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Additional Information

Optimal Iterative Methods for Finding Multiple Roots of Nonlinear Equations Using Weight Functions and Dynamics [☆]

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Abstract

In this paper, we propose a family of iterative methods for finding multiple roots, with known multiplicity, by means of the introduction of four univariate weight functions. With the help of these weight functions, that play an important role in the development of higher order convergent iterative techniques, we are able to construct three-point eight-order optimal multiple-root finders. Also, numerical experiments have been applied to a number of test equations for different special schemes from this family satisfying the conditions given in the convergence analysis. We have also compared the basins of attraction of some proposed and known methods in order to check the wideness of the sets of converging initial points for each problem.

Keywords: Nonlinear equations; optimal iterative methods; multiple root; basin of attraction; stability.

1. Introduction

Newton's method has been widely used to investigate simple or multiple zeros of a nonlinear equation. If the given function involves only a simple zero, then Newton's method converges quadratically to the exact solution provided that a proper initial guess is selected close to the exact solution. Newton's method has a drawback that it converges linearly when a given function has multiple roots. For a nonlinear equation of the form $f(x) = 0$, which involves multiple roots with multiplicity $m > 1$ a priori, modified Newton's method [14, 15] is given as:

$$x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (1.1)$$

It efficiently locates the desired multiple zero with quadratic order convergence. The numerical scheme (1.1) is a second-order, one-point, optimal method on the basis of Kung-Traub's conjecture [9] which states that any multipoint method without memory can reach its convergence order of at most 2^{r-1} for r functional evaluations per iteration. In the recent past, many researchers like Li et al. [11] in (2009), Sharma and Sharma [19] and Li et al. [10] in (2010), Zhou et al. [23] in (2011), Sharifi et al. [16] in (2012), Soleymani et al. [17], Soleymani and Babajee [18], Liu and Zhou et al. [12] and Zhou et al. [24] in (2013), Thukral [20] in (2014), Behl et al. [1] and Hueso et al. [7] in (2015) and Behl et al. [2] in (2016) have presented optimal fourth-order iterative methods for multiple zeros. In addition, Li et al. [10] and Neta [13] presented optimal and non-optimal fourth-order iterative methods. Thukral [21] and Geum et al. [5, 6] have been able to present sixth order convergent methods for finding multiple roots.

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In 2013, Thukral [21] presented a multi-point iterative method with sixth-order convergence, which is given by

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - m \frac{f(x_n)}{f'(x_n)} \sum_{i=1}^3 i \left(\frac{f(y_n)}{f(x_n)} \right)^{\frac{i}{m}}, \\ x_{n+1} &= z_n - m \frac{f(x_n)}{f'(x_n)} \left(\frac{f(z_n)}{f(x_n)} \right)^{\frac{1}{m}} \left\{ \sum_{i=1}^3 i \left(\frac{f(y_n)}{f(x_n)} \right)^{\frac{i}{m}} \right\}^2. \end{aligned} \quad (1.2)$$

In 2015, Geum et al. [5], have given the following two-point, sixth-order iterative scheme, for $m > 1$:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - Q(p_n, s_n) \frac{f(y_n)}{f'(y_n)}, \end{aligned} \quad (1.3)$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$ and $Q : \mathbb{C}^2 \rightarrow \mathbb{C}$ is a holomorphic function in the neighbourhood of origin $(0, 0)$.

In 2016, Geum et al. [6], have given a three-point iterative scheme with sixth-order convergence for multiple zeros involving weight function approach as follows:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ w_n &= x_n - mG(p_n) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - mK(p_n, t_n) \frac{f(x_n)}{f'(x_n)}, \end{aligned} \quad (1.4)$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $s_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$. The weight functions $G : \mathbb{C} \rightarrow \mathbb{C}$ is analytic in a neighborhood of 0 and $K : \mathbb{C}^2 \rightarrow \mathbb{C}$ is holomorphic in a neighborhood of $(0, 0)$. All of above three schemes (1.2), (1.3), and (1.4) require four functional evaluations in order to produce sixth-order convergence with the efficiency index $6^{\frac{1}{4}} = 1.5650$.

Recently, in [3] Behl et al. have developed a family optimal eighth order iterative methods given as:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ z_n &= y_n - u_n Q(h_n) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - u_n t_n G(h_n, t_n) \frac{f(x_n)}{f'(x_n)}, \end{aligned} \quad (1.5)$$

where $u_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $h_n = \frac{u_n}{a_1 + a_2 u_n}$, $t_n = \sqrt[m]{\frac{f(z_n)}{f(y_n)}}$ and $Q : \mathbb{C} \rightarrow \mathbb{C}$ is analytic in a neighbourhood of 0 and $G : \mathbb{C}^2 \rightarrow \mathbb{C}$ is holomorphic in the neighbourhood of $(0, 0)$.

Motivated by the need to present a family of optimal higher order convergent methods for finding multiple roots, we present a family of optimal eighth order convergence method using only four function evaluations. Section 2 provides the methodology and convergence analysis, for the proposed optimal eight-order scheme. In Section 3, some special cases of the new scheme are considered. Section 4 includes the numerical experiments and comparisons of different multiple zero finders using test functions. Finally, conclusions are given in Section 5.

2. Construction of optimal scheme with eight-order convergence

Let us consider the following scheme involving univariate weight functions for solving the root-finding problem:

$$\begin{aligned}
y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 0, \\
z_n &= y_n - m u_n H(u_n) \frac{f(x_n)}{f'(x_n)}, \\
x_{n+1} &= z_n - u_n P(u_n) G(v_n) L(w_n) \frac{f(x_n)}{f'(x_n)},
\end{aligned} \tag{2.1}$$

where the weight functions $H : \mathbb{C} \rightarrow \mathbb{C}$, $P : \mathbb{C} \rightarrow \mathbb{C}$, $G : \mathbb{C} \rightarrow \mathbb{C}$ and $L : \mathbb{C} \rightarrow \mathbb{C}$ are analytic functions in a neighborhood of 0 with $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$ and $w_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$. In the next Theorem 1, we show that the order of convergence of the proposed scheme will reach at optimal eight order without using additional functional evaluations.

Theorem 1. *Let us consider $x = \gamma$ (say) be a multiple zero with multiplicity $m > 1$ of the involved function f . In addition, we assume that $f : \mathbb{C} \rightarrow \mathbb{C}$ is an analytical function in the region enclosing the multiple zero γ . The proposed scheme defined by (2.1) has optimal eight-order convergence, if the weight functions satisfy:*

$$\begin{aligned}
H_0 &= H(0) = 1, \quad H_1 = H'(0) = 2, \quad H_2 = H''(0), \quad H_3 = H'''(0) \text{ are free real numbers} \\
P_0 &= P(0), \quad L_0 = L(0), \quad P_1 = P'(0) = 2P_0, \quad P_2 = P''(0) = P_0(2 + H_2), \\
L_1 &= L'(0) = 2L_0, \quad P_3 = P'''(0) = P_0(-24 + 6H_2 + H_3), \\
G_0 &= G(0) = 0, \quad G_1 = G'(0) = \frac{m}{L_0 P_0}, \quad G_2 = G''(0) = \frac{2m}{L_0 P_0}, \\
P_0, L_0 &\text{ are also free real numbers.}
\end{aligned}$$

The error equation is given as:

$$\begin{aligned}
e_{n+1} &= \frac{1}{48m^8} c_1 (c_1^2 (m - H_2 + 9) - 2mc_2) \left[(14m^3 - G_3 L_0 P_0 (H_2 - 9))^2 \right. \\
&\quad - m^2 (G_3 L_0 P_0 + 12H_2 - 144) \\
&\quad + 2m (161 - 48H_2 + 3H_2^2 + 4H_3 - 9G_3 L_0 P_0 + G_3 H_2 L_0 P_0) c_1^4 \\
&\quad - (4m12m^2 + G_3 (H_2 - 9) L_0 P_0 - m (-72 + 6H_2 + G_3 L_0 P_0)) c_1^2 c_2 \\
&\quad \left. + 4m^2 (6m - G_3 L_0 P_0) c_2^2 + 24m^3 c_1 c_3 \right] e_n^8 + O(e_n^9),
\end{aligned} \tag{2.2}$$

where $c_k = \frac{m!}{(m+k)!} \frac{f^{(m+k)}(\gamma)}{f^{(m)}(\gamma)}$, $k = 1, 2, 3, \dots$, and $e_n = x_n - \gamma$.

Proof. By expanding $f(x_n)$ and $f'(x_n)$ by the Taylor's series expansion on γ , we have

$$f(x_n) = \frac{f^{(m)}(\gamma)}{m!} e_n^m (1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9)) \tag{2.3}$$

and

$$\begin{aligned}
f'(x_n) &= \frac{f^{(m)}(\gamma)}{m!} e_n^{m-1} (m + c_1(m+1)e_n + c_2(m+2)e_n^2 + c_3(m+3)e_n^3 \\
&\quad + c_4(m+4)e_n^4 + c_5(m+5)e_n^5 + c_6(m+6)e_n^6 + c_7(m+7)e_n^7 \\
&\quad + c_8(m+8)e_n^8 + O(e_n^9)),
\end{aligned} \tag{2.4}$$

respectively. By inserting the above expressions (2.3) and (2.4), in the first sub-step of our method (2.1), we obtain:

$$y_n - \gamma = \frac{c_1 e_n^2}{m} + \frac{(2mc_2 - (m+1)c_1^2) e_n^3}{m^2} + \sum_{i=0}^4 A_i e_n^{i+4} + O(e_n^9), \tag{2.5}$$

where $A_i = A_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, \dots$, are given in term of m, c_1, c_2, \dots, c_8 with explicitly written two coefficients

$$A_0 = \frac{1}{m^3} (3c_3m^2 + c_1^3(m+1)^2 - c_1c_2m(3m+4))$$

and

$$A_1 = \frac{-1}{m^4} (c_1^4(m+1)^3 - 2c_2c_1^2m(2m^2+5m+3) + 2c_3c_1m^2(2m+3) + 2m^2(c_2^2(m+2) - 2c_4m)).$$

