A new graphical visualization of n-dimensional Pareto front for decision-making in multiobjective optimization

X. Blasco\textsuperscript{a,}, J.M. Herrero\textsuperscript{a}, J. Sanchis\textsuperscript{a}, M. Martínez\textsuperscript{a}

\textsuperscript{a}Predictive Control and Heuristic Optimization Group
Department of Systems Engineering and Control
Universidad Politécnica de Valencia.
Camino de Vera S/N, 46022-Valencia, Spain.

Preprint submitted to Elsevier 19 December 2007
Abstract

New challenges in engineering design lead to multiobjective ( multicriteria ) problems. In this context, the Pareto front supplies a set of solutions where the designer ( decision-maker ) has to look for the best choice according to his preferences. Visualization techniques often play a key role in helping decision-makers, but they have important restrictions for more than two-dimensional Pareto fronts. In this work, a new graphical representation, called Level Diagrams, for n-dimensional Pareto front analysis is proposed. Level Diagrams consists of representing each objective and design parameter on separate diagrams. This new technique is based on two key points: classification of Pareto front points according to their proximity to ideal points measured with a specific norm of normalized objectives ( several norms can be used ); and synchronization of objective and parameter diagrams. Some of the new possibilities for analyzing Pareto fronts are shown. Additionally, in order to introduce designer preferences, Level Diagrams can be coloured, so establishing a visual representation of preferences that can help the decision-maker. Finally, an example of a robust control design is presented - a benchmark proposed at the American Control Conference. This design is set as a six-dimensional multiobjective problem.

Key words: multiobjective optimization, multidimensional visualization, decision making tools, high dimensional Pareto front

* Corresponding author. DISA-Universidad Politécnica de Valencia.

Camino de Vera S/N, 46022-Valencia, Spain.

Email address: xblasco@isa.upv.es (X. Blasco).

URL: ctl-predictivo.upv.es (X. Blasco).
1 Motivation

In numerous engineering areas, the task of obtaining suitable designs becomes a multiobjective (or multicriteria) problem. This means it is necessary to look for a solution in the design space that satisfies several specifications (objectives) in the performance space. Generally, these specifications are conflicting, that is, there is no simultaneous optimal solution for all of them. In this context, the solution is not unique, instead there is a set of possible solutions where none is best for all objectives. This set of optimal solutions in the design space is called the Pareto set. The region defined by the performances (the value of all objectives) for all Pareto set points is called the Pareto front.

The exact determination of the Pareto front is unrealistic for real-world problems, as it is usually an infinite set. Therefore, it is usual to focus on obtaining a discrete approximation. A common step for solving a multiobjective optimization problem is to obtain the discrete approximation of the Pareto front. This is an open research field where numerous techniques have already been developed [19] and where new techniques are being constantly developed [17,14]. An alternative, and very active research line, is Multiobjective Evolutionary Algorithms [5,9]. In general, these algorithms supply reasonable solutions for Pareto front approximations. Once obtained, the next step for the designer is to select one, or more, solutions inside the Pareto front approximation.

The final solution is often selected using methodologies that normally include designer preferences. Different approaches to introducing preferences can be found in the literature [19,4,25]. The usual classification is based on when the Decision Maker (DM) is consulted: a priori, a posteriori, and progressive (or interactive) decision; and the DM has to introduce preferences before, after,
or during the optimization process respectively.

Decision-making techniques (decision support systems), or simply tools for helping decision-makers, is a field in constant development with interesting and successful solutions in:

- A priori methodology [16].
- Progressive methodology [22].
- A posteriori methodologies [7,23,6,21,26,13,12].

It is widely accepted that visualization tools are valuable and provide decision-makers with a meaningful method to analyze the Pareto set and select good solutions. For two-dimensional problems (and sometimes for three-dimensional) it is usually straightforward to make an accurate graphical analysis of the Pareto set point possibilities, but this becomes more difficult for higher dimensions. Several of the techniques proposed for multidimensional visualization can be consulted at [2]. The most common are:

- Scatter diagrams: The visualization consists of an array of scatter diagrams arranged in the form of an \( n \times n \) matrix. Each dimension of the original data defines one row and column of the matrix. The complexity of the representation increases notably with the dimension.

- Parallel coordinates: A multidimensional point is plotted in a two-dimensional graph. Each dimension of original data is translated to an \( x \) coordinate in the two-dimensional plot. This is a very compact way of presenting multidimensional information, but with large sets of data it loses clarity and the analysis becomes difficult to perform.

Other more complex, but interesting, alternatives offering graphical represen-
This work contributes a new alternative, called *Level Diagrams*. It enables easier analysis of the Pareto set (and Pareto set approximations supplied by multiobjective optimization techniques) and so may become a useful tool for decision-makers. *Level diagrams* can be used in an a priori and progressive methodology to help the DM. *Level Diagrams* tries to be a geometrical visualization of the Pareto front and set, which when combined with a colouring methodology of the points based on preferences, can be a powerful tool to help DM’s make decisions.

