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Remarks on fixed point assertions in digital topology, 3

Laurence Boxer

Department of Computer and Information Sciences, Niagara University, NY 14109, USA; and Department of Computer Science and Engineering, State University of New York at Buffalo, USA. (boxer@niagara.edu)

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Abstract

We continue the work of [5] and [3], in which are considered papers in the literature that discuss fixed point assertions in digital topology. We discuss published assertions that are incorrect or incorrectly proven; that are severely limited or reduce to triviality under "usual" conditions; or that we improve upon.

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1. Introduction

The topic of fixed points in digital topology has drawn much attention in recent papers. The quality of discussion among these papers is uneven; while some assertions have been correct and interesting, others have been incorrect, incorrectly proven, or reducible to triviality. In [5] and [3], we have discussed many shortcomings in earlier papers and have offered corrections and improvements. We continue this work in the current paper.

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2. Preliminaries

We use $\mathbb{N}$ to represent the natural numbers, $\mathbb{Z}$ to represent the integers, and $\mathbb{R}$ to represent the reals.

A *digital image* is a pair (X,κ), where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation on X. Thus, a digital image is a graph. In order to model the "real world," we usually take X to be finite, although there are several papers that consider infinite digital images. The points of X may be thought of as the "black points" or foreground of a binary, monochrome "digital picture," and the points of $\mathbb{Z}^n \setminus X$ as the "white points" or background of the digital picture.

2.1. Adjacencies, connectedness, continuity.

In a digital image (X,κ), if $x, y \in X$, we use the notation $x \leftrightarrow_\kappa y$ to mean x and y are κ-adjacent; we may write $x \leftrightarrow y$ when κ can be understood. We write $x \leftrightarroweq_\kappa y$, or $x \leftrightarroweq y$ when κ can be understood, to mean $x \leftrightarrow_\kappa y$ or $x = y$.

The most commonly used adjacencies in the study of digital images are the c_u adjacencies. These are defined as follows.

Definition 2.1. Let $X \subset \mathbb{Z}^n$. Let $u \in \mathbb{Z}$, $1 \le u \le n$. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Then $x \leftrightarrow_{c_u} y$ if

- for at most u distinct indices i, $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Definition 2.2 ([13])**.** A digital image (X,κ) is κ-*connected*, or just *connected* when κ is understood, if given $x, y \in X$ there is a set $\{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x_i \leftrightarrow_\kappa x_{i+1}$ for $0 \le i < n$, and $x_n = y$.

Definition 2.3 ([13, 1])**.** Let (X,κ) and (Y,λ) be digital images. A function $f : X \to Y$ is (κ,λ)-*continuous*, or κ-*continuous* if $(X,\kappa) = (Y,\lambda)$, or *digitally continuous* when κ and λ are understood, if for every κ-connected subset X' of X, $f(X')$ is a λ-connected subset of Y.

Theorem 2.4 ([1])**.** *A function $f : X \to Y$ between digital images (X,κ) and (Y,λ) is (κ,λ)-continuous if and only if for every $x, y \in X$, if $x \leftrightarrow_\kappa y$ then $f(x) \leftrightarroweq_\lambda f(y)$.*

Theorem 2.5 ([1])**.** *Let $f : (X,\kappa) \to (Y,\lambda)$ and $g : (Y,\lambda) \to (Z,\mu)$ be continuous functions between digital images. Then $g \circ f : (X,\kappa) \to (Z,\mu)$ is continuous.*

2.2. Fixed, approximate fixed points.

A *fixed point* of a function $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. If (X,κ) is a digital image, an *almost fixed point* [13] or *approximate fixed point* [4] of $f : X \to X$ is a point $x \in X$ such that $f(x) \leftrightarroweq_\kappa x$.

2.3. Digital metric spaces.

A *digital metric space* [8] is a triple (X, d, κ), where (X,κ) is a digital image and d is a metric on X. We are not convinced that this is a notion worth developing; under conditions in which a digital image models a "real world" image, X is finite or d is (usually) an ℓ_p metric, so that (X, d, κ) is discrete as a topological space. Typically, assertions in the literature do not make use of both d and κ, so that this notion has an artificial feel. E.g., for a discrete topological space, all self-maps are continuous, although on digital images, self-maps are often not digitally continuous.

