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Design, analysis and stability of iterative methods for solving nonlinear problems

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Abstract

In every science discipline, from engineering to economics, finding solutions of an equation is essential. As iterative methods allow us to find solutions of nonlinear equations, there exist in the literature plenty of studies about these methods, such us Newton, Traub or Chebyshev. In this work, we present and design different families of methods depending on parameters. By using complex dynamics tools, we will compare several methods in order to find those ones with a good and stable behavior, by means of the properties of the rational function obtained when they are applied on quadratic polynomials. The stability of the methods plays an important role in their reliability when they are applied on different problems. It is also important to focus on their order of convergence, what means the speed at which the method reaches the solution. The study of fixed points, together with the critical points and the development of their respective parameter and dynamical planes, represent the variety of the presented classes and enable to select the best elements of the families.

Resumen

En todas las disciplinas de la ciencia, desde la ingeniería hasta la economía, encontrar las soluciones de una ecuación es esencial. Puesto que los métodos iterativos nos permiten encontrar dichas soluciones de ecuaciones no lineales, existen en la literatura numerosos estudios sobre estos métodos, entre los que cabe destacar los de Newton, Traub o Chebyshev. En este trabajo, presentamos y diseñamos diferentes familias de métodos dependientes de ciertos parámetros. Empleando herramientas de dinámica compleja, se compararán varias clases con el fin de encontrar aquellas con un comportamiento bueno y estable, mediante las propiedades de la función racional obtenida cuando se aplican a polinomios cuadráticos. La estabilidad de los métodos juega un papel importante en su fiabilidad cuando son aplicados en distintos problemas. Asimismo, es importante centrarse en su orden de convergencia, el cual indica la velocidad a la que se alcanza la solución. El estudio de los puntos fijos, junto con los puntos críticos y sus planos de parámetros asociados, muestran la riqueza de las clases presentadas y nos permiten seleccionar los mejores elementos de las familias.

Resum

En totes les disciplines de la ciència, des de l'enginyeria fins a l'economia, trobar les solucions d'una equació és essencial. Ja que els mètodes iteratius ens permeten trobar les dites solucions d'equacions no lineals, existixen en la literatura nombrosos estudis sobre estos mètodes, entre els que cal destacar els mètodes de Newton, Traub o Chebyshev. En este treball, presentem i dissenyem diferents famílies de mètodes dependents de certs paràmetres. Emprant ferramentes de dinàmica complexa, es compararan unes quantes classes a fi de trobar aquelles amb un comportament bo i estable, per mitjà de les propietats de la funció racional obtinguda quan s'apliquen a polinomis quadràtics. L'estabilitat dels mètodes juga un paper important en la seua fiabilitat quan són aplicats en distints problemes. Així mateix, és important centrar-se en la seua orde de convergència, el qual indica la velocitat a què s'aconseguix la solució. L'estudi dels punts fixos, junt amb els punts crítics i els seus plans de paràmetres associats, mostren la riquesa de les classes presentades i ens permeten seleccionar els millors elements de les famílies.

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Chapter 1

Introduction

Many real problems in Science and Engineering require the resolution of equations or systems of equations that, in general, are nonlinear. Nonlinear equations have generally analytical solution. That is why the manner to solve them is by approximating their solutions through iterative techniques. There are plenty of different types of fields in science whose study is developed through nonlinear equations and system. For example, related to Engineering of Telecommunications, these equations are found in electronics and space communications, among others. For instance, we can find the following examples of nonlinear equations $f(z) = 0$ or systems $F(z) = 0$:

The flow of electrical current in a circuit comprised by a resistor R, an inductance L and a capacitor C is defined by

$$i(t) = e^{-Rt/2L} cos\left(\sqrt{(4L/C) - R^2t/(2L)}\right). \tag{1.1}$$

In which instant t does the intensity of the current take a concrete value?

The current i (in microamperes μAs) in a diode is related to the voltage v (in volts) by the following equation:

$$i = I_s(e^{v/\theta} - 1), \tag{1.2}$$

where I_s is the saturation current in microamperes and θ the diode variable. For example, a diode with $I_s = 20$ and $\theta = 5.2$ is connected to a circuit in which v and i must satisfy $v + i = 4$. We want to determine all the possible solutions of the equation.

In order to study the movement of celestial bodies with elliptic orbits, the well-known Kepler equation must be solved:

$$M = E - e \cdot sinE, \tag{1.3}$$

where e is the orbit eccentricity, M the average anomaly and the unknown variable is the eccentric anomaly E. For instance, it is used for solving the equation for the eccentricity and the average anomaly of Halley's Comet.

Furthermore, Global Positioning System (GPS) obtains the position (x,y,z) of the observer and his clock bias by solving a system of equations defined by four different satellites:

$$\rho_i = \sqrt{(x_i - x)^2 + (y_i - y)^2 + (z_i - z)^2} + c \cdot dt, \qquad i = 1, 2, 3, 4. \tag{1.4}$$

Therefore, four satellites are needed. In the case that there are only three, the system of equations to solve is the following:

$$\rho_j = \sqrt{(\Delta_j^1)^2 + (\Delta_j^2)^2 + (\Delta_j^3)^2} + c \cdot t_u, \quad j = 1, 2, 3 \tag{1.5}$$

$$\frac{x^2 + y^2}{(a+h)^2} + \frac{z^2}{(b+h)^2} = 1 \tag{1.6}$$

Where $\Delta_j^1 = x_j - x$, $\Delta_j^2 = y_j - y$ and $\Delta_j^1 = x_j - x$.

Usually, these problems can not be analytically solved, so we use iterative schemes. The creation of iterative methods for solving equations and systems is a relevant and challenging task in the field of numerical analysis. The best known is Newton's method, whose characteristics have been improved in many researches (see, for instance, the texts [1], [2] and the references therein).

Many different techniques have been employed to design these new methods, which can be classified by different criterions. They can be categorized, for instance, by their order of convergence, that is the velocity at which methods converge to a concrete point, or by Kung-Traub conjecture [3], which characterize methods by optimal or not optimal.

In recent times, an utilized technique to enlarge the order of convergence consists on methods composition, resulting in multistep schemes with order of convergence as product of both individual methods' order. In these cases, the challenge is to decrease the number of evaluations resulting in an optimal method. Related to this field, analysis tools of complex dynamics associated to iterative methods has been object of many works (see [4], [5], [6], [7], [8]...). This analysis include the asymptotic behavior of fixed points, being these ones roots of the equation under study with addition to different points, the basins of attraction associated to each one of the attractive fixed points and graphic representations, as dynamic and parameter planes. Due to these tools, we will be able to choose the member of the family with more stability.

Throughout this work, families of iterative methods for solving nonlinear equations and systems found in the literature will be studied by applying complex dynamic tools. We will also design new families and carry out their respective study in order to obtain optimal methods, taking into account the order of convergence and computational efficiency.

The structure performed for this Final Degree Project is the following:

First, in this section it has been introduced the necessity and importance of this work, related to the relevance of the resolution of iterative methods for solving nonlinear equations. Afterwards, the objectives and methodology of this work will be explained.

Chapter 2 shows previous notions necessary to understand and follow the study carried out in this project. These basic concepts will be used and developed later in this work regarding to the application of complex dynamic in iterative methods, such as the analysis of fixed and critical points and dynamical and parameter planes.

In Chapter 3 a study of a family of iterative methods designed by Kou ([9]) will be carried out. Its dynamics will be developed and analyzed through dynamic techniques explained in the section of Basic Concepts, with the aim of selecting the most stable members of the family by applying those dynamic tools.

Chapter 4 develops the design and analysis of a parametric class of iterative methods. The dynamical study is carried out differentiating two schemes of the same family. The difference between them is their order of convergence, which has been increased from four to eight in order to improve the speed at which the method reaches the solution of the equation to which it is applied. The dynamical study developed in this chapter allows to choose the more efficient and stable elements of each one of the schemes.

In Chapter 5 we construct a new class of iterative methods for solving nonlinear problems, based on a weight function. This weight function leads to many well-known schemes, varying the value of the parameter of the family. The dynamical behavior of this family will also be studied, resulting in the selection of the best schemes of the family, in terms of stability and efficiency.

At the final point, the overall conclusions obtained after examining all the results achieved with the development of the work will be shown. Additionally, new ideas that came up throughout the project will be named as further work. Finally, the bibliography used will be desplayed.

1.1 Objectives

In this Final Degree Project, the main objective is to develop a deep analysis of parametric families of iterative methods for solving nonlinear equations and systems that enables to understand the behavior of the class and also to obtain the best and most stable schemes of each family.

To accomplish this general objective, specific goals should be arranged, as previous steps to reach that main objective. Within these specific aims, the following are to be found:

- Study of complex dynamic existing tools in order to achieve the successive analysis. These resources are used for classifying and comparing iterative schemes with the same order of convergence.
- Research of existing families of iterative methods in the literature, so as to know and understand the latest findings.
- Design of new schemes employing knowledge acquired. The aim of constructing these methods is to obtain the best behavior possible, either by accelerating their convergence or by improving the computational efficiency.
- Application of skills of dynamic analysis to study the parametric families.
- Selection of the best elements of each family, as a result of the analysis made, which will be the ones with the best characteristics, in terms of complex dynamics.

1.2 Methodology

In this section, methodology employed in this work is described, with the goal of achieving the objectives mentioned before. This Final Degree Project has been written in accordance with the applicable regulations of Higher Technical School of Telecommunications Engineering of Polytechnic University of Valencia. The procedure carried out to analyze and select information in order to develop this project has been the following:

First of all, it has been conducted a research, acquisition and selection of information in the literature related to the subject of this work. All these data were deeply read in order to extract the most relevant facts and understand what has been achieved in the last years. Several sources of information have been used, being most of them articles published in scientific journals. When any peace of information from these sources has been used within the work, it will be correctly cited.

From last studies found in the literature, it has been developed a study of discovered methods and, after their analysis and comprehension, conclusions obtained have allow to design new iterative schemes, whose behavior has been determined as stable and optimal by complex dynamic tools.

In order to develop numerical and symbolic dynamic analysis, several programs have been used, such as Matlab and Mathematica. Mathematica enable to develop numerical analysis, as finding the rational function of a method applied on quadratic polynomials, along with their fixed and critical points. In Matlab we have generated the code for the representation of dynamic and parameter planes of the different families.

Chapter 2

Basic concepts

2.1 Iterative methods

With the aim of obtaining an estimated solution of a nonlinear equation, iterative methods are used. Nevertheless, not all of them work the same way. The analysis of the stability, the order of convergence, the computational efficiency of the method, among others, characterize the different iterative classes. The work will be developed around iterative methods which, in some circumstances, find an approximation of a root α from a nonlinear equation $f(z) = 0$.

Iterative methods can be classified according to different criterions. They can be called with or without memory depending on the information needed with the objective of obtaining the following iteration. In this way, a method without memory can be described as it follows:

$$z_{k+1} = \Phi(z_k), k = 0, 1, 2, \dots \tag{2.1}$$

Whereas one with memory will have the expression

$$z_{k+1} = \Phi(z_k, z_{k-1}, z_{k-2}, ...), k = 0, 1, 2, \dots \tag{2.2}$$

In addition, each method can be classified in one-point and multipoint schemes. One-point iterative schemes are schemes in which the (k+1)th-iterate is achieved using evaluations only of kth-iterate, following the expression (2.1).

Newton's method is one of the most popular schemes for solving nonlinear equations, being its iterative expression:

$$z_{k+1} = z_k - \frac{f(z_k)}{f'(z_k)}, \quad k = 0, 1, \dots \tag{2.3}$$

The maximum order of convergence that can be reached by a one-point schem that uses d evaluations per step is $p = d$. Nevertheless, Traub [10] proved that, so as to design a one-point method of order p, it is necessary that the iterative expression contains derivatives at least of order $p - 1$. Therefore, it is interesting to enlarge the number of steps with the objective of increasing the order of convergence and the computational efficiency. In multipoint schemes, also known as predictor/corrector methods, we achieve the (k+1)th-iterate using functional evaluations of the kth-iterate and additionally different points, having an expression similar to the following:

$$y_k = \Psi(z_k), z_{k+1} = \Phi(z_k, y_k), \quad k = 0, 1, \ldots \tag{2.4}$$

These way, the order of convergence is enlarged without increasing functional evaluations.

The last criterion to classify iterative methods is the presence or absence of derivatives. Free derivative schemes allow that any method can be used. To that end, an habitual technique is the replacement of derivative by divided differences. Throughout the work, multipoint methods without memory and with derivatives will be subject of study.

The order of convergence, already mentioned before, is the speed at which methods reach the root α, solution of $f(z) = 0$. Let $\{z_k\}_{k\geq 0}$ be a sequence obtained through an iterative scheme, the sequence converges to the root α with order of convergence p if

$$\lim_{k\to\infty} \frac{|z_{k+1} - \alpha|}{|z_k - \alpha|^p} = C, \tag{2.5}$$

being C is the asymptotic error constant, $C > 0$.

The approximation's error in the *k*th-iteration can be denoted as $e_k = z_k - \alpha$. In this case, the error equation of a method of order *p* is the following:

$$e_{k+1} = e_k^p + O(e_k^{p+1}). \tag{2.6}$$

In order to compare iterative procedures, there exist different measures. Traub in [10] introduced the *informational efficiency* of an iterative method, with order of convergence *p* and *d* number of functional evaluations, as

$$I = p/d. \tag{2.7}$$

Otherwise, Ostrowski in [11] defined the *efficiency index*:

$$EI = p^{1/d}. \tag{2.8}$$

In this vein, Kung and Traub establish in [3] the definition of optimal method. Traub and Kung conjecture denotes that the order of convergence of method without, which has *d* functional evaluations per iteration satisfies

$$p \leq 2^{d-1}, \tag{2.9}$$

being the optimal method the one which satisfies the equality.

A conventional technique to enlarge the order of convergence of a method consists on *composition of methods*, described in [12]. Let p_1 and p_2 the order of convergence of two different methods, it can be obtain a new method with order $p = p_1 \cdot p_2$. Nevertheless, this composition increases the number of functional evaluations, affecting the value of the efficiency index.

On the other hand, weight function procedure is another way to enlarge the order of convergence, but without adding evaluations. Functions with one or several variables can be used.

2.2 Complex dynamics

The utilization of techniques of complex dynamics is a useful way to compare iterative procedures. From the application of discrete dynamics techniques to the associated fixed point operator of iterative methods, we will conduct the dynamical study. This study consists on the analysis of the rational function, result of the fixed point operator applied on a polynomial function. After the analysis, we will obtain fixed points that, generally, will coincide with the roots of the polynomial. Dynamics of these points will allow us to determine the stability of the iterative method. Graphic representation of the methods will allow to acquire conclusions related to method's properties.

In this Final Degree Project, different families (3.1, 4.1, 5.1) will be analyzed through complex dynamic tools applied on quadratic polynomials. It can be seen in the literature ([13]) that using an affine map it is possible to transform the roots of a polynomial without modifying the qualitative dynamic behavior of the family. Therefore, it can be applied with $p(z) = (z-a)(z-b)$. By this quadratic polynomial, the families have operators which are rational functions (3.2, 4.2, 5.6), depending on parameter of the families and also on the roots of the polynomial a and b.

The rational map $h(z) = \dfrac{z-a}{z-b}$ was used by Blanchard in [14]. This is a Möbius transformation that satisfies:

$$\text{i) } h(\infty) = 1, \quad \text{ii) } h(a) = 0, \quad \text{iii) } h(b) = \infty.$$

Blanchard also verified that the well-known Newton's operator on quadratic polynomials is conjugate to z^2. Likewise, the operator of the families under study on quadratic polynomials are conjugated to different operators (3.3, 4.2, 5.7). In these new operators, the roots of the polynomial a and b do not appear.

It is convenient to mention basic notions of complex dynamics (fully developed in [13]) that will appear in this project.

With a rational function $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, in which $\hat{\mathbb{C}}$ is the Riemann sphere, the *orbit of a point* $z_0 \in \hat{\mathbb{C}}$ is the following:

$$\{z_0,\, R(z_0),\, R^2(z_0), \ldots, R^n(z_0), \ldots\}.$$

The map R is studied making a classification of the initial points depending on the orbits' asymptotic character.