The following sections describe the proposed graphical representation and show simple examples. Subsequently, this representation is used in a more complex problem that involves choosing an adequate solution to a multiobjective problem with six dimensions in performance space and six dimensions in parameter space. Additionally, a method to show designer preferences in the *Level Diagrams* is enabled by colouring the points.

### 2 Level Diagrams for Pareto front

Multiobjective problems can be formalized as follows:

\[
\begin{align*}
\theta &= [\theta_1, \ldots, \theta_l] \in \mathcal{D} \\
\mathbf{J}(\theta) &= [J_1(\theta), \ldots, J_s(\theta)] \\
\min_{\theta \in \mathcal{D}} \mathbf{J}(\theta)
\end{align*}
\]

where \(\theta\) is the decision vector, \(\mathcal{D}\) is the decision space, and \(\mathbf{J}(\theta)\) is the objective vector.
Without loss of generality, a multiobjective minimization problem is considered. This involves the simultaneous minimization of all objectives \( J_i(\theta) \). In general, there is no single solution: in fact, there is a set of solutions where none is better than others. Using a definition of dominance, the Pareto set \( \Theta_p \) is the set of every non-dominated solution. Pareto dominance is defined as follows.

A solution \( \theta^1 \) dominates another solution \( \theta^2 \), denoted by \( \theta^1 \prec \theta^2 \), if

\[
\forall i \in \{1, \ldots, s\}, J_i(\theta^1) \leq J_i(\theta^2) \land \exists k \in \{1, \ldots, s\} : J_k(\theta^1) < J_k(\theta^2) \tag{2}
\]

Therefore, the Pareto optimal set \( \Theta_p \), given by

\[
\Theta_p = \{ \theta \in D \mid \nexists \tilde{\theta} \in D : \tilde{\theta} \prec \theta \}. \tag{3}
\]

\( \Theta_p \) is unique, and normally includes infinite solutions. Hence, a set \( \Theta^*_p \), with a finite number of elements from \( \Theta_p \), should be obtainable, and this should be the goal of multiobjective algorithms. In fact, the realistic goal of a multi-objective algorithms is to obtain a discrete approximation of the Pareto front. In the following, this discrete approximation is referenced as \( \Theta^*_p \).

At this point, the decision-maker has a set \( \Theta^*_p \subset \mathcal{R}^l \), that constitutes the Pareto set discrete approximation and an associated set of objective values for every point that constitutes an approximation to the Pareto front \( \mathbf{J}(\Theta^*_p) \subset \mathcal{R}^s \).

The Level Diagrams tool is based on the classification of the Pareto front.

\(^1\) A maximization problem can be easily converted to a minimization problem, for instance for each one of the objectives that have to be maximized, the following transformation could be applied: \( \max J_i(\theta) = -\min(-J_i(\theta)) \)

\(^2\) Notice that \( \Theta^*_p \) is not unique.
approximation \( (J(\Theta^*_p)) \) according to the proximity to the ideal point \(^3\).

For this classification, every objective \((J_i(\theta), i = 1 \ldots s)\) is normalized with respect to its minimum and maximum values on the Pareto front approximation, \( J_i(\theta), i = 1 \ldots s: \)

\[
J_i^M = \max_{\theta \in \Theta^*_p} J_i(\theta); \quad J_i^m = \min_{\theta \in \Theta^*_p} J_i(\theta); \quad i = 1 \ldots s \tag{4}
\]

\[
\bar{J}_i(\theta) = \frac{J_i(\theta) - J_i^m}{J_i^M - J_i^m} \rightarrow 0 \leq \bar{J}_i(\theta) \leq 1 \tag{5}
\]

A norm is applied to evaluate the distance to the ideal point. Different norms could be used to obtain different characteristics of the diagrams, the most common being:

- 1-norm: \(||\bar{J}(\theta)||_1 = \sum_{i=1}^{s} |\bar{J}_i(\theta)|\)
- Euclidean norm (2-norm): \(||\bar{J}(\theta)||_2 = \sqrt{\sum_{i=1}^{s} J_i(\theta)^2}\)
- Infinite norm (\(\infty\)-norm): \(||\bar{J}(\theta)||_\infty = \max\{\bar{J}_i(\theta)\}\)

Depending on the selected norm and the dimension of the objective vector, the value for each is:

\[
0 \leq ||\bar{J}(\theta)||_1 \leq s \tag{6}
\]
\[
0 \leq \sqrt{\sum_{i=1}^{s} J_i(\theta)^2} \leq \sqrt{s} \tag{7}
\]
\[
0 \leq ||\bar{J}(\theta)||_\infty \leq 1 \tag{8}
\]

Each norm gives a different point of view of the Pareto front shape, for instance:

- Euclidean norms supply an accurate evaluation of the conventional geomet-

\(^3\) The ideal point \([19]\) is a point with the minimum value of the Pareto front at each objective.
rical distance to the ideal point, and then offer a better view of the ‘real’ shape.