We say a sequence $\{x_n\}_{n=0}^{\infty}$ is *eventually constant* if for some $m > 0$, $n > m$ implies $x_n = x_m$.

Proposition 2.6 ([10]). *Let (X, d, κ) be a digital metric space. If for some $a > 0$ and all distinct $x, y \in X$ we have $d(x, y) > a$, then any Cauchy sequence in X is eventually constant, and (X, d) is a complete metric space.*

Note that the hypotheses of Proposition 2.6 are satisfied if X is finite or if d is an ℓ_p metric.

3. Universal functions and AFPP

A digital image (X, κ) has the *approximate fixed point property* (AFPP) if every κ-continuous $f : X \to X$ has an approximate fixed point.

We can paraphrase Theorem 3.3 of [13] as follows.

Theorem 3.1. *A digital interval $([a, b]_{\mathbb{Z}}, c_1)$ has the AFPP.* □

Definition 3.2 ([4]). Let (X, κ) and (Y, λ) be digital images. A (κ, λ)-continuous function $f : X \to Y$ is *universal for* (X, Y) if given a (κ, λ)-continuous function $g : X \to Y$ such that $g \neq f$, there exists $x \in X$ such that $f(x) \leftrightarrow_{\lambda} g(x)$.

It was shown in [4] that there is a relationship between the AFPP and universal functions. In this section, we show there are advantages in the study of the AFPP to replacing the notion of universal function with a similar notion of a "weakly universal function." This enables us to make several improvements on results of [4].

The following assertion, one implication of which is incorrect, appears as Proposition 5.5 of [4].

> Let (X, κ) be a digital image. Then (X, κ) has the AFPP if and only if the identity function 1_X is universal for (X, X).

The implication of this assertion that is correct is stated in the following with its proof as given in [4].

Proposition 3.3. *Let (X, κ) be a digital image. If the identity function 1_X is universal for (X, X), then (X, κ) has the AFPP.*

Proof. If 1_X is universal for (X, X), then for $1_X \neq f : X \to X$, f being κ-continuous, there exists $x \in X$ such that $f(x) \leftrightarrow_{\kappa} 1_X(x) = x$. Thus X has the AFPP. □

However, the converse of Proposition 3.3 is not generally true, as shown be the following.

Example 3.4. Let $f : ([-1,1]_{\mathbb{Z}}, c_1) \to ([-1,1]_{\mathbb{Z}}, c_1)$ be the map $f(z) = -z$. Then f is c_1-continuous, and there is no $z \in [-1,1]_{\mathbb{Z}}$ such that $f(z) \leftrightarrow_{c_1} z$. Hence $1_{[-1,1]_{\mathbb{Z}}}$ is not a universal function for $([-1,1]_{\mathbb{Z}}, [-1,1]_{\mathbb{Z}})$. However, by Theorem 3.1, $([-1,1]_{\mathbb{Z}}, c_1)$ has the AFPP. □

Definition 3.5. Let (X, κ) and (Y, λ) be digital images. Let $f : X \to Y$ be (κ, λ)-continuous. Then f is a *weakly universal function* for (X, Y) if for every (κ, λ)-continuous $g : X \to Y$ such that $g \neq f$ there exists $x \in X$ such that $f(x) \leftrightarrows_\lambda g(x)$. □

Notice the difference between Definitions 3.2 and 3.5: the former requires $f(x)$ and $g(x)$ to be adjacent, while the latter requires $f(x)$ and $g(x)$ to be adjacent or equal.

Proposition 3.6. *A universal function between digital images is weakly universal.*

Proof. This is immediate from Definitions 3.2 and 3.5. □

For a graph $G = (V, E)$ (V is the vertex set; E is the edge set), a subset D of V is called *dominating* if for every $v \in V$, either $v \in D$ or there is a $w \in D$ such that $\{v, w\} \in E$ [6]. The following generalizes a result of [4].

