is to say the coordinates of these center must fulfill the condition:

$$\frac{b_1}{a_1} > \frac{b_2}{a_2} > \dots > \frac{b_n}{a_n}$$

Our analytical development, outlined in Fig. 7, has allowed us to model with Mathematica the oval arch of several centers and, in
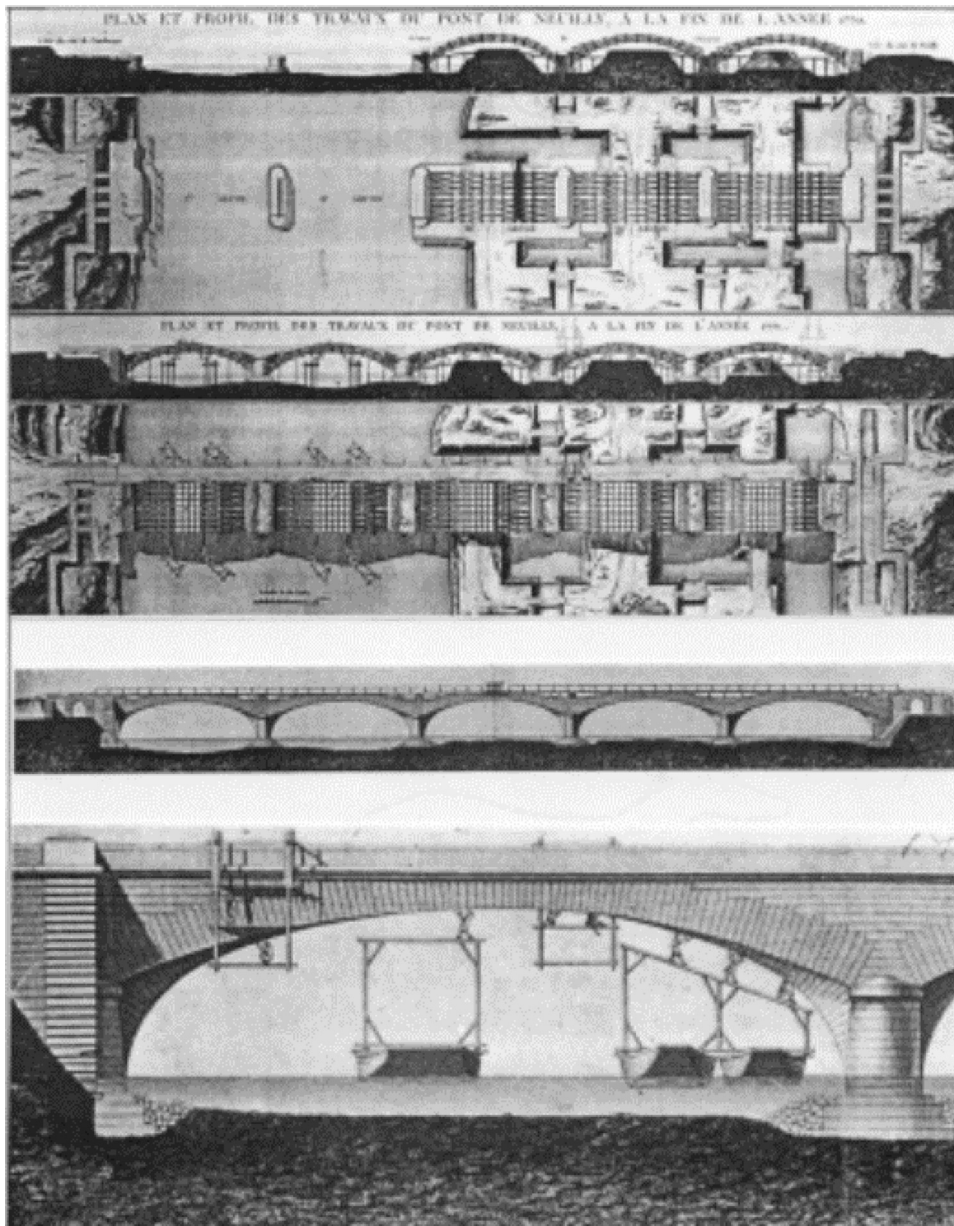
Fig. 8. Neuilly Bridge by Jean Rodolphe Perronet, made of oval arches of 11 centers (Huerta, [13] (p. 367)).

particular, to model and analyze the geometry of the Neuilly Bridge, which we present below.

