Interval-valued 2-tuple hesitant fuzzy linguistic term set and its application in multiple attribute decision making

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Abstract. The hesitant fuzzy linguistic term sets can retain the completeness of linguistic information elicitation by assigning a set of possible linguistic terms to a qualitative variable. However, sometimes experts cannot make sure that the objects attain these possible linguistic terms but only provide the degrees of confidence to express their hesitant cognition. Given that the interval numbers can denote the possible membership degrees that an object belongs to a set, it is suitable and convenient to provide an interval-valued index to measure the degree of a linguistic variable to a given hesitant fuzzy linguistic term set. Inspired by this idea, we introduce the concept of interval-valued 2-tuple hesitant fuzzy linguistic term set (IV2THFLTS) based on the interval number and the hesitant fuzzy linguistic term set. Then, we define some interval-valued 2-tuple hesitant fuzzy linguistic aggregation operators. Afterwards, to overcome the instability of subjective weights, we propose a method to compute the weights of attributes. For the convenience of application, a method is given to solve the multiple attribute decision making problems with IV2THFLTSs. Finally, a case study is carried out to validate the proposed method, and some comparisons with other methods are given to show the advantages of the proposed method.

Keywords: Hesitant fuzzy linguistic term sets, interval numbers, interval-valued 2-tuple hesitant fuzzy linguistic term set, aggregation operators, weight determining method, oversea investment evaluation

1. Introduction

Torra [1] introduced the hesitant fuzzy set (HFS) to express the membership degrees that an element belongs to a set as some discrete values in [0, 1]. The HFS is useful in representing the hesitancy of decision-makers (DMs)’ cognition when determining the evaluation values [2, 3]. It has attracted many researchers’ attention [4, 5, 6]. However, the HFS can only be used to represent the quantitative information. To retain the completeness of linguistic elicitation based on the fuzzy linguistic approach [7], motivated by the HFS, Rodríguez et al. [8] introduced the hesitant fuzzy linguistic term set (HFLTS) as an ordered finite subset of a consecutive linguistic term set (LTS). Liao et al. [9] redefined the HFLTS in mathematical representation and called its elements as hesitant fuzzy linguistic elements (HFLEs). Since the HFLTS can retain the completeness of linguistic information elicitation, it has been a hot research topic [10]. Wei et al. [11] defined the operations on HFLEs based on the convex combination and compared the HFLEs based on the possibility degree formulas. To enhance the applicability of HFLTSs, different types of distance and similarity measures between HFLTSs were investigated [12, 13]. Liao et al. [14] developed a hesitant linguistic VIKOR method to solve the multiple attribute decision making (MADM) problems within the context of HFLTSs and the criteria conflict with each other. Zhang et al. [15] applied the hesitant linguistic VIKOR method to the inpatient admission assessment process in West China Hospital. Rodríguez et al. [16] proposed a new linguistic group decision model to promote the elicitation of flexible
and rich linguistic information based on the HFLTS. Liao et al. [17] developed two methods for hesitant linguistic MADM problems based on the ELECTRE II method.

Although the HFLTS is useful in representing the complex linguistic expressions, it is limited in some cases to represent comprehensive linguistic information [18,19]. Thus, many scholars extended the HFLTS into different variations. Wang [20] generalized the HFLTS by enabling any non-consecutive linguistic terms in them, and referred it as the extended HFLTS (EHFLTS). Zhang and Wu [21] proposed the concept of the hesitant fuzzy linguistic set (HFLS) by combining the HFS and the fuzzy linguistic approach. Chen et al. [22] proposed the proportional HFLTS, which includes the proportional information of each generalized linguistic term. Lin et al. [23] proposed the concepts of HFLS and hesitant fuzzy uncertain linguistic set (HFULS), but the concept of HFLS they introduced is different from that defined by Zhang and Wu [21]. Wei [24] proposed the concept of interval valued HFULS based on the HFS and the uncertain LTS. Wang et al. [25] proposed the concept of interval-valued HFS (IVHFLS) based on the interval-valued HFS. Due to the hesitancy and uncertainty of DMs’ cognition, Meng et al. [26] introduced the linguistic interval HFS (LHFS) based on the linguistic hesitant fuzzy set (LHFS) [27], where the membership degrees of linguistic terms are intervals rather than real numbers. The LHFS not only gives the possible linguistic terms of a linguistic variable but also considers the possible membership degree of each linguistic term. To extend the applicability of LHFSs, Zhu et al. [28] proposed the concept of the comprehensive cloud of LHFSs. As shown in Table 1, a wide range of concepts were proposed in the literature.

Considering the powerfulness of HFLTS, it is flexible for DMs to provide their opinions by HFLTS, but sometimes they cannot make sure that the objects attain these possible linguistic terms but only provide the degrees of confidence to express their hesitant cognition. Due to the complexity of MADM problems and the subjective uncertainty of DMs, it is difficult for DMs to express membership degree with the precise values. Xu and Da [29] first introduced the concept of interval number and defined the operations of interval numbers. As Chen et al. [30] noted, the precise membership degrees in the form of some discrete values in $[0,1]$ are sometimes hard to be obtained. It may be flexible for DMs to express the membership degrees with an interval number within $[0,1]$. Given that the interval numbers can denote the possible membership degrees that an element belongs to a given set, it is suitable and convenient to provide an interval-valued index to measure the degree or intensity of a linguistic variable to a given HFLTS. Inspired by this idea, in this paper, we extend the HFLTS to the interval-valued 2-tuple hesitant fuzzy linguistic term set (IV2THFLTS) based on the interval number and the HFLTS. Then, we define several IV2THFL aggregation operators for solving MADM problems. To overcome the instability of subjective weights, a method is proposed to compute the weights of attributes. For the convenience of application, a method is given to solve the MADM problems with IV2THFLTSs.