With the help of Taylor's series expansion and expression (2.5), we have

$$f(y_n) = f^{(m)}(\gamma)e_n^{2m} \left[\frac{\left(\frac{c_1}{m}\right)^m}{m!} + \frac{(2mc_2 - (m+1)c_1^2)\left(\frac{c_1}{m}\right)^m e_n}{m!} + \sum_{i=0}^6 \bar{A}_i e_n^{k+2} + O(e_n^9) \right], \quad (2.6)$$

where $\bar{A}_i = \bar{A}_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, \dots$, and the first two coefficients are

$$\begin{aligned} \bar{A}_0 &= \frac{1}{2m!c_1^3} \left(\frac{c_1}{m}\right)^{m+1} [c_1^4(m^3+3m^2+3m+3) - 2mc_1^2c_2(2m^2+3m+2) \\ &\quad + 4c_2^2m^2(m-1) + 6c_1c_3m^2], \end{aligned}$$

and

$$\begin{aligned} \bar{A}_1 &= \frac{1}{6m^2m!c_1^3} \left(\frac{c_1}{m}\right)^m [6mc_1^4c_2(m^4+4m^3+7m^2+8m+4) \\ &\quad - c_1^6(1+m)^2(m^3+4m^2+5m+8) + 8m^3c_2^3(m^2-3m+2) \\ &\quad - 6m^2c_1^3c_3(3m^2+4m+3) + 36(m-1)m^3c_1c_2c_3 \\ &\quad - 12m^3c_1^2(c_2^2(m^2+m+1) - 2c_4)]. \end{aligned}$$

By using expressions (2.3) and (2.6), we get

$$u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}} = \frac{c_1e_n}{m} + \frac{(2mc_2 - (m+2)c_1^2)e_n^2}{m^2} + \sum_{i=0}^4 \widetilde{A}_i e_n^{i+3} + O(e_n^8), \quad (2.7)$$

where $\widetilde{A}_i = \widetilde{A}_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, \dots$, whose first few coefficients are

$$\widetilde{A}_0 = \frac{1}{2m^3} \{c_1^3(2m^2+7m+7) - 2mc_1c_2(3m+7) + 6m^2c_3\},$$

$$\begin{aligned} \widetilde{A}_1 &= \frac{-1}{6m^4} [c_1^4(6m^3+29m^2+51m+34) - 6mc_1^2c_2(4m^2+16m+17) \\ &\quad + 12m^2c_1c_3(2m+5) + (c_2^2(m+3) - 2mc_4)]. \end{aligned}$$

Now, inserting (2.5)-(2.7) in the second sub-step of scheme (2.1), where $H(u_n) \approx H_0 + H_1u_n + \frac{H_2}{2}u_n^2 + \frac{H_3}{6}u_n^3$, we obtain

$$z_n - \gamma = \frac{\{c_1^3(m+9-H_2) - 2mc_1c_2\}e_n^4}{2m^3} \sum_{i=0}^4 \widehat{A}_i e_n^{i+4} + O(e_n^9), \quad (2.8)$$

where $\widehat{A}_i = \widehat{A}_i(m, c_1, c_2, \dots, c_8)$, $i = 0, 1, \dots$, are given in terms of m, c_1, c_2, \dots, c_8 , being the two first coefficients:

$$\widehat{A}_0 = \frac{-1}{6m^4} [c_1^4\{7m^2+84m+H_3+125-3H_2(3m+7)\} - 6mc_1^2c_2\{4(m+7)-3H_2\} + 12m^2c_2^2 + 12m^2c_1c_3],$$

$$\begin{aligned} \widehat{A}_1 &= \frac{1}{24m^5} [c_1^5\{46m^3+677m^2+1850m+1507+4H_3(4m+9)-6H_2(12m^2+53m+59)\} \\ &\quad - 4mc_1^3c_2(13m-9H_2+83) - 168m^3c_2c_3 + 12m^2c_1(c_2^2(17m-12H_2+115) - 6mc_4)]. \end{aligned}$$

Again, using Taylor's series expansion for $f(z_n)$, we get

$$f(z_n) = f^{(m)}(\gamma)e_n^{4m} \left[\frac{2^{-m} \left(\frac{c_1^3(m+9-H_2) - 2mc_1c_2}{m^3} \right)^m}{m!} - \frac{1}{3(m^3m!)} \left[2^{-m} \alpha_0 \left(\frac{c_1^3(m+9-H_2) - 2mc_1c_2}{m^3} \right)^{m-1} \right] + \sum_{i=0}^2 A_i' e_n^{i+2} + O(e_n^5) \right], \quad (2.9)$$

where $A_i' = A_i'(m, c_1, c_2, \dots, c_8)$ and

$$\alpha_0 = c_1^4 \{7m^2 + 84m + H_3 + 125 - 3H_2(3m+7)\} - 6mc_1^2c_2 \{4(m+7) - 3H_2\} + 12m^2c_2^2 + 12m^2c_1c_3.$$

By using (2.6) and (2.9), we get

$$v_n = \left(\frac{f(z_n)}{f(y_n)} \right)^{\frac{1}{m}} = \frac{(c_1^2(m+9-H_2) - 2mc_2) e_n^2}{2m^2} + \sum_{i=0}^3 B_i e_n^{i+3} + O(e_n^7), \quad (2.10)$$

where $B_i = B_i(m, c_1, c_2, \dots, c_8)$ and the first two terms are

$$\begin{aligned} B_0 &= \frac{-1}{6m^3} [c_1^3 \{4m^2 + 54m + H_3 + 98 - 6H_2(m+3)\} \\ &\quad - 12mc_1c_2(m+9-H_2) + 12m^2c_3], \\ B_1 &= \frac{1}{24m^4} [c_1^4 (18m^3 + 313m^2 + 1002m + 899 + 4H_3(3m+8) - 6H_2(6m^2 + 33m + 43)) \\ &\quad - 12mc_1^2c_2(6m^2 + 87m + 2H_3 + 167 - H_2(10m + 33)) \\ &\quad + 24m^2c_1c_3(3m - 3H_2 + 26) + 12m^2(c_2^2(3m + 35 - 4H_2) - 6mc_4)]. \end{aligned}$$

By using (2.3) and (2.9), we get:

$$w_n = \left(\frac{f(z_n)}{f(x_n)} \right)^{\frac{1}{m}} = \frac{1}{2m^3} \{c_1^3(m-H_2+9) - 2mc_1c_2\} e_n^3 + \sum_{i=0}^3 \widetilde{B}_i e_n^{i+4} + O(e_n^8), \quad (2.11)$$

where $\widetilde{B}_i = \widetilde{B}_i(m, c_1, c_2, \dots, c_8)$ and first two coefficients are

$$\begin{aligned} \widetilde{B}_0 &= \frac{-1}{6m^4} [c_1^4 (7m^2 + 87m + H_3 + 152 - 3H_2(3m+8)) - 6mc_1^2c_2(4m - 3H_2 + 29) \\ &\quad + 12m^2c_2^2 + 12m^2c_1c_3], \\ \widetilde{B}_1 &= \frac{1}{24m^5} [c_1^5 (46m^3 + 711m^2 + 2246m + 2061 + 8H_3(2m+5) - 12H_2(6m^2 + 30m + 37)) \\ &\quad - 4mc_1^3(53m^2 + 624m + 8H_3 + 1123 - 9H_2(7m+20)) + 12m^2c_1^2c_3(13m - 9H_2 + 87) \\ &\quad - 168m^3c_2c_3 + 12m^2c_1(c_2^2(17m - 12H_2 + 121) - 6mc_4)]. \end{aligned}$$

Next, we expand weight functions $P(u_n)$, $G(v_n)$ and $L(w_n)$ in the neighborhood of origin by Taylor's series expansion up to third-order terms as follows:

$$\begin{aligned} P(u_n) &\approx P_0 + P_1u_n + P_2\frac{u_n^2}{2} + P_3\frac{u_n^3}{6}, \\ G(v_n) &\approx G_0 + G_1v_n + G_2\frac{v_n^2}{2} + G_3\frac{v_n^3}{6}, \\ L(w_n) &\approx L_0 + L_1w_n + L_2\frac{w_n^2}{2} + L_3\frac{w_n^3}{6}. \end{aligned}$$

Now by using (2.3) – (2.11) in the last sub-step of suggested method (2.1), we have

$$x_{n+1} = \frac{-G_0L_0P_0c_1e_n^2}{m^2} + \frac{G_0L_0 [c_1^2((m+3)P_0 - P_1) - 2mP_0c_2] e_n^3}{m^3} + \sum_{i=0}^4 \widetilde{B}_i e_n^{i+4} + O(e_n^9). \quad (2.12)$$

We can easily obtain at-least fourth-order convergence, if we put $G_0 = 0$ in (2.12), so we get

$$x_{n+1} = \frac{(m - G_1 L_0 P_0) c_1 ((m - H_2 + 9) c_1^2 - 2m c_2) e_n^4}{2m^4} + \sum_{i=0}^3 B_i' e_n^{i+5} + O(e_n^9). \quad (2.13)$$

Again, we will put $G_1 = \frac{m}{L_0 P_0}$ in (2.13) equation for obtaining at-least fifth-order convergence:

$$x_{n+1} = \frac{(2P_0 - P_1) c_1^2 ((m - H_2 + 9) c_1^2 - 2m c_2) e_n^5}{2m^4 P_0} + \sum_{i=0}^2 D_i e_n^{i+6} + O(e_n^9), \quad (2.14)$$

where $D_i = D_i(m, c_1, c_2, \dots, c_8, G_2, G_3, L_0, L_1, L_2, L_3, P_0, P_1, P_2, P_3, H_2, H_3)$, $i = 0, 1, \dots$