2.4. Neuilly Bridge over the Seine River. Eleven-centered oval arch

Neuilly Bridge (Fig. 8) was built in the 18th century over the river Seine between the cities of Neuilly-sur-Seine and Courbevoie belonging to the metropolitan area of Paris. As Perronet [26] (p. 1–4), describes in 1766 the project began giving the bridge 5 arches, each one with a span of 100 ft one hundred, this being the total length of all the bridge arches of 10 toises (195 m) and a height from the starting point of the vaults to the keystone of 30 feet (9.75 m) plus 7 ft more under the arches to prevent large increases. "The width of the bridge is 45 ft (14.6 m) from one side to the other of which 29 (9.4 m) is made available for the passing of horse drawn

carts and 6 ft 3in (5.2 m) for each pathway"[3]. In 1942, the bridge by Perronet was replaced with a metallic one.

As Huerta shows in [13] (p. 367): "the bridge adopts two great innovations by Perronet: the use of very surbased oval arches of several centers and the notable reduction in the thickness of the columns which could not resist the push of the arches and during the construction, centerings were necessary in every opening. The simultaneous un-centering was carried out in the presence of the King of France in (Perronet 1788)."

Perronet justifies that the original curve of the vaults was designed based on eleven centered oval arches "in such a way that it made the flow of water easier than if it had been semi-elliptical".

Through the use of oval arches, Perronet intended to replace the old design of bridges consisting in the use of semicircles in which

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[3] Perronet narrates that it is the swell of the river and in 1740, the Neuilly rose to 23ft. The equivalent in toise and feet is: 1 toise = 1.95 m; 1 foot = 32.5 cm.
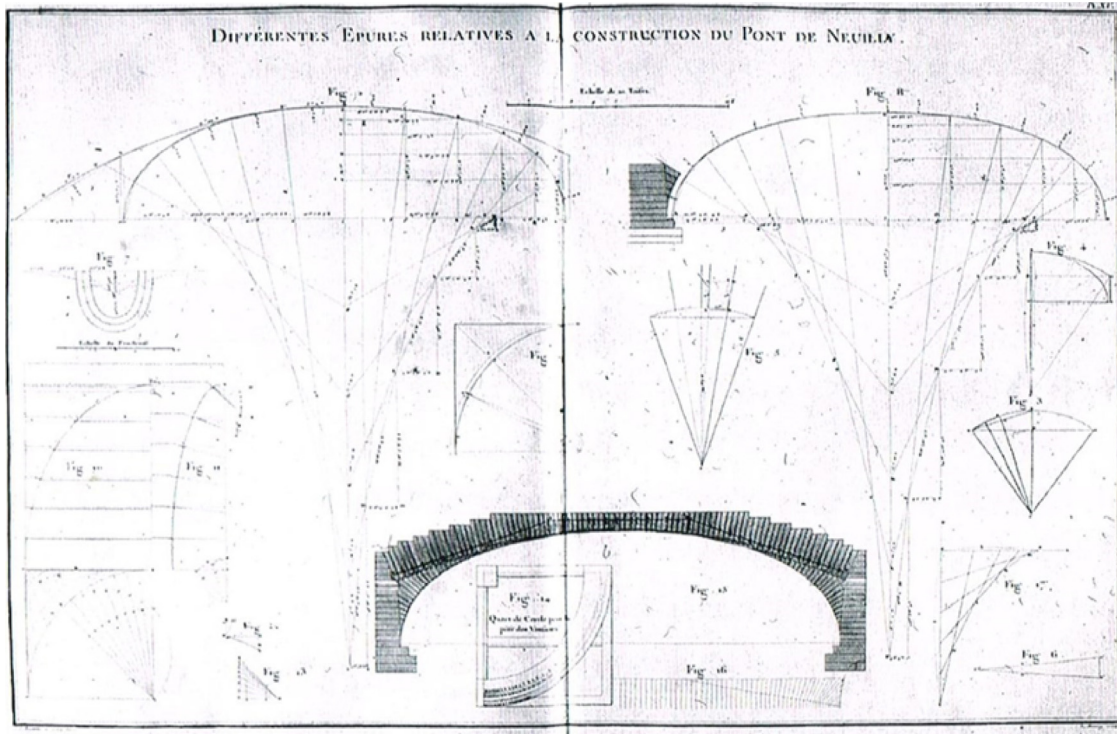
Fig. 9. Curvatures and sagita of the arches Neuilly Bridge, according to a sketch by Jean Rodolphe Perronet [26] (p. 50).