### Table 1. A comparison of different extended concepts

<table>
<thead>
<tr>
<th>Concepts</th>
<th>Year</th>
<th>Reference</th>
<th>Representation form</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>HFLTS</td>
<td>2012</td>
<td>[8]</td>
<td>$(x_i, s_{x_i}) &gt; x \in X$</td>
<td>$(s_i, s_{x_i})$</td>
</tr>
<tr>
<td>PHFLTS</td>
<td>2016</td>
<td>[22]</td>
<td>$(x_i, \rho_i) \in S, i = 0,1, \ldots, g$</td>
<td>$(s_i, 0.3), (s_i, 0.5)$</td>
</tr>
<tr>
<td>HFS</td>
<td>2014</td>
<td>[23]</td>
<td>$(x_i, s_{x_i}) &gt; x \in X$</td>
<td>$(s_i, 0.3, 0.5)$</td>
</tr>
<tr>
<td>HFULS</td>
<td>2014</td>
<td>[23]</td>
<td>$(x_i, s_{x_i}) &gt; x \in X$</td>
<td>$(s_i, 0.3, 0.5)$</td>
</tr>
<tr>
<td>IVHFLS</td>
<td>2014</td>
<td>[25]</td>
<td>$(x_i, s_{x_i}) &gt; x \in X$</td>
<td>$(s_i, 0.3, 0.5)$</td>
</tr>
<tr>
<td>IVHFULS</td>
<td>2016</td>
<td>[24]</td>
<td>$(x_i, s_{x_i}) &gt; x \in X$</td>
<td>$(s_i, 0.3, 0.5)$</td>
</tr>
<tr>
<td>LHFS</td>
<td>2014</td>
<td>[27]</td>
<td>$[(s_{x_i}, \rho_{s_{x_i}}) \in S }</td>
<td>(s_i, 0.2, 0.3)</td>
</tr>
<tr>
<td>LHFS</td>
<td>2016</td>
<td>[26]</td>
<td>$[(s_{x_i}, \rho_{s_{x_i}}) \in S</td>
<td>(s_i, 0.2, 0.3), (0.4, 0.5)$</td>
</tr>
</tbody>
</table>

The main contributions of this paper are summarized as follows:

1. We extend the HFLTS to the IV2THFLTS which expresses the evaluation information more flexibly by depicting the interval-valued membership degrees in the form of interval numbers. It can retain the completeness of linguistic information given by the DMs and thus effectively enhance the accuracy of
decision results.

(2) We define some generalized aggregation operators for IV2THFLEs. With these operators, DMs can choose different values of parameter $\lambda$ to express their preference.

(3) We propose a method to obtain the weights of attributes to overcome the instability of subjective weights based on the differences of these attributes. The weights obtained in this paper can improve the accuracy of decision results.

(4) We propose a method to solve the MADM problems with the IV2THFLTSs. We illustrate the procedure by a case study concerning the overseas investment evaluation.

The remainder of this paper is organized as follows. In Section 2, some basic knowledge of interval number, HFLTS and HFULS are reviewed. In Section 3, we define the concept of IV2THFLTSs and their operations. In Section 4, some aggregation operators for IV2THFLEs are defined. We propose a method to obtain the weights of attributes and then develop a method to solve the MADM problems under IV2THFL environment. Section 5 illustrates the applicability of the proposed method. Some conclusions are drawn in Section 6.

2. Preliminaries

2.1. Interval numbers

It may be difficult for DMs to give the precise membership degree of an element to a set. However, it is easy to give the interval-valued membership degree $\tilde{\mu} = \mu^U - \mu^L$. Especially, $\tilde{\mu}$ is a real number if $\mu^L = \mu^U$. Let $\tilde{\mu} = [\mu^L, \mu^U]$ and $\tilde{\nu} = [\nu^L, \nu^U]$ be two interval numbers and $\mu^L \geq \nu^L$. Then,

(1) $\tilde{\mu} + \tilde{\nu} = [\mu^L + \nu^L, \mu^U + \nu^U]$;
(2) $\lambda \tilde{\mu} = [\lambda \mu^L, \lambda \mu^U]$;
(3) $\tilde{\mu} \cdot \tilde{\nu} = [\mu^L \cdot \nu^L, \mu^U \cdot \nu^U]$;
(4) $\tilde{\mu}^\lambda = [(\mu^L)^\lambda, (\mu^U)^\lambda]$.

The possibility degree of $\tilde{\mu} \leq \tilde{\nu}$ is defined as [29]:

$$p(\tilde{\mu} \leq \tilde{\nu}) = \max \left\{ 1 - \max \left( \frac{\nu^L - \mu^L}{\nu^U - \mu^L + \nu^U - \mu^U}, 0 \right), 0 \right\}$$ (1)

2.2. HFLTS

Let $S = \{s_i | i = -t, ..., 0, ..., t \}$ be a finite LTS, satisfying: $s_i > s_j$, if $i > j$. To retain the completeness of information, $S$ is extended to $\bar{S} = \{s_i | i \in [-t, t] \}$ [29]. To improve the accuracy of linguistic information representation, Rodríguez et al. [8] proposed the HFLTS, which is an ordered finite subset of consecutive linguistic terms of $S$. Later, Liao et al. [9] redefined mathematically as

$$H_S = \{< x, h_\delta(x) > | x \in X \}$$ (2)

where $h_\delta(x) = \{s_\delta(x) | s_\delta(x) \in S, l = 1, ..., L \}$ denotes the possible membership degrees of $x$ to $S$. For convenience, $h_\delta(x)$ is called the HFLE.