• $\infty$-norm can supply information about the worst objective for a specific point, and is useful for trade-off analysis between different objectives. An increment in this norm directly reveals a worsening of at least one of the objectives. The use of $\infty$-norm is the generalization of the representation by layers described at [3].

To plot Level Diagrams, the points of the Pareto front are sorted in ascending order of the value of the $|| \bar{J}(\theta)||_x$. Once every point is classified, the graphical representation of the Pareto front, and the Pareto set, is performed with the following methodology. Each objective ($J_i$) and decision variable ($\theta_j$) has its own graphical representation. Axis Y on all the graphs corresponds to the value of $|| \bar{J}(\theta)||_x$, and this means that all graphs are synchronized with respect to this axis. Axis X corresponds to values of the objective, or decision variables, in physical units.

It is important to remark that for an adequate interpretation of Level Diagrams, each objective and component of a point is represented at the same Y position (level) for all graphs, and this means all information for a single point is drawn at the same position on the Y axis for all graphs $J_i$ and $\theta_j$.

The following simple example of Level Diagrams is shown to clarify this new alternative. A classical 2D problem is selected, although Level Diagrams is not necessary for a 2D problem (classical representation is sufficient).

Characteristics of the multiobjective problem 1 (MOP1) are:

$$-10^5 \leq \theta \leq 10^5,$$
\[ J_1(\theta) = \theta^2, \]
\[ J_2(\theta) = (\theta - 2)^2, \]  

Fig. 1. Pareto front classical 2D representation of \( \Theta^*_P \) for MOP1. Normalized 2D representation with isolines for 1-Norm.

Fig. 2. 1-norm Level Diagrams representation for MOP1 problem.

Figures 1 and 2 show a 2D classical representation and Level Diagrams respectively\(^4\) for a discrete set of Pareto points of the MOP1 problem. Figure 1

\(^4\) Objective and parameter axes with a physical meaning are marked with \( J_i \) (\( J_i \) Units) and \( \theta \) (\( \theta \) Units) respectively. For a real problem, these ‘Units’ have a specific meaning and offer a real meaningfulness that gives valuable information to the designer.
shows the most common type of representation of a 2D Pareto front. Isolines of 1-norm on the same figure are shown.

Each point of the Pareto front $J(\Theta_p^*)$ corresponds to a point on each graph $(J_1$ and $J_2)$ on Level Diagrams (see figure 2). For instance, point A at figure 1 corresponds to points A on all graphs $(J_1$, $J_2)$ of figure 2, point B of figure 1 corresponds to points B in figure 2, etc.

The Pareto set $\Theta_p^*$ is drawn in a similar way. The classification of the Pareto front points is maintained, and then for each component of a Pareto set there is an associated graph. A point is drawn at the same level (Y coordinate) on each graph and this level is the same for the associated Pareto front point (see graph $\theta$ at figure 2).

In the Level Diagram representation, point A of the Pareto front of figure 1 is represented with three points in figure 2. One point is shown in the $J_1$ graph, one point in the $J_2$ graph, and another in the $\theta$ graph. All these points are at the same position on the Y axis, and this position shows the distance to the ideal point with a particular norm, (in the example, 1-norm).

3 Pareto front graphical analysis

The proposed graphical representation enables a new analysis of large sets of Pareto points (or Pareto point approximation obtained from a multiobjective optimization algorithm). Pareto fronts and sets can also be analyzed simultaneously. This new alternative gives the DM valuable information about the

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5 Each objective on one axis. This type of representation is only possible for 2D and 3D cases.
characteristics of the different zones of the Pareto front. Remember, that all points of the Pareto front are optimum in some sense, and could be useful for different design alternatives depending on DM preferences. Some of the characteristics that can be analyzed are:

- The points situated at the lower levels corresponding to the zones of the Pareto front nearer to the ideal point.
- Qualitative analysis of Pareto front or set discontinuities: these are visible when a vertical band at *Level Diagrams* is empty, or when there are discontinuities in the norm axes. These empty bands give an unreachable range of values for the objective function and parameter design. Remember, that X axes are represented in physical units, so the DM can quickly obtain an unreachable range in physical units.
- Analysis of trade-off between several objectives.

Two simple examples are presented to show the use of this visualization. Both examples correspond to typical test functions in multiobjective literature.