A $z_0 \in \hat{\mathbb{C}}$ is named *fixed point* when $R(z_0) = z_0$. A *periodic point* z_0 with period $p > 1$ satisfies $R^p(z_0) = z_0$ and $R^k(z_0) \neq z_0$, for $k < p$. A *pre-periodic point* is different from a periodic point but there is a $k > 0$ that makes $R^k(z_0)$ periodic. A *critical point* z_0 applied to the derivative of the rational function makes it null, $R'(z_0) = 0$. Furthermore, fixed points z_0 can be named *attractive* when $|R'(z_0)| < 1$, *superattractive* when $|R'(z_0)| = 0$, *repulsive* when $|R'(z_0)| > 1$ and *parabolic* when $|R'(z_0)| = 1$.

The basin of attraction of an attractive point α can be defined as:

$$\mathcal{A}(\alpha) = \{z_0 \in \hat{\mathbb{C}} \;:\; R^n(z_0) \to \alpha,\; n \to \infty\}.$$

The *immediate basin of attraction* of an attractive point is the connected component of its basin of attraction that holds the attractor.

Additionally, an assortment of points $z \in \hat{\mathbb{C}}$, whith orbits that tend to an attractive point is named the *Fatou set* , $\mathcal{F}(R)$. The *Julia set*, $\mathcal{J}(R)$ is its complementary, since the boundaries of the basins of attraction of the fixed points belong to this set.

The successive Theorem establishes a classical result of Fatou and Julia that we use in the study of parameter space related to the family.

Theorem 1 *[14] Let R be a rational function. The immediate basin of attraction of an attracting fixed or periodic point holds, at least, a critical point.*

By using this result, one can be sure to find all the stable behavior associated with a rational function R, by analyzing the performance of R on the set of critical points.

In this section, some basic notions relating to iterative methods and complex dynamics have been explained. These concepts will be used throughout the following chapters and, additionally, they will be complemented going into detail on more dynamic techniques.

Chapter 3

Choosing the most stable members of Kou's family of iterative methods

Based on [15]: "Choosing the most stable members of Kou's family of iterative methods", Journal of Computational and Applied Mathematics.
Presented at "Mathematical Modelling in Engineering and Human Behaviour 2016" Congress, IMM, Valencia (Spain).

3.1 Introduction

In this chapter, a family of iterative methods and its dynamics are considered in order to find the solutions of a nonlinear equation $f(z) = 0$. Specifically, the family of iterative methods designed by Kou [9] is presented. As it has been mentioned, dynamical study of the rational function of an iterative scheme provides relevant data related to the convergence and behavior of the system. Regarding this facts, Amat et al. in [16] analyzed the dynamical character of different families.

These research show different numerical behavior, for instance, periodic orbits, attracting fixed points, free critical points, etc. Certainly, the parameter space related to a family of schemes is capable of explaining the behavior of the different members of the family, allowing to select the best choices.

The chapter is divided as follows: in this Introduction, we will present the family of iterative methods. In Section 3.2, fixed and critical points of the rational function of the family will be analyzed, showing the stability of these fixed points in Section 3.3. We will depict parameter and dynamical planes of the family in Sections 3.4 and 3.6, respectively. Additionally, orbits of period two found in the family will be described in Section 3.5. The theoretical results will be verified with numerical results in Section 3.8. Finally, the chapter presents some notes and conclusions.

As we have mentioned, Kou's family of iterative methods is presented, having the following iterative expression:

$$z_{k+1} = z_k - \left(1 - \frac{3}{4}\frac{t_k - 1}{\gamma t_k + 1 - \gamma}\right)\frac{f(z_k)}{f'(z_k)}, \quad n = 0, 1, \ldots, \tag{3.1}$$

where $y_k = z_k - \dfrac{2}{3}\dfrac{f(z_k)}{f'(z_k)}$, $t_k = \dfrac{f'(y_k)}{f'(z_k)}$ and γ is a free parameter.

The order of convergence of (3.1) is proven by the authors in [9].

Theorem 2 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently derivable function in the open interval I and let $\alpha \in I$ be a simple solution of the nonlinear equation $f(z) = 0$. We consider that z_0 is an initial approximation close enough to α. Then, the sequence $\{z_k\}_{k\geq 0}$ obtained by using Kou's family converges to α with order of convergence three, being the error equation*

$$e_{k+1} = \frac{2}{3}(3-2\gamma)c_2^2 e_k^3 + O(e_k^4).$$

Besides, if $\gamma = \frac{3}{2}$, the method has order four and the following error equation:

$$e_{k+1} = \left(c_2^3 - c_2c_3 + \frac{c_4}{9}\right) e_k^4 + O(e_k^5),$$

where $c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}$, $j = 2, 3, \ldots$ and $e_k = x_k - \alpha$.

When $p(z) = (z-a)(z-b)$ is used, we obtain the following operator of the family:

$$T_{p,\gamma,a,b}(z) = z + \frac{(a-z)(b-z)(3a^2+3b^2+b(-15+4\gamma)z+(15-4\gamma)z^2+a(b(9-4\gamma)+(-15+4\gamma)z))}{(a+b-2z)(3a^2+3b^2+4b(-3+\gamma)z-4(-3+\gamma)z^2+a(b(6-4\gamma)+4(-3+\gamma)z))}. \quad (3.2)$$

This operator can be conjugated to operator $O_\gamma(z)$ on quadratic polynomials, with the Möbius transformation,

$$O_\gamma(z) = \left(h \circ T_{p,\gamma,a,b} \circ h^{-1}\right)(z) = -z^3\frac{6-4\gamma+3z}{-3-6z+4\gamma z}. \quad (3.3)$$

It can be observed that the parameters a and b do not appear in $O_\gamma(z)$.

3.2 Study of the fixed and critical points

Now, a dynamical analysis of the members of the described family will show their behavior. In first place, the object of study will be the fixed points of the operator $O_\gamma(z)$ which are not the solutions of $p(z)$, naming these points as *strange fixed points*. Afterwards, *free critical points* will be analyzed, which are the critical points of $O_\gamma(z)$ different from zero and ∞, associated with the solutions of the polynomial.

Fixed points of the operator $O_\gamma(z)$ are the solutions of $O_\gamma(z) = z$. Specifically, they are zero, infinity and the strange fixed points

- $ex_1(\gamma) = 1$,
- $ex_2(\gamma) = \dfrac{1}{6}(-9+4\gamma-\sqrt{45-72\gamma+16\gamma^2})$,
- $ex_3(\gamma) = \dfrac{1}{6}(-9+4\gamma+\sqrt{45-72\gamma+16\gamma^2})$.

In the following lemma, relations among strange fixed points are shown.

Lemma 1 *There are three strange fixed points of operator $O_\gamma(z)$ apart from the following cases:*

i) If the parameter $\gamma = \frac{3}{4}$, the rational function is $O_{3/4}(z) = z^3$, and as a result, there exist no strange fixed points.

ii) When $\gamma = \frac{9}{4}$, the rational function is $O_{9/4}(z) = -z^3$, and there exist no strange fixed points.

iii) When $\gamma = \frac{15}{4}$, there exist only one strange fixed point, $ex_2 = ex_3 = 1$, since $O_{15/4}(z) = -z^3 \frac{-3+z}{-1+3z}$.

iv) When $\gamma = \frac{3}{2}$, the rational function is $O_{3/2}(z) = z^4$, and there exist no strange fixed points.

With the objective of determining the critical points, the operator $O_\gamma(z)$ has to be derivated:

$$O'_\gamma(z) = 2z^2 \frac{16\gamma^2 z + 27(1+z)^2 - 6\gamma(3 + z(8+3z))}{(-3+(-6+4\gamma)z)^2}.$$

As it was claimed, the points $z = 0$ and $z = \infty$, which are associated with the solutions of the polynomial by means of Möbius map, are critical points. However, there are free critical points within the family, which must be analyzed, in terms of complex dynamics.

Lemma 2 *From $O'_\gamma(z) = 0$ it can be determined that:*

a) When $\gamma = \frac{3}{4}$, $\gamma = \frac{9}{4}$ or $\gamma = \frac{3}{2}$, free critical points of operator $O_\gamma(z)$ don't exist.

b) If $\gamma = 0$, the only free critical point is $z = -1$.

c) When $\gamma = 3$, the only free critical point is $z = 1$.

d) In the rest of the cases,

$$cr_1(\gamma) = \frac{27 - 24\gamma + 8\gamma^2 - 2\sqrt{-81\gamma + 171\gamma^2 - 96\gamma^3 + 16\gamma^4}}{9(-3+2\gamma)}$$

and

$$cr_2(\gamma) = \frac{27 - 24\gamma + 8\gamma^2 + 2\sqrt{-81\gamma + 171\gamma^2 - 96\gamma^3 + 16\gamma^4}}{9(-3+2\gamma)} = \frac{1}{cr_1(\gamma)},$$

are free critical points.

Hence, the conclusions of the preceding result are the follwoing:

- If $\gamma = 0$, $cr_1(0) = cr_2(0) = -1$, which is a pre-image of $z = 1$. Therefore, it is not a fixed point and the rational function of this point is $O_0(z) = z^3 \frac{2+z}{1+2z}$.

- When $\gamma = 3$, $cr_1(3) = cr_2(3) = 1$, which is a superattractor with an associated operator of $O_3(z) = -z^3 \dfrac{-2+z}{-1+2z}$.
- There is up to one independent free critical point. Consequently, only $cr_1(\gamma)$ will be taken into account.

In the coming section it can be verified that the parameter of the family affects the amount of fixed points and their stability. This fact is relevant because the presence of attractive or supperattractive strange fixed points could lead to the iterative method converging to a false solution.

3.3 Stability of the fixed points

Since the family has order of convergence three, we know that the origin and ∞ (related to the roots of $p(z)$) are superattractors. Nevertheless, the behavior of the other fixed points provides relevant information. Now, stability of these strange fixed points will be developed.

Theorem 3 *The behavior of $ex_1(\gamma) = 1$, $\gamma \neq \dfrac{9}{4}$, is the following:*

i) When $\left|\gamma - \dfrac{13}{4}\right| < \frac{1}{2}$, $ex_1(\gamma) = 1$ is attractive, being superattractive if $\gamma = 3$.

ii) When $\left|\gamma - \dfrac{13}{4}\right| = \dfrac{1}{2}$, $ex_1(\gamma) = 1$ is a parabolic point.

iii) If $\left|\gamma - \dfrac{13}{4}\right| > \dfrac{1}{2}$, then $ex_1(\gamma) = 1$ is a repulsor.

Proof. We can prove that

$$O'_\gamma(1) = \frac{8(-3+\gamma)}{-9+4\gamma}.$$

So,

$$\left|\frac{8(-3+\gamma)}{-9+4\gamma}\right| \leq 1 \qquad \text{is equivalent to} \qquad 8|-3+\gamma| \leq |-9+4\gamma|\,.$$

Taking into account that $\gamma = a + ib$, being an arbitrary number. Therefore,

$$8^2(3^2 - 6a + a^2 + b^2) \leq 9^2 - 72a + 16a^2 + 16b^2.$$

By simplifying

$$495 - 312a + 48a^2 + 48b^2 \leq 0,$$

that is,

$$\left(a - \frac{13}{4}\right)^2 + b^2 \leq \frac{1}{4}.$$

Hence,

$$\left|O'_\gamma(1)\right| \leq 1 \qquad \text{if and only if} \qquad \left|\gamma - \frac{13}{4}\right| \leq \frac{1}{2}.$$

■

Theorem 4 *The study of the stability of strange points $ex_2(\gamma)$ and $ex_3(\gamma)$ concludes that:*

i) When $\left|\gamma - \frac{9}{2}\right| < \frac{3}{4}$, both points are attractive points, becoming superattractive if $\gamma = \frac{9}{2}$.

ii) When $\left|\gamma - \frac{9}{2}\right| = \frac{3}{4}$, then $ex_2(\gamma)$ and $ex_3(\gamma)$ are parabolic points.

iii) In any other circumstance, both points are repulsors.

Proof. It can be proved that

$$O'_\gamma\left(ex_i\right) = 6 - \frac{4\gamma}{3}, \quad i = 2, 3$$

So,

$$\left|\frac{18 - 4\gamma}{3}\right| \leq 1 \qquad \text{is equivalent to} \qquad |18 - 4\gamma| \leq 3.$$

If we consider $\gamma = a + ib$ an arbitrary complex number, we obtain that

$$18^2 - 144a + 4^2a^2 + 4^2b^2 \leq 9.$$

By simplifying

$$315 - 144a + 16a^2 + 16b^2 \leq 0,$$

that is,

$$\left(a - \frac{9}{2}\right)^2 + b^2 \leq \left(\frac{3}{4}\right)^2.$$

Therefore,

$$\left|O'_\gamma\left(ex_i\right)\right| \leq 1 \qquad \text{if and only if} \qquad \left|\gamma - \frac{9}{2}\right| \leq \frac{3}{4}.$$

■

In Figure 3.1, stability regions of $ex_i(\gamma)$, $i = 1, 2, 3$ are depicted, obtained from Theorem 3 and Theorem 4.

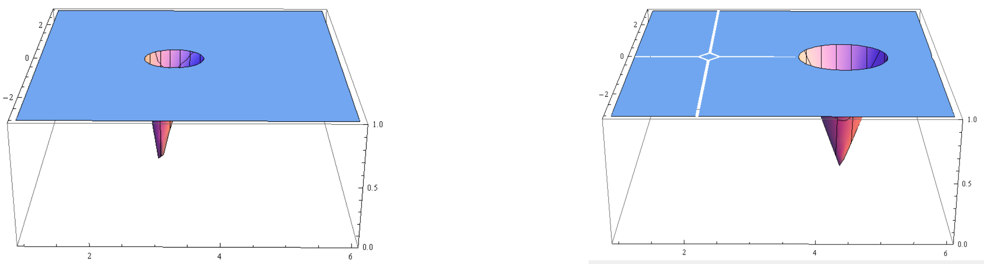


Figure 3.1: Stability regions of $ex_1(\gamma)$ (left) and $ex_i(\gamma)$, $i = 2, 3$ (right).

3.4 The parameter plane

It has been proved that the character of $O_\gamma(z)$, in terms of complex dynamics, relies on the parameter of the family γ. Considering Theorem 1 , it would be interesting to know the behavior of free critical points, for example, do some of them belong to a basin of attraction different to those of zero and infinity? In order to answer that, we represent the parameter space related to the family (3.1).

The parameter plane associated with an independent free critical point of the rational function (3.3) is achieved by linking every point of the plane with a complex value of the parameterγ, that is, with an element of family (3.1). Each value of the parameter which belongs to the same point of the parameter plane result in subsets of elements of (3.1) which have an analogous character. Therefore, we are interested in finding areas in parameter space with the more stable behavior, as these parameters would provide the more efficient elements of the family in reference to numerical stability.

It was mentioned before that $cr_1(\gamma) = \dfrac{1}{cr_2(\gamma)}$, therefore, there is up to one free independent critical point and different parameter planes can be developed, with complementary information. Taking into account that the free critical point $z = cr_1(\gamma)$ is the initial point of the iterative scheme of the family, the point of the complex mesh related to each value of the parameter γ is colored in red when the scheme reaches the solutions of the polynomial, zero and infinity, being black in any other circumstance. The brightness shows the amount of iterations needed, the lower, the brighter. This way, parameter space P_1 is represented, appearing in Figure 3.2. The procedure carried out to create this parameter space appears in [17]. The parameters used have been the following: a mesh of 1000×1000 points, a maximum of 500 iterations and a tolerance of 10^{-3} as a stopping criterium.

From now on, the study will focused on P_1, in order to analyze its dynamical variety.

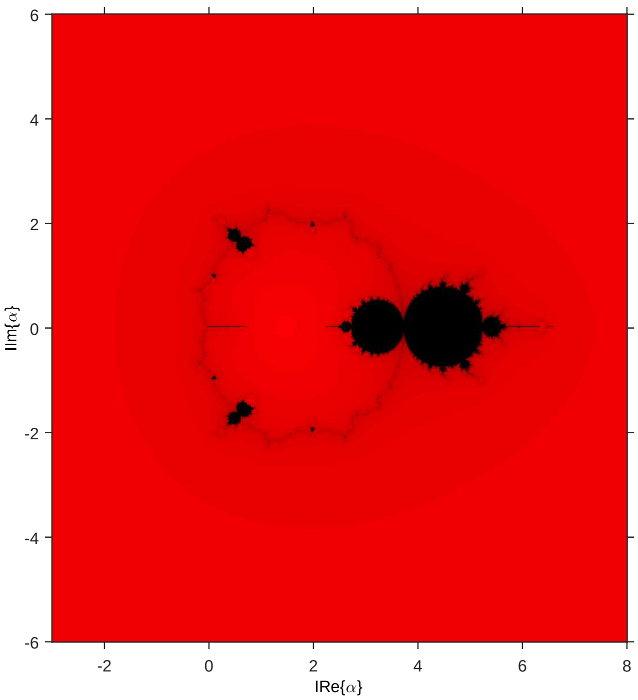


Figure 3.2: Parameter plane P_1 related to $cr_i(\gamma)$, $i = 1, 2$

It can be checked that members of family (3.1) are, overall, greatly stable, since the red area is quite large. Nevertheless, there exist small black areas that inform us about different pathological behavior of some elements of the family.