For getting at-least sixth-order convergence, we choose $P_1 = 2P_0$ and $G_2 = \frac{2m}{L_0 P_0}$ in equation (2.14):

$$x_{n+1} = \frac{((H_2 + 2) P_0 - P_2) c_1^3 (c_1^2 (m - H_2 + 9) - 2m c_2) e_n^6}{4m^5 P_0} + \sum_{i=0}^1 D_i^\circ e_n^{i+7} + O(e_n^9), \quad (2.15)$$

where $D_i^\circ = D_i^\circ(m, c_1, c_2, \dots, c_8, G_3, L_0, L_1, L_2, L_3, P_0, P_2, P_3, H_2, H_3)$, $i = 0, 1, \dots$

For obtaining at least seventh-order of convergence, we substitute $P_2 = P_0(H_2 + 2)$ and $L_1 = 2L_0$ in equation (2.15) obtaining:

$$x_{n+1} = \frac{((H_3 + 6H_2 - 24) P_0 - P_3) c_1^4 ((m + 9 - H_2) c_1^2 - 2m c_2) e_n^7}{12m^6 P_0} + D_i' e_n^8 + O(e_n^9). \quad (2.16)$$

Finally, we will get eighth-order convergence by putting $P_3 = P_0(H_3 + 6H_2 - 24)$ in (2.16). So,

$$\begin{aligned} x_{n+1} = & \frac{1}{48m^8} c_1 (c_1^2 (m - H_2 + 9) - 2m c_2) \left[\left(14m^3 - G_3 L_0 P_0 (H_2 - 9)^2 - m^2 (G_3 L_0 P_0 + 12H_2 - 144) \right. \right. \\ & + 2m (161 - 48H_2 + 3H_2^2 + 4H_3 - 9G_3 L_0 P_0 + G_3 H_2 L_0 P_0) c_1^4 \\ & - 4m (12m^2 + G_3 (H_2 - 9) L_0 P_0 - m (-72 + 6H_2 + G_3 L_0 P_0)) c_1^2 c_2 \\ & \left. \left. + 4m^2 (6m - G_3 L_0 P_0) c_2^2 + 24m^3 c_1 c_3 \right] e_n^8 + O(e_n^9). \end{aligned} \quad (2.17)$$

This expression shows that our suggested scheme (2.1) reaches optimal eight-order convergence by using only four functional evaluations i.e. $f(x_n)$, $f'(x_n)$, $f(z_n)$ and $f(y_n)$ per iteration. Hence this completes the proof. ■

3. Some special cases of weight functions

In this section, we discuss some special cases of our suggested method (2.1) by using different types of weight functions P , H , G and L . These special cases are as follows:

Case 1. When all the weight functions are polynomials:

$$\begin{aligned} H(u_n) &= 1 + 2u_n + \frac{H_2}{2} u_n^2 + \frac{H_3}{6} u_n^3, \\ P(u_n) &= P_0 + 2P_0 u_n + \left(1 + \frac{H_2}{2} \right) P_0 u_n^2 + \left(-4 + H_2 + \frac{H_3}{6} \right) P_0 u_n^3, \\ G(v_n) &= \frac{m}{L_0 P_0} v_n + \frac{m}{L_0 P_0} v_n^2 + G_3 \frac{v_n^3}{6} \\ L(w_n) &= L_0 + 2L_0 w_n + L_2 \frac{w_n^2}{2} + L_3 \frac{w_n^3}{6}, \end{aligned} \quad (3.1)$$

so for $H_2 = H_3 = G_3 = L_2 = L_3 = 0$ the method (2.1) becomes:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - m u_n (1 + 2u_n) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - m u_n v_n \frac{f(x_n)}{f'(x_n)} \left[(1 + 2u_n + u_n^2 - 4u_n^3) (1 + v_n) (1 + 2w_n) \right]. \end{aligned} \quad (3.2)$$

Case 2. Let us consider (3.1) for the choice of $H_2 = H_3 = 1, L_2 = L_3 = G_3 = 0$ as follows:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_n \left(1 + 2u_n + \frac{u_n^2}{2} + \frac{u_n^3}{6} \right) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - mu_n v_n \frac{f(x_n)}{f'(x_n)} \left[\left(1 + 2u_n + \frac{3}{2}u_n^2 - \frac{17}{6}u_n^3 \right) (1 + v_n) (1 + 2w_n) \right]. \end{aligned} \quad (3.3)$$

Case 3. When the exponential weight function is $H(u)$, polynomial and rational weight function as $P(u)$, $G(u)$ and $L(w)$:

$$\begin{aligned} H(u_n) &= e^{-\frac{1}{6}u_n(-16u_n^2 - H_3u_n^2 + 6H_2u_n^2 - 3u_nH_2 + 12u_n - 12)}, \\ P(u_n) &= \frac{1}{6} \frac{P_0(-24 - 36u_n + 6H_2u_n + 108u_n^3 - 12H_2u_n^3 + 3H_2^2u_n^3 - 4H_3u_n^3)}{-4 + 2u_n + H_2u_n}, \\ G(v_n) &= \frac{m}{L_0P_0}v_n + \frac{m}{L_0P_0}v_n^2 + G_3 \frac{v_n^3}{6}, \\ L(w_n) &= -\frac{2(-3L_2 + 24L_0 + L_3w_n - 12L_2w_n + 48L_0w_n)L_0^2}{6L_0L_2 - 48L_0^2 - 2L_0L_3w_n + 12L_0L_2w_n - 3L_2^2w_n^2 + 4L_3L_0w_n^2}. \end{aligned}$$

For $H_2 = H_3 = L_2 = G_3 = 0, L_0 = L_3 = 1$ our proposed method (2.1) becomes

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_n e^{-\frac{1}{6}u_n(-16u_n^2 + 12u_n - 12)} \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - mu_n v_n \frac{f(x_n)}{f'(x_n)} \left[\left(\frac{-2 - 3u_n + 9u_n^3}{-2 + u_n} \right) (1 + v_n) \left(\frac{24 + 49w_n}{24 + w_n - 2w_n^2} \right) \right]. \end{aligned} \quad (3.4)$$

Case 4. When the combination of polynomial and rational functions are used:

$$\begin{aligned} H(u_n) &= \frac{1}{6} \left(\frac{-24 - 48u_n + 6H_2u_n - 4u_n^3H_3 + 3u_n^3H_2^2}{-4 + H_2u_n} \right), \\ P(u_n) &= \frac{1}{6} \frac{P_0(-24 - 36u_n + 6H_2u_n + 108u_n^3 - 12H_2u_n^3 + 3H_2^2u_n^3 - 4H_3u_n^3)}{-4 + 2u_n + H_2u_n}, \\ G(v_n) &= \frac{m}{L_0P_0}v_n + \frac{m}{L_0P_0}v_n^2 + G_3 \frac{v_n^3}{6}, \\ L(w_n) &= -\frac{2(-3L_2 + 24L_0 + L_3w_n - 12L_2w_n + 48L_0w_n)L_0^2}{6L_0L_2 - 48L_0^2 - 2L_0L_3w_n + 12L_0L_2w_n - 3L_2^2w_n^2 + 4L_3L_0w_n^2}. \end{aligned}$$

For $L_2 = G_3 = 0, H_2 = H_3 = L_0 = L_3 = 1$ the method (2.1) becomes:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - mu_n \frac{1}{6} \left(\frac{-24 - 42u_n - u_n^3}{-4 + u_n} \right) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - mu_n v_n \frac{f(x_n)}{f'(x_n)} \left[\frac{1 - 24 - 30u_n + 95u_n^3}{6} (1 + v_n) \left(\frac{24 + 49w_n}{24 + w_n - 2w_n^2} \right) \right]. \end{aligned} \quad (3.5)$$

In a similar way, we can find a new optimal eight order convergence iterative schemes for multiple zero by simply assigning different values of $P_0, L_0, G_3, L_2, L_3, H_2, H_3$, or by considering a new weight function that satisfies all the conditions of Theorem 1.

4. Numerical Experiments

In this section, we show the efficiency, convergence behavior and effectiveness of our suggested scheme. For this purpose, we consider some of the special cases of our proposed schemes namely, (3.2)-(3.5), denoted by $S1$, $S2$, $S3$ and $S4$ respectively. In addition, we choose a total number of eight test problems for comparison given by Examples 1-8. Now we compare our methods with the other existing ones of order six and eight on the basis of iteration error $|x_n - \gamma|$, computational order of convergence σ_n and residual errors in the function $|f(x_n)|$. We compare our proposed method with a family of two-point sixth-order method, which were presented by Guem et al. [5], out of them we choose the following:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1 \\ x_{n+1} &= y_n - m \left[1 + 2(m-1)(p_n - s_n) - 4p_n s_n + s_n^2 \right] \frac{f(y_n)}{f'(y_n)}, \end{aligned} \quad (4.1)$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, and $s_n = \sqrt[m-1]{\frac{f'(y_n)}{f'(x_n)}}$, named as $GM1$.

We also compare our scheme with another non-optimal method with sixth-order convergence based on weight function approach suggested by the same author Geum et al. [6]. Among their special cases, we select the following:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ w_n &= x_n - m \left[1 + p_n + 2p_n^2 \right] \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - m \left[1 + p_n + 2p_n^2 + (1 + 2p_n)t_n \right] \frac{f(x_n)}{f'(x_n)}, \end{aligned} \quad (4.2)$$

where $p_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$ and $s_n = \sqrt[m]{\frac{f(w_n)}{f(x_n)}}$, denoted by $GM2$.

Finally, we choose a special case of the optimal eighth order method given by Behl et al. [3] for $a_1 = a_2 = 1$ represented by OM given as:

$$\begin{aligned} y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1, \\ z_n &= y_n - m u_n (1 + 2h_n + 3h_n^2) \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - m u_n t_n \frac{1 + 2t_n + 3h_n^2 + h_n(2 + 6t_n + h_n)}{1 + t_n} \frac{f(x_n)}{f'(x_n)}, \end{aligned} \quad (4.3)$$

where $u_n = \sqrt[m]{\frac{f(y_n)}{f(x_n)}}$, $h_n = \frac{u_n}{1+u_n}$ and $t_n = \sqrt[m]{\frac{f(z_n)}{f(y_n)}}$.