### 3.1 Test problem MOP2

This test problem has the following characteristics: both Pareto front and set are bi-dimensional and discontinuous. Functions to optimize are \( J(\theta) = [J_1(\theta), J_2(\theta)] \), \( \theta = [\theta_1, \theta_2] \) where:

\[
J_1(\theta) = \left(1 + (A_1 - B_1)^2 + (A_2 - B_2)^2\right), \quad \text{(11)}
\]

\[
J_2(\theta) = (\theta_1 + 3)^3 + (\theta_2 + 1)^2, \quad \text{(12)}
\]

\[
A_1 = 0.5 \sin(\theta_1) - 2 \cos(\theta_1) + \sin(\theta_2) - 1.5 \cos(\theta_2),
\]

\[
A_2 = 1.5 \sin(\theta_1) - \cos(\theta_1) + 2 \sin(\theta_2) - 0.5 \cos(\theta_2),
\]
\[ B_1 = 0.5 \sin(\theta_1) - 2 \cos(\theta_1) + \sin(\theta_2) - 1.5 \cos(\theta_2), \]
\[ B_2 = 1.5 \sin(\theta_1) - \cos(\theta_1) + 2 \sin(\theta_2) - 0.5 \cos(\theta_2), \]
\[-\pi \leq \theta_1 \leq \pi; \quad -\pi \leq \theta_2 \leq \pi.\]

For the MOP2 problem, the multiobjective evolutive algorithm \(\varepsilon\)-MOGA gives 868 points as the Pareto front approximation. The graphics in figure 3 are obtained with the classical representation.

![Classical representation of Pareto front and set for the MOP3 problem.](image)

Fig. 3. Classical representation of Pareto front and set for the MOP3 problem.

The graph shown in figure 4 is obtained with the new representation.

For this problem, the principal characteristics of the Pareto front and set can be observed with both visual representations - classical and Level Diagrams.

Let’s analyze the Pareto front first:

---

6 The \(\varepsilon\)-MOGA variable (\(\varepsilon\)-MOGA) \([8,10]\) is an elitist multiobjective evolutionary algorithm based on the concept of \(\varepsilon\)-dominance \([20]\). \(\varepsilon\)-MOGA obtains an \(\varepsilon\)-Pareto set, \(\Theta^*_P\), that converges towards the Pareto optimal set \(\Theta_P\) in a distributed manner around Pareto front \(J(\Theta_P)\), with limited memory resources. It also dynamically adjusts the limits of the Pareto front \(J(\Theta^*_P)\) and avoids losing the Pareto front extremities.
Fig. 4. 1-norm *Level Diagrams* representation of Pareto front and set approximation for the MOP3 problem.

- The whole range of $J_1$ between the maximum and minimum can be reached by points of the Pareto front. This does not happen for $J_2$ where there are unreachable values of the Pareto front (approximately between 4 and 21). This means there are discontinuities along the front (or at least along its discrete approximation).
- There are points near the ideal point and so it is relatively simple for the designer (or DM) to choose a unique solution.
- It is quite simple to maintain a low value of $J_2$, but if a value of $J_1 < 2$ is required then $J_2$ has to be greatly increased.

In a similar way, the Pareto set can be analyzed:

- Values of the Pareto set are localized at two particular zones: $\theta_1 \approx -3.1$, $\theta_2 \in [-1, 0.75]$ and $\theta_1 \in [0.8, 1]$, $\theta_2 \in [1.55, 2]$. Notice that there are discontinuities in the Pareto set.

With the new visual representation, it is possible to view the same characteristics, and the precision of the range of values is probably improved because the Pareto front and set are better related to the physical unit range. For
instance, it is easier to see the range value of the nearest point to the ideal point: \( \theta_1 \approx -3.1, \theta_2 \approx 0.5 \). In the classical representation, the Pareto set and Pareto front are unsynchronized.

### 3.2 Test problem MOP3

Although the new visual representation could seem less intuitive than the classical view for bi-dimensional problems, it offers huge possibilities as the dimension of the optimization problem grows because it enables a graphical analysis that is very difficult to achieve with other methods.

The MOP3 test problem is not yet a very high dimensional problem, but presents some characteristics that complicate analysis in the classical way. The MOP3 problem has the following characteristics: the Pareto front is a line in the three-dimensional objective space, and the Pareto set is bi-dimensional and discontinuous.

Functions to minimize are \( \mathbf{J}(\theta) = [J_1(\theta), J_2(\theta), J_3(\theta)] \), \( \theta = [\theta_1, \theta_2] \) where:

\[
\begin{align*}
J_1(\theta) &= 0.5(\theta_1^2 + \theta_2^2) + \sin(\theta_1^2 + \theta_2^2), \\
J_2(\theta) &= \frac{(3\theta_1 - 2\theta_2 + 4)^2}{8} + \frac{(\theta_1 - \theta_2 + 1)^2}{27}, \\
J_3(\theta) &= \frac{1}{(\theta_1^2 + \theta_2^2 + 1)} - 1.1e^{(-\theta_1^2-\theta_2^2)} + 0.2,
\end{align*}
\]

\(-3 \leq \theta_1 \leq 3; \ -3 \leq \theta_2 \leq 3.\)

The approximation to the Pareto front is obtained with \( \varepsilon \)MOGA and has 775 points. The 3D visual representation is shown in figure 5. As can be seen, it is very difficult to obtain useful conclusions about the principal characteristics of the Pareto front; and it is difficult to obtain range values for each objective.
It is also very difficult to see the nearest zone to the ideal point, and it is not clear if there are discontinuities, etc.