The upper bound $H_{x^U} = \max \{s_i \}$ and the lower bound $H_{x^L} = \min \{s_i \}$ of $H_S$ are introduced to define the envelope of $H_S$ [8]. The envelope of a HFLTS, $env(H_S)$, is a linguistic interval, where

$$env(H_S) = [H_{x^L}, H_{x^U}], H_{x^L} \leq H_{x^U}.$$ (3)

2.3. HFULS

Xu [31] proposed the concept of uncertain linguistic variable as interval linguistic terms. Inspired by the idea of HFS which represents the membership degree of an element to a set in multiple values, Lin et al. [23] proposed the HFULS by combining the uncertain linguistic variable and HFS. Let $\tilde{S}$ be a set of uncertain linguistic terms. A HFULS on $X$ is in form of:

$$\tilde{A} = (x, < \tilde{s}_\delta(x), h_\delta(x) > | x \in X )$$ (4)

where $h_\delta(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees that element $x$ belongs to an uncertain linguistic term $\tilde{s}_\delta(x)$. We call $< \tilde{s}_\delta(x), h_\delta(x) > = < [s_\delta(x), s_\delta(x)'], 0 >$.

$h_\delta(x)$ is the hesitant uncertain linguistic element (HFULE).

3. The IV2THFLTSs and their operations

By combining interval number with HFLTE, we can introduce the concept of IV2THFLTS. Then we shall define the operations and comparison laws of IV2THFLEs in this section.
3.1. IV2THFLTS

Considering that DMs cannot make sure that an object belongs to a HFLE, we introduce an interval-valued index to measure the degree of a linguistic variable to a given HFLE. In this sense, it is natural to define the IV2THFLTS.

**Definition 1.** Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a reference set and \( h_i(x) \) be a HFLE on \( X \). The IV2THFLTS \( A \) in \( X \) is defined as:

\[
A = \{< x, h_i(x), \tilde{I}_A(x) > | x \in X \}
\]

where \( \tilde{I}_A(x) \) is a closed subinterval of \([0,1]\), denoting the possible interval-valued membership degree of \( x \) to \( h_i(x) \). For convenience, \( e = < h_i, \tilde{I}_A > \) is called an interval-valued 2-tuple hesitant fuzzy linguistic element (IV2THFLE). \( A \) is the collection of all IV2THFLTS. When \( \tilde{I}_A = [1,1] \), the IV2THFLTS is reduced to the HFFLTS.

The HFULS is composed of the HFS and the uncertain linguistic variable, where the membership degree of linguistic variable \( x \in X \) to the uncertain linguistic set \( \tilde{s}_{h(x)} \) is represented by the HFS. Compared with the HFULS, the proposed IV2THFLTS consists of HFLEs and interval number. Experts are often unable to determine some precise membership degrees in the form of HFS. However, the interval number can accurately denote the membership degrees of linguistic variables to the HFLEs, which is convenient for DMs to provide the membership degrees. In addition, the HFLETS can avoid linguistic information loss in decision process.

3.2. Operations of IV2THFLTS

Motivated by the operations of HFULSs [21] and interval numbers [29], we develop some operations of IV2THFLTS.

**Definition 2.** Let \( e = < h_i, [r^\mu, r^\nu] > \), \( e_1 = < h_1^\mu, [r_1^\mu, r_1^\nu] > \) and \( e_2 = < h_2^\mu, [r_2^\mu, r_2^\nu] > \) be three IV2THFLTSs and \( \lambda \geq 0 \). Then, we have

1. \( e_1 \oplus e_2 = < U_{s_{1\lambda}, s_{2\lambda}}(s_{1\beta}, s_{2\beta}), [r_1^\mu + r_2^\mu - r_1^\nu \cdot r_2^\nu, r_1^\nu + r_2^\nu - r_1^\mu \cdot r_2^\mu] > \);
2. \( e_1 \otimes e_2 = < U_{s_{1\lambda}, s_{2\lambda}}(s_{1\beta}, s_{2\beta}), [r_1^\mu \cdot r_2^\mu, r_1^\nu \cdot r_2^\nu] > \);
3. \( \lambda e = < U_{s_{1\lambda}, s_{2\lambda}}(s_{1\beta}, s_{2\beta}), [1 - (1 - r^\mu)^\lambda, 1 - (1 - r^\nu)^\lambda] > \);
4. \( e^\lambda = < U_{s_{1\lambda}, s_{2\lambda}}(s_{1\beta}, s_{2\beta}), [(r^\mu)^\lambda, (r^\nu)^\lambda] > \).

**Example 1.** Let \( e = < s_2, s_3, [0.4, 0.6] > , e_1 = < s_1, s_2, s_3, s_4, [0.2, 0.6] > , e_2 = < s_1, s_2, ... > \) be three IV2THFLEs and \( \lambda = 2 \). Then, we have

1. \( e_1 \oplus e_2 = < s_1, s_2, s_3, s_4, s_5, s_6, s_7, [0.52, 0.92] > \);
2. \( e_1 \otimes e_2 = < s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}, [0.08, 0.48] > \);
3. \( \lambda e = < s_1, s_2, [0.64, 0.84] > \);
4. \( e^\lambda = < s_1, s_2, [0.16, 0.36] > \).

3.3. Comparison laws of IV2THFLTSs

Note that an IV2THFLE consists of a HFLE and an interval number. The HFLE denotes DMs’ linguistic evaluation index, while the interval number denotes the possible membership degree that an object belongs to the HFLE. For convenience, we transform both the HFLE and the interval membership degree into linguistic preference values. Then, we compare the IV2THFLTSs by comparing these linguistic preference values.