In Tables 1–16, the comparison of the iteration error $|x_i - \gamma|$, $i = 1, 2, 3$, the absolute residual error $|f(x_i)|$ for $i = 1, 2, 3$, and the computational order of convergence σ_n of our methods $S1$, $S2$, $S3$ and $S4$ with $GM1$, $GM2$ and OM is shown. We have done our calculations with several number of significant digits (minimum 1000 significant digits) to minimize the round of error but due to limitations we show the results up to nine decimal places only. All computations have been performed using the programming package Mathematica 9 and Maple 16, with multiple precision arithmetic. Further, in the tables $a \times 10^{(\pm b)}$ is written as $a(\pm b)$. We use the formula given by Jay [8]:

$$\sigma_n \approx \frac{\log |f(x_{n+1})/f(x_n)|}{\log |f(x_n)/f(x_{n-1})|},$$

in order to calculate computational order of convergence.

The dynamical behavior of the test functions is presented in Figures 1-16. The dynamical planes that appear in this section have been generated by using the routines published in [4]. We have used a mesh of 400×400 points in an appropriate region of the complex plane. We paint in orange the points whose orbit converges to the multiple root and in black whose points whose orbit converges to another thing (strange fixed points, cycles, etc.) or diverges. We work with a tolerance of 10^{-3} and a maximum number of 40 iterations. The multiple root is represented in the different figures by a white star.

Example 1. Let us consider the following standard nonlinear test function:

$$f_1(x) = \left[\cos\left(\frac{\pi x}{2}\right) + x^2 - \pi \right]^5. \quad (4.4)$$

The above function has a multiple zero at $\gamma \approx 2.03472489627912661035$ of multiplicity $m = 5$ with initial guess $x_0 = 2.5$.

$f_1(x), x_0 = 2.5, m = 5, \gamma \approx 2.034724896$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
GM1	5.220762470(-4)	8.422456031(-21)	1.488718831(-121)	5.999888304
GM2	1.111141891(-3)	2.533864417(-18)	3.588609343(-106)	5.999684855
OM	2.762711102(-4)	1.295945853(-28)	3.044163730(-223)	7.999943046
S1	1.228789153(-4)	6.745130071(-32)	5.565313341(-250)	7.999977076
S2	1.055907181(-4)	1.450243196(-32)	1.837681327(-255)	7.999981851
S3	1.332830873(-4)	1.679366805(-31)	1.068023512(-246)	7.999973241
S4	1.109542612(-4)	2.207929169(-32)	6.019291728(-254)	7.999980201

Table 1: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_1(x)$

$f_1(x), x_0 = 2.5, m = 5, \gamma \approx 2.034724896$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
GM1	4.810380435(-14)	5.249218640(-98)	9.056581698(-602)	5.999888304
GM2	2.103992164(-12)	1.293652895(-85)	7.371091759(-525)	5.999684855
OM	1.994805140(-15)	4.527257350(-137)	3.237732291(-1110)	7.999943046
S1	3.470811790(-17)	1.729229137(-153)	6.612246055(-1244)	7.999977076
S2	1.626120684(-17)	7.945204563(-157)	2.595684552(-1271)	7.999981851
S3	5.211125989(-17)	1.654355054(-151)	1.721095493(-1227)	7.999973241
S4	1.954088073(-17)	6.498701433(-156)	9.786510375(-1264)	7.999980201

Table 2: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_1(x)$

In Tables 1 and 2 we observe that our method has an error in the third iteration $|x_3 - \gamma|$ or residual error $|f(x_3)|$ smaller than those of the known schemes. The computational order of convergence confirms the theoretical results.

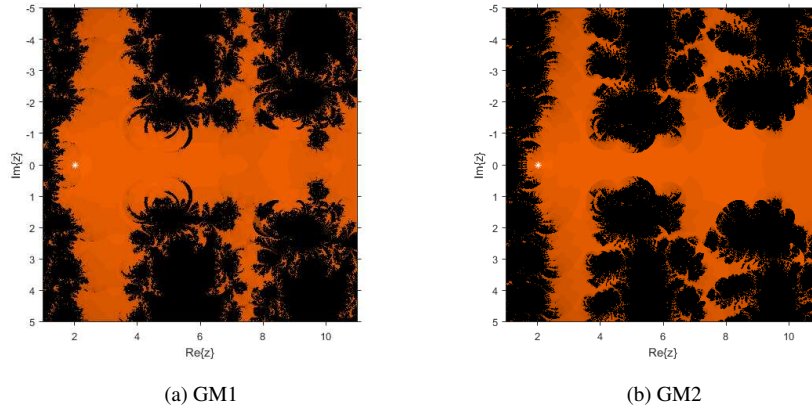


Figure 1: Basins of attraction for $f_1(x)$ of known methods

As can be seen at Figures 1 and 2, the basins of attraction, that is, the sets of convergent initial points for each scheme are similar for all the used methods.

Example 2. We assume another standard test problem which is defined by:

$$f_2(x) = (e^x + x - 20)^2. \quad (4.5)$$

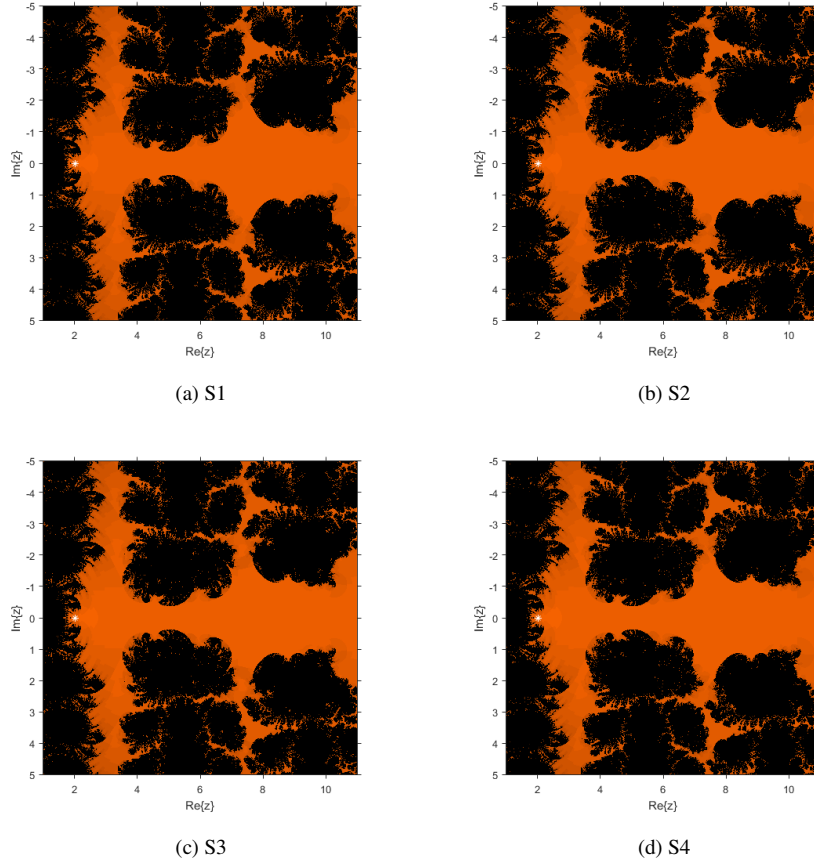


Figure 2: Basins of attraction for $f_1(x)$ of new methods

This function f_2 has multiple zero at $\gamma \approx 2.84243895378444706781$ of multiplicity $m = 2$. The initial approximation is taken as $x_0 = 0.6$.

The conclusions that can be obtained from Tables 3–4 and Figures 3–4 are similar to those of the previous example.

$f_2(x), x_0 = 3, m = 2, \gamma \approx 2.842438953$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
GM1	1.184940430(-6)	2.622019043(-37)	3.078026494(-221)	5.999999934
GM2	5.274534652(-6)	1.152582670(-32)	1.254891624(-192)	5.999999512
OM	2.076361288(-7)	3.013378012(-54)	5.930063701(-429)	7.999999999
S1	6.016268247(-8)	4.258443058(-59)	2.683120238(-468)	7.999999997
S2	4.141469194(-8)	1.343043245(-60)	1.642761529(-480)	7.999999998
S3	8.415591301(-8)	9.648357994(-58)	2.880074485(-457)	7.999999995
S4	4.900052028(-8)	6.593652507(-60)	7.088086766(-475)	7.999999997

Table 3: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_2(x)$

Example 3. Let us assume another nonlinear test problem which is given below:

$$f_3(x) = \left[\ln(x) + \sqrt{x^4 + 1} - 2 \right]^9. \quad (4.6)$$

The function f_3 has a zero of multiplicity $m = 9$ at $\gamma \approx 1.22281396362897310432$ and initial guess $x_0 = 10.5$.

The numerical results shown in Tables 5–6 are qualitatively similar to the previous ones and, in Figures 5–6, the basins of attraction have analogous aspects.