Let us remark the two balls with centers $(13/4, 0)$ and $(9/2, 0)$, which will be called D_1 and D_2, respectively. The first ball is related to values of the parameter γ for which $ex_1(\gamma)$ is attractive or superattractive (see Theorem 3). The second one belongs to values of the parameter for which $ex_2(\gamma)$ and $ex_3(\gamma)$ are simultaneously attractors or superattractors (see Theorem 4). Besides, different black areas and bulbs can be observed, which are related to attractive orbits of different periods. In the next section, orbits of period two are going to be analyzed.

3.5 Orbits of period two

With the aim of obtaining the elements of the family with orbits of period two, $O_\gamma(O_\gamma(z))$ has been obtained, also named as $O_\gamma^2(z)$:

$$O_\gamma^2(z) = \frac{z^9(6-4\gamma+3z)^3(18-12\gamma+36z-48\gamma z+16\gamma^2 z+18z^3-12\gamma z^3+9z^4)}{(-3-6z+4\gamma z)^3(-9-18z+12\gamma z-36z^3+48\gamma z^3-16\gamma^2 z^3-18z^4+12\gamma z^4)}.$$

Figure 3.3: Stability areas of the orbits with period two $pe_i(\gamma),\ i=1,2$ (left) and $pe_i(\gamma),\ i=3,4$ (right)

The two-periodic points of $O_\gamma(z)$ can be obtained as the solution of $O_\gamma^2(z) = z$, what leads to fixed points showed in previous sections, and also to the following two-periodic points:

$$pe_{\{1,2\}}(\gamma) = \tfrac{1}{12}(-9+4\gamma-r(\gamma) \pm \sqrt{2}\sqrt{-9+16\gamma^2+9r(\gamma)-4\gamma(12+r(\gamma))},$$
$$pe_{\{3,4\}}(\gamma) = \tfrac{1}{12}(-9+4\gamma+r(\gamma) \pm \sqrt{2}\sqrt{-9+16\gamma^2-9r(\gamma)+4\gamma(-12+r(\gamma))},$$

where $r(\gamma) = \sqrt{45-24\gamma+16\gamma^2}$.

In Figure 3.3, stability regions of the orbits $pe_{1,2}$ and $pe_{3,4}$ are depicted. There exist small areas where these orbits are attractive. Additionally, we can observe that there are several values of parameter γ in which the two-periodic orbits become superattractive, what means that they satisfy $O_\gamma'^2(z) = 0$.

In Figure 3.4 all the stability regions are represented, comprised by those of the strange fixed points and the two-periodic orbits. This 3D-plot enable to identify many of the different stability regions appearing in the parameter plane P_1 (see Figure 3.2) as black regions.

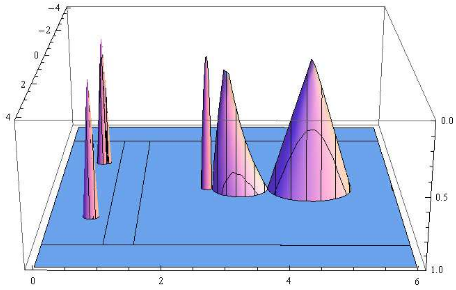
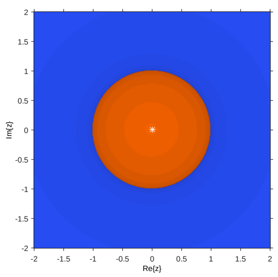


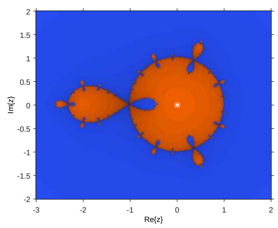
Figure 3.4: Stability areas of strange fixed points and points with two-periodic orbits

3.6 Dynamical Planes

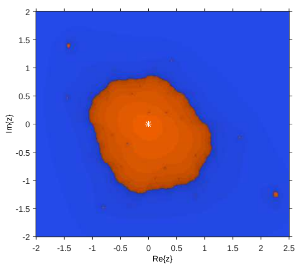
Now, the performance of the different elements of the described family (3.1) will be analyzed. These elements will be chosen after studying the parameter space of the family and the stability of the fixed points and the two-periodic orbits.



(a) $\gamma = \frac{3}{4}$ and $\gamma = \frac{3}{2}$



(b) $\gamma = 0$



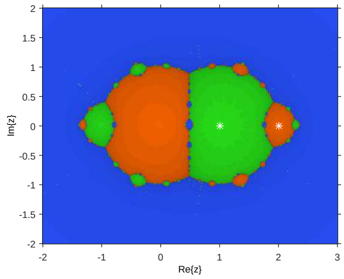
(c) $\gamma = 2 - 4i$

Figure 3.5: Stable dynamical planes

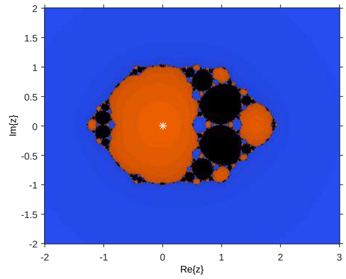
Similar to the obtaining of the parameter space, we have created some dynamical planes following a procedure explained in [17]. These planes are related to a value of the parameter of the family, resulting in a specific scheme, which is iterated. This iteration is made selecting initial estimations as the points of the complex plane. It has been utilized a squared mesh of 400 points, in which points with an orbit converging to infinity are painted in blue, in orange those whose orbits converges to zero and in different colors, such as green or red, those points with an orbit converging to one of the strange fixed points, which are represented as a white star in the figures. Additionally, some points are colored with black when they reach the maximum forty iterations without reaching any of the fixed points. It has been used a tolerance of 10^{-3}.

In the parameter plane P_1, we can observe some areas related to elements of the family with stable and efficient numerical behavior. These areas can be found selecting values of the parameter γ that are colored in red (Figure 3.2). In Figure 3.5, several stable behavior related to different values of γ in this red area are shown. Specifically, it has been used $\gamma = 0$, $\gamma = \frac{3}{4}$, $\gamma = \frac{3}{2}$ and $\gamma = 2 - 4i$.

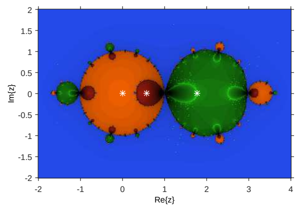
Besides that, we can depict unstable dynamical planes if we select the parameter γ being located in the black area of P_1. We can see some of these dynamical planes in Figure 3.6, corresponding to $\gamma = -2 \in D_2$, where the presence of four different basins of attraction can be seen, being two of them of the points zero and infinity, and the rest of the superattractive points $ex_2(-2)$ and $ex_3(-2)$.



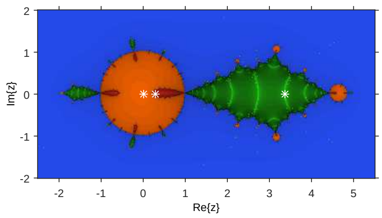
(a) $\gamma = 3$, in D_1



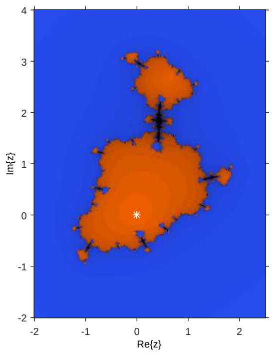
(b) $\gamma = 2.7$, in a bulb to the left of D_1



(c) $\gamma = 4$, in D_2



(d) $\gamma = 5$, in D_2



(e) $\gamma = 2 + 2i$, in a Maldelbrot type set

Figure 3.6: Dynamical planes with unstable behavior

3.7 Numerical results

In this section, we will check the theoretical results obtained previously, in terms of validity and effectiveness. Throughout these numerical experiments, we have used Matlab R2013b with double precision arithmetics. It has been selected stopping criterium $|z_{k+1} - z_k| < tol$ or $|f(z_{k+1})| < tol$, with a tolerance $tol = 10^{-12}$. With the objective of verifying the theoretical order of convergence, it has been used the approximated computational order of convergence ACOC introduced in [18] as

$$p \approx ACOC = \frac{\ln\left(|z_{k+1} - z_k|/|z_k - z_{k-1}|\right)}{\ln\left(|z_k - z_{k-1}|/|z_{k-1} - z_{k-2}|\right)}.$$

The numerical results are shown in the following Tables, from 3.2 to 3.4, where '-' means that ACOC is not stable along the iterative process. Besides, if the scheme does not converge, it is represented as 'nc'.

These numerical results are achieved by solving the nonlinear functions appearing in Table 3.1, with some elements of Kou's family in comparison to several well-known iterative schemes as Newton', Traub' and Homeier's. In the case of Kou's family, we have selected some elements with good stability properties and different ones with bad behavior. Now, we are going to recall the following known iterative expressions:

$$z_{k+1} = z_k - \frac{f(z_k) + f(y_k)}{f'(z_k)}, \tag{3.4}$$

$$z_{k+1} = z_k - \frac{1}{2}\left(1 + \frac{f'(z_k)}{f'(y_k)}\right)\frac{f(z_k)}{f'(z_k)}, \tag{3.5}$$

where $y_k = z_k - \frac{f(z_k)}{f'(z_k)}$ is Newton's step, corresponding to Traub' [10] and Homeier's [19] procedures, respectively.

Test functions	Zeros
$f_1(z) = \arctan(z)$	$\alpha = 0$
$f_2(z) = e^{z^2-3z}\sin z + \ln(z^2+1)$	$\alpha = 0$
$f_2(z) = e^z - 1.5 - \arctan(z)$	$\alpha_1 \approx 0.767653,\ \alpha_2 \approx -14.101270$

Table 3.1: Test functions and their zeros

With regards to Table 3.2, it should be mentioned that classical methods have convergence issues when the initial estimation is far from the solution. On the other hand, stable members of Kou's family have a better numerical behavior.

	z_0	α	$\lvert z_{k+1} - z_k\rvert$	$\lvert f(z_{k+1})\rvert$	$Iter$	$ACOC$
Newton	1	0.0	7.96e-10	0.0	5	2.9937
	1.8	nc				
	-1.9	nc				
Traub	1	0.0	3.63e-10	0.0	4	-
	1.8	nc				
	-1.9	nc				
Homeier	1	3.31e-24	1.97e-8	3.31e-24	4	2.9951
	1.8	nc				
	-1.9	nc				
Kou $\gamma = \frac{3}{2}$	1	2.12e-22	1.65e-6	2.12e-22	3	4.6015
	1.8	0.0	5.80e-10	0.0	5	-
	-1.9	nc				
$\gamma = \frac{3}{4}$	1	0.0	2.89e-6	0.0	3	4.7770
	1.8	0.0	8.14e-6	0.0	4	-
	-1.9	0.0	1.27e-7	0.0	5	-
$\gamma = 0$	1	0.0	3.80e-7	0.0	4	-
	1.8	-1.78e-16	7.95e-4	1.78e-16	3	2.7746
	-1.9				> 1000	
$\gamma = 3$	1	-6.09e-15	0.014	6.09e-15	3	3.4617
	1.8	nc				
	-1.9	nc				
$\gamma = 5$	1	0.0	2.38e-10	0.0	4	-
	1.8	nc				
	-1.9	nc				
$\gamma = 2.7$	1	-2.62e-16	7.86e-4	2.62e-16	3	3.6158
	1.8	nc				
	-1.9	nc				

Table 3.2: Numerical results for $f_1(z)$

In the following Tables, 3.3 and 3.4, Newton', Traub' and Homeier's methods present a stable behavior. We can observe that the elements of Kou's family defined as stable on quadratic polynomials present as good behavior as classical methods. Nevertheless, if the values of the parameter γ of Kou's family are selected among the unstable ones, the numerical behavior is not appropriate.

	z_0	α	$\lvert z_{k+1}-z_k\rvert$	$\lvert f(z_{k+1})\rvert$	$Iter$	$ACOC$
Newton	-1.5	-6.23e-17	7.90e-9	6.23e-17	13	2.0065
	2.8	-1.06e-14	7.26e-8	1.06e-14	12	2.0128
	-3	-2.76e-23	5.26e-12	2.76e-23	21	2.0008
Traub	-1.5	-2.16e-13	3.00e-5	2.16e-13	9	3.0901
	2.8	nc				
	-3	-8.52e-20	2.20e-7	2.52e-20	15	3.0219
Homeier	-1.5	9.29e-24	3.05e-12	9.29e-24	8	3.0000
	2.8	2.38e-14	2.07e-5	2.38e-14	4	-
	-3	-2.71e-13	4.67e-5	2.71e-13	8	3.1501
Kou $\gamma=\frac{3}{2}$	-1.5	8.57e-17	5.04e-7	8.57e-17	6	-
	2.8	3.64e-17	2.87e-6	3.64e-17	5	6.8344
	-3	7.93e-15	2.57e-4	7.93e-15	8	5.8084
$\gamma=\frac{3}{4}$	-1.5	-5.03e-13	5.01e-5	5.03e-13	8	3.2028
	2.8	1.73e-19	9.16e-10	1.73e-19	7	3.0223
	-3	2.42e-22	1.56e-11	2.42e-22	13	3.0135
$\gamma=0$	-1.5	-3.32e-13	3.46e-5	3.32e-13	9	3.0961
	2.8	7.90e-19	2.33e-7	7.90e-19	7	3.0234
	-3	-1.79e-13	2.82e-5	1.79e-13	14	3.0901
$\gamma=3$	-1.5	nc				
	2.8	nc				
	-3	nc				
$\gamma=5$	-1.5				>1000	
	2.8				>1000	
	-3				>1000	
$\gamma=2.7$	-1.5	nc				
	2.8	nc				
	-3	nc				

Table 3.3: Numerical results for $f_2(z)$

3.8 Conclusions

We have presented the dynamical study applied on quadratic polynomials of a parametric family of iterative methods for solving nonlinear equations, designed by Kou et al. With the parameter plane obtained from the family, we have been able to verify the presence of numerous values of γ, what means diverse elements of the family, with stable behavior, but existing other ones without convergence to the solutions of the polynomial. It has been also shown the presence of periodic orbits with period two, analyzing their analytical expressions. Finally, the family under study has been applied with non-polynomial equations, achieving the numerical results that prove that the information given by the theoretical dynamical study in terms of the good or bad numerical behavior of the different elements of the family was correct.

	z_0	α	$\|z_{k+1} - z_k\|$	$\|f(z_{k+1})\|$	$Iter$	$ACOC$
Newton	4	0.7677	1.56e-12	0.0	9	2.0001
	2	0.7677	1.34e-7	0.0	6	2.0021
	-2	-14.1013	8.80e-8	0.0	7	2.0037
Traub	4	0.7677	3.83e-7	0.0	5	2.8767
	2	0.7677	5.55e-5	0.0	4	2.7523
	-2	-14.1013	2.45e-6	0.0	5	2.9919
Homeier	4	0.7677	2.10e-8	0.0	5	3.1308
	2	0.7677	2.19e-10	0.0	4	3.0927
	-2	-14.1013	1.36e-4	0.0	4	3.1931
Kou $\gamma = \frac{3}{2}$	4	0.7677	5.18e-5	2.22e-16	4	3.3038
	2	0.7677	9.68e-5	0.0	3	3.2606
	-2	-14.1013	0.0078	4.44e-16	3	6.8668
$\gamma = \frac{3}{4}$	4	0.7677	8.49e-12	2.22e-16	6	2.9780
	2	0.7677	1.68e-6	1.11e-16	4	2.8713
	-2	-14.1013	9.77e-10	0.0	5	3.0043
$\gamma = 0$	4	0.7677	4.12e-7	1.11e-16	6	
	2	0.7677	5.64e-5	4.49e-13	4	
	-2	-14.1013	2.98e-6	2.22e-16	5	
$\gamma = 3$	4	nc				
	2	nc				
	-2	-	5.94e-13	0.0849	35	0.9723
$\gamma = 5$	4	nc				
	2				> 1000	
	-2				> 1000	
$\gamma = 2.7$	4	nc				
	2	nc				
	-2	nc				

Table 3.4: Numerical results for $f_3(z)$

Chapter 4

Stability of a parametric class of iterative methods of fourth and eighth order

Based on [20]: "On two classes of fourth- and seventh-order vectorial methods with stable behavior", Journal of Mathematical Chemistry.
Partial results were shown in different congresses: "International conference Computational and Mathematical Methods in Science and Engineering 2017", Rota (Spain) and "Mathematical Modelling in Engineering and Human Behaviour 2017", IMM, Valencia (Spain).