**Definition 3.** Let \( e_1 = < h_1^\mu, \tilde{I}(e_1) > \) and \( e_2 = < h_2^\mu, \tilde{I}(e_2) > \) be two IV2THFLTSs with \( \tilde{I}(e_1) = [r_1^\mu, r_1^\nu] \), \( \tilde{I}(e_2) = [r_2^\mu, r_2^\nu] \). Let \( env(h_1) \) and \( env(h_2) \) be the envelopes of \( h_1^\mu \) and \( h_2^\mu \), respectively, where \( env(h_1) = [s_{1\lambda}, s_{2\lambda}] \) and \( env(h_2) = [s_{1\beta}, s_{2\beta}] \). For convenience, \( LP_{e_1} = [\alpha_1, \mu_1] = [\alpha_1 \cdot r_1^\mu, \alpha_1 \cdot r_1^\nu] \) and \( LP_{e_2} = [\delta_2, \mu_2] = [\beta_2 \cdot r_2^\mu, \beta_2 \cdot r_2^\nu] \) are called the linguistic preference values. The possibility degree of \( e_1 \geq e_2 \) can be defined as:

\[
p(e_1 \geq e_2) = \max \left\{ 1 - \max \left( \frac{\delta_2 - \mu_2}{l_1 + l_2}, 0 \right) \right\}
\]

where \( l_1 = \mu_2 - \mu_1 \). Thus, \( n_1 = \delta_2 - \delta_1 \).

According to Definition 3, we obtain

1. \( 0 \leq p(e_1 \geq e_2) \leq 1 \);
2. \( p(e_1 \geq e_2) + p(e_2 \geq e_1) = 1 \);
3. \( p(e_1 \geq e_1) = 0.5 \).
To rank the IV2THFLEs $e_i (i=1,...,m)$, we transform them into corresponding linguistic preference values $LP_i (i=1,2,...,m)$, and then compare $LP_i (i=1,2,...,m)$ by Eq. (6). Let $p_y = p (LP_i \geq LP_j)$. Then, a complementary matrix $P=\left(p_{ij}\right)_{m \times m}$ can be constructed with $p_y \geq 0$, $p_y + p_x = 1$, $p_x = 0.5$, $i = 1,2,...,m$. Summing all elements in each line of $P$, we have $p_y = \sum_{i=1}^{m} p_{yi}$. Finally, we can rank the IV2THFLEs in descending order of the values of $p_y (i=1,2,...,m)$.

Example 2. Let $e_1 = \langle \{s_1, s_2\}, \{0.4,0.6\} \rangle$, $e_2 = \langle \{s_1, s_2\}, \{0.2,0.8\} \rangle$, $e_3 = \langle \{s_1, s_2\}, \{0.4,0.8\} \rangle$ and $e_4 = \langle \{s_1, s_2\}, \{0.5,0.8\} \rangle$ be four IV2THFLEs. A complementary matrix can be obtained as:

$$P = \begin{bmatrix}
0.50 & 0.80 & 0.16 & 0.11 \\
0.20 & 0.50 & 0 & 0.45 \\
0.89 & 1.00 & 0.50 & 0.56 \\
0.89 & 1.00 & 0.45 & 0.50
\end{bmatrix}$$

Summing all values in each line of $P$, we have $p_1 = 1.57$, $p_2 = 0.70$, $p_3 = 2.90$, $p_4 = 2.83$. Since $p_3 > p_4 > p_2 > p_1$, we have $e_3 > e_4 > e_1 > e_2$.

4. Aggregation operators for IV2THFLEs

To solve various MADM problems under the IV2THFL environment, we define some aggregation operators of IV2THFLEs to obtain the overall linguistic aggregation information. In addition, a weighting method is proposed to overcome the instability of subjective weights. Finally, we develop a method to solve the MADM problems with IV2THFLEs.

4.1. The IV\text{2THFLWA}, IV\text{2THFLOWA}, and IV\text{2THFLHA} operators

Definition 4. Let $e_i (i=1,2,...,n)$ be a collection of IV2THFLEs. Then an IV2THFLWA operator can be defined as:

$$IV2THFLWA_{\omega}(e_1,e_2,\ldots,e_n) = \sum_{i=1}^{n} (\omega_i e_i)$$  (7)

where $\omega = (\omega_1,\omega_2,\ldots,\omega_n)^T$ is the weight vector of $e_i (i=1,2,...,n)$ with $\omega > 0$, $\sum_{i=1}^{n} \omega_i = 1$.

According to Definition 2, Theorem 1 can be derived.

Theorem 1. The aggregated result obtained by the IV2THFLWA operator is also an IV2THFLE, and

$$IV2THFLWA_{\omega}(e_1,e_2,\ldots,e_n) = \bigcup_{s \in A_{\omega}} \{s_a, s_b, \ldots, s_{\omega} \}$$

$$\{s_{\omega}, (1-[1-s^{(1)}_a]^{\nu_1})^{\nu_2}, (1-[1-s^{(1)}_b]^{\nu_1})^{\nu_2} \} > (8)$$

Proof. It can be proved by the mathematical induction on $n$.