$f_2(x), x_0 = 3, m = 2, \gamma \approx 2.842438953$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
<i>GM1</i>	4.629227753(-10)	2.266661714(-71)	3.123631064(-439)	5.999999934
<i>GM2</i>	9.172452910(-9)	4.379849588(-62)	5.191913723(-382)	5.999999512
<i>OM</i>	1.421415208(-11)	2.993796359(-105)	1.159401194(-854)	7.999999999
<i>S1</i>	1.193354388(-12)	5.978837017(-115)	2.373533119(-933)	7.999999997
<i>S2</i>	5.654886771(-13)	5.946960028(-118)	8.897419619(-958)	7.999999998
<i>S3</i>	2.334986277(-12)	3.069176359(-112)	2.734780442(-911)	7.999999995
<i>S4</i>	7.916193999(-13)	1.433399132(-116)	1.656429957(-946)	7.999999997

Table 4: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_2(x)$

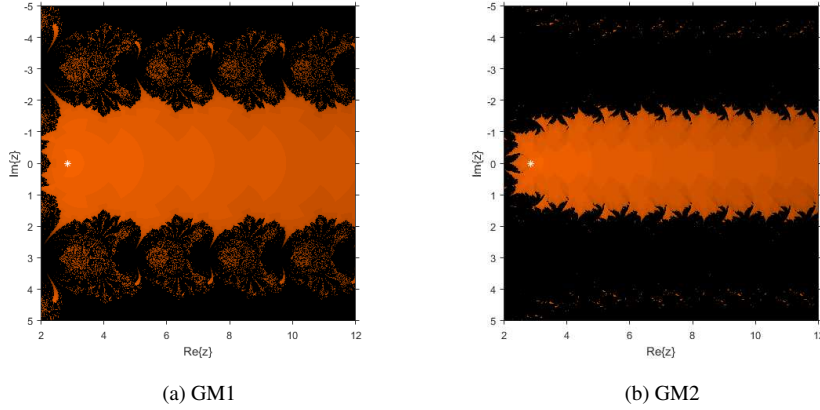


Figure 3: Basins of attraction for $f_2(x)$ of known methods

$f_3(x), x_0 = 10.5, m = 9, \gamma \approx 1.222813963$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
<i>GM1</i>	1.100433815	5.166891715(-3)	7.860206039(-16)	5.194457048
<i>GM2</i>	1.371432030	1.959291353(-2)	8.637659416(-12)	4.655699798
<i>OM</i>	1.118274420	3.884821127(-3)	9.873012471(-21)	6.765739642
<i>S1</i>	3.501464637(-8)	7.483773054(-4)	6.114496475(-27)	7.205549180
<i>S2</i>	8.787056593(-1)	5.446311777(-4)	3.397621553(-28)	7.282937131
<i>S3</i>	9.192937469(-1)	8.477827672(-4)	2.211285047(-26)	7.157013052
<i>S4</i>	8.860182249(-1)	5.890154435(-4)	7.334736681(-28)	7.257262688

Table 5: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_3(x)$

$f_3(x), x_0 = 10.5, m = 9, \gamma \approx 1.222813963$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
<i>GM1</i>	5.376098681(5)	3.317195453(-17)	1.423830394(-132)	5.194457048
<i>GM2</i>	6.949710765(6)	5.630018548(-12)	3.327496198(-96)	4.655699798
<i>OM</i>	6.462280745(5)	2.536717042(-18)	1.108153842(-176)	6.765739642
<i>S1</i>	6.313838272(4)	9.176461916(-25)	1.485248686(-232)	7.205549180
<i>S2</i>	4.284891437(4)	5.250854749(-26)	7.501151077(-244)	7.282937131
<i>S3</i>	7.072239735(4)	2.820257458(-24)	1.571648600(-227)	7.157013052
<i>S4</i>	4.696527720(4)	1.062940955(-25)	7.638597576(-241)	7.257262688

Table 6: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_3(x)$

Example 4. Let us suppose another standard test function which is defined below:

$$f_4(x) = (\cos x - x)^3. \quad (4.7)$$

The function f_4 has a multiple zero with multiplicity $m = 3$ at $\gamma \approx 0.73908513321516064165$ and initial guess $x_0 = 1$.

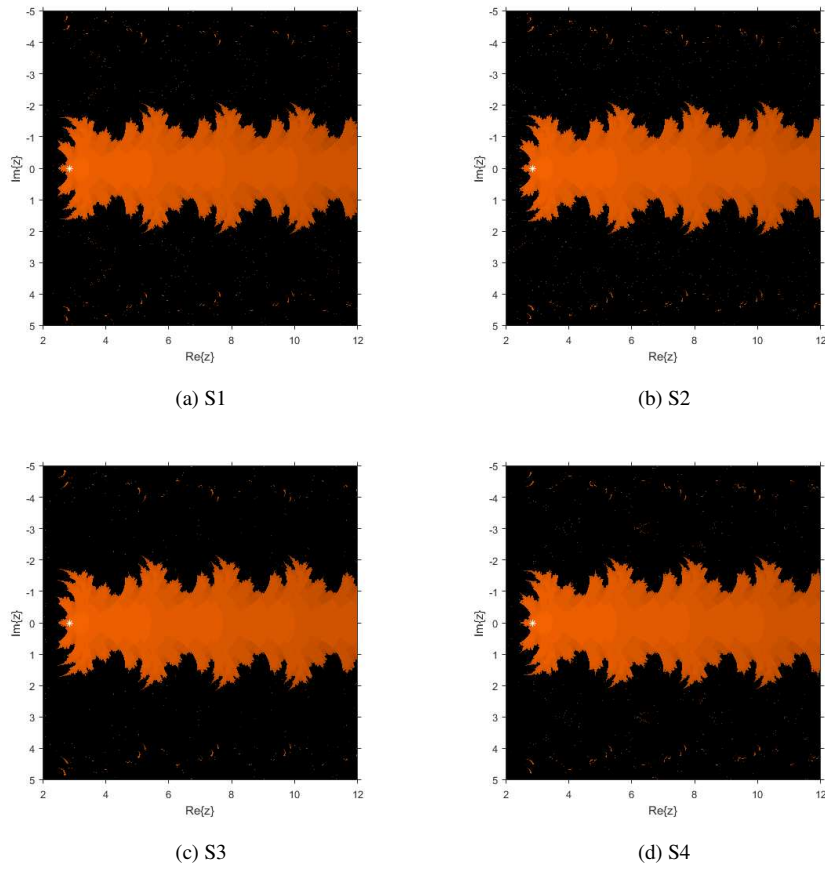


Figure 4: Basins of attraction for $f_2(x)$ of new methods

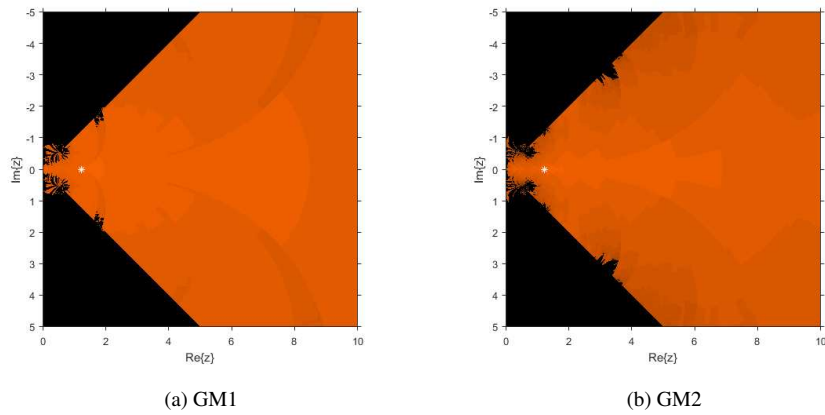


Figure 5: Basins of attraction for $f_3(x)$ of known methods

In this example $f_4(x)$, the basins of attraction of methods GM1 and GM2 (see Figure 7) are wider than those corresponding to S1, S2, S3 and S4 (see Figure 8). The numerical results shown in Tables 7–8 show the good performance of the new schemes.

Example 5. *Let us consider*

$$f_5(x) = (\sin^2 x - x^2 + 1)^2. \tag{4.8}$$

The function f_5 has a multiple zero at $\gamma \approx 1.40449164821534122603$ with multiplicity $m = 2$, and initial guess $x_0 = 2$.

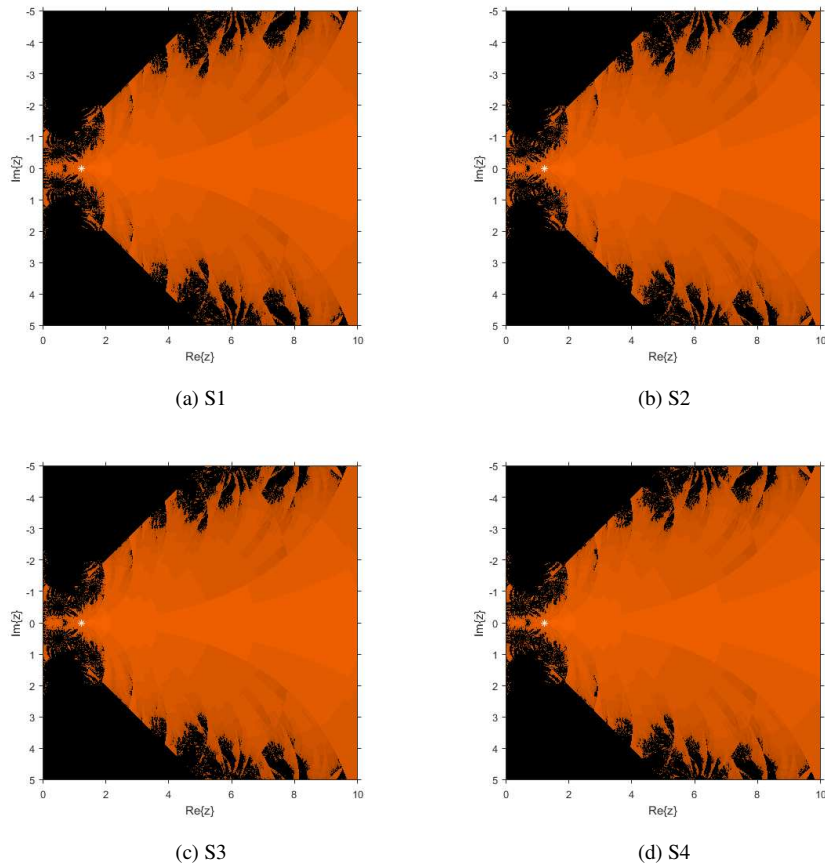


Figure 6: Basins of attraction for $f_3(x)$ of new methods

$f_4(x), x_0 = 1, m = 3, \gamma \approx 0.739085133$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
<i>GM1</i>	1.143472639(-6)	2.264182681(-38)	1.364669639(-228)	5.999999918
<i>GM2</i>	2.553088759(-6)	6.835881398(-36)	2.518668790(-213)	5.999999784
<i>OM</i>	8.481354395(-8)	4.488396983(-59)	2.761212765(-469)	7.999999995
<i>S1</i>	3.501464637(-8)	1.454164026(-62)	1.286834499(-497)	7.999999998
<i>S2</i>	3.007233122(-8)	3.434924257(-63)	9.952169910(-503)	7.999999998
<i>S3</i>	3.761173109(-8)	2.969574320(-62)	4.483955610(-495)	7.999999998
<i>S4</i>	3.094738237(-8)	4.594985643(-63)	1.085340805(-501)	7.999999998