Other common alternatives for graphical representation are:

- **Parallel coordinates.** Each objective is represented in a vertical axis as shown in figure 6. However, when the Pareto front is described with a large number of points, then interpretation with this visual technique becomes very complicated.

- **Scatter plot.** Consists in the projection of all the pairs of objectives as shown in figure 7. In this case, analysis is difficult.

In both cases, the Pareto front and Pareto set could be synchronized by drawing each point with different colours, but this method is less direct than the synchronization performed in Level Diagrams.

![Fig. 5. Classical representation of the Pareto front and set for the MPO3 problem.](image)

Figure 8 shows the Level Diagrams representation for the MPO3 Pareto solution. Some interesting conclusions, that may help the DM, can be made:

- It is a Pareto front with numerous points at the lower levels, and that means there are numerous points relatively close to the ideal point. Therefore, it is a front where a reasonable compromise between objectives can be found.
Fig. 6. Parallel coordinate (with normalized axes) representation of Pareto front and set for the MPO3 problem.

Fig. 7. Scatter plot representation of Pareto set for MOP3 problem.

- The nearest points to the ideal are at: $J_1 \in [0.4, 0.6]$, $J_2 \in [0.35, 0.55]$ and $J_3 \in [0.15, 0.18]$. These values mark the order of magnitude for an adequate compromise between all objectives.
- A similar analysis could be made with the Pareto set. The values of the front nearest to the ideal point, have the following range of values: $\theta_1 \approx -0.5$, $\theta_2 \approx 0.5$. 
Fig. 8. 1-norm *Level Diagrams* representation of the Pareto set and front for the MPO3 problem.

\[ \theta_2 \approx 0.3. \]

- If the Pareto front has no discontinuities at any of the objectives, then all values inside the range of the Pareto front can be achieved for each objective separately. If required by the DM, a specific value of one objective function can always be obtained and it is optimum in the Pareto sense. For instance, if the designer thinks that \( J_1(\theta) \) has to be in the range \([2, 4]\), after seeing *Level Diagrams*, then it is necessary to choose a point of 1-norm in range \([1.2, 1.35]\).

At this level, the other objectives and design parameters are situated at the ranges: \( J_2(\theta) \approx 0, 0.3 \leq J_3(\theta) \leq 0.35, -2.5 \leq \theta_1 \leq -2 \) and \(-1.7 \leq \theta_2 \leq -1\).

Qualitatively, \( J_2 \) is improved and \( J_3 \) is worsened - compared with the zone nearest to the ideal point. The design parameters are in a zone clearly different from the one nearest to the ideal point. Therefore, the designer can have an idea of the value for each objective and design parameter.

- A discontinuity can be observed in the design parameters. There is a a gap in the ranges: \(-2 \leq \theta_1 \leq -1.5\) and \(-1 \leq \theta_2 \leq -0.2\).
To illustrate the Level Diagrams utility, a multiobjective design of a controller is described. The problem is based on a robust control benchmark proposed at the American Control Conference (ACC). Wie and Bernstein [28] proposed a series of problems for robust control - in which the controller designer must achieve a trade-off between maximizing stability and robust system performance, and minimizing control effort.

Fig. 9. A two mass and spring system with uncertainties in the parameters.

Figure 9 shows the process described in the benchmark. It is a flexible structure of two masses connected by a spring. $x_1$ and $x_2$ indicate mass 1 and mass 2 positions. The nominal values for the two masses and for the spring are $m_1 = m_2 = k = 1$. Control action $u$ is the strength applied to mass 1, and controlled variable $y$ is the mass 2 position $x_2$ affected by noise measurements $v$. Moreover, there is a disturbance $w$ on mass 2.

### 4.1 Design objectives

The specific problem is to obtain a coefficient of the controller transfer function with six degrees of freedom. The numerator and denominator coefficients give the parameter vector to be obtained by optimization $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$:

$$G_{controller}(s) = \frac{\theta_1 s^2 + \theta_2 s + \theta_3}{s^3 + \theta_4 s^2 + \theta_5 s + \theta_6} \quad (16)$$
Design objectives \((J_i(\theta))\) have to be quantities that the designer wishes to minimize. For the robust control benchmark, six functions that supply specification values for controller design will be used (also used in [15] and [18]):