(1) For $n=2$. Since

$$a_0 e_1 < a_0 h_{\omega} \{1-[1-s^{(1)}_a]^{\nu_1}, 1-[1-s^{(1)}_b]^{\nu_2} \}$$

$$a_0 e_2 < a_0 h_{\omega} \{1-[1-s^{(1)}_a]^{\nu_2}, 1-[1-s^{(1)}_b]^{\nu_2} \}$$

Then

$$IV2THFLWA_{\omega}(e_1,e_2) =$$

$$< U_{\{s_1, s_2\}, \{0.5,0.8\}}, M_1, N_1 > = < U_{\{s_1, s_2\}, \{0.5,0.8\}}, (1-[1-s^{(1)}_a]^{\nu_2} \cdot (1-s^{(1)}_b)^{\nu_2}), 1 - (1-s^{(1)}_a)^{\nu_2} \cdot (1-s^{(1)}_b)^{\nu_2} >$$

where

$$M_1 = 2-(1-s^{(1)}_a)^{\nu_1} - (1-s^{(1)}_b)^{\nu_1} - (1-(1-s^{(1)}_a)^{\nu_1}) \cdot (1-(1-s^{(1)}_b)^{\nu_1}), N_1 = 2-(1-s^{(1)}_a)^{\nu_2} - (1-s^{(1)}_b)^{\nu_2} - (1-(1-s^{(1)}_a)^{\nu_2}) \cdot (1-(1-s^{(1)}_b)^{\nu_2})$$

(2) If Eq. (8) holds for $n = k$, that is

$$IV2THFLWA_{\omega}(e_1,e_2,\ldots,e_{k+1}) = < U_{\{s_1, s_2\}, \{0.5,0.8\}},$$

$$\{s_{\omega}, (1-[1-s^{(1)}_a]^{\nu_1})^{\nu_2}, \ldots, (1-[1-s^{(1)}_b]^{\nu_2})^{\nu_2} \} >$$

$$1-\prod_{i=1}^{k} ([1-s^{(1)}_i]^{\nu_1})^{\nu_2} >$$

Then, when $n = k+1$, by the operations in Definition 2, we have

$$IV2THFLWA_{\omega}(e_1,e_2,\ldots,e_{k+1}) = < U_{\{s_1, s_2\}, \{0.5,0.8\}},$$

$$\{s_{\omega}, (1-[1-s^{(1)}_a]^{\nu_1})^{\nu_2}, \ldots, (1-[1-s^{(1)}_{k+1}]^{\nu_2})^{\nu_2} \} >$$

$$1-\prod_{i=1}^{k+1} ([1-s^{(1)}_i]^{\nu_1})^{\nu_2} >$$

(9)
where $M_2 = 2 - \prod_{i=1}^{k} \left( 1 - r_{i+1}^2 \right)^{\alpha_i} - \left( 1 - r_{k+1}^2 \right)^{\alpha_{k+1}} - \prod_{i=1}^{k} \left( 1 - r_i^2 \right)^{\alpha_i} - \left( 1 - r_k^2 \right)^{\alpha_k} \right)$, $N_2 = 2 - \prod_{i=1}^{k} \left( 1 - r_i^2 \right)^{\alpha_i} - \left( 1 - r_k^2 \right)^{\alpha_k} \right)$. Therefore, Eq. (8) holds for $n = k + 1$. Thus, Eq. (8) holds for all $n$.

Inspired by the ordered weighted averaging (OWA) operator [32], we define the IV2THFLOWA operator according to the OWA operator.

**Definition 5.** Let $e_i$ $(i=1,2,\ldots,n)$ be a collection of IV2THFLEs, $e_{\omega(i)}$ be the $i^{th}$ largest of them, $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation associated vector such that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then an IV2THFLOWA operator is defined as:

$$IV2THFLOWA_\omega (e_1, e_2, \ldots, e_n) = \sum_{i=1}^{n} (\omega_i e_{\omega(i)})$$

(9)

**Theorem 2.** The aggregated result obtained by the IV2THFLOWA operator is also an IV2THFLE, and $IV2THFLOWA_\omega (e_1, e_2, \ldots, e_n)$ is a collection of IV2THFLEs, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the aggregation associated vector such that $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. Then an IV2THFLE operator is defined as:

$$IV2THFLE_\omega (e_1, e_2, \ldots, e_n) = \sum_{i=1}^{n} (\omega_i e_{\omega(i)})$$

(10)

The proof of Theorem 2 is similar to that of Theorem 1.

**Definition 6.** Let $e_i$ $(i=1,2,\ldots,n)$ be a set of IV2THFLEs, $w = (w_1, w_2, \ldots, w_n)^T$ be the aggregation associated vector such that $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$. $\alpha$ is the balancing coefficient. Then we define an IV2THFL hybrid averaging (IV2THFLHA) operator as follows:

$$IV2THFLHA_\omega (e_1, e_2, \ldots, e_n) = \sum_{i=1}^{n} (\omega_i e_{\omega(i)})$$

(11)
\[ \alpha_e^i \leq \alpha_{e_1}^i \leq \cdots \leq \alpha_{e_n}^i \]

Then

\[ \text{GIV2THFLOW}_\omega (e_1, e_2) = \leq \cup_{s, t, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

where \( M_\omega = (2 - (1 - (r_i^j)^{\alpha_i})^{\alpha_i}) \left( 1 - (r_i^j)^{\alpha_i} \right)^{\alpha_i} \), \( N_\omega = (2 - (1 - (r_i^j)^{\alpha_i})^{\alpha_i}) \left( 1 - (r_i^j)^{\alpha_i} \right)^{\alpha_i} \), \( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \).

(2) If Eq. (14) holds for \( n = k \), then

\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_k) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{k} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{k} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

...\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

(2) If Eq. (14) holds for \( n = k + 1 \), then

\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_{k+1}) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{k+1} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{k+1} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

...\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

(2) If Eq. (14) holds for \( n = k + 1 \), then

\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_{k+1}) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{k+1} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{k+1} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

...\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

Theorem 5. The aggregated value of the GIV2THFLOW operator is also an IV2THFLE, and

\[ \text{GIV2THFLOW}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

The proof of Theorem 5 is similar to that of Theorem 4.