Table 7: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_4(x)$

$f_4(x), x_0 = 1, m = 3, \gamma \approx 0.7390851332$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
<i>GM1</i>	7.008773523(-18)	5.441250983(-113)	1.191370810(-683)	5.999999918
<i>GM2</i>	7.801225670(-17)	1.497434806(-105)	7.489926365(-638)	5.999999784
<i>OM</i>	2.859960418(-21)	4.238753403(-175)	9.868791727(-1406)	7.999999995
<i>S1</i>	2.012396762(-22)	1.441466509(-185)	9.989240126(-1491)	7.999999998
<i>S2</i>	1.274869299(-22)	1.899834898(-187)	4.620805750(-1506)	7.999999998
<i>S3</i>	2.494217627(-22)	1.227571969(-184)	4.226182817(-1483)	7.999999998
<i>S4</i>	1.389428292(-22)	4.547962158(-187)	5.993255053(-1503)	7.999999998

Table 8: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_4(x)$

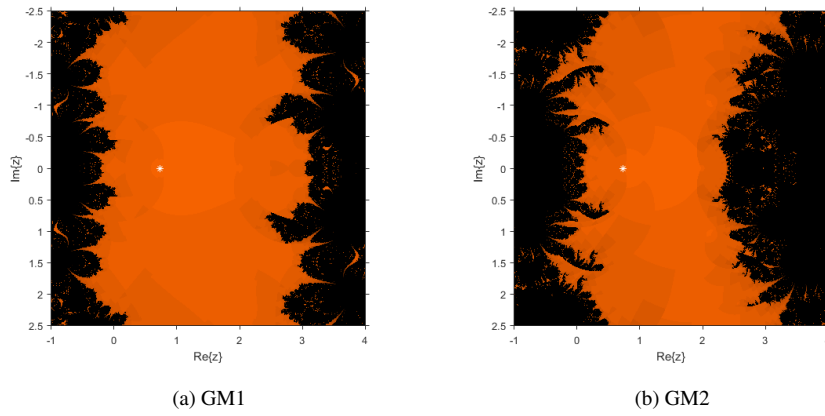


Figure 7: Basins of attraction for $f_4(x)$ of known methods

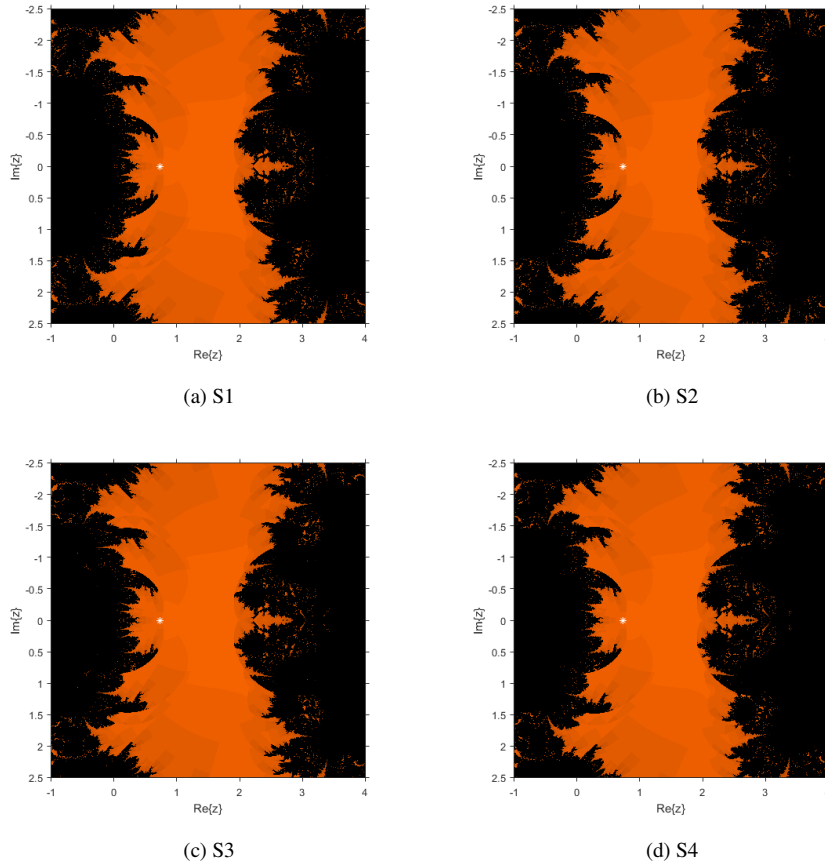


Figure 8: Basins of attraction for $f_4(x)$ of new methods

Example 6. Let us define:

$$f_6(x) = (x^4 - 2x^2 + 1)^3. \quad (4.9)$$

The multiple zero of function f_6 , $\gamma = 1$, has multiplicity $m = 6$ and initial guess $x_0 = 2.2$.

Example 7. Let us consider

$$f_7(x) = (x^3 + 4x^2 - 10)^6. \quad (4.10)$$

The function f_7 has a zero of multiplicity $m = 6$ at $\gamma \approx 1.36523001341409684576$ and initial guess $x_0 = 3$.

$f_5(x), x_0 = 2, m = 2, \gamma \approx 1.404491648$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
GM1	4.870872815(-3)	3.238330049(-14)	2.878860879(-81)	5.997983222
GM2	8.364077694(-3)	2.686261557(-12)	3.157588207(-69)	5.995064976
OM	3.733996390(-3)	1.735208010(-18)	3.905151979(-144)	7.998368341
S1	1.989845761(-3)	3.773039180(-21)	6.424492009(-163)	7.999233393
S2	1.782676109(-3)	1.110205003(-21)	2.550303431(-167)	7.999374198
S3	2.089149870(-3)	7.429195419(-21)	1.942062269(-160)	7.999127318
S4	1.828746306(-3)	1.569755717(-21)	4.705271277(-166)	7.999319213

Table 9: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_5(x)$

$f_5(x), x_0 = 2, m = 2, \gamma \approx 1.404491648$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
GM1	1.473306752(-4)	6.462658702(-27)	5.107523315(-161)	5.997983222
GM2	4.368017549(-4)	4.446984291(-23)	6.144407130(-137)	5.995064976
OM	8.642820503(-5)	1.855547555(-35)	9.398204311(-281)	7.998368341
S1	2.447717915(-5)	8.773070744(-41)	2.543586949(-324)	7.999233393
S2	1.963932585(-5)	7.595832192(-42)	4.008232718(-333)	7.999374198
S3	2.698542362(-5)	3.401359733(-40)	2.324316702(-319)	7.999127318
S4	2.066902234(-5)	1.518564168(-41)	1.364389398(-330)	7.999319213

Table 10: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_5(x)$

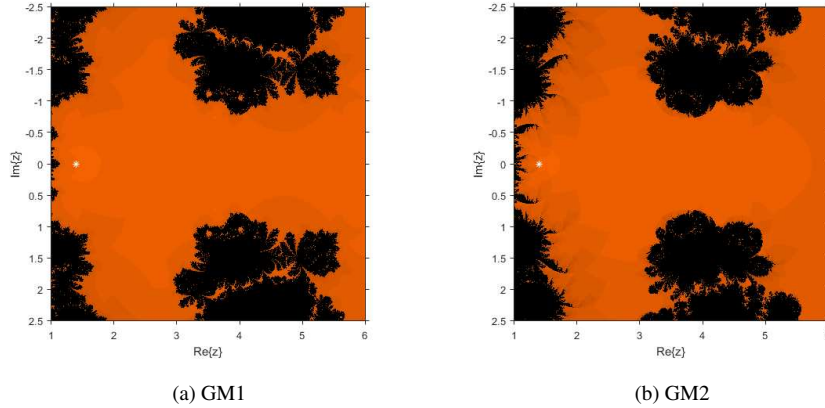


Figure 9: Basins of attraction for $f_5(x)$ of known methods

$f_6(x), x_0 = 2.2, m = 6, \gamma = 1.0$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
GM1	1.723204281(-2)	7.057617821(-12)	3.614720256(-68)	5.993840804
GM2	2.889915685(-2)	4.612259665(-10)	9.025136370(-57)	5.985806935
OM	1.509607903(-2)	5.545623631(-15)	2.040685202(-114)	7.994270715
S1	8.716435190(-3)	2.448353847(-17)	1.008749898(-133)	7.997132194
S2	7.876424462(-3)	7.960208732(-18)	9.115369008(-138)	7.997616679
S3	9.113663383(-3)	4.508564491(-17)	1.733962385(-131)	7.996782482
S4	8.077031167(-3)	1.099421716(-17)	1.369556332(-136)	7.997436201