the span is double the height above the keystone, which forced the introduction of a larger number of columns in bridges of certain magnitude which, in turn, was an obstacle to floods and navigation. Perronet justifies the use of oval arches because it is easier and less arduous to execute than an ellipse, since each portion of the ellipse has a different curvature in addition to the areas close to the springs were an obstacle in time of floods since its height not proportional to that of the keystone. Although there are numerous three-centered oval arches used on bridges setting 60-degree angles for each arch or other greater angles in the lower arches to improve the height at the springs, Perronet justifies the use of an eleven-centered oval arch in order to improve the visual and geometric effect. He explains that the three-centered oval arch is unpleasant to the eye for "the sudden move of a curvature of small radius to a curvature of great radius"; However, in an oval arch of several centers, composed of a greater number of different circles, that defect is corrected. Therefore, he decides to construct the Neuilly Bridge with five oval arches each one with eleven centers.

Once decided on the construction of the vaults for the Neuilly Bridge through the design of the eleven-centered oval arch and the span and sagita of the vault are known, the ways to independently design the number of centers is infinite, Perronet imposed conditions on the design (Fig. 9) which we will analyze below.

2.5. Mathematical analysis of the eleven-centered oval arch in the Neuilly Bridge

In the design of the Neuilly Bridge with an eleven-centered oval arch [26], Perronet considers the span, 2ℓ, and the sagita f, and obtains the centers B_i with two additional conditions. The first one is the sum $A = a_1 + a_2 + a_3 + a_4 + a_5$ which is the third part of the distance $B = b_1 + b_2 + b_3 + b_4 + b_5$. The second one is that the intervals between the points of the intersection of the arch radius with the axis of the span "...*are among them 1, 2,3,4,5.* . ." (Here these points will be referred to as D_i) and that the intersections with the vertical axis of the arch are equidistant, "... and by extending them we find

the extension of the small axis in the points *i, 4, 5, 6, 7, B equidistant among them.* . . (These points will be referred to as E_i).

We will generalize a construction of this type of oval arch of $2n + 1$ centers, considering the span 2ℓ, and the sagita f, using the note and calculations that we obtained in our previous mathematical development (Fig. 10) and with help of Mathematica software.

Being $A = a_1 + a_2 + \ldots + a_n$ and $B = b_1 + b_2 + \ldots + b_n$ we suppose that $B = mA$ (in the bridge of Perronet it is iqual to 3). The points E_i (Fig. 10) will be:

$$E_i = \left(0, \quad \frac{m}{n}iA\right), \quad i = 0, \quad 1, \quad 2, \quad \ldots, \quad n$$

The intersections of the radius with the axis span maintain between them the relation 1, 2, ..., n, in a way that the point D_i is within a distance $(n + (n-1) + \ldots + (n-i+1))\frac{A}{1+2+\ldots+n}$ of the center of the arch. Given that

$$1 + 2 + \ldots + n = \frac{(n+1)n}{2} = \frac{n^2+n}{2},$$

$$n + (n-1) + \ldots + (n-i+1) = \frac{2ni - i^2 + i}{2}$$

it is deduced that

$$D_i = \left(\frac{2ni - i^2 + i}{n^2 + n}A, \quad mA\right)$$

The center B_i is the intersection of the line that meets the points D_{i+1} and E_i with the lines that passes through D_i and E_{i-1}. The equation of the first line is:

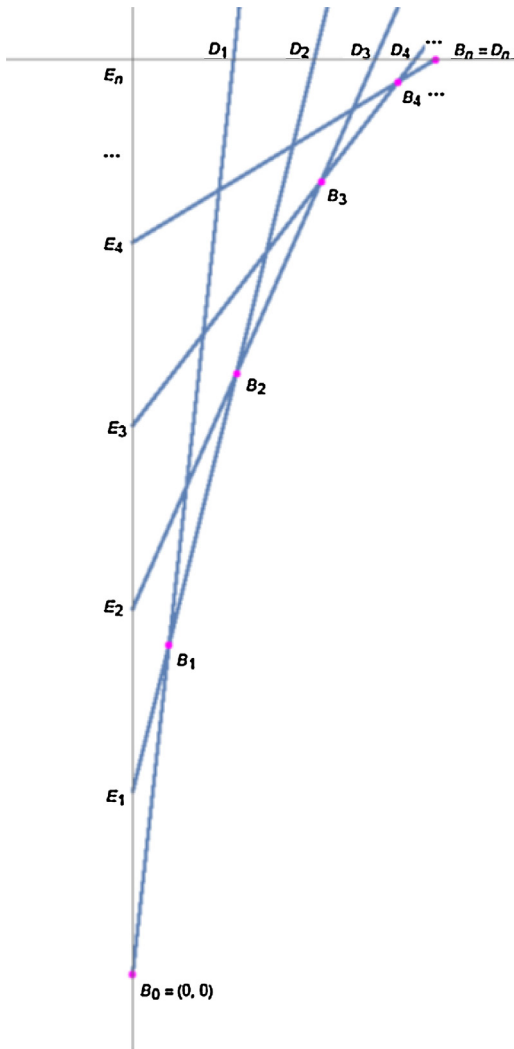$$y = \frac{m}{n}\left(Ai + \frac{(i-n)\left(n^2+n\right)}{(i+1)(i+2n)}x\right)$$

Fig. 10. Construction scheme of the oval arch of $2n+1$ centers based on the one carried out by Perronet Neuilly Bridge.

Substituting i for $i-1$ in this equation we obtain the expression of the other line. The coordinates of the intersection point $B_i = (x_i, y_i)$ that are obtained are:

$$x_i = \frac{Ai}{(n(n+1))} \frac{(i+1)(i-1-2n)(i-2n)}{i^2 + 2(n^2+n) - i(1+2n)}$$

$$y_i = \frac{Aim}{n} \frac{2i^2 + n(3+4n) - i(2+5n)}{i^2 + 2(n^2+n) - i(1+2n)}$$

$$a_i = x_i - x_{i-1}, \quad b_i = y_i - y_{i-1}$$

With these values, using the formula developed in the previous section we will calculate the points of tangency. First of all, we need to determine the value A. In the design of the Neuilly Bridge it has $m = 3$, $n = 5$ so we have $B = 3A$ and $r_n = \ell - A$. According to condition (1)

$$\sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2} + r_n = \sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2} + \ell - A$$

$$= f + \sum_{i=1}^{n} b_i = f + B = f + 3A$$

that is

$$A = \frac{1}{4}\left(\sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2} + \ell - f\right)$$

And so, from the Neuilly Bridge's data $2\ell = 120ft \approx 39m$, $f = 30ft \approx 9.75m$, we deduce the value $A = 13.015m$.