4.1 Introduction

Along this chapter we are going to conduct an exhaustive study of a new designed family of iterative methods. Firstly, the dynamical analysis of the family of order four will be analyzed, explaining its fixed and critical points and the parameter and dynamical planes, through sections from 4.2 to 4.4. Afterwards, we will enlarge the order of convergence of the proposed family in order to improve its characteristics. We will develop the dynamical analysis of this new scheme in sections from 4.6 to 4.8. We will finish with some conclusions obtained.

The family under study in this chapter is based on the family presented in [21]. It is constructed by adding a new step to Newton's method. This way, the following two-step scheme is obtained

$$\begin{aligned} y_k &= z_k - \frac{f(z_k)}{f'(z_k)}, \\ z_{k+1} &= y_k - \left(\alpha_1 + \alpha_2 \frac{f'(z_k)}{f'(y_k)} + \alpha_3 \left(\frac{f'(z_k)}{f'(y_k)}\right)^2\right) \frac{f(y_k)}{f'(y_k)}, \quad k = 0, 1, \ldots, \end{aligned} \tag{4.1}$$

where α_1, α_2 and α_3 are free parameters.

The following result establishes the convergence of family (4.1), which was proved in [21].

Theorem 5 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently differentiable function in an open interval I and $z^* \in I$ a root of equation $f(z) = 0$. Choosing an initial approximation z_0 close enough to z^*, the iterative scheme defined by (4.1) has fourth-order convergence when $\alpha_2 = 3 - 2\alpha_1$ and $\alpha_3 = -2 + \alpha_1$, being α_1 a free parameter. In particular, if $\alpha_1 = \frac{5}{4}$ then method (4.1) has order five.*

An analysis of the convergence of the family presented (4.1) on quadratic polynomials will be carried out applying tools of complex dynamics. This would enable to find connections between the values of the parameter of the family α_1 and the stability of the iterative scheme obtained with that value. It is known that, if the iterative method satisfies the scaling theorem (what successfully happens with family (4.1)), the roots of a polynomial can be transformed by an affine map without modifying the dynamic character of the family. Therefore, and as in the previous Chapter, we use a generic polynomial $p(z) = (z - a)(z - b)$. Applying this polynomial to the family (4.1), we obtain the following rational operator:

$$T_{p,\alpha_1,a,b}(z) = \frac{(a-z)(b-z)}{a+b-2z} + z + (a-z)^2(b-z)^2 \left[\frac{(a^4 + b^4 - 4a^3z - 4b^3z + 4(1+\alpha_1)b^2z^2 - 8\alpha_1 bz^3}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^3} \right.$$
$$\left. + \frac{4\alpha_1 z^4 - 4az((-1+2\alpha_1)b^2 + (2-4\alpha_1)bz + 2\alpha_1 z^2)}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^3} + \frac{a^2((-2+4\alpha_1)b^2 + (4-8\alpha_1)bz + 4(1+\alpha_1)z^2))}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^3} \right],$$

which depends on parameters α_1, a and b, being the last ones the solutions of $p(z)$.

If the conjugacy map $h(z) = \dfrac{z-a}{z-b}$ is considered ([14]), the operator $T_{p,\alpha_1,a,b}(z)$ on quadratic polynomials is conjugated to the rational function $O_{\alpha_1}(z)$,

$$O_{\alpha_1}(z) = \left(h \circ T_{p,\alpha_1,a,b} \circ h^{-1}\right)(z) = -z^4 \frac{5 - 4\alpha_1 + 2z^2 + z^4}{-1 - 2z^2 + -5z^4 + 4\alpha_1 z^4}, \tag{4.2}$$

where the parameters a and b have been obviated.

In the following sections we analyze the strange fixed and critical points of the rational function $O_{\alpha_1}(z)$, the character of the fixed points, the parameter plane related to the family and some dynamical planes describing different behavior, such as stability and periodic orbits.

4.2 Study of the fixed and critical points

Now, the *strange fixed points* and the *free critical points* will be analyzed.

In this family, the fixed points of $O_{\alpha_1}(z)$ are zero, infinity and the strange fixed points: $ex_1(\alpha_1) = 1$ and the roots of the polynomial

$$r(\alpha_1, z) = 1 + z + 3z^2 + (-2 + 4\alpha_1)z^3 + 3z^4 + z^5 + z^6,$$

that are denoted by $ex_i(\alpha_1)$, $i = 2, 3, 4, 5, 6, 7$.

Therefore, there are seven strange fixed points, except in the following cases:

i) When $\alpha_1 = 1$, the operator of the family is $O_1(z) = z^4$. As a result, there exists no strange fixed points.

ii) If $\alpha_1 = 2$, the operator is $O_2(z) = -z^4 \frac{3+z^2}{1+3z^2}$. There are only six strange fixed points as $ex_2(\alpha_1) = ex_3(\alpha_1) = -1$.

iii) When $\alpha_1 = -2$, there are only five strange fixed point as $ex_2(\alpha_1) = ex_3(\alpha_1) = 1$.

On the other hand, the first derivative of $O_{\alpha_1}(z)$ is calculated so we can obtain the critical points:

$$O'_{\alpha_1}(z) = -4z^3 \frac{(1+z^2)^2(-5+4\alpha_1+2(1-2\alpha_1)z^2+(-5+4\alpha_1)z^4)}{(1+2z^2+(5-4\alpha_1)z^4)^2}.$$

Proposition 1 *We achieve, by examining the equation $O'_{\alpha_1}(z) = 0$ of the family:*

a) When $\alpha_1 = 1$, operator $O_{\alpha_1}(z)$ has no free critical points.

b) If $\alpha_1 = 2$ or $\alpha_1 = \frac{5}{4}$, we only obtain $z = -i$ and $z = i$ as free critical points.

c) In other circumstances,

$$cr_1(\alpha_1) = -i,$$

$$cr_2(\alpha_1) = i,$$

$$cr_3(\alpha_1) = -\sqrt{\frac{1-2\alpha_1+2\sqrt{3}\sqrt{-2+3\alpha_1-\alpha_1^2}}{5-4\alpha_1}},$$

$$cr_4(\alpha_1) = \sqrt{\frac{1-2\alpha_1+2\sqrt{3}\sqrt{-2+3\alpha_1-\alpha_1^2}}{5-4\alpha_1}} = -cr_3,$$

$$cr_5(\alpha_1) = -\sqrt{\frac{-1+2\alpha_1+2\sqrt{3}\sqrt{-2+3\alpha_1-\alpha_1^2}}{-5+4\alpha_1}} = \frac{1}{cr_3},$$

and

$$cr_6(\alpha_1) = \sqrt{\frac{-1+2\alpha_1+2\sqrt{3}\sqrt{-2+3\alpha_1-\alpha_1^2}}{-5+4\alpha_1}} = -cr_5 = \frac{1}{cr_4},$$

are free critical points.

We should mention that $cr_1(\alpha_1)$ and $cr_2(\alpha_1)$ are pre-images of $z = 1$ and $cr_3(\alpha_1)$ and $cr_5(\alpha_1)$ are conjugated, as well as $cr_4(\alpha_1)$ and $cr_6(\alpha_1)$. Therefore, there is only one independent free critical point.

4.3 Stability of the fixed points

Of course, $z = 0$ and $z = \infty$ are superattracting fixed points, however, the character of the rest of fixed points must be analyzed.

The behavior of the strange fixed point $ex_1(\alpha_1) = 1$ of the family, $\alpha_1 \neq 2$, is the following:

i) If $|\alpha_1 - 2| > 4$, then $ex_1(\alpha_1) = 1$ is an attractor.

ii) When $|\alpha_1 - 2| = 4$, $ex_1(\alpha_1) = 1$ is a parabolic point.

iii) If $|\alpha_1 - 2| < 4$, then $ex_1(\alpha_1) = 1$ is a repulsor.

In Figure 4.1, the stability regions of all strange fixed points $ex_i(\alpha_1)$, $i = 1, 2, 3, 4, 5, 6, 7$´are shown.

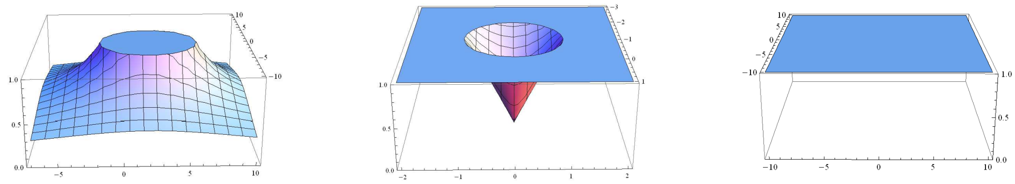


Figure 4.1: Stability regions of $ex_1(\alpha_1)$ (left), $ex_i(\alpha_1)$, $i = 6, 7$ (middle) and $ex_i(\alpha_1)$, $i = 2, 3, 4, 5$ (right).

The study of the stability of strange fixed points $ex_i(\alpha_1), i = 2, 3, 4, 5$ allows us to determine that they are repulsors for any value of α_1.

4.4 The parameter and dynamical planes

We can represent the parameter plane related to an independent free critical point of operator as it has been explained in the previous chapter. We obtain, therefore, the most stable regions which will lead to the finest elements of the proposed family.

The points painted in red are those in which the scheme converges to one of the solutions (zero and infinity), being black in any other circumstance. We obtain P_1 if the initial point of the iterative scheme is considered as a independent free critical point of $O_{\alpha_1}(z)$. The maximum number of iterations utilized has been 500, with a mesh of 1000×1000 points and being 10^{-3} the tolerance.

In this family only one parameter space is obtained, since $cr_4(\alpha_1)$ is equal in module to $cr_6(\alpha_1)$ and the operator's powers are even numbers. From P_1 it can be verified that we can find the best elements of the family choosing values of α_1 between 1 and 2, in terms of real values.

We obtain the dynamical planes linked to different values of α_1, using a mesh of 400×400 points.

Firstly, we select values of the parameter α_1 belonging to red regions of the parameter plane, with stable numerical behavior. These dynamical planes are found in Figure 4.3, selecting the values of $\alpha_1 = 1$, $\alpha_1 = 2$ and $\alpha_1 = 0.5$.

Secondly, unstable behavior is shown in Figure 4.4, corresponding to values of α_1 in the black region, specifically, $\alpha_1 = 3$, $\alpha_1 = 3.5$ and $\alpha_1 = -1.5$.

In Figure 4.4a and 4.4b the two-periodic orbits are represented, whereas in Figure 4.4c we can observe 4 basins of attraction, two of them from the solutions of $p(z)$ and the other ones related to the basins of attraction of the strange fixed points $ex_i(\alpha_1)$, $i = 5, 6$.

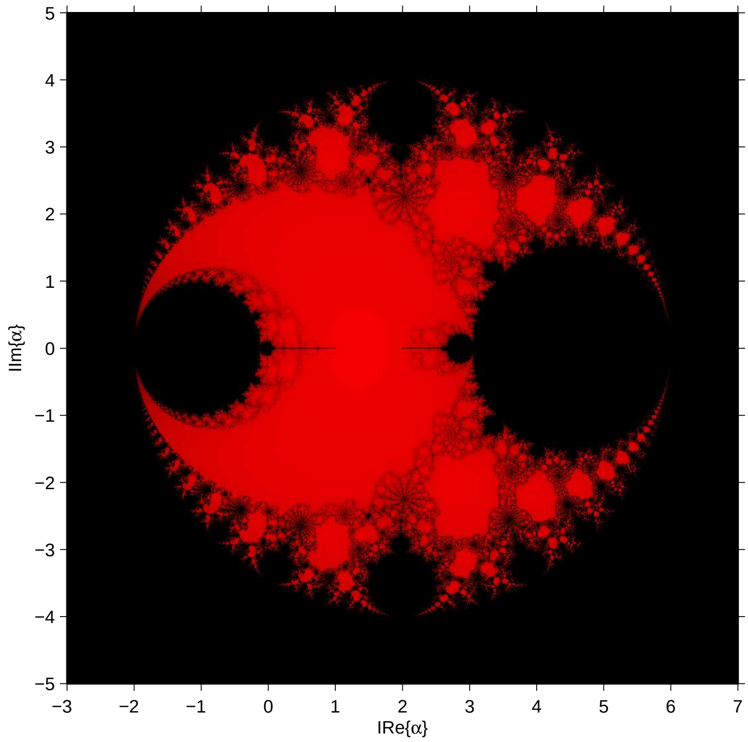
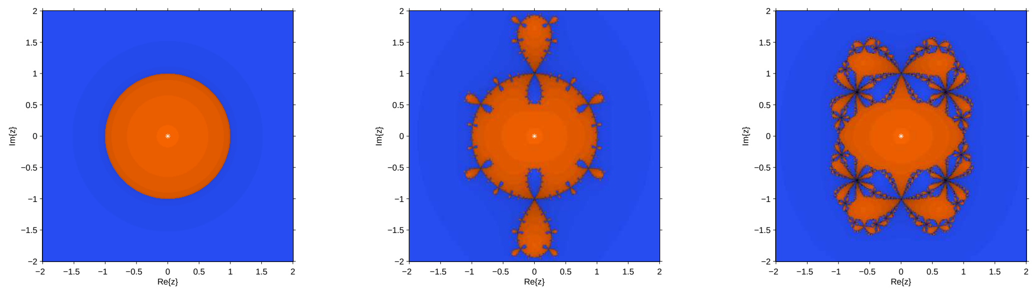
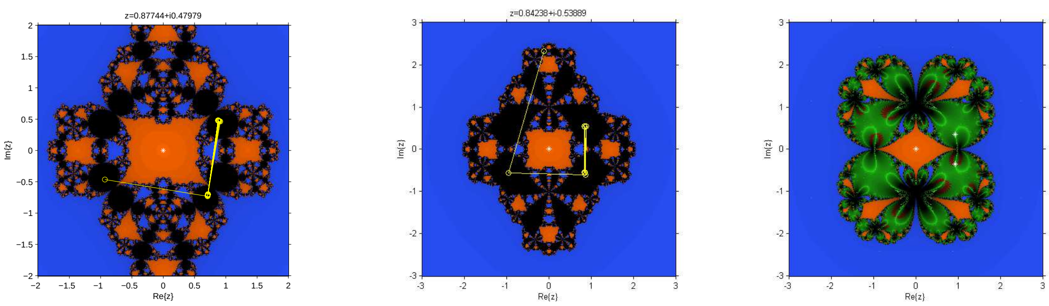


Figure 4.2: Parameter plane P_1 associated with $cr_i(\alpha_1)$, $i = 3, 4, 5, 6$



(a) $\alpha_1 = 1$ **(b)** $\alpha_1 = 2$ **(c)** $\alpha_1 = 0.5$

Figure 4.3: Some dynamical planes with stable behavior



(a) $\alpha_1 = 3$ **(b)** $\alpha_1 = 3.5$ **(c)** $\alpha_1 = -1.5$

Figure 4.4: Dynamical planes with unstable behavior

4.5 Increasing the order

Now, our objective is to increase the order of convergence of the family and to analyze its dynamics with the objective of finding the best elements of the resulting scheme. When we take $\alpha_1 = \frac{5}{4}$ and add a step in order to improve the order to eight, we obtain the iterative expression:

$$z_{k+1} = t_k - \left(\beta_1 + \beta_2 \frac{f'(z_k)}{f'(y_k)} + \beta_3 \left(\frac{f'(z_k)}{f'(y_k)}\right)^2\right) \frac{f(t_k)}{f'(y_k)}, \quad n = 0, 1, \ldots, \tag{4.3}$$

where $y_k = z_k - \dfrac{f(z_k)}{f'(z_k)}$, $t_k = y_k - \left(\alpha_1 + \alpha_2 \dfrac{f'(z_k)}{f'(y_k)}\right) + \alpha_3 \left(\dfrac{f'(z_k)}{f'(y_k)}\right)^2 \dfrac{f(y_k)}{f'(y_k)}$, $\beta_2 = -2(-1+\beta_1)$, $\beta_3 = -1 + \beta_1$ and β_1 is a free parameter.