Proposition 3.7. *Let (X, κ) and (Y, λ) be digital images. If $f : X \to Y$ is a weakly universal function, then $f(X)$ is λ-dominating in Y.*

Proof. Let $y \in Y$ and let $\tilde{y} : X \to Y$ be the constant function with image $\{y\}$. Since f is weakly universal, there exists $x \in X$ such that $f(x) \leftrightarrows_\lambda \tilde{y}(x) = y$. Since y is an arbitrary member of Y, the assertion follows. □

Theorem 3.8. *The digital image (X, κ) has the AFPP if and only if 1_X is weakly universal for (X, X).* □

Proof. (X, κ) has the AFPP if and only if given a (κ, κ)-continuous $f : X \to X$, for some $x \in X$ we have $f(x) \leftrightarrows_\kappa 1_X(x) = x$; i.e., if and only if 1_X is universal. □

The following is suggested by Theorem 5.7 of [4].

Proposition 3.9. *Let (W, κ), (X, λ), and (Y, μ) be digital images. Let $f : W \to X$ be (κ, λ)-continuous and let $g : X \to Y$ be (λ, μ)-continuous. If $g \circ f$ is weakly universal, then g is weakly universal.*

Proof. Let $h : X \to Y$ be (λ, μ)-continuous. Since $g \circ f$ is weakly universal, there exists $w \in W$ such that $g \circ f(w) \leftrightarrows_\mu h \circ f(w)$, i.e., for $x = f(w)$ we have $g(x) \leftrightarrows_\mu h(x)$. Since h was arbitrarily chosen, the assertion follows. □

The following is suggested by Theorem 5.8 of [4].

Theorem 3.10. *Let $g : (U, \mu) \to (X, \kappa)$ and $h : (Y, \lambda) \to (V, \nu)$ be digital isomorphisms. Let $f : X \to Y$ be (κ, λ)-continuous. Then the following are equivalent.*

(1) *f is a weakly universal function for (X, Y).*
(2) *$f \circ g$ is weakly universal.*
(3) *$h \circ f$ is weakly universal.*

Proof. (1 implies 2): Let $k : U \to Y$ be (μ, λ)-continuous. Since f is weakly universal, there exists $x \in X$ such that $(k \circ g^{-1})(x) \leftrightarrows_\lambda f(x)$, i.e., for $u = g^{-1}(x)$ we have

$$k(u) = k(g^{-1}(x)) \leftrightarrows_\lambda (f \circ g)(g^{-1}(x)) = (f \circ g)(u).$$

Since k is arbitrary, $f \circ g$ is weakly universal.

(2 implies 1): This follows from Proposition 3.9.

(1 implies 3): Let $m : X \to V$ be (κ, ν)-continuous. Since f is weakly universal, there exists $x \in X$ such that $(h^{-1} \circ m)(x) \leftrightarrows_\lambda f(x)$. By continuity,

$$m(x) = h \circ (h^{-1} \circ m)(x) \leftrightarrows_\nu h \circ f(x).$$

Since m is arbitrary, $h \circ f$ is weakly universal.

(3 implies 1): Let $r : X \to Y$ be (κ, λ)-continuous. Since $h \circ f$ is weakly universal, there exists $x \in X$ such that $h \circ f(x) \leftrightarrows_\nu h \circ r(x)$. Thus,

$$f(x) = h^{-1} \circ h \circ f(x) \leftrightarrows_\lambda h^{-1} \circ h \circ r(x) = r(x).$$

Since r is arbitrary, f must be weakly universal. $\square$

Corollary 5.9 of [4] claims that an isomorphism $f : (X, \kappa) \to (Y, \lambda)$ is universal for (X, Y) if and only if (X, κ) has the AFPP. Example 3.4 above shows that this assertion is incorrect. However, we have the following.