The values obtained for the centres are:

$a_1 = 1.562$; $a_2 = 2.889$; $a_3 = 3.644$; $a_4 = 3.28$; $a_5 = 1.64$;

$b_1 = 14.056$; $b_2 = 11.583$; $b_3 = 8.2$; $b_4 = 4.295$; $b_5 = 0.911$;

$B_0 = (0,0)$; $B_1 = (1.562, 14.056)$; $B_2 = (4.462, 25.658)$;

$B_3 = (8.098, 33.839)$; $B_4(11.388, 38.069)$; $B_5 = (13.015, 39.045)$

And the radius:

$r_0 = 48.795$; $r_1 = 34.652$; $r_2 = 22.714$;

$r_3 = 13.74$; $r_4 = 8.337$; $r_5 = 6.485$

The value of A used by Perronet was obtained in an ingeniously manner. He explains (adapting the nomenclature of Perronet to our notes), "... *the state of the question gives* $(z + \ell - A = B + f)$;" where

$$z = \sum_{i=1}^{n} \sqrt{a_i^2 + b_i^2}$$

Perronet calculates the value of z for a standard figure with the proportion of 1/3 (this value we call S) and get the value A, taking into account that by similarity it has $1/S = A/z$, "... it was calculated with the table of sines the value of S and then A or i12, that are found at 39 ft 10 in 8 lines...". We can observe that this value A used by Perronet (39 ft 10 in 8 lines) is very close to the one ($A = 13.015m$) we have obtained with Mathematica tools. We have applied this way of building an eleven-centered oval arch to design with Mathematica the interactive graph in Fig. 11, where we can clearly see that, applying the data previously obtained, our model corresponds to the drawing by Perronet for the Neuilly Bridge. In addition, this construction with Mathematica models any eleven-centered oval arch, varying the data of the span and sagita.

A construction of this type for a three-centered oval arch would correspond to the values of $n = 1$, $b = ma$ (Table 3). That is to say we would be in case 3, of obtaining the arch given the span and the sagita.

Conclusions

The mathematical analysis of a three-centered oval arch which we have carried out in this work, has allowed us to obtain expressions of all the elements that operate in the construction of said arches. To make the reading and handling of said data easier, we have presented in table form these expressions, in the following cases:

- Case 1 knowing the span and radius of the lesser circumference;
- Case 2 knowing the sagita and radius of the greater circumference;
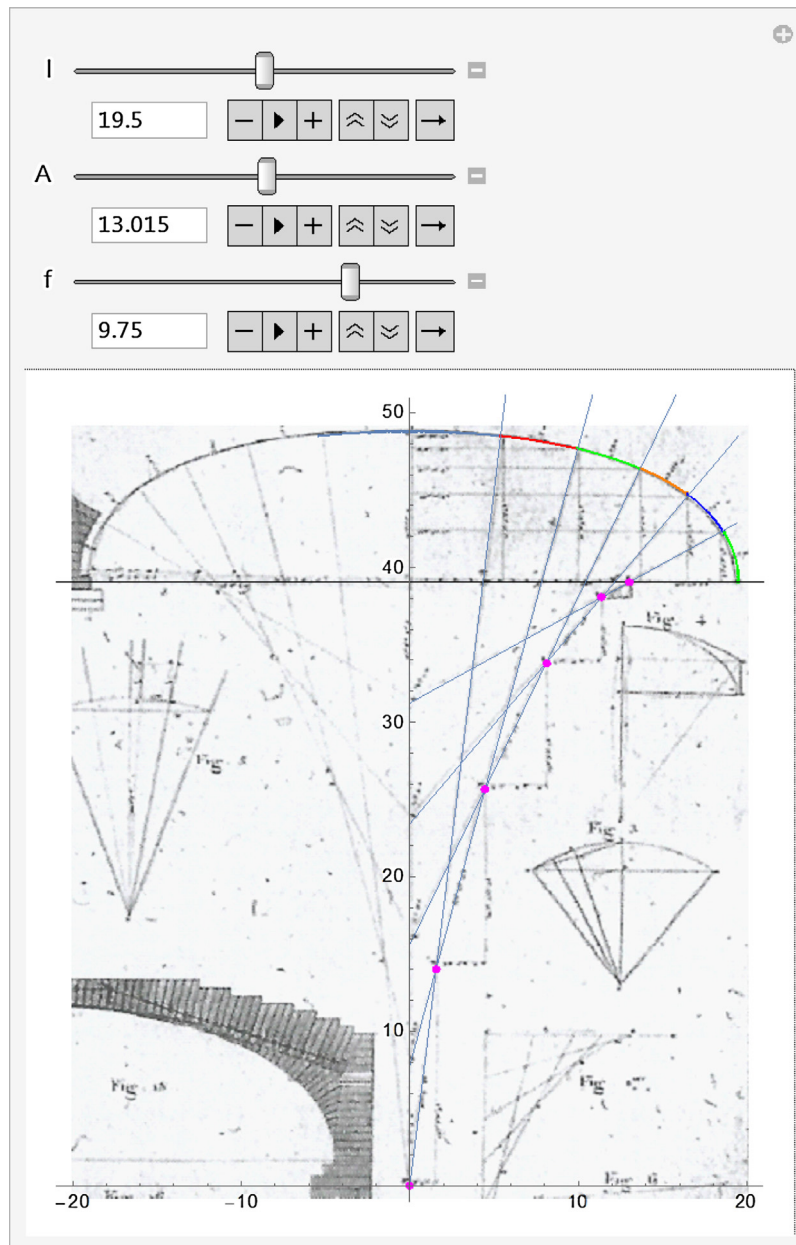- Case 3 knowing the sagita and the span.