The rational function of the operator is the following:

$$\begin{aligned}
T_{p,\beta_1,a,b}(z) = & \frac{(a-z)(b-z)}{a+b-2z} + z + \frac{(a-z)^2(b-z)^2}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^3} \\
& * \frac{\gamma - 4b^3 z 9b^2 z^2 - 10bz^3 + 5z^4 + 3a^2(b^2-2bz+3z^2) - 2az(3b^2-6bz+5z^2)}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^3} \\
& + \frac{(a-z)^6(b-z)^6(a^2+2b^2-2az-4bz+3z^2)(2a^2+b^2-4az-2bz+3z^2)}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^9} \\
& * \frac{\gamma - 4b^3 z + 4b^2(1+\beta_1)z^2 - 8b\beta_1 z^3 + 4\beta_1 z^4 - 4az(b^2(-1+2\beta_1) + b(2-4\beta_1)z + 2\beta_1 z^2) + a^2\delta}{(a+b-2z)(a^2+b^2-2az-2bz+2z^2)^9},
\end{aligned}$$

where $\gamma = a^4 + b^4 - 4a^3 z$ and $\delta = (b^2(-2+4\beta_1) + b(4-8\beta_1)z + 4(1+\beta_1)z^2)$.

This rational function relies on parameter β_1 and additionally on the parameters a and b, from polynomial $p(z)$.

The operator $T_{p,\beta_1,a,b}(z)$ applied on quadratic polynomials is conjugated to the rational function $O_{\beta_1}(z)$,

$$O_{\beta_1}(z) = \left(h \circ T_{p,\alpha_1,a,b} \circ h^{-1}\right)(z) = -z^8 \frac{(2+z^2)r(z)}{(1+2z^2)s(z)}, \tag{4.4}$$

where $r(z) = 6 + 18z^2 + 18z^4 + 15z^6 + 6z^8 + z^{10} - 4\beta_1(1+2z^2)$ and $s(z) = -1 - 6z^2 - 15z^4 - 18z^6 + 2(-9+4\beta_1)z^8 + (-6+4\beta_1)z^{10}$.

We can observe that the roots of the polynomial, a and b, do not appear.

4.6 Study of the fixed and critical points

Now, we obtain fixed points of the operator, which are the solutions of $O_{\beta_1}(z) = z$ (zero and infinity) and the strange fixed points $ex_1(\beta_1) = 1$ and also the roots of the following polynomial

$$\begin{aligned}
r(\beta_1, z) \quad = \quad & 1 + z + 9z^2 + 9z^3 + 36z^4 + 36z^5 + 84z^6 + (72+8\beta_1)z^7 + 126z^8 + (84+20\beta_1)z^9 + \\
& 126z^{10} + (72+8\beta_1)z^{11} + 84z^{12} + 36z^{13} + 36z^{14} + 9z^{15} + 9z^{16} + z^{17} + z^{18}.
\end{aligned}$$

Hence, there are nineteenth strange fixed points. However, in the following case:

i) If $\beta_1 = -\frac{208}{9}$, there are sixteen strange fixed points.

The following step in our study is to calculate the first derivative of $O_{\beta_1}(z)$:

$$O'_{\beta_1}(z) = -4z^7 \frac{(1+z^2)^8(2\beta_1(8+9z^2-16z^4+9z^6+8z^8)-3(8+19z^2+10z^4+19z^6+8z^8))}{(1+2z^2)^2(-1+z^2(2z+z^2)(-3-6z^2-6z^4+(-6+4\beta_1)z^6))^2}.$$

We already have mention that the points zero and infinity, which are associated to the solutions of the polynomial, are critical points.

Proposition 2 *For this family of order eight, we obtain the following free critical points:*

$$cr_1(\beta_1) = -i,$$

$$cr_2(\beta_1) = i,$$

$$cr_3(\beta_1) = -\frac{1}{4}\sqrt{\frac{1}{-6+4\beta_1}\left(57-\gamma+3\sqrt{6}\sqrt{\frac{\theta}{(3-2\beta_1)^2}}-2\beta_1(9+\sqrt{6}\sqrt{\frac{\theta}{(3-2\beta_1)^2}})\right)},$$

$$cr_4(\beta_1) = -cr_3(\beta_1),$$

$$cr_5(\beta_1) = -\frac{1}{4}\sqrt{\frac{1}{-6+4\beta_1}\left(57-\gamma-3\sqrt{6}\sqrt{\frac{\theta}{(3-2\beta_1)^2}}+2\beta_1(-9+\sqrt{6}\sqrt{\frac{\theta}{(3-2\beta_1)^2}})\right)} = \frac{1}{cr_3},$$

$$cr_6(\beta_1) = -cr_5(\beta_1) = \frac{1}{cr_4},$$

$$cr_7(\beta_1) = -\frac{1}{4}\sqrt{\frac{1}{-6+4\beta_1}\left(57+\gamma+3\sqrt{6}\sqrt{\frac{\delta}{(3-2\beta_1)^2}}-2\beta_1(9+\sqrt{6}\sqrt{\frac{\delta}{(3-2\beta_1)^2}})\right)},$$

$$cr_8(\beta_1) = -cr_7(\beta_1),$$

$$cr_9(\beta_1) = -\frac{1}{4}\sqrt{\frac{1}{-6+4\beta_1}\left(57+\gamma-3\sqrt{6}\sqrt{\frac{\delta}{(3-2\beta_1)^2}}+2\beta_1(-9+\sqrt{6}\sqrt{\frac{\delta}{(3-2\beta_1)^2}})\right)} = \frac{1}{cr_7},$$

and

$$cr_{10}(\beta_1) = -cr_9(\beta_1) = \frac{1}{cr_8},$$

where $\gamma = \sqrt{4977-9348\beta_1+4420\beta_1^2}$, $\theta = -165+108\beta_1^2-19\gamma+2\beta_1(74+3\gamma)$ *and* $\delta = -165+108\beta_1^2-19\gamma+\beta_1(148-6\gamma)$.

We observed that $cr_1(\beta_1)$ and $cr_2(\beta_1)$ are pre-images of z=1, and the following pairs are conjugated: $cr_3(\beta_1)$ and $cr_5(\beta_1)$, $cr_4(\beta_1)$ and $cr_6(\beta_1)$, $cr_7(\beta_1)$ and $cr_9(\beta_1)$, $cr_8(\beta_1)$ and $cr_{10}(\beta_1)$. Hence, there are only two independent free critical points.

a) If $\beta_1 = 1$, then $cr_1(\beta_1) = cr_3(\beta_1) = -i$ and $cr_2(\beta_1) = cr_4(\beta_1) = i$. Then, the number of free critical points is six.

b) If $\beta_1 = \frac{16}{3}$, $cr_5(\beta_1) = cr_7(\beta_1) = -1$ and $cr_6(\beta_1) = cr_8(\beta_1) = 1$, there exist six free critical points.

4.7 Stability of the fixed points

Proposition 3 *In this case, the behavior of the strange fixed point $ex_1(\beta_1) = 1$, $\beta_1 \neq \frac{16}{3}$ is the following:*

i) When $\left|\beta_1 - \frac{32}{6}\right| > \frac{256}{9}$, $ex_1(\beta_1) = 1$ is attractive.

ii) If $\left|\beta_1 - \frac{32}{6}\right| = \frac{256}{9}$, $ex_1(\beta_1) = 1$ is a parabolic point.

iii) When $\left|\beta_1 - \frac{32}{6}\right| < \frac{256}{9}$, then $ex_1(\beta_1) = 1$ is a repulsor.

Proof. We can prove that

$$O'_{\beta_1}(1) = \frac{256}{48 - 9\beta_1}.$$

So,

$$\left|\frac{256}{48 - 9\beta_1}\right| \leq 1 \quad \text{is analogous to} \quad 256 \leq |48 - 9\beta_1|.$$

If $\beta_1 = a + ib$ is considered an arbitrary complex number. Hence,

$$256^2 \leq 48^2 - 864a + 81a^2 + 81b^2.$$

By simplifying

$$81a^2 - 864a + 81b^2 - 63232 \geq 0,$$

that is,

$$\left(a - \frac{32}{6}\right)^2 + b^2 \geq \frac{65536}{81}.$$

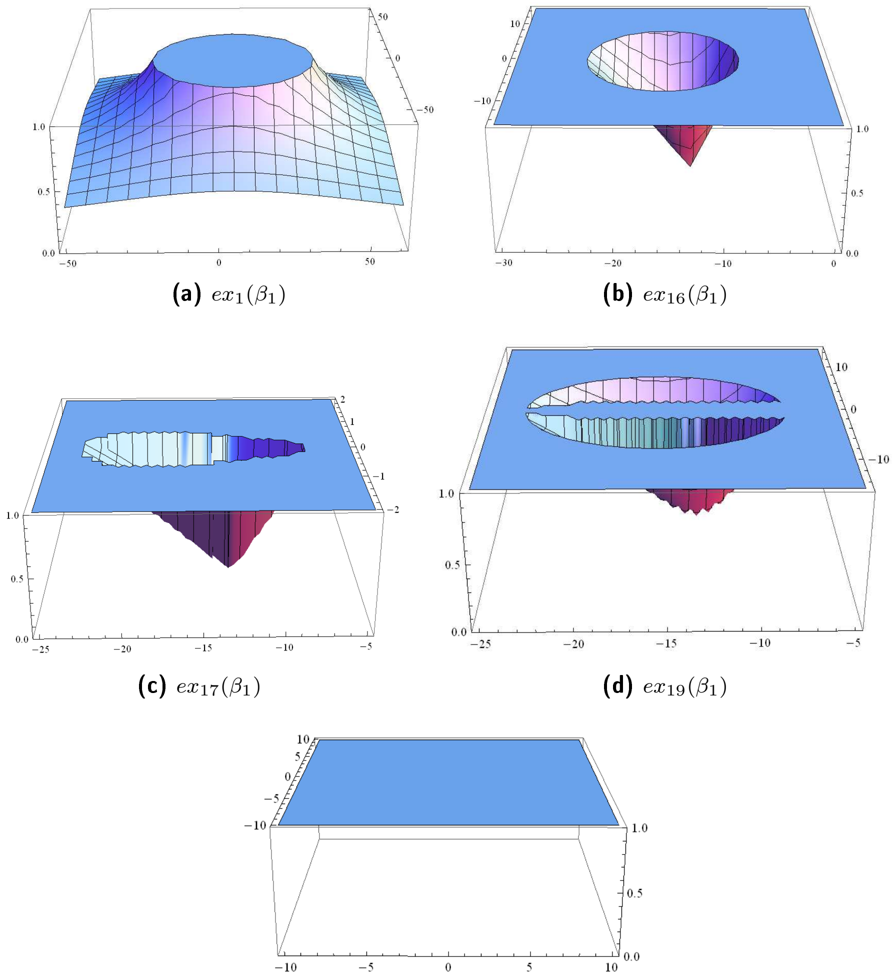
Therefore,

$$\left|O'_{\beta_1}(1)\right| \leq 1 \quad \text{if and only if} \quad \left|\beta_1 - \frac{32}{6}\right| \geq \frac{256}{9}.$$

■

In Figure 4.5, the stability areas of $ex_i(\beta_1)$, $i = 1, 2, 3, ..., 19$ are represented.

We observe that strange points $ex_i(\beta_1), i = 2, 3, ..., 14, 15$ and $ex_{18}(\beta_1)$ are repulsors in any case.



(a) $ex_1(\beta_1)$

(b) $ex_{16}(\beta_1)$

(c) $ex_{17}(\beta_1)$

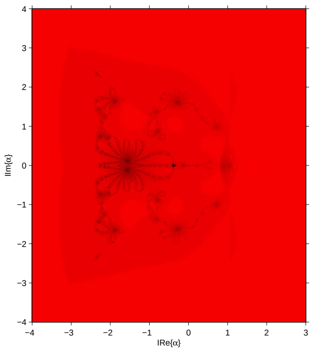
(d) $ex_{19}(\beta_1)$

(e) $ex_i(\beta_1), i = 2, 3, ..., 14, 15$ and $ex_{18}(\beta_1)$

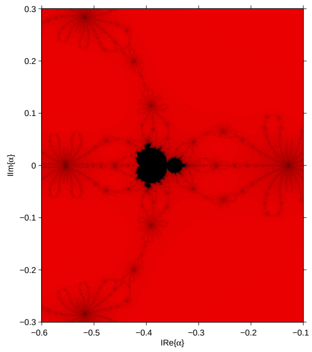
Figure 4.5: Stability areas of $ex_i(\beta_1),\ i = 1, 2, 3, ..., 19$

4.8 The parameter space and dynamical planes

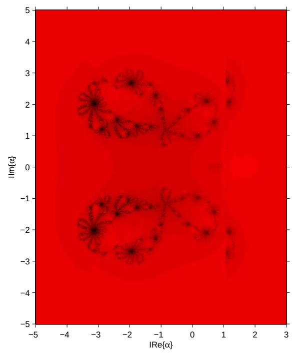
Taking into account the free critical points of the family, the parameter space P_2 can be obtained (for $cr_i(\beta_1), i = 3, 4, 5, 6$) and P_3 ($cr_i(\beta_1), i = 7, 8, 9, 10$), as we can see in Figure 4.6. It has been used a mesh of 1000×1000 points, with a maximum number of iterations of 500 and a tolerance of 10^{-3}.



(a) Parameter plane P_2



(b) Parameter plane P_2 magnified



(c) Parameter plane P_3

Figure 4.6: Parameter planes of the family of order eight

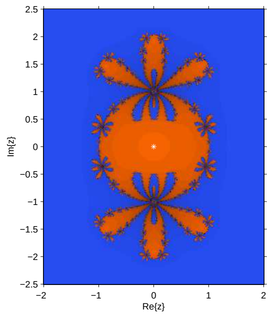
We obtain two parameter planes because of the four pairs of conjugated critical points and since the operator has only pair powers.

The best real values of the parameter β_1 are found around the following regions:

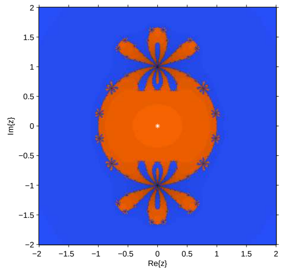
- For P_2: $\beta_1 < -2.2$ and $\beta > 1.2$,
- For P_3: $\beta_1 < 0.5$ and $\beta > 1$.

Let us result that the number of best values of the parameter is much bigger with order eight than order four, as its red area is larger.

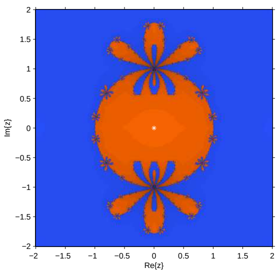
We also show in Figure 4.7 the dynamical planes with good characteristics, taking into account their stability and efficiency. They are related to values of the parameter colored in red in the parameter planes. The last two ones (Figures 4.7e and 4.7f) correspond to dynamical planes with slow convergence. Nevertheless, we represent in Figure 4.8 unstable behavior selecting the parameter β_1 in the black area of parameter spaces.



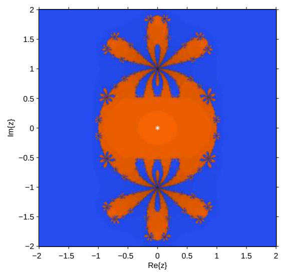
(a) $\beta_1 = \frac{16}{3}$



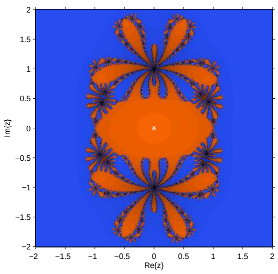
(b) $\beta_1 = \frac{3}{2}$



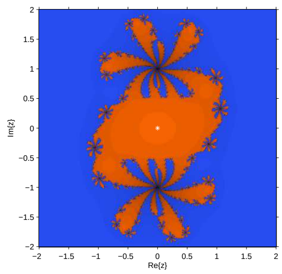
(c) $\beta_1 = 2$



(d) $\beta_1 = 3$

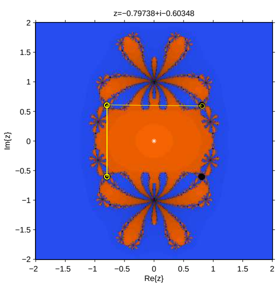


(e) $\beta_1 = -1.55 + 0.12i$

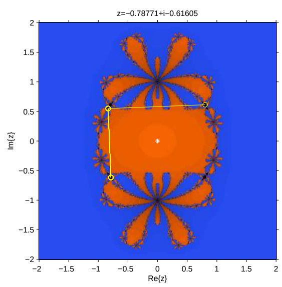


(f) $\beta_1 = -0.278 + 1.616i$

Figure 4.7: Stable dynamical planes



(a) $\beta_1 = -0.4$



(b) $\beta_1 = -0.34$

Figure 4.8: Dynamical planes with unstable behavior

4.9 Conclusions

In this chapter we studied the dynamical analysis developed on quadratic polynomials of a different parametric family of iterative methods with two varieties: fourth and eighth order. We have been able to verify, due to the parameter plane, that there are more values of the parameter of the family with stable behavior once that we enlarge the order of convergence of this family.