Definition 9. Let \( e_i (i=1, 2, \ldots, n) \) be a set of IV2THFLEs, \( w = (w_1, w_2, \ldots, w_n)^T \) be the aggregation associated vector such that \( w_i \in [0, 1] \) and \( \sum_{i=1}^{n} w_i = 1 \). \( n \) is the balancing coefficient. Then a GIV2THFLHA operator is defined as:

\[ \text{GIV2THFLHA}_\omega (e_1, e_2, \ldots, e_n) = \left( \sum_{i=1}^{n} (\alpha_{e(i)}^{\omega(i)}) \right)^{\frac{1}{\sum_{i=1}^{n} (\alpha_{e(i)}^{\omega(i)})}} \]

where \( \omega = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T \) is the aggregation associated vector with \( \alpha_i \in [0, 1] \), \( \sum_{i=1}^{n} \alpha_i = 1 \), and \( e_{(i)} \) is the \( i \) th largest element of \( e_i, (i = 1, 2, \ldots, n) \).

Theorem 6. The aggregated result of the GIV2THFLHA operator is also an IV2THFLE, and

\[ \text{GIV2THFLHA}_\omega (e_1, e_2, \ldots, e_n) = \leq \cup_{s, a, b} \{s, \alpha_{(a)}^i \} \].

\[ \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}}, \left( 1 - \prod_{i=1}^{n} \left( (1 - (r_i^j)^{\alpha_i})^{\alpha_i} \right) \right)^{\frac{1}{\alpha_i}} \]

The proof of Theorem 6 is similar to that of Theorem 4.
4.3. A method to determine the weights of attributes

The subjective weights of attributes lead to the instability of decision results. To overcome this problem, in this subsection, we propose a method to obtain the weights of attributes under the IV2THFL environment.

The differences of the evaluation information between attributes have an influence on the accuracy of decision results. The smaller the differences are, the more precise the decision results would be. To compute the difference between two IVHFES, Chen et al. [33] defined the distance measures for the IVHFES. Motivated by this idea, we can define the distance measures for IV2THFLEs. We first transform the evaluation information of alternative \( A \) with respect to attribute \( C_j \) into the linguistic preference value \( LP_{ij} \) based on Definition 3. Then we obtain the subscript of \( LP_{ij} \), i.e., \( sub(LP_{ij}) = [LP_{ij}^1, LP_{ij}^2] \). We compute the difference between attributes \( C_i \) and \( C_k \) using the distance measure proposed in Ref. [33] and thus obtain

\[
D_{ik} = d(LP_{ij}^1, LP_{ik}^1) = \frac{1}{2m} \sum_{l=1}^{m} \left( |LP_{ij}^1 - LP_{ik}^1| + |LP_{ij}^2 - LP_{ik}^2| \right)
\]

Let \( D_{ik} = \sum_{l=1}^{m} D_{ik} \) be the deviation of attribute \( C_i \) from the remaining attributes. The smaller \( D_{ik} \) is, the closer attribute \( C_i \) is to that of the rest attributes, and hence the more valuable evaluation information of attribute \( C_i \) provides. Thus \( C_i \) should be assigned a large weight. The weights can be calculated as:

\[
\omega_l = \frac{(D_{ik})^{-1}}{\sum_{l=1}^{n}(D_{ik})^{-1}} \quad (l = 1, 2, \ldots, n)
\]  

(20)

4.4. A method for MADM with IV2THFLEs

For a MADM problem with uncertain linguistic information, let \( A = \{A_1, A_2, \ldots, A_m\} \) be a discrete collection of variables, \( C = \{C_1, C_2, \ldots, C_n\} \) be a discrete collection of attributes, whose weight vector is \( \omega = (\omega_1, \omega_2, \ldots, \omega_n) \) with \( \omega_j \geq 0, \ j = 1, 2, \ldots, n, \)

\[
\sum_{j=1}^{n} \omega_j = 1 .
\]

Suppose that \( T = (t_{ij})_{m \times n} \) is the decision matrix, where \( t_{ij} \) is the preference information in the form of IV2THFLE. In the following, the GIV2THFLWA operator is used to develop a method to solve the MADM problems under the IV2THFL environment.

Algorithm 1

Step 1. Compute the difference between any two attributes \( C_i \) and \( C_k \) and then determine the weights of attributes by Eq. (20).

Step 2. Utilize the GIV2THFLWA operator to aggregate the evaluation arguments in the decision matrix \( T \), and obtain the overall IV2THFL preference values \( t_i (i = 1, 2, \ldots, m) \) of the alternative \( A_i \). We have

\[
t_i = GIV2THFLWA_i (t_{i1}, t_{i2}, \ldots, t_{im} = \left( \sum_{j=1}^{n} (\omega_j t_{ij}) \right)^{1/\omega_j} \quad (i = 1, 2, \ldots, m)
\]

(21)

Step 3. Transform the collective preference information \( t_i (i = 1, 2, \ldots, m) \) into the corresponding linguistic preference values \( LP_i (i = 1, 2, \ldots, m) \) based on Definition 3.

Step 4. Compare each value \( LP_i \) with all values of \( LP_j (j = 1, 2, \ldots, m) \) by Definition 3. For simplicity, a complementary matrix \( P = (p_{ij})_{m \times n} \) is constructed, where \( p_{ij} \geq 0 \), \( p_{ij} + p_{ji} = 1, \ p_{ij} = 0.5 \). Summing all the values in each line of \( P \), we have \( p_i = \sum_{j=1}^{n} p_{ij} (i = 1, 2, \ldots, m) \).

Step 5. Select the best alternative according to the values of \( p_i (i = 1, 2, \ldots, m) \).

In Algorithm 1, Step 1 is to obtain the objective weights according to the differences of attributes. Step 2 is to derive the collective IV2THFL preference values of alternatives by the GIV2THFLWA operator. Step 3 is to transform the overall preference information into the corresponding linguistic preference values. Step 4 is to compare the linguistic preference values and establish the complementary matrix. Step 5 is to rank all the alternatives according to the values of each line of the complementary matrix.
5. Case study: Global mineral investment evaluation

In this section, we apply the proposed MADM method in a practical example concerning the global mineral investment evaluation (adapted from Ref. [34]). Then the proposed method is compared with other existing methods.