Table 11: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_6(x)$

Example 8. We define another function as:

$$f_8(x) = (8xe^{-x^2} - 2x - 3)^8. \quad (4.11)$$

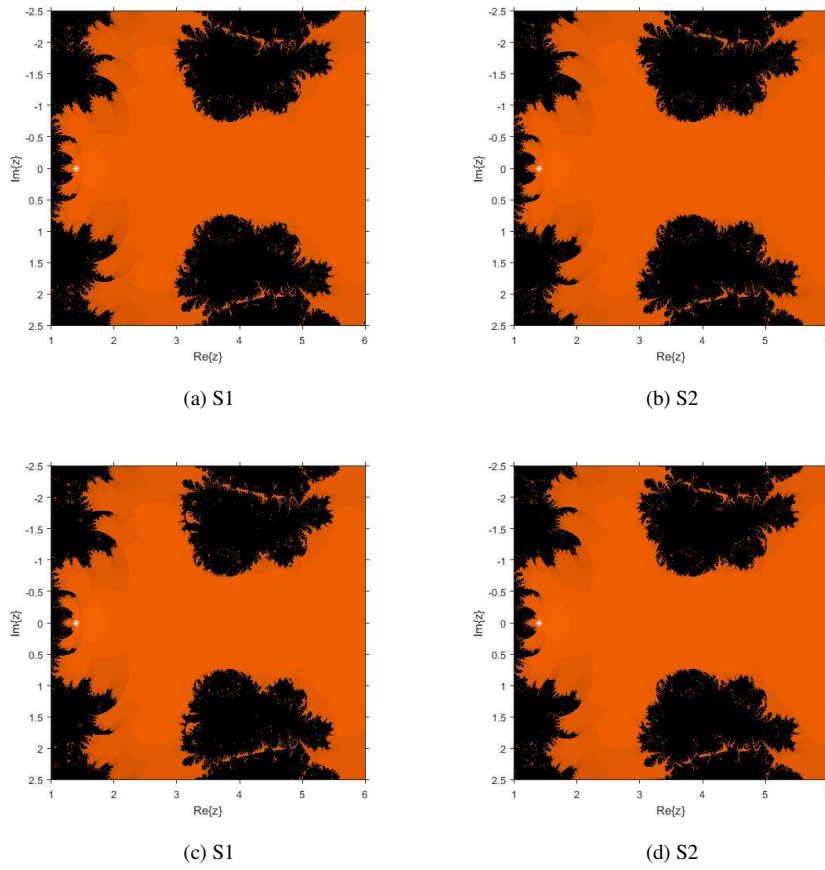


Figure 10: Basins of attraction for $f_5(x)$ of new methods

$f_6(x), x_0 = 2.2, m = 6, \gamma = 1.0$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
<i>GM1</i>	1.764232175(-9)	7.909131501(-66)	1.427670977(-403)	5.993840804
<i>GM2</i>	4.063252417(-8)	6.161159788(-55)	3.458618109(-335)	5.985806935
<i>OM</i>	7.924274533(-10)	1.861582656(-84)	4.622065179(-681)	7.994270715
<i>S1</i>	2.881013079(-11)	1.378558116(-98)	6.743432263(-797)	7.997132194
<i>S2</i>	1.564575613(-11)	1.628271081(-101)	3.671347563(-821)	7.997616679
<i>S3</i>	3.768642533(-11)	5.375386437(-97)	1.739474264(-783)	7.996782482
<i>S4</i>	1.820508721(-11)	1.130227420(-100)	4.223372379(-814)	7.997436201

Table 12: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_6(x)$

$f_7(x), x_0 = 3, m = 6, \gamma \approx 1.365230013$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
<i>GM1</i>	6.204268880(-2)	9.647881165(-9)	1.711905253(-49)	5.974013174
<i>GM2</i>	1.018176714(-1)	5.373771561(-7)	1.861210469(-38)	5.937107558
<i>OM</i>	6.625432825(-2)	4.264387084(-10)	1.801130693(-75)	7.967277307
<i>S1</i>	4.209676560(-2)	4.184325630(-12)	5.021948915(-92)	7.982862600
<i>S2</i>	3.849107043(-2)	1.484407431(-12)	8.773237057(-96)	7.985864599
<i>S3</i>	4.382099514(-2)	7.585129608(-12)	7.980285872(-90)	7.980549102
<i>S4</i>	3.951876180(-2)	2.094114395(-12)	1.608068505(-94)	7.984566726

Table 13: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_7(x)$

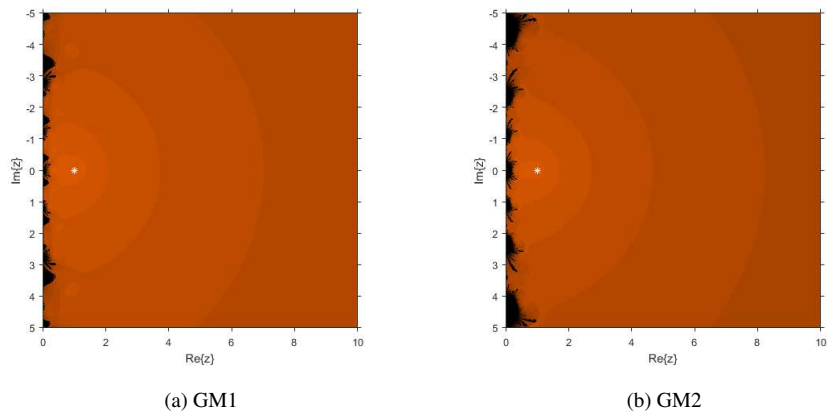


Figure 11: Basins of attraction for $f_6(x)$ of known methods

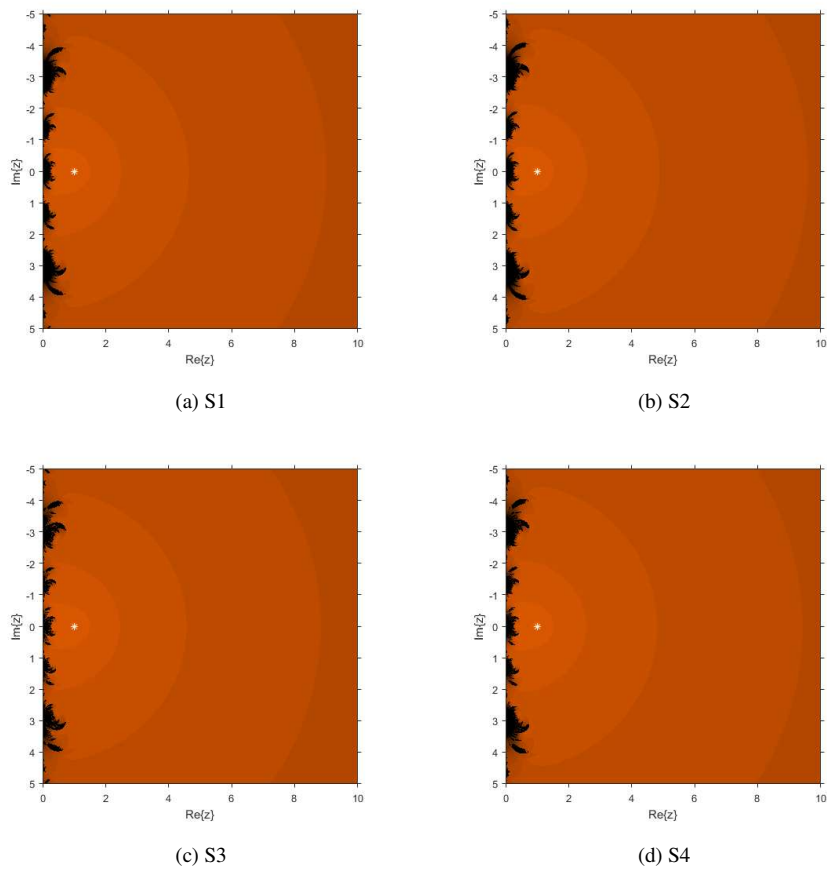


Figure 12: Basins of attraction for $f_6(x)$ of new methods

The function f_8 has a multiple zero of multiplicity $m = 8$ at $\gamma \approx -1.79035317915895441218$ and initial guess $x_0 = -1.2$.

Regarding Examples 5-8, it can be deduced from Tables 9–16 and Figures 9–16, that the proposed schemes show better error estimations in very similar basins of convergence. So, in the same set of initial estimations, the final results are better than those provided by existing schemes.

$f_7(x), x_0 = 3, m = 6, \gamma \approx 1.365230013$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
<i>GM1</i>	1.386208354	1.635349358(-41)	5.103841843(-286)	5.974013174
<i>GM2</i>	3.036987379 (1)	4.883090237(-31)	8.429305217(-220)	5.937107558
<i>OM</i>	2.080980548	1.219434775(-49)	6.922934486(-442)	7.967277307
<i>S1</i>	1.276484149 (-1)	1.088357832(-61)	3.252764362(-541)	7.982862600
<i>S2</i>	7.381106939(-2)	2.169389728(-64)	9.246514296(-564)	7.985864599
<i>S3</i>	1.632286362 (-1)	3.861863288(-60)	5.237566543(-528)	7.980549102
<i>S4</i>	8.671256861 (-2)	1.710099080 (-63)	3.506276298 (-556)	7.984566726

Table 14: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_7(x)$

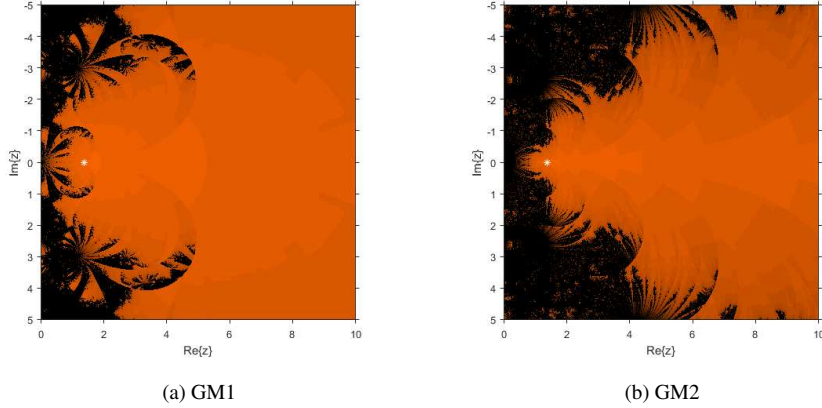


Figure 13: Basins of attraction for $f_7(x)$ of known methods

$f_8(x), x_0 = -1.2, m = 8, \gamma \approx -1.790353179$				
	$ x_1 - \gamma $	$ x_2 - \gamma $	$ x_3 - \gamma $	σ_n
<i>GM1</i>	4.307854679(-4)	9.207188540(-22)	8.779275351(-128)	5.999958813
<i>GM2</i>	2.151830489(-3)	9.635795059(-17)	7.819790959(-97)	5.999566387
<i>OM</i>	6.590079930(-4)	7.233132734(-26)	1.525428460(-201)	7.999918618
<i>S1</i>	3.130404132(-4)	4.981126746(-29)	2.048317073(-227)	7.999966699
<i>S2</i>	2.924745644(-4)	1.771835647(-29)	3.214839223(-231)	7.999976637
<i>S3</i>	3.179611169(-4)	8.782382901(-29)	2.978315561(-225)	7.999957888
<i>S4</i>	2.946011389 (-4)	2.423576514 (-29)	5.086910238 (-230)	7.999969754