When we raise the order of convergence to eight, from the parameter space it can be verified that unstable schemes of the family are obtained selecting values of the parameter placed in small areas of the parameter space. Apart from these areas, the family and its schemes has a stable behavior.

Chapter 5

Fixed point root-finding methods of fourth-order of convergence

Based on [22]: "Fixed Point Root-Finding Methods of Fourth-Order of Convergence", Symmetry. Presented at "Seventh Conference on Finite Difference Methods 2018", Lozenetz (Bulgary). Partial results are based on [23]: "Stability of a family of iterative methods of fourth-order", Lecture Notes in Computer Science.

5.1 Introduction

In this chapter, we have designed a family of iterative methods for solving nonlinear problems, applying the weight-function technique. This family includes several well-known schemes in the literature that can be achieved by varying weigh functions. The weight function of the family depends on two different evaluations of the derivative, being this one the only difference between the two steps of each method, leads to an uncommon scheme. We will realize the study of the family applying tools of complex dynamics on quadratic polynomials, so as to select the most stable elements, since every possible scheme is optimal in the sense of Kung-Traub's conjecture.

Throughout this Introduction, the scheme of the family is presented and the order of convergence is proven. Later, in Section 5.2 the behavior of the class will be analyzed, based on the study of the fixed points and critical points in Section 5.2. Additionally, the analysis of the parameter space will enable to choose the most efficient schemes of the family, in relation to their stability. Finally, in Section 5.6 some conclusions are mentioned.

In our study, through weight function procedure and Newton's scheme, a new family of iterative schemes is presented, with the following iterative expression:

$$\begin{aligned} y_n &= z_n - \gamma \frac{f(z_n)}{f'(z_n)}, \\ z_{n+1} &= z_n - H(t_n)\frac{f(z_n)}{f'(z_n)}, \quad n = 0, 1, \ldots, \end{aligned} \tag{5.1}$$

where γ is a real parameter and the variable of the weight function $H(t)$ is $t = \frac{f'(y)}{f'(z)}$.

Now, we present the convergence result of this family, describing the conditions that function $H(t)$ must satisfy for reaching order four.

Theorem 6 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently derivable function in an open interval D and let $\bar{z} \in D$ be a simple solution of the nonlinear equation $f(z) = 0$. Starting from a known initial estimation z_0 close enough to $\bar{z}$, if $\gamma = 2/3$ and function H satisfies $H(1) = 1$, $H'(1) = -3/4$, $H''(1) = 9/4$ and $|H'''(1)| < +\infty$, then sequence $\{z_n\}_{n\geq 0}$ obtained from (5.1) converges to $\bar{z}$ with order of convergence four, being the error equation*

$$e_{n+1} = (5c_2^3 - c_2c_3 + c_4/9)e_n^4 + O(e_n^5),$$

where $c_j = \frac{f^{(j)}(\bar{z})}{j!f'(\bar{z})}$, $j = 2, 3, \ldots$ and $e_n = z_n - \bar{z}$.

Proof. It is known that $f(z_n)$ and $f'(z_n)$ can be expressed as

$$f(z_n) = f'(\bar{z})[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + O(e_n^5)]$$

and

$$f'(z_n) = f'(\bar{z})[1 + 2c_2e_n^2 + 3c_3e_n^3 + 4c_4e_n^4 + O(e_n^5)].$$

By direct division,

$$\frac{f(z_n)}{f'(z_n)} = e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + [-8c_2^3 + c_2(4c_2^2 - 3c_3) + 10c_2c_3 - 3c_4]e_n^4 + O(e_n^5).$$

Then,

$$y_n = z_n - \gamma f(z_n)f'(z_n) = (1-\gamma)e_n + \gamma c_2e_n^2 - 2[\gamma(c_2^2 - c_3)]e_n^3 + \gamma(4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + O(e_n^5).$$

Again, by expanding in Taylor series,

$$\begin{aligned} f'(y_n) &= f'(\bar{z})\left[1 - 2[(-1+\gamma)c_2]e_n + [2\gamma c_2^2 + 3(-1+\gamma)^2c_3]e_n^2 \right. \\ &\quad \left. +2[-2\gamma c_2^3 + (5-3\gamma)\gamma c_2c_3 - 2(-1+\gamma)^3c_4]e_n^3\right] + O(e_n^4). \end{aligned}$$

Therefore,

$$\begin{aligned} t_n = \frac{f'(y_n)}{f'(z_n)} &= 1 - 2\gamma c_2e_n + 3\gamma[2c_2^2 + (-2+\gamma)c_3]e_n^2 \\ &\quad -4[\gamma(4c_2^3 + (-7+3\gamma)c_2c_3 + (3-3\gamma+\gamma^2)c_4)]e_n^3 + O(e_n^4) \end{aligned}$$

and, taking into account that t tends to 1 when n tends to infinity, the weight function can be expressed as

$$\begin{aligned} H(t_n) &= h_0 + h_1(t_n - 1) + \frac{1}{2}h_2(t_n-1)^2 + \frac{1}{6}h_3(t_n-1)^3 + O((t_n-1)^4) \\ &= h_0 - 2\gamma h_1c_2e_n + [2\gamma^2h_2c_2^2 + 3\gamma h_1(2c_2^2 + (-2+\gamma)c_3)]]e_n^2 \\ &\quad + \left[-\frac{4}{3}\gamma^3h_3c_2^3 - 6\gamma^2h_2c_2(2c_2^2 + (-2+\gamma)c_3) - 4\gamma h_1(4c_2^3 + (-7+3\gamma)c_2c_3\right. \\ &\quad \left. +(3-3\gamma+\gamma^2)c_4)\right] e_n^3 + O(e_n^4), \end{aligned}$$

where h_0, h_1, h_2 and h_3 denotes $H(1)$, $H'(1)$, $H''(1)$ and $H'''(1)$, respectively. Finally,

$$e_{n+1} = z_{n+1} - \bar{z} = z_n - \bar{z} - H(t_n)\frac{f(z_n)}{f'(z_n)} =$$

$$= (1-h_0)e_n + (h_0+2\gamma h_1)c_2e_n^2 + [-2(h_0+\gamma(4h_1+\gamma h_2))c_2^2 + (2h_0-3(-2+\gamma)\gamma h_1)c_3]e_n^3$$
$$+[(4h_0+26\gamma h_1+14\gamma^2h_2+\frac{4}{3}\gamma^3h_3)c_2^3+(-7h_0+\gamma((-38+15\gamma)h_1+6(-2+\gamma)\gamma h_2))c_2c_3$$
$$+(3h_0+4\gamma(3-3\gamma+\gamma^2)h_1)c_4]e_n^4+O(e_n^5).$$

Therefore, in order to eliminate first order error,

$$1-h_0=0, \quad \text{which means that} \quad h_0=H(1)=1.$$

For the second order error:

$$h_0+2\gamma h_1=0,$$

being $h_0=1$

$$h_1=H'(1)=\frac{-1}{2\gamma}.$$

With these values of h_0 and h_1, the error equation is now

$$\begin{aligned} e_{n+1} &= [(2-2\gamma^2h_2)c_2^2+\frac{1}{2}(-2+3\gamma)c_3]e_n^3 \\ &\quad +[(-9+14\gamma^2h_2+\frac{4\gamma^3h_3}{3})c_2^3\frac{3}{2}(8-5\gamma-8\gamma^2h_2+4\gamma^3h_2)c_2c_3 \\ &\quad +(-3+6\gamma-2\gamma^2)c_4]e_n^4+O(e_n^5). \end{aligned}$$

So, we can eliminate the third order error if

$$2-2\gamma^2h_2=0 \quad \text{and} \quad -2+3\gamma=0,$$

which means that

$$h_2=H''(1)=\frac{1}{\gamma^2} \quad \text{and} \quad \gamma=\frac{2}{3}.$$

Hence it is proven that if $\gamma = 2/3$ and function H satisfies $H(1)=1$, $H'(1)=-3/4$ and $H''(1)=9/4$, the order of convergence is four. ■

Now, we analyze how to extend the structure (5.1) for designing derivative-free methods and for solving nonlinear systems, $n>1$.

If we want to design a derivative-free scheme with a similar structure as (5.1), we change $f'(z_n)$ by the divided difference $f[z_n,w_n]$, where $w_n=z_n+\rho f(z_n)$ with ρ a real parameter, obtaining the following iterative expression:

$$\begin{aligned} y_n &= z_n-\gamma\frac{f(z_n)}{f[z_n,w_n]}, \\ z_{n+1} &= z_n-H(t_n)\frac{f(z_n)}{f[z_n,w_n]}, \quad n=0,1,\ldots, \end{aligned} \tag{5.2}$$

being in this case $t=\dfrac{f[y,w]}{f[z,w]}$ the variable of the weight function. Unfortunately, family (5.2) does not reach order four. However, if we use as variable of the weight function $t=\dfrac{f[y,z]}{f[z,w]}$, being $w_n=z_n+\rho f(z_n)^2$, the class

$$\begin{aligned} y_n &= z_n-\gamma\frac{f(z_n)}{f[z_n,w_n]}, \\ z_{n+1} &= z_n-H(t_n)\frac{f(z_n)}{f[z_n,w_n]}, \quad n=0,1,\ldots, \end{aligned} \tag{5.3}$$

reaches order four under certain conditions as we establish in the following result.

Theorem 7 *Let $f : D \subseteq \mathbb{R} \to \mathbb{R}$ be a sufficiently derivable function in an open interval D and let $\bar{z} \in D$ be a simple solution of the nonlinear equation $f(z) = 0$. Starting from a known initial estimation z_0 close enough to $\bar{z}$, if $\gamma = 1$ and function H satisfies $H(1) = 1$, $H'(1) = -1$, $H''(1) = 4$ and $|H'''(1)| < +\infty$, then sequence $\{z_n\}_{n\geq 0}$ obtained from (5.3) converges to $\bar{z}$ with order of convergence four, being the error equation*

$$e_{n+1} = (-f'(\bar{z})^2\rho c_2^2 + \frac{1}{6}(30 + h_3)c_2^3 - c_2c_3)e_n^4 + O(e_n^5),$$

where $c_j = \frac{f^{(j)}(\bar{z})}{j!f'(\bar{z})}$, $j = 2, 3, \ldots$, $e_n = z_n - \bar{z}$ and $h_3 = H'''(1)$.

On the other hand, family (5.1) can be extended to the multidimensional case, $n > 1$, so we accomplish a family of iterative methods for solving nonlinear systems $F(z) = 0$, where $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$. In this case, the iterative expression is

$$\begin{array}{rcl} y^{(n)} & = & z^{(n)} - \gamma[F'(z^{(n)})]^{-1}F(z^{(n)}), \\ z^{(n+1)} & = & z^{(n)} - H(t^{(n)})[F'(z^{(n)})]^{-1}F(z^{(n)}), \quad n = 0, 1, \ldots, \end{array} \tag{5.4}$$

where $F'(z^{(n)})$ is the Jacobian matrix of F evaluated in the iteration $z^{(n)}$. The variable of weight is $t = [F'(z)]^{-1}F'(y)$ and $H(t)$ is a matrix function $H : X \to X$, where $X = \mathbb{R}^{n\times n}$, such that

(i) $H'(u)(v) = h_1uv$, being H' the first derivative of H, $H' : X \to \mathcal{L}(X)$, $h_1 \in \mathbb{R}$ and $\mathcal{L}(X)$ denotes the space of linear mappings from X to itself.

(ii) $H''(u, v)(w) = h_2uvw$, being H'' the second derivative of H, $H'' : X \times X \to \mathcal{L}(X)$ and $h_2 \in \mathbb{R}$.

Then, the Taylor expansion of H around the identity matrix I gives

$$H(t^{(n)}) \approx H(I) + h_1(t^{(n)} - I) + \frac{1}{2}h_2(t^{(n)} - I)^2.$$

A similar result to Theorem 6 can be establish for class (5.4), obtaining conditions for function $H(t)$ to reach order four. In the proof we use some tools and notations introduced in [24].

Theorem 8 *Let $F : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be a sufficiently differentiable function in an convex set D and let $\bar{z} \in D$ be a solution of the nonlinear system $F(z) = 0$, such that $F'(\bar{z})$ is nonsingular. Starting from a known initial estimation $z^{(0)}$ close enough to $\bar{z}$, if $\gamma = 2/3$ and function H satisfies $H(I) = I$, $h_1 = -3/4$ and $h_2 = 9/4$, then sequence $\{z_n\}_{n\geq 0}$ obtained from (5.4) converges to $\bar{z}$ with order of convergence four, being the error equation*

$$e_{n+1} = \left(\frac{26}{9}C_4 - 6C_2C_3 - C_3C_2 - 7C_2^3 - 2C_2^2\right)e_n^4 + O(e_n^5),$$

where $C_j = \frac{1}{j!}[F'(\bar{z})]^{-1}F^{(j)}(\bar{z})$, $j = 2, 3, \ldots$ and $e_n = z^{(n)} - \bar{z}$.

Proof. By using Taylor expansion of $F(z^{(n)})$ and $F'(z^{(n)})$ around $\bar{z}$,

$$F(z^{(n)}) = F'(\bar{z})[e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4] + O(e_n^5),$$

$$F'(z^{(n)}) = F'(\bar{z})[I + 2C_2e_n + 3C_3e_n^2 + 4C_4e_n^3] + O(e_n^4).$$

From the above expression, we conjecture

$$F'(z^{(n)})^{-1} = [I + X_2e_n + X_3e_n^2 + X_4e_n^3]F'(\bar{z})^{-1} + O(e_n^4)$$

and, from $F'(z^{(n)})^{-1}F'(z^{(n)}) = F'(z^{(n)})F'(z^{(n)})^{-1} = I$, we have

$$F'(z^{(n)})^{-1} = [I - 2C_2e_n + (4C_2^2 - 3C_3)e_n^2 + (-4C_4 + 6C_2C_3 + 6C_3C_2 - 8C_2^3)e_n^3]F'(\bar{z})^{-1} + O(e_n^4).$$

Then,

$$F'(z^{(n)})^{-1}F(z^{(n)}) = e_n - C_2e_n^2 + 2(C_2^2 - C_3)e_n^3 + (-3C_4 + 4C_2C_3 + 3C_3C_2 - 4C_2^3)e_n^4 + O(e_n^5).$$

Similarly, we calculate

$$y^{(n)} - \bar{z} = (1-\gamma)e_n + \gamma C_2e_n^2 - 2\gamma(C_2^2 - C_3)e_n^3 - \gamma(-3C_4 + 4C_2C_3 + 3C_3C_2 - 4C_2^3)e_n^4 + O(e_n^5).$$

So,

$$\begin{aligned}
F'(y^{(n)}) &= F'(\bar{z})\left[I + 2C_2(y^{(n)} - \bar{z}) + 3C_3(y^{(n)} - \bar{z})^2 + 4C_4(y^{(n)} - \bar{z})^3\right] + O(y^{(n)} - \bar{z})^4) \\
&= F'(\bar{z})\left[I + 2C_2(1-\gamma)e_n + (2\gamma C_2^2 + 3C_3(1-\gamma)^2)e_n^2 + (-4\gamma(C_2^2 - C_2C_3)\right. \\
&\quad +6\gamma(1-\gamma)C_3C_2 + 4(1-\gamma)^3C_4)e_n^3 + (-2\gamma(-3C_2C_4 + 4C_2^2C_3 + 3C_2C_3C_2 - 4C_2^4) \\
&\quad \left.+3(5\gamma^2 - 4\gamma)C_3C_2^2 + 12(\gamma - \gamma^2)C_3^2 + 12\gamma(1-\gamma)^2C_4C_2)e_n^4\right] + O(e_n^5).
\end{aligned}$$