5.1. Case description

ABC Nonferrous Metals Co. Ltd. is a large state-owned company whose main business is producing and selling nonferrous metals. The company evaluates the global mineral investment business according to the oversea investment department, which consists of executive managers and several experts in this field. Recently, this department decided to select several alternatives from some foreign countries based on preliminary survey. After detailed analysis, four countries \{A_1, A_2, A_3, A_4\} are taken into consideration. Three factors are finally considered, including \(C_1\) : resources, \(C_2\) : politics and policy and \(C_3\) : infrastructure.

To obtain the decision information, the LTS \(S = \{s_0: nothing, s_1: very \ low, s_2: low, s_3: medium, s_4: high, s_5: very \ high, s_6: perfect\}\) is used. The decision information takes the form of IV2THFLEs where \(C_j(A_i)\) is the evaluation argument of alternative \(A_i\) on criterion \(C_j\). In \(C_j(A_i)\) there is a consensus on the chosen LTS and each DM can use a value to express his/her opinions, i.e., the value \(C_j(A_i)\) denotes to what degree \(A_i\) matches this given linguistic terms under \(C_j\). DMs gave their own evaluation values in the form of IV2THFLEs based on the survey of these four countries as well as their knowledge and experience. Consequently, following a heated discussion, they came to a consensus on the final decision as shown in decision matrix \(T\).

5.2. Application of the proposed method

To overcome the instability of subjective weights, the weights of criteria are calculated by Eq. (20) and we obtain \(\omega_1 = 0.34, \omega_2 = 0.36, \omega_3 = 0.29\). To get the best alternative, we let \(\lambda = 0.1\) and utilize the GIV2THFLWA operator to aggregate all the linguistic evaluation information \(t_j(i=1,2,3,4, j=1,2,3)\) in decision matrix \(T\). Then we can obtain the overall preference values \(t(i=1,2,3,4)\) as

\[
\begin{align*}
\text{\(T_1\)} &= \langle \{s_{2.58,2.87,2.99,3.28,2.76,3.16,3.31,3.51}, \{s_{2.92,3.24,3.34,3.70}\}\rangle, \quad \text{\(T_2\)} &= \langle \{s_{3.41,3.69,3.92,3.67,3.27,3.21,3.02,3.28,3.00,3.33,3.61,3.61\}\rangle, \\
\text{\(T_3\)} &= \langle \{s_{0.41,0.39,0.28,0.05,0.26,0.28,0.38,0.37,0.43,0.71}\rangle, \quad \text{\(T_4\)} &= \langle \{s_{0.10,0.19,0.17,0.19,0.20,0.20,0.36,0.36,0.38,0.38,0.40,0.50}\rangle.
\end{align*}
\]

Subsequently, we transform the overall preference values into their corresponding linguistic preference values as \(L_P(i=1,2,3,4)\) and obtain the complementary matrix \(P\) as follows:

\[
\begin{align*}
P &= \begin{bmatrix}
0.50 & 0.43 & 0.72 & 0.43 \\
0.57 & 0.50 & 0.77 & 0.51 \\
0.28 & 0.23 & 0.50 & 0.23 \\
0.57 & 0.49 & 0.77 & 0.50 
\end{bmatrix}
\end{align*}
\]

Then, we have \(p_1 = 2.08, p_2 = 2.35, p_3 = 1.24, p_4 = 2.33\). We rank \(p_i(i=1,2,3,4)\) and obtain \(A_2 \succ A_1 \succ A_3 \succ A_4\). Thus, the best alternative is \(A_2\).

In the above example, we only give the ranking order according to the GIV2THFLWA operator with \(\lambda = 0.1\). As the parameter \(\lambda\) changes, different
results can be obtained, shown as Table 2. By Table 2, we can find that the decision results are highly related to $\lambda$. As $\lambda$ increases, $A_2$ is the best choice first, and then $A_4$ becomes the best choice in the case $\lambda \geq 0.5$. The DMs can choose the values of $\lambda$ according to their preferences.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$LP_1$</th>
<th>$LP_2$</th>
<th>$LP_3$</th>
<th>$LP_4$</th>
<th>Ranking</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>$[s_{1.08}, s_{1.28}]$</td>
<td>$[s_{1.17}, s_{1.42}]$</td>
<td>$[s_{0.02}, s_{1.35}]$</td>
<td>$[s_{1.15}, s_{1.35}]$</td>
<td>$A_2 &gt; A_4 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>0.2</td>
<td>$[s_{1.08}, s_{1.28}]$</td>
<td>$[s_{1.20}, s_{1.57}]$</td>
<td>$[s_{0.15}, s_{1.93}]$</td>
<td>$[s_{1.21}, s_{1.48}]$</td>
<td>$A_2 &gt; A_4 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>0.5</td>
<td>$[s_{1.14}, s_{2.43}]$</td>
<td>$[s_{1.27}, s_{2.68}]$</td>
<td>$[s_{0.55}, s_{2.02}]$</td>
<td>$[s_{1.32}, s_{2.61}]$</td>
<td>$A_2 &gt; A_2 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>1.0</td>
<td>$[s_{1.18}, s_{2.52}]$</td>
<td>$[s_{1.14}, s_{2.74}]$</td>
<td>$[s_{0.87}, s_{2.09}]$</td>
<td>$[s_{1.40}, s_{2.70}]$</td>
<td>$A_2 &gt; A_2 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>2.0</td>
<td>$[s_{1.25}, s_{2.67}]$</td>
<td>$[s_{1.44}, s_{2.83}]$</td>
<td>$[s_{1.71}, s_{2.72}]$</td>
<td>$[s_{1.76}, s_{1.71}]$</td>
<td>$A_2 &gt; A_2 &gt; A_2 &gt; A_3$</td>
</tr>
<tr>
<td>5.0</td>
<td>$[s_{1.41}, s_{1.00}]$</td>
<td>$[s_{1.66}, s_{1.05}]$</td>
<td>$[s_{1.37}, s_{2.46}]$</td>
<td>$[s_{1.36}, s_{1.11}]$</td>
<td>$A_2 &gt; A_2 &gt; A_2 &gt; A_3$</td>
</tr>
</tbody>
</table>

5.3. Comparison analyses

To illustrate the advantages, the presented method is compared with other two representative MADM methods.