Table 15: Comparison in terms of iterative error $|x_n - \gamma|$ for $f_8(x)$

$f_8(x), x_0 = -1.2, m = 8, \gamma \approx -1.790353179$				
	$ f(x_1) $	$ f(x_2) $	$ f(x_3) $	σ_n
<i>GM1</i>	4.696176340(-23)	2.041187296(-164)	1.394877046(-1012)	5.999958813
<i>GM2</i>	1.833474373(-17)	2.937426449(-124)	5.526214182(-765)	5.999566387
<i>OM</i>	1.409937170(-21)	2.961270251(-197)	1.158780890(-1602)	7.999918618
<i>S1</i>	3.649585210(-24)	1.497915470(-222)	1.224740400(-1809)	7.999966699
<i>S2</i>	2.118884994(-24)	3.839317649(-226)	4.509627098(-1840)	7.999976637
<i>S3</i>	4.134675040(-24)	1.398831837(-220)	2.446990311(-1792)	7.999957888
<i>S4</i>	2.245338546 (-24)	4.704575558 (-225)	1.772144614 (-1830)	7.999969754

Table 16: Comparison in terms of absolute residual errors $|f(x_n)|$ for $f_8(x)$

5. Conclusion

In this paper, we have proposed a family of iterative methods for solving nonlinear equations for multiple roots with known multiplicity. The family of schemes include univariate weight functions involving function-to-function ratio. The

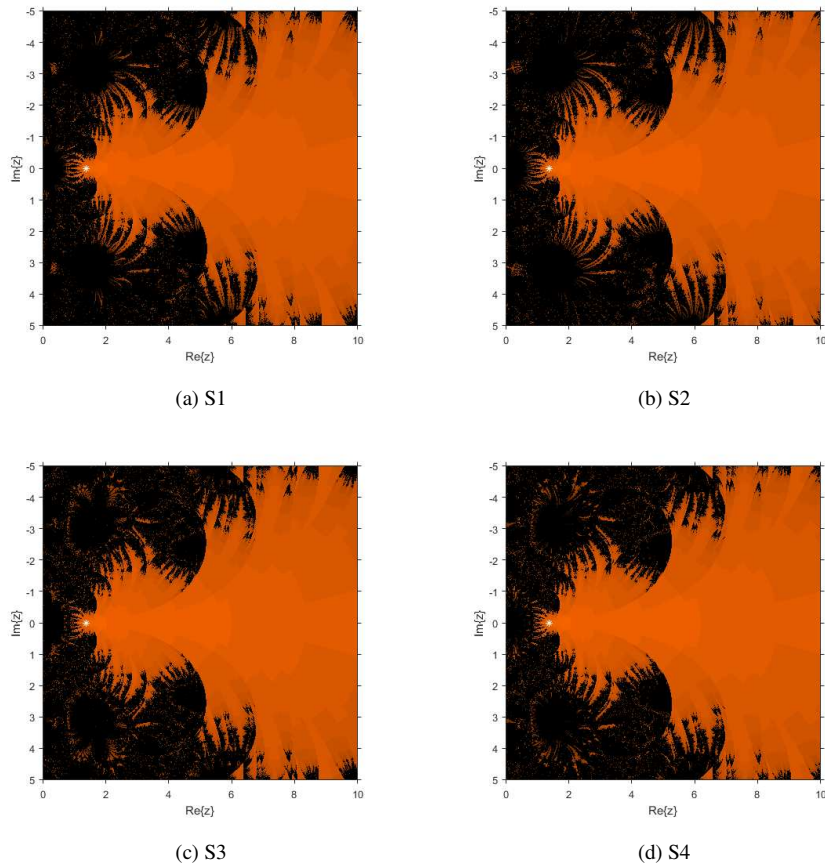


Figure 14: Basins of attraction for $f_7(x)$ of new methods

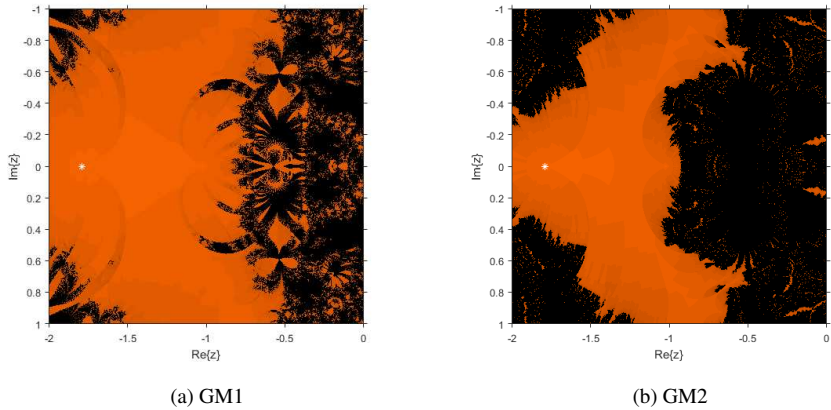


Figure 15: Basins of attraction for $f_8(x)$ of known methods

methods involve only one derivative evaluation. The selection of weight functions yield optimal eighth order convergent methods for multiple roots. In addition, Tables 1-16 show that the proposed methods namely S1-S4 have better performance as compared to other similar known ones. The basins of attraction shown in the numerical section allow us to compare the set of initial approximations of the different methods used in the numerical section.

Acknowledgments The authors would like to thank the anonymous reviewers for their comments and suggestions that have improved the final version of the manuscript.

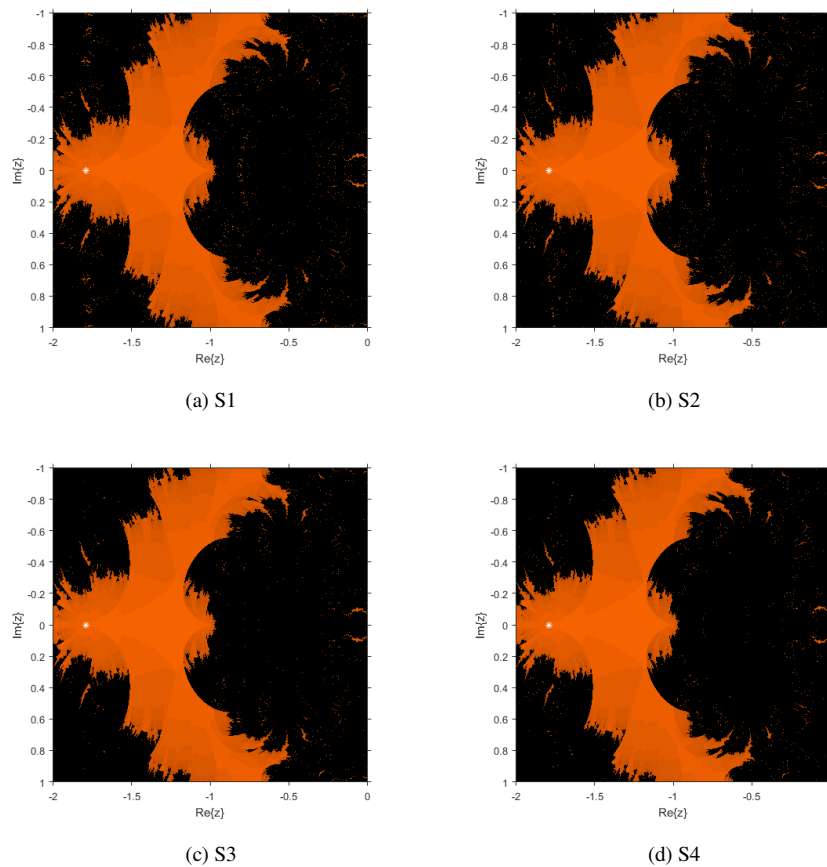


Figure 16: Basins of attraction for $f_8(x)$ of new methods

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Answer to reviewer #2

Reviewer #2: The work is interesting as it proposes an eight-order optimal scheme to find multiple roots. The proposed methods show an interesting dynamical behaviour.

In the proof of Theorem 1 it's a typo with the spacing of the left (, but I wouldn't use the expression "with the help of Mathematica 9"

After equation 2.5 there is a typo "ldots" twice

After equation 3.5 "a" is missing

In a similar way, we can find "A" new optimal eight

or by considering "A" new weight function that satisfy "IES"

In the new version, we have corrected all these typos.

In the beginning of section 4. What is a vigorous method?

We have deleted this adjective.

It's desirable to be more precise with the number of digits used after equation 4.4 "We have done our calculations with several number of significant digits (minimum 1000 significant digits)"

Maybe it would be better to say it converges to something different from the root instead of another thing in the last paragraph of page 8.

After Table 2 it should be confirmS

The way in which Tables, dynamical planes and the text appears after example 3 is a little bit confusing.

References appear mixed up with dynamical planes

The conclusion is too brief in my opinion

In the corrected version, we have taken into account all these suggestions.

Thank you very much for your comments and suggestions that have improved this manuscript.

The authors