Therefore, variable t of the weight function is described as

$$\begin{aligned}
t = F'(z_n)^{-1}F'(y_n) &= I - 2\gamma C_2e_n + (6\gamma C_2^2 + 3C_3(\gamma^2 - 2\gamma))e_n^2 \\
&\quad +(-4\gamma C_2^2 + (6\gamma^2 - 8\gamma)C_2C_3 + (12\gamma - 6\gamma^2)C_3C_2 \\
&\quad +(-4\gamma^3 + 12\gamma^2 - 12)C_4 - 12\gamma C_2^3)e_n^3 + O(e_n^4)
\end{aligned}$$

and

$$\begin{aligned}
H(t) &= H(I) + h_1(t-I) + \frac{1}{2}h_2(t-I)^2 \\
&= H(I) - 2\gamma C_2h_1e_n + (6\gamma h_1C_2^2 + 3h_1C_3(\gamma^2 - 2\gamma) + 2h_2\gamma^2C_2^2)e_n^2 \\
&\quad + \left[-4\gamma h_1C_2^2 + h_1(6\gamma^2 - 8\gamma)C_2C_3 + h_1(12\gamma - 6\gamma^2)C_3C_2\right. \\
&\quad \left.+h_1(-4\gamma^3 + 12\gamma^2 - 12)C_4 - 12h_1\gamma C_2^3 - 2\gamma h_2(6\gamma C_2^3 + 3C_2C_3(\gamma^2 - 2\gamma))\right] e_n^3 + O(e_n^4).
\end{aligned}$$

Therefore,

$$\begin{aligned}
e_{k+1} &= (I - H(I))e_n + (H(I)C_2 + 2\gamma h_1C_2)e_n^2 + \left(-2H(I)(C_2^2 - C_3) - 2\gamma h_1C_2^2\right. \\
&\quad \left.-6\gamma h_1C_2^2 - 3h_1C_3(\gamma^2 - 2\gamma) - 2h_2\gamma^2C_2^2\right) e_n^3 + O(e_n^4).
\end{aligned}$$

In order to reach order four, the coefficients of e_n, e_n^2 and e_n^3 must be zero, so

$$H(I) = I, \qquad h_1 = \frac{-1}{2\gamma}, \qquad h_2 = \frac{9}{4} \qquad and \qquad \gamma = \frac{2}{3}.$$

With these values, the error equation is

$$e_{n+1} = \left(\frac{26}{9}C_4 - 6C_2C_3 - C_3C_2 - 7C_2^3 - 2C_2^2\right) e_n^4 + O(e_n^5)$$

and the proof is finished. ■

Many known schemes designed for solving nonlinear equations or nonlinear systems can be obtained as particular cases of (5.4) by using different weight functions satisfying the conditions of Theorem 8. For example, the classical Jarratt's scheme [25]

$$\begin{aligned} y_n &= z_n - \tfrac{2}{3}\frac{f(z_n)}{f'(z_n)}, \\ z_{n+1} &= z_n - \tfrac{3f'(y_n)+f'(z_n)}{6f'(y_n)-2f'(z_n)}\frac{f(z_n)}{f'(z_n)}, \quad n=0,1,\ldots, \end{aligned} \tag{5.5}$$

is obtained from (5.1) for equations and (5.4) for systems, by using

$$H(t) = (6t-2I)^{-1}(3t+I).$$

In a similar way, the method constructed by Khattri and Abbasbandy in [26] is an element of (5.4), $n \geq 1$, by using the weight function

$$H(t) = I + \frac{21}{8}t - \frac{9}{2}t^2 + \frac{15}{8}t^3.$$

The parametric family of iterative methods for solving nonlinear equations or systems, designed by Kanwar et al. in [27], is a particular case of (5.1) or (5.4) using the weight function

$$H(t) = \frac{1}{2}\frac{\alpha_1^2 - 22\alpha_1\alpha_2 - 27\alpha_2^2 + 3(\alpha_1^2 + 10\alpha_1\alpha_2 + 5\alpha_2^2)}{(\alpha_1+3\alpha_2 t)(3(\alpha_1+\alpha_2)t - \alpha_1 - 5\alpha_2)}t,$$

where α_1 and α_2 are free real parameters such that $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq 3\alpha_2$.

Sharma and Arora published in [28] a method for solving nonlinear systems which we can obtain from (5.4) by using

$$H(t) = \frac{23}{8}I + t\left(-3I + \frac{9}{8}t\right).$$

Hueso et al. presented in [29] a parametric family of iterative schemes, which is obtained from the weight function

$$H(t) = \frac{5-8\alpha}{8}I + \alpha t^{-1} + \frac{\alpha}{3}t + \frac{9-8\alpha}{24}t,$$

where α is a real free parameter.

Ghorbanzadeh and Soleymani presented in [30] an iterative method solving nonlinear equations or nonlinear systems, which is a particular case of our scheme using the weight function

$$H(t) = 4(I+3t)^{-1}\left[I + \frac{9}{16}(t-I)\right]^2.$$

Finally, the class of iterative schemes proposed by Argyros et al. in [31] for solving nonlinear equations can be deduced by using the weight function

$$H(t) = M(t)\frac{\beta+1}{\beta + [1-(3/2)(\beta-1)(1-t)]^{1/2}},$$

where β is a free parameter and $M(t)$ is a function satisfying $M(1)=1$, $M'(1)=0$ and $M''(1) = \frac{-9}{16}(\beta-1)$.

One of the most simpler sub-class of family (5.1) is obtained when $H(t)$ is the Taylor polynomial of third degree, satisfying $\gamma = 2/3$, $H(1) = 1$, $H'(1) = -3/4$ and $H''(1) = 9/4$, that is, the weight function is:

$$H(t) = 1 - \frac{3}{4}(t-1) + \frac{9}{8}(t-1)^2 + \frac{1}{6}\gamma(t-1)^3,$$

being $\gamma = H'''(1)$ the free parameter. In this way, we obtain a parametric family of iterative methods of order four, denoted by CGTγ. In the next sections, we are going to analyze de dynamical behavior of this class in terms of parameter γ, for finding the methods with good stability properties and to avoid the elements of the family with chaotic behavior.

Using the generic quadratic polynomial $p(x) = (x-a)(x-b)$, the following function is the fixed point operator of the proposed family:

$$T_{p,\gamma,a,b}(z) = z + \frac{(-a+z)(-b+z)(1 + \frac{(a-z)(b-z)}{(a+b-2z)^2} + \frac{2(a-z)^2(b-z)2}{(a+b-2z)^4} + \frac{22\gamma(a-z)^3(-b+z)^3}{81(a+b-2z)^6})}{a+b-2z}, \quad (5.6)$$

which depends on parameters γ, a and b.

This operator on quadratic polynomials is conjugated to operator $O_\gamma(z)$,

$$O_\gamma(z) = \left(\phi \circ T_{p,\gamma,a,b} \circ \phi^{-1}\right)(z) = z^4 \frac{405 + 32\gamma + 1134z + 1134z^2 + 486z^3 + 81z^4}{81 + 486z + 1134z^2 + 1134z^3 + 405z^4 + 32\gamma z^4}. \quad (5.7)$$

In this new operator O_γ, parameters a and b have been obviated.

5.2 Study of the fixed points

Stability and reliability of the members of the family are analyzed in the rest of the chapter, regarding the properties of its associate rational function when the class is applied on polynomial $p(z)$. Firstly, fixed points of the rational function $O_\gamma(z)$ are calculated. Specifically, we focus on the points that are not related to the original roots of polynomial $p(z)$, the *strange fixed points*.

In the next result, some properties of the strange fixed points are described.

Theorem 9 *Fixed points of the rational function $O_\gamma(x)$ are the roots of equation $O_\gamma(z) = z$. Therefore, we obtain $z = 0$, $z = \infty$ (corresponding to the roots of $p(z)$), and the following strange fixed points:*

- $ex_1(\gamma) = 1$,
- *the roots of symmetric sixth-degree polynomial* $r(t) = 81 + 567t + 1701t^2 + (2430 - 32\gamma)t^3 + 1701t^4 + 567t^5 + 81t^6$, *or analogously,*

$$\begin{aligned} ex_{2,3}(\gamma) &= \frac{s_1(\gamma) \pm \sqrt{s_1(\gamma)^2 - 4}}{2}, ex_2(\gamma) = \frac{1}{ex_3(\gamma)}, \\ ex_{4,5}(\gamma) &= \frac{s_2(\gamma) \pm \sqrt{s_2(\gamma)^2 - 4}}{2}, ex_4(\gamma) = \frac{1}{ex_5(\gamma)}, \\ ex_{6,7}(\gamma) &= \frac{s_3(\gamma) \pm \sqrt{s_3(\gamma)^2 - 4}}{2}, ex_6(\gamma) = \frac{1}{ex_7(\gamma)}, \end{aligned}$$

where $s_1(\gamma)$, $s_2(\gamma)$ and $s_3(\gamma)$ are the roots of the third-degree polynomial $s(t) = 81t^3 + 567t^2 + 1458t + 1296 - 32\gamma$, that is,

$$\begin{aligned}
s_1(\gamma) &= \frac{1}{9}\left(-21 - \frac{15\ 3^{1/3}}{(\Phi(\gamma))^{1/3}} + (9\Phi(\gamma))^{1/3}\right), \\
s_2(\gamma) &= \frac{1}{18}\left(-42 + \frac{15\ 3^{1/3}\left(1+i\sqrt{3}\right)}{(\Phi(\gamma))^{1/3}} + i\left(i+\sqrt{3}\right)(9\Phi(\gamma))^{1/3}\right), \\
s_3(\gamma) &= \frac{1}{18}\left(-42 + \frac{15\ 3^{1/3}\left(1-i\sqrt{3}\right)}{(\Phi(\gamma))^{1/3}} - \left(1+i\sqrt{3}\right)(9\Phi(\gamma))^{1/3}\right),
\end{aligned}$$

being $\Phi(\gamma) = 24 + 16\gamma + \sqrt{1701 + 768\gamma + 256\gamma^2}$.

It is possible to find values of parameter γ where two or more strange fixed points coincide. Consequently, operator $O_\gamma(x)$ provides seven strange fixed points, except in the following cases:

i) If $\gamma = 0$ or $\gamma = \frac{891}{4}$, there are only five strange fixed points.

ii) If $\gamma = \frac{3}{16}i\left(8i + 5\sqrt{5}\right)$ or $\gamma = -\frac{3}{16}i\left(-8i + 5\sqrt{5}\right)$ the strange fixed points are $ex_1 = 1$ and the roots of a sixth-degree polynomial, which are two simple and two double roots.

5.3 Stability of the fixed points

In this section, we observe that the number of fixed points is not the only characteristic that depends on the parameter, since the stability of these points relies on the parameter, as well. This fact can lead to the existence of attracting strange fixed points, which can make the iterative scheme converge to a false solution.

It is known that $z = 0$ and $z = \infty$ are always superattracting fixed points, as the order of convergence of the class is greater than 2. However, relevant numerical information is provided by the stability of the other fixed points (for example, $z = 1$ corresponds to the divergence of the original method). Therefore, we determine this stability in the next results.

Theorem 10 *Strange fixed point $ex_1(\gamma) = 1$, with $\gamma \neq -\dfrac{405}{4}$, has the following character:*

i) When $\left|\gamma + \frac{405}{4}\right| < 324$, $ex_1(\gamma) = 1$ is a repulsor.

ii) If $\left|\gamma + \frac{405}{4}\right| = 324$, $ex_1(\gamma) = 1$ is a parabolic point.

iii) When $\left|\gamma + \frac{405}{4}\right| > 324$, then $ex_1(\gamma) = 1$ is an attractor.

Proof. We can prove that

$$O'_\gamma(1) = \frac{1296}{405 + 4\gamma}.$$

So,

$$\left|\frac{1296}{405+4\gamma}\right| \leq 1 \qquad \text{is equivalent to} \qquad 1296 \leq |405 + 4\gamma|.$$

If we consider $\gamma = c + id$ an arbitrary complex number. Therefore,

$$1296^2 \leq 405^2 + 16c^2 + 16d^2 + 3240c.$$

By simplifying

$$1515591 - 3240c - 16c^2 - 16d^2 \leq 0,$$

that is,

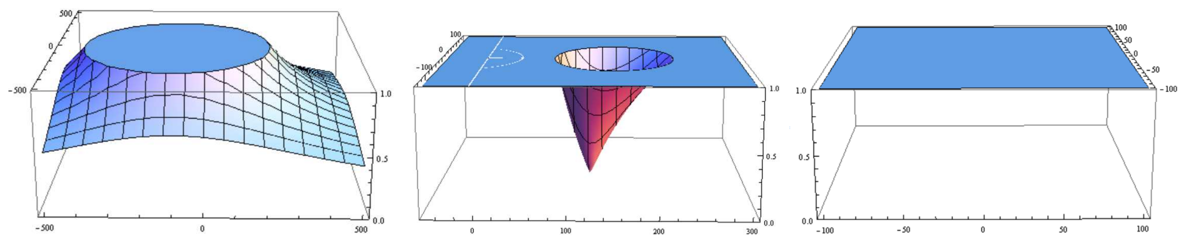
$$\left(c + \frac{405}{4}\right)^2 + d^2 \leq 324^2.$$

Therefore,

$$\left|O'_\gamma(1)\right| \leq 1 \quad \text{if and only if} \quad \left|\gamma + \frac{405}{4}\right| \leq 324 .$$

■

Stability regions of every strange fixed point ($ex_i(\gamma)$, $i = 1, 2, \ldots, 7$) are shown in Figure 5.1.



(a) $ex_1(\gamma)$ **(b)** $ex_i(\gamma),\ i = 2, 3$ **(c)** $ex_i(\gamma),\ i = 4, 5, 6, 7$

Figure 5.1: 3D-view of stability functions of strange fixed points

We can observe from the stability of $ex_i(\gamma), i = 4, 5, 6, 7$ that these strange fixed points are repulsive for any value of parameter γ.

5.4 Analysis of the critical points

Regarding the dynamics of the family, it is significant to analyze the critical points of the rational function $O_\gamma(z)$ different from 0 and ∞, the *free critical points*.

With the purpose of calculating the critical points, we study which points make null the derivative of $O_\gamma(z)$.

$$O'_\gamma(z) = (1+z)^6 \frac{324z^3 405(1+z)^2 + 16\gamma\left(2 - 3z + 2z^2\right)}{\left(81 + 486z + 1134z^2 + 1134z^3 + (405 + 32\gamma)z^4\right)^2}.$$

We know that $z = 0$ and $z = \infty$, which are linked to the roots of the polynomial through Möbius map, are the critical points that lead to their Fatou components. Nevertheless, several free critical points can be obtained, some of them depending on the value of the parameter γ.

Proposition 4 *The number of critical points of the rational function $O_\gamma(z)$ depend on the value of the parameter γ:*

a) If $\gamma = 0$, there exist one only free critical point, $z = -1$, which is a pre-image of the fixed point $z = 1$.

b) In all other cases, the free critical points are $cr_1(\gamma) = -1$,

$$cr_2(\gamma) = \frac{-405 + 24\gamma - 4\sqrt{7}\sqrt{-\gamma(405 + 4\gamma)}}{405 + 32\gamma} = \frac{1}{cr_3},$$

which means that there exists only one independent free critical point.

Proof. We obtain the previous critical points, since

$$O'_\gamma(z) = (1+z)^6 \frac{324z^3 405(1+z)^2 + 16\gamma\left(2 - 3z + 2z^2\right)}{(81 + 486z + 1134z^2 + 1134z^3 + (405 + 32\gamma)z^4)^2} =$$

$$= z(1+z)^6\left(-\frac{-405 + 24\gamma - 4\sqrt{7}\sqrt{-\gamma(405 + 4\gamma)}}{405 + 32\gamma} + z\right)\left(-\frac{-405 + 24\gamma + 4\sqrt{7}\sqrt{-\gamma(405 + 4\gamma)}}{405 + 32\gamma} + z\right).$$

■

In the case of strange fixed point $ex_i(\gamma), i = 2, 3$, we can see that there exist a ball in which the strange fixed points are attractors, whereas outside this ball, $ex_i(\gamma), i = 2, 3$ are repulsive.

5.5 The parameter space

We have observed that the dynamical behavior of operator $O_\gamma(x)$ depends on the values of the parameter γ. Taking into account Theorem 1, we are interested in knowing what happens with the free critical points, and if any of them gives rise to a basin of attraction different from those of zero and infinity. In order to have knowledge of this, we obtain the parameter plane associated to the family.

We depict the parameter space associated with a free critical point of operator (5.7) by linking the parameter plane's points with a complex value of the parameter with an element of family (5.1). The values of the parameterγ which belongs to the same component of the parameter plane lead to sets of schemes of the family (5.1) with an analogous dynamical behavior. Therefore, our objective is to find stable areas in the parameter space, due to the fact that the values of γ in these regions will provide the best members of the family in terms of numerical stability.