- \(J_1(\theta)\): Robust stability and robust performances \((\text{Re}(\lambda)_{\text{max}})\).
- \(J_2(\theta)\): Maximum control effort \((u_{\text{max}})\).
- \(J_3(\theta)\): Worst case settling time \((t_{\text{est}}^{\text{max}})\)
- \(J_4(\theta)\): Noise sensitivity \((\text{noise}_{\text{max}})\).
- \(J_5(\theta)\): Nominal control effort \((u_{\text{nom}})\).
- \(J_6(\theta)\): Nominal settling time \((t_{\text{nom}}^{\text{est}})\)

To prevent instability problems when nominal or worst case poles are unstable:

- Nominal control effort and settling time \((J_5(\theta)\) and \(J_6(\theta)\)) are coerced to \(\infty\).
- Maximum control effort \((J_2(\theta))\), the worst case settling time \((J_3(\theta))\) and noise sensitivity \((J_4(\theta))\) are coerced to \(\infty\).

### 4.2 Graphical analysis of Pareto front

With the multiobjective evolutionary algorithm \(\varepsilon^2\text{MOGA} \) [10] a Pareto front approximation of 2328 points is obtained\(^7\). Visual representation of the approximations of the Pareto front and the Pareto set is shown in figures 10, 11 and 12. These figures show the Level Diagrams method with Euclidean norm, 1-norm and \(\infty\)-norm respectively.

\(^7\) An imposed designer constraint is that the search range for each parameter is limited to \(\theta_i \in [-10, 25]\). To obtain this solution, the algorithm has made 52100 evaluations of objective vectors.
As can be seen, the volume of data hinders graphical analysis. Parallel coordinates, or scattered plots, are inadequate for this case. However, with the new representation method some valuable information can be interpreted.

Using *Level Diagrams* the nearest zone to the ideal point of the Pareto front can be approximately established (table 1). It is worth remarking that the values differ depending on the norm used. For a geometrical interpretation, the Euclidean norm gives a better representation. The other norms give useful information for the decision-maker and depend on the preferences. It is recommendable to represent the same Pareto front with different norms to see the differences. Usually for DM purposes, it is sufficient to plot with the $\infty$-norm - as can be seen in the following analysis.

Using the Euclidean norm (figure 10), the lowest value is always over 0.9, and that means the Pareto front is relatively far from the ideal point. A sign of probable nonconvexity is that all Pareto points are above the hyperspheres of the 0.9 radius and below 1.75. Remember, that all objectives are normalized and the best value for a single objective is 0 and the worst is 1, meaning all points with a euclidean norm of 1 are on the hypersphere surface of radius 1. This hypersphere is nonconvex, and so it is certain that points above this hypersphere correspond to points of nonconvex zones of the Pareto front.

Another sign of possible nonconvexity can be seen at $J_4$, where the nearest zone is horizontal and this means all these points have the same Euclidean distance (they are part of a hypersphere) so producing a nonconvex zone.

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8 A more accurate, even an exact, position of the nearest zone to the ideal point can be achieved by increasing the zoom ratio in the Level Diagrams, but this paper only intends showing how a DM could use the method in an easy and intuitive way.
Table 1

Values of objectives and parameter for the nearest points to the ideal.

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<tr>
<td>Values with Euclidean norm</td>
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</tr>
<tr>
<td>$J_i$</td>
<td>$[-0.03, -0.01]$</td>
<td>$[0.15, 0.22]$</td>
<td>$\approx 80$</td>
<td>$[0.5, 4]$</td>
<td>$[0.12, 0.2]$</td>
<td>$\approx 80$</td>
</tr>
<tr>
<td>$\theta_i$</td>
<td>$[-5, -0.5]$</td>
<td>$[3, 5]$</td>
<td>$\approx 0.15$</td>
<td>$[16, 24]$</td>
<td>$[12, 25]$</td>
<td>$[14, 21]$</td>
</tr>
</tbody>
</table>

Values with 1-norm

| $J_i$ | $\approx -0.015$ | $\approx 0.17$ | $\approx 90$ | $\approx 0.1$ | $\approx 0.12$ | $\approx 70$ |
| $\theta_i$ | $\approx -0.2$ | $[3, 4]$ | $\approx 0.15$ | $[22, 24]$ | $[12, 18]$ | $[15, 18]$ |

Values with $\infty$-norm

| $J_i$ | $\approx -0.45$ | $\approx 0.3$ | $\approx 60$ | $\approx 5$ | $\approx 0.25$ | $\approx 50$ |
| $\theta_i$ | $\approx -5$ | $\approx 6$ | $\approx 0.4$ | $\approx 17.5$ | $\approx 20.5$ | $\approx 19$ |

A quick analysis of design parameters shows that parameter $\theta_5$ easily reaches its highest limit of 25. Therefore, good solutions can probably be found by increasing this range. For the other parameters, it is possible to better adjust the range of values for a new Pareto front search. For instance, all of the parameters (except $\theta_1$) are positive.