Fig. 11. Interactive graph with Mathematica superimposed on the drawing by Perronet for the Neuilly Bridge.

With the methodology used and with the help of the Mathematica program, we have studied the design of the Orléans Bridge by Perronet. We conclude with three possible construction hypotheses, graphically very similar, with a value of angle α equal to or very close to 60° in all of them.

On the other hand, we have developed a geometric-mathematical construction for the oval arch of $2n+1$ centers, obtaining the expressions of the points of tangency. Applying this study to the an eleven-centered oval arch and imposing the same conditions that Perronet used in the construction of the Bridge of Neuilly, we were able to deduce mathematically all the necessary data for its construction, starting from the drawings of author.

This way of approaching the geometric problem of the construction of oval arches has allowed us to carry out with Mathematica interactive graphs that can be used in two ways: to mathematically model an existing arch or to design the construction of a new one (Figs. 5 and 11).

References

[1] P. González Serrano, Prehistoria y primeras civilizaciones, Historia Universal del Arte, Tomo 1, ed. Espasa Calpe, Madrid, 1996.
[2] S. Serlio, Tercero y cuarto libro de Architectura de Sebastiano Serlio Boloñes. Translated by Francisco de Villalpando Architecto, Casa de Iván de Ayala, Toledo, 1552 (Edición facs. Barcelona: Serie Arte y Arquitectura, Editorial Alta Fulla, 1990) (Edición digitalizada en Huerta, S. (2004) Selección de Tratados españoles de Arquitectura y Construcción de los siglos XVI al XX).
[3] H. Ruiz el Joven, El libro de arquitectura de Hernán Ruiz el Joven. Study and critical edition by Pedro Navascués Palacio, ETSA, Madrid, 1974.
[4] A. Vandelvira, Libro de cortes de cantera de Alonso de Vandelvira, arquitecto. Sacado a la luz por Philipe Lázaro de Goiti, arquitecto, Biblioteca Nacional de España, 1646 (Edición digitalizada).

[5] J. de Torija Breue, Tratado de todo genero de bobedas, Pablo de Val, Madrid, 1661 (Edición facsímil de 1981, Valencia: Colección Juan de Herrera dirigida por Luis Cervera, Vera Albatros Ediciones).

[6] Simonin, Tratado Elemental de los Cortes de Canteria, o Arte de la Montea, Imprenta de la viuda de Josef Garcia, 1795 (Edición facsimil. Zaragoza: Arquitecturas de los Precursores, Comisión de Cultura COA de Aragón, Gráficas Mola, 1994.) (Edición digitalizada en Huerta, S. (2004). Selección de Tratados españoles de Arquitectura y Construcción de los siglos XVI al XX).

[7] A. Choisy, Histoire de l'architecture, Gauthier-Villars, imprimeur-libraire, traducción propia, Paris, 1899.

[8] E. Viollet-Le-Duc, Dictionnaire raisonné de l'architecture française du XIe au XVIe siècle, 10 vol., B. Bance, Paris, 1854–1868 (Digital version in https://www.fr.wikisource.org).