Case 1. Comparison with the MADM method with HFLTSs

Rodríguez et al. [8] developed a MADM model with HFLTS, and utilized min-upper and max-lower operators to obtain the linguistic evaluation intervals. For comparison, we utilize the MADM method in Ref. [8] to solve the illustrative example above. To begin with, we obtain the product of the subscripts corresponding to linguistic terms and the discrete values within the interval-valued membership degrees. Then the IV2THFLEs are transformed into the HFLEs by integrating the possible interval membership degrees and each HFLTS. For example, an IV2THFLE $\langle s_{1.08}, \ldots, s_{0.3}, [0.3, 0.5] \rangle$ can be replaced by a HFLE $\{s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}\}$. The transformed decision matrix $TD$ under hesitant fuzzy linguistic environment is shown as follows:

\[
TD = \begin{bmatrix}
A_1 & A_2 & A_3 & A_4 \\
\{s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}\} & \{s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}\} & \{s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}\} & \{s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}, s_{1.08}\}
\end{bmatrix}
\]

According to the min-upper and max-lower operators, we obtain the linguistic intervals $L_i$ ($i = 1, 2, 3, 4$) as $L_i = [s_{1.08}, s_{1.08}]$, $L_i = [s_{1.08}, s_{1.08}]$, $L_i = [s_{1.08}, s_{1.08}]$, $L_i = [s_{1.08}, s_{1.08}]$. Then, the nondominance degrees of alternatives are obtained as: $NDD_1 = 0$, $NDD_2 = 0.46$, $NDD_3 = 0$, $NDD_4 = 0.92$. The ranking of the alternatives is: $A_3 > A_1 > A_2 > A_1$. From Table 2, when $\lambda = 0.5$, $1,2,5$, obviously, the most desirable alternative is consistent with that obtained by the proposed method. However, in other cases, the ranking results obtained by our method is inconsistent with that obtained by the method in Ref. [8]. The main reason is that the proposed method can effectively retain the completeness of decision information in operation process.

Case 2. Comparison with the MADM method with HFULSs

Lin et al. [23] proposed a MADM method based on HFULS. To compare it with the proposed method, we need to transform the IV2THFLEs into the HFLEs. We obtain the mean value of the subscripts corresponding to all linguistic terms in the IV2THFLEs, and then transform the interval membership degree into some discrete values in it. For example, an IV2THFLE $\langle s_{1.08}, [0.3, 0.5] \rangle$ can be replaced by a HFLE $\{s_{1.08}, [0.3, 0.4, 0.5] \}$. By the approach in Ref. [23], we get $s(A_1) = 1.8698$, $s(A_2) = 1.9927$, $s(A_3) = 1.4585$, $s(A_4) = 2.136$. Since $s(A_1) > s(A_2) > s(A_3) > s(A_4)$, the ranking of alternatives is $A_1 > A_2 > A_3 > A_4$. Obviously, when $\lambda = 0.5, 1, 2, 5$, the above ranking is the same.
as that obtained by the proposed method. When \( \lambda = 0.1,0.2 \), the above ranking is different from that obtained by the method in this paper. Thus, DMs can flexibly select the value of parameter \( \lambda \) to make decisions according to their preferences.

Compared with the above methods within different contexts, the advantages of the proposed method for MADM problems under IV2THFL environment are listed as follows:

1. The IV2THFLEs can provide a flexible choice for DMs and closely depict the precise membership degrees of a linguistic variable to HFLTS. The IV2THFLEs not only give the possible linguistic terms but also consider the possible membership degrees. In addition, the IV2THFLEs can retain the completeness of decision information, which are more precise than HFLTS.

2. We define some generalized aggregation operators for IV2THFLEs. Different decision results can be obtained when different values of \( \lambda \) are used. Thus, DMs can flexibly select the value of \( \lambda \) according to their preferences.

3. We propose a method to obtain the weights of attributes based on the differences of attributes, which avoids the instability of subjective weights. In addition, compared with the MADM methods in the literature, the proposed MADM method can get more accurate decision results.

### 6. Conclusions

In this paper, we extended the HFLTS to the IV2THFLTS. Then we defined some aggregation operators for IV2THFLEs. To overcome the instability of subjective weights, a method was proposed to obtain the weights of attributes based on the differences between attributes. Moreover, we applied the GIV2THFLWA operator to develop a method for MADM problems under the IV2THFL environment. We applied the proposed method to solve an illustrative example where different values of \( \lambda \) were used. Finally, the proposed method was compared with other two representative MADM methods. The results showed that the method we presented can avoid information loss and enhance the accuracy of decision results. What is more, it allows DMs to choose different values of \( \lambda \) to aggregate linguistic information. Thus, DMs can select the most appropriate parameter according to their preference.

In the future, the proposed MADM method can be utilized in various fields, such as transportation, logistics, and artificial intelligence. Some new weight determination methods may be developed.

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### References