With 1-norm Level Diagrams (figure 11) it is possible to see indices of nonconvexity of the Pareto front. In the example, it is easy to see that lower values of 1-norm are near the extremes of the Pareto front: the lower 1-norm value of $J_1$ is near its worst value, and for $J_2$, $J_3$, $J_4$, $J_5$ and $J_6$ the lower 1-norm values are near the low $J_i$ ($i = 2, \ldots, 6$) value.
∞-norm Level Diagrams (figure 12) better show the worst objective - and give a quick view of the weakness of each solution. It is usual to see V layouts in several of the objectives, when an objective increases then others must decrease. For this problem, the vertices of the V (the nearest point to ideal) at objective $J_1$, $J_2$, $J_4$ and $J_5$ are around 50% of the scale of each function. For $J_3$ and $J_6$ the points are mostly concentrated under values of 150 units. That means these values are easily attainable - independently of the values of the other objectives.

It can also be seen with all the norms that there are no important discontinuities in any of the objectives. Reachable ranges for all objectives include the complete range of the Pareto front (see axes X). This tells the designer (DM) the order of magnitude that the objectives can achieve.

An interesting remark can be made for table 1: the nearest values with each norm are quite different, this is due to the nonconvexity (with a convex Pareto, the results are more similar). For instance, by comparing the Euclidean norm and ∞-norm it is clear that the geometrical proximity to the ideal point (measured with the Euclidean norm) is not the best choice from the trade-off point of view (measured with ∞-norm).

Generally, a nonconvex Pareto front and, in particular, a Pareto front, which is far from the ideal point, represents a challenging problem for the DM. It is difficult to select a single solution because there is not a clear trade-off solution - and so the DM has to select according to his preferences and experience. Therefore, ∞-norm Level diagrams offer better alternatives for these problems.

In summary, a quick intuitive and quantitative approach to the performance
attainable with the solution of the Pareto front can be made with this graphical representation. To extract more quantitative information, these diagrams can be zoomed and coloured according to designer preferences - as shown in the following sections.

4.3 Including design preferences

To conclude with a specific solution, the designer or decision-maker (DM) has to establish a set of preferences. As a default, it is possible to select points nearest the ideal, but it may be that this is not the preferred solution.

As mentioned in the motivation section, different approaches to introducing preferences can be found in the literature. The Level Diagrams can be used in a posteriori and progressive methodology. This graphical representation, combined with a colouring methodology of the points based on preference, can be a powerful tool to help the DM make a decision.

The benchmark problem [28] established a specific requirement:

- The maximum settling time for the nominal system \( (m_1 = m_2 = k = 1) \) must be 15 seconds for unit impulse in perturbation \( w \) at time \( t = 0 \).

This constraint is translated to the objectives as \( J_6(\theta) < 15 \text{ sec} \). The points that satisfy the constraints are coloured in dark red (figure 13, \( J_6 \) diagram has been zoomed between 11 and 40 to highlight zones of interest). It is now easier for the DM to choose an adequate solution. Among these points, a good choice is the point associated with the lowest norm, that is, the nearest one to the ideal point, for instance, if \( \infty \)-norm is preferred (see worst performance)
the solution is (see squared point at figure 13):

\[
J_1 = -0.04855; \ J_2 = 0.449; \ J_3 = 21.9;
J_4 = 4.57; \ J_5 = 0.348; \ J_6 = 14.9;
\theta_1 = -4.643; \ \theta_2 = 9.57347; \ \theta_3 = 1.49719;
\theta_4 = 18.7568; \ \theta_5 = 22.7352; \ \theta_6 = 17.7596
\]  

(17)

Table 2
Preferences for the controller design.

<table>
<thead>
<tr>
<th></th>
<th>$J_i^{HD}$</th>
<th>$J_i^D$</th>
<th>$J_i^U$</th>
<th>$J_i^T$</th>
<th>$J_i^{HU}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_1$</td>
<td>-0.01</td>
<td>-0.005</td>
<td>-0.001</td>
<td>-0.0005</td>
<td>-0.0001</td>
</tr>
<tr>
<td>$J_2$</td>
<td>0.85</td>
<td>0.90</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
</tr>
<tr>
<td>$J_3$</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>21</td>
<td>25</td>
</tr>
<tr>
<td>$J_4$</td>
<td>0.5</td>
<td>0.9</td>
<td>1.2</td>
<td>1.4</td>
<td>1.5</td>
</tr>
<tr>
<td>$J_5$</td>
<td>0.5</td>
<td>0.7</td>
<td>1</td>
<td>1.5</td>
<td>2</td>
</tr>
<tr>
<td>$J_6$</td>
<td>10</td>
<td>11</td>
<td>12</td>
<td>14</td>
<td>15</td>
</tr>
</tbody>
</table>

More sophisticated preferences can be considered, a good and intuitive way to set preferences is the idea proposed by Messac [16] with the range of preferences and the type of optimization to perform (see original source for more information).