[9] J.M. Gentil Baldrich, Planta oval y traza eliptica en la arquitectura: Consideraciones geométricas y un ejemplo español, in: E. Carazo, J.M. Otxotorena (Eds.), Arquitecturas Centralizadas. El espacio sacro de planta central: diez ejemplos en Castilla y León, Universidad de Valladolid, Valladolid, 1994.

[10] J.M. Gentil Baldrich, La traza oval y la Sala Capitular de la catedral de Sevilla. Una aproximación geométrica, in: J.A. Ruiz de la Rosa, et al. (Eds.), Quatro edificios sevillanos, Sevilla, Colegio Oficial de Arquitectos de Andalucia, Demarcación Occidental, 1996, pp. 73–147.

[11] M. Fernández Gómez, La planta oval. Traza y simbolo. Loggia, Arquitectura & Restauración 3 (1996) 16–21.

[12] P. Rosin, On Serlio's construction of ovals, Math. Intell. 23 (1) (2001) 58–69.

[13] S. Huerta Fernández, Arcos, bóvedas y cúpulas. Geometria y equilibrio en el cálculo tradicional de estructuras de fábrica, Instituto Juan de Herrera, Madrid, 2004.

[14] S. Huerta Fernández, Oval Domes: history, geometry and mechanics, Nexus Netw. J. 9 (2007) 211–248.

[15] F. Garcia Jara, Las cúpulas de la arquitectura religiosa de la provincia de Alicante: del Renacimiento al siglo XIX (Ph.D. Thesis), 2008.

[16] F. Garcia Jara, La estereotomía de las cúpulas sobre base oval, in: X Congreso Internacional de Expresión Gráfica aplicada a la Edificación, APEGA, 2010, pp. 767–780.

[17] A. López Mozo, Ovals for any given proportion in architecture: a layout possibly known in the sixteenth century, Nexus Netw. J. 13 (2011) 569–597.

[18] J. Barrallo, Ovals and ellipses in architecture. Proceedings of ISAMA 2011, Columbia College, Chicago, Illinois, 2011, pp. 9–18.

[19] A. Mazzotti, A euclidean approach to eggs and polycentric curves, Nexus Netw. J. 16 (2) (2014) 345–387.

[20] A. Mazzotti, What Borromini might have known about ovals. Ruler and Compass Constructions, Nexus Netw. J. (2014) 389–415.

[21] A. Mazzotti, All sides to an oval. Properties, parameters and Borromini's mysterious construction, Springer, 2017.

[22] E. Capilla Tamborero, V. Calvo Roselló, Chapter 8: La cúpula interior de la Baslica de la Virgen de los Desamparados de Valencia. Análisis geométrico y matemático, in: J.C. Navarro (Ed.), Bóvedas Valencianas. Arquitecturas ideales, reales y virtuales en época medieval y moderna, UPV (Scientia), Valencia, 2014, pp. 196–237 (ISBN: 978-84-9048-188-2).

[23] E. Capilla Tamborero, Geometra, arte y construcción. Las bóvedas de los siglos XIII a XVI en el entorno valenciano (Ph.D. thesis), Universitat Politècnica de València, 2016 http://www.hdl.handle.net/10251/63672.

[24] S. Huerta Fernández, R. Hernando De La Cuerda, La teora de bóvedas en el siglo XVIII: la contribución de Philippe de la Hire, in: F. Bores, J. Fernández, S. Huerta, E. Rabasa (Eds.), Actas del Segundo Congreso Nacional de Historia de la Construcción, Instituto Juan de Herrera, SEdHC, U. Coruña, CEHOPU, Madrid, 1998, pp. 233–244.

[25] A. Rovira y Rabassa, Estereotomia de la piedra, Libreria y estamperia artsticia. 1ª parte, Barcelona, 1897.

[26] J.R. Perronet, La construcción de puentes en el siglo XVIII, ed. Instituto Juan de Herrera, Ministerio de Fomento-Cedex-Cehopu, Madrid, 2005.