Since $cr_2(\gamma) = \dfrac{1}{cr_3(\gamma)}$, we have at most one free independent critical point, consequently, there exist only one parameter plane of the family. A in previous chapters, if we consider the independent free critical point as a starting point of the iterative schemes of the family associated to each complex value of γ, this point of the complex plane is painted in red if the method converges to any of the roots and they are black in other cases. Following this procedure, we obtain the parameter plane presented in Figure 5.2, by using the processes described in [17]. We have used a mesh of 2000×2000 points, 500 maximum iterations and 10^{-3} as the tolerance used in the stopping criterium. We also show a zoom of this parameter plane in Figure 5.3 (a), focusing on the biggest red area, where the members of family (5.1) are, in general, very stable.

Nevertheless, we can see black regions that inform us about different pathological behavior of the elements of the family. The black ball represented in Figure 5.3 (b), correspond to values of parameter γ for which $ex_i(\gamma), i = 2, 3$ are attracting. Besides, the big black ball that surrounds the parameter plane in Figure 5.2 correspond to values of the parameter for which $ex_1(\gamma)$ is attracting.

In addition, we can analyze the remaining black regions by drawing dynamical planes with the parameter γ corresponding to values inside these black regions.

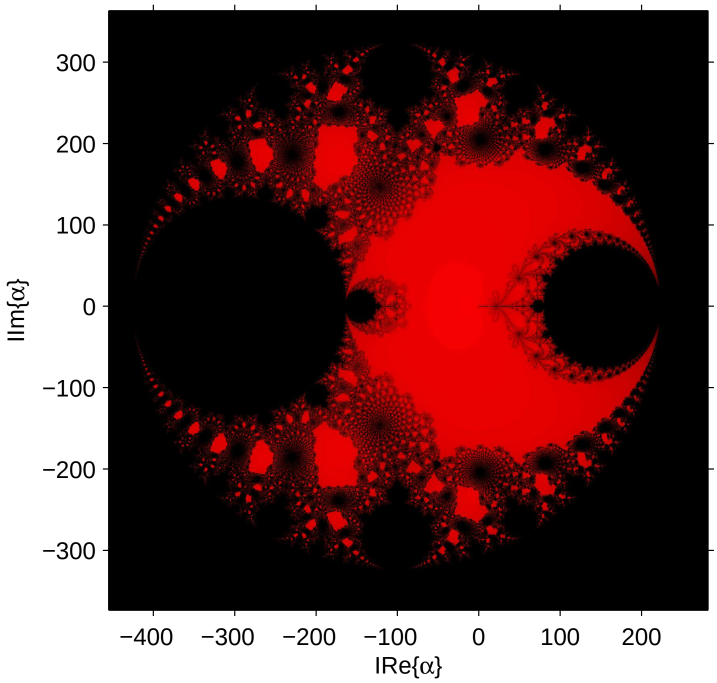


Figure 5.2: Parameter plane associated to $cr_i(\gamma)$, $i = 2, 3$.

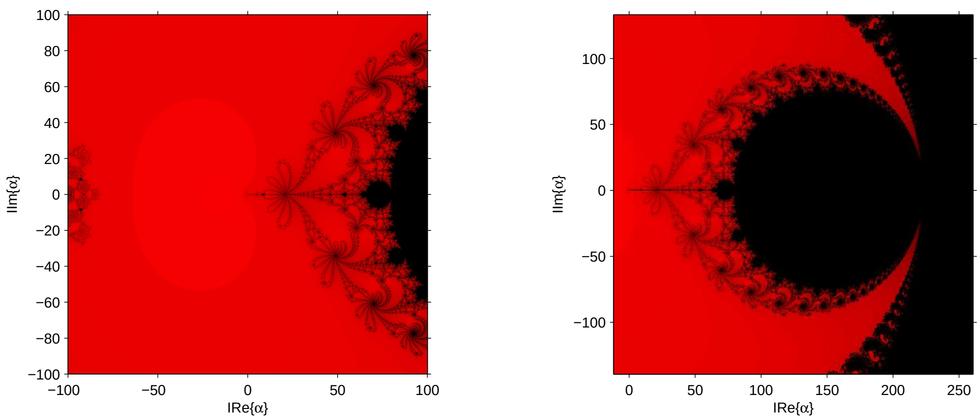
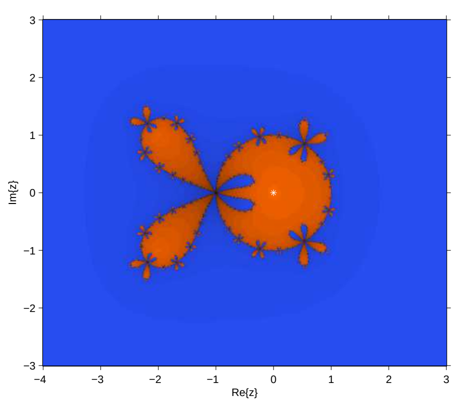


Figure 5.3: Details of the parameter plane

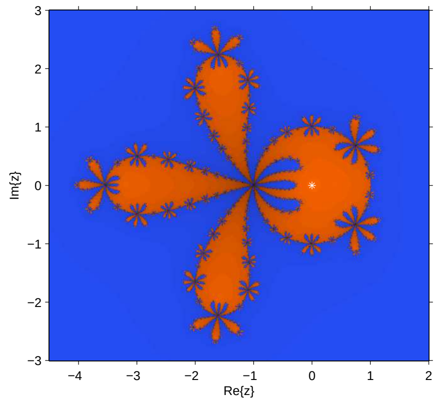
5.6 Dynamical Planes

Now, we will analyze the qualitative behavior of the different elements of family through the dynamical planes. These elements are selected taking into account the conclusions obtained by studying the parameter plane of the family. Dynamical planes with stable behavior are depicted by using points in the red region of the parameter plane, whereas dynamical planes with unstable performance are calculated with points in the black region of the parameter plane of the family.

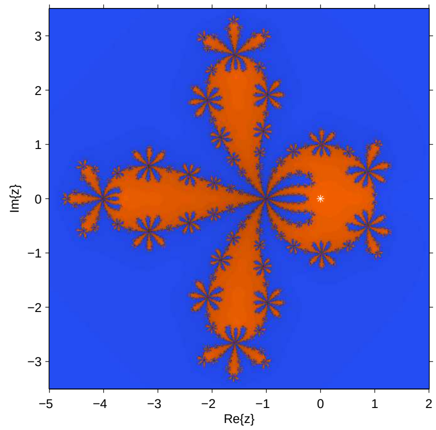
Every dynamical plane presented here has been generated by using the routines appearing in [17]. The dynamical plane related to a value of parameter γ is obtained by iterating an element of family (5.1). Initial estimations are based on each point of the complex plane. The color of each point



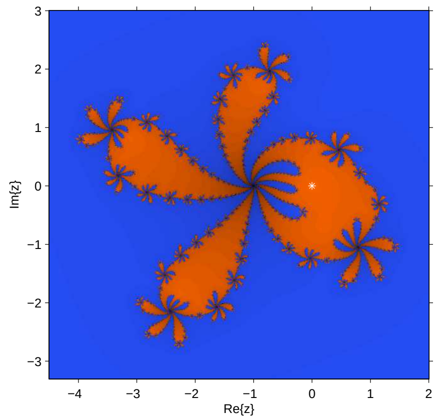
(a) $\gamma = 0$



(b) $\gamma = -30$



(c) $\gamma = -60$



(d) $\gamma = 0 + 30i$

Figure 5.4: Some dynamical planes with stable behavior

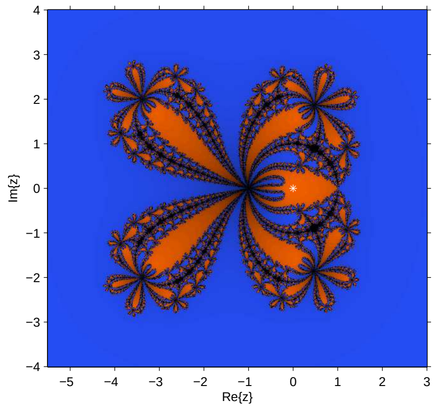
represents their convergence, following the same pattern that in the previous chapters. In this case, a mesh of 600×600 points has been used and a tolerance of 10^{-3}).

Some dynamical planes are shown in Figure 5.4. These planes correspond to values of the parameter γ which, from parameter plane, give us elements of the family with stable behavior. Therefore, we can see only two basins of attraction, that correspond to zero (orange basin) and infinity (blue basin).

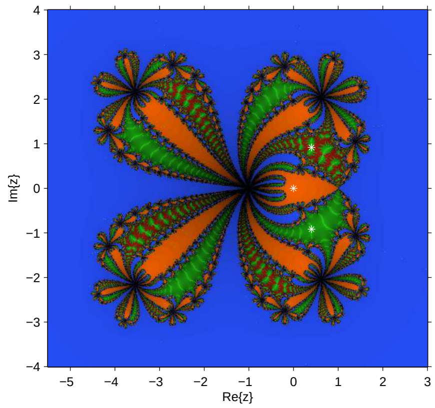
On the other hand, as it has been said, unstable behavior is found when we choose values of the parameter in the black region of parameter plane. These dynamical planes are shown in Figure 5.5.

In Figures 5.5 (a), (c), (e) and (f) the dynamical planes of iterative schemes related to $\gamma = 65$, $\gamma = 85{+}35i$, $\gamma = -140$ and $\gamma = -200$, respectively, are presented, with regions of slow convergence. Figure 5.5 (b), corresponding to $\gamma = 90$, shows four basins of attraction, being two of them of the superattractive points 0 and ∞, and the other ones related to strange fixed points.

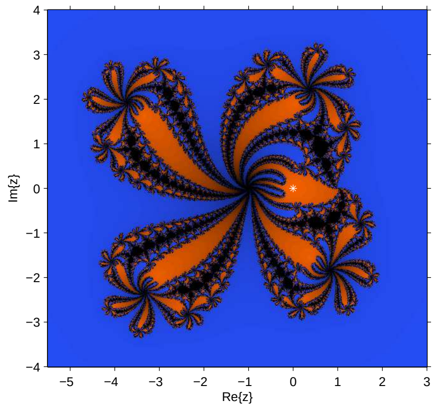
Finally, in Figure 5.5 (d), corresponding to $\gamma = \frac{891}{4}$, we can observe an orbit of period two.



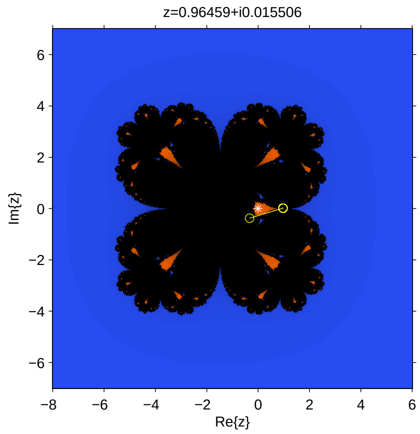
(a) $\gamma = 65$



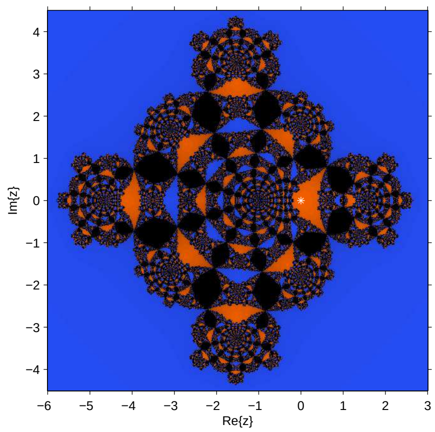
(b) $\gamma = 90$



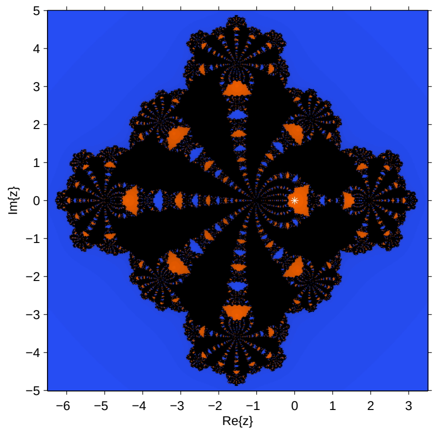
(c) $\gamma = 83 + i35$



(d) $\gamma = \dfrac{891}{4}$



(e) $\gamma = -140$



(f) $\gamma = -200$

Figure 5.5: Some dynamical planes with unstable behavior

5.7 Conclusions

In this chapter, a class of optimal iterative methods for solving nonlinear equations is presented, which holds many known methods as particular elements. This family is extended in two different directions: a class of derivative-free methods for solving nonlinear equations and a multidimensional family for nonlinear systems. Both classes preserve the order of convergence of the initial family. The dynamical analysis in these areas will be object of study in future works.

Moreover, a complex dynamical study for a parametric sub-family applied on quadratic polynomials has been presented. We have been able to prove, in terms of the parameter space, that there are many values of γ with good stability properties, which means that there exist plenty elements of the family with stable behavior. Nevertheless, there are other values of γ with convergence anomalies that must be avoided in practical applications.

Chapter 6

Conclusions and further work

6.1 Conclusions

In the course of this Final Degree Project, a deep analysis of iterative methods for solving nonlinear equations and systems has been carried out. The relevance of these methods in many fields of science has been detailed, presenting several examples related to those different fields. Additionally, the importance of achieving the improvement of them has also been explained, creating new schemes in order to accelerate the convergence or to improve the computational efficiency.

With the main objective of design an iterative method with the best characteristics, such as a high order of convergence and a stable behavior in terms of complex dynamics, throughout the project, the following actions have been developed:

It has been carried out the analysis of complex dynamics associated to the family of iterative methods created by Kou. The study of the operator of the family, along with the strange fixed points and the free critical points was developed, followed by the representation and interpretation of parameter and dynamical planes. Those tools and their analysis led to the selection of the most stable elements of the presented family.

A new family of methods for the resolution of nonlinear equations, with order of convergence four, has been designed in Chapter 4, based on a family presented in [21] for solving nonlinear systems. Additionally, the presented family has been modified, increasing the order to eight. This modifications enabled the family to reach the solution of the equation applied faster, what is truly important in the application of iterative schemes. The dynamical analysis carried out proved that the dynamical characteristics of the family of order eight had improved, compared with the fourth-order family, having more elements of the family with stable behavior.

In Chapter 5, a new family of fourth-order has also been created, on the basis of previous studies of existing families of iterative methods. The family under study includes known schemes in the literature selecting some specific values of the parameter of the family, fact that makes it a relevant family for researching. The analysis of this family and its dynamics also enable to choose the schemes with the best characteristics, in terms of efficiency and stability.

Dynamical characteristics of every family mentioned have been obtained as a result of complex dynamics. Since all the families under study are parametric, their behavior has been analyzed according to their respective parameters and with the objective of finding the best elements of each one of the presented families. These optimal elements have been shown in the different chapters of the respective families.

Routines in Matlab have been implemented, so as to depict dynamic and parameter planes. Through these representations, it has been realized the analysis of the stability of the different methods, allowing the fulfilment of the main objective of the selection of the most efficient members of the families.

All these actions and studies, led to the following conclusions:

- Parameter planes enable the fulfilment of a previous study of the dynamical behavior of a parametric family of iterative methods.
- As a result of the analysis of the parameter planes, it can be achieved the selection of a member of the family whose stability is guaranteed.
- Dynamic planes show the behavior of a large amount of initial estimations, providing an idea of the orbit of each one of those points.
- The characteristics of the basins of attractions of the dynamical planes contribute with relevant information related to the properties of each method.

6.2 Further work

The development of this Final Degree Project has been the result of several years of research, thanks to the development of three collaboration scholarship in Applied Mathematics Department from Polytechnic University of Valencia. After the research and the results obtained, more studies will be carried out, since the analysis of iterative methods for solving nonlinear problems is a field in continuous development due to the relevance of improving these methods. Further work would be placed in the following lines:

- Development of the dynamical analysis of the class of derivative-free methods for solving nonlinear equations and the multidimensional family for nonlinear systems showed in Section 5.1.
- Design of optimal iterative methods with higher orders of convergence.
- Design of iterative methods adapting them for the resolution of nonlinear matrix equations.
- Implementation of new iterative methods for solving nonlinear systems.
- Study of the dynamics associated to the created iterative methods.

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