In this problem, all objectives have to be minimized and the ranges of preferences can be established as shown in table 2. The designer has to choose the values $J_i^x$ to define the ranges for each objective according to the following classification:

- Highly desirable (HD): $J_i \leq J_i^{HD}$
Desirable (D): $J_{i}^{HD} < J_{i} \leq J_{i}^{D}$
Tolerable (T): $J_{i}^{D} < J_{i} \leq J_{i}^{T}$
Undesirable (U): $J_{i}^{T} < J_{i} \leq J_{i}^{U}$
Highly undesirable (HU): $J_{i}^{U} < J_{i} \leq J_{i}^{HU}$
Unacceptable (UNA): $J_{i} > J_{i}^{HU}$

With the table of preferences, it is possible to classify each point of the Pareto set according to designer preferences. For example, with table 2, a point is classified as follows:

$$J(\theta) = (-0.0032, 0.95, 22, 0.4, 2.1, 14.5)$$

\( \downarrow \)

(D, T, HU, HD, UNA, HU) \hspace{1cm} (18)

A scoring system has to be established to colour Level diagrams according to this classification. The proposed system follows the ‘ones vs others’ criteria\(^9\) rule (OVO rule) established by Messac [16]:

Full reduction for one criterion across a given region is preferred over full reduction for all the other criteria across the next best region.

In other words: \((U, U, U, U, U)\) is preferred over \((T, T, T, T, HU)\). A possible scoring system follows below:

- A vector of scores is generated (score), and each position of the vector corresponds to the score for each range of preferences. For the example of the six ranges \((HD, D, T, U, HU\) and \(UNA)\):

$$\text{score} = (\text{score}_{HD}, \text{score}_{D}, \text{score}_{T}, \text{score}_{U}, \text{score}_{HU}, \text{score}_{UNA})$$

\(^9\) criterion is equivalent to objective
An initial value is assigned for the two first preference ranges:

- \( \text{score}(0) = \text{score}_{HD} = 0 \)
- \( \text{score}(1) = \text{score}_{D} = 1 \)

The following ranges are scored as follows, for \( i = 2 \ldots N_{obj} - 1 \) (\( N_{obj} \) is the number of objectives):

\[
\text{score}(i) = N_{obj} \times \text{score}(i - 1) + 1
\]

Then, for the example with six ranges of preferences, the score vector is:

\[
\text{score} = (0, 1, 7, 43, 259, 1555)
\]

A point \( A \) with \( (U, U, U, U, U, U) \) has a total score of \( 6 \times \text{score}_U = 258 \) and a point \( B \) with \( (T, T, T, T, T, HU) \) has a total score of \( 5 \times \text{score}_T + \text{score}_{HU} = 294 \).

As lower scores are better, so point \( A \) has a better score than point \( B \) satisfying the OVO rule.

Once the score system is established, the colour for the Level diagrams is assigned according to the score of each point. Then for the six-dimensional problem, if the preferences are those defined in table 2, the resulting coloured Level Diagrams are shown in figure 14. A darker colour means a lower score and so a better point. The DM can now choose one of the darker points, for instance, the point with the lowest norm (squared point at figure 14).

5 Conclusions

A new alternative visualization methodology for Pareto front representation is presented and called Level Diagrams. It enables analysis of large sets of a
high dimensional Pareto fronts and sets. The fundamental idea is classification by layers, and synchronous representation of all objectives and parameters. It is shown that this Level Diagrams representation enables a good analysis of the Pareto front, and so provides an excellent tool to help decision-making. In this article, only some of the Pareto front characteristics have been evaluated (discontinuities, closeness to ideal point, ranges of attainable values), but it already offers valuable information and seems to be open to other characteristic evaluations. New possibilities for incorporating designer preferences in this representation have been developed (based on scoring and colouring points of Level Diagrams) and will contribute to improving decision-making tools for multiobjective problems.

Acknowledgments

Partially supported by MEC (Spanish Government) and FEDER funds: projects DPI2005-07835, DPI2004-8383-C03-02 and GVA-026.

References


Fig. 10. Euclidean norm *Level Diagrams* representation of the approximation of the Pareto front and set for the benchmark problem.
Fig. 11. 1-norm Level Diagrams representation of the approximation of the Pareto front and set for the benchmark problem.
Fig. 12. $\infty$-norm *Level Diagrams* representation of the approximation of the Pareto front and set for the benchmark problem.
Fig. 13. $\infty$-norm *Level Diagrams* representation of the Pareto front and set for the benchmark problem. Points that satisfy $J_6 < 15$ sec are coloured in red (dark colour). The selected point is squared. $J_6$ diagram has been zoomed between 11 and 40.
Fig. 14. Coloured $\infty$-norm Level Diagrams representation of the Pareto front and set for the benchmark problem. Better points in darker colour. The selected point (darker with lower norm) is marked with a square.