Corollary 3.11. *Let $f : (X, \kappa) \to (Y, \lambda)$ be an isomorphism. The following are equivalent.*
(1) *f is weakly universal for (X, Y).*
(2) *(X, κ) has the AFPP.*
(3) *(Y, λ) has the AFPP.*

Proof. (1) $\Leftrightarrow$ (2): By Theorem 3.10, f is weakly universal if and only if $f \circ f^{-1} = 1_X$ is weakly universal, which, by Theorem 3.8 is true if and only if (X, κ) has the AFPP.

(1) $\Leftrightarrow$ (3): By Theorem 3.10, f is weakly universal if and only if $f^{-1} \circ f = 1_Y$ is weakly universal, which, by Theorem 3.8 is true if and only if (Y, λ) has the AFPP. $\square$

The following generalizes Theorem 5.10 of [4] and corrects its proof (stated in terms of Proposition 5.5 of [4], which, we noted above, is incorrect).

Theorem 3.12. *Let (X_i, κ_i) be digital images, $1 \leq i \leq v$. Let $X = \Pi_{i=1}^v X_i$. If $(X, NP_v(\kappa_1, \ldots, \kappa_v))$ has the AFPP, then each (X_i, κ_i) has the AFPP.*

Proof. Let $f_i : X_i \to X_i$ be (κ_i, κ_i)-continuous, $1 \leq i \leq v$. Then the product function $f = \Pi_{i=1}^v f_i : X \to X$ is $(NP_v(\kappa_1, \ldots, \kappa_v), NP_v(\kappa_1, \ldots, \kappa_v))$-continuous [2]. By Theorem 3.8, 1_X is weakly universal, so there exists $p = (x_1, \ldots, x_v) \in X$, $x_i \in X_i$, such that

$$p \leftrightarrows_{NP_v(\kappa_1, \ldots, \kappa_v)} f(p) = (f_1(x_1), \ldots, f_v(x_v)),$$

hence $x_i \Leftrightarrow_{\kappa_i} f_v(x_i)$ for all i. Since f_i was taken arbitrarily, the conclusion follows. □

4. Digital expansions in [12]

The paper [12] contains several assertions that are incorrect or incorrectly proven, limited, or can be improved.

4.1. Digital expansive mappings.

Definition 4.1 ([12])**.** Let (X, d, κ) be a complete digital metric space. Let $T : X \to X$. If $d(T(x), T(y)) \geq k\, d(x, y)$ for all $x, y \in X$ and some $k > 1$, then T is a *digital expansive mapping*.

Theorem 4.2 ([12])**.** *If $T : X \to X$ is a digital expansive mapping on complete digital metric space (X, d, κ) and T is onto, then T has a fixed point.*

However, in practice, the hypotheses of Theorem 4.2 often cannot be satisfied, as shown in the following, which combines Theorems 4.8 and 4.9 of [5].

Theorem 4.3. *Let (X, d, κ) be a digital metric space of more than one point. If there exist $x_0, y_0 \in X$ such that either*

- $d(x_0, y_0) = diam\, X > 0$, *or*
- $d(x_0, y_0) = \min\{d(x, y) \mid x, y \in X, x \neq y\}$,

then there is no self-map $T : X \to X$ that is a digital expansive mapping and is onto.

In practice, a digital image (X, κ) typically consists of a finite set of more than 1 point; or, should a metric d be used, it is typically an ℓ_p metric. Under such circumstances, by Theorem 4.3 a digital metric space (X, d, κ) cannot have a self-map that is both a digital expansive mapping and onto.

4.2. 1st generalization of expansive mappings.

Theorem 3.4 of [12] states the following.

Theorem 4.4. *Let, (X, d, κ) be a complete digital metric space and $T : X \to X$ be an onto self map. Let T satisfy $d(T(x), T(y)) \geq k\, [d(x, T(x)) + d(y, T(y))]$ where $k \geq 1/2$, for all $x, y \in X$. Then T has a fixed point.*

But Theorem 4.4 reduces to a trivial statement, as we see in the following.

Proposition 4.5. *A map T as in Theorem 4.4 must be the identity map.*

Proof. For such a map, we have

$$0 = d(T(x), T(x)) \geq k\, [d(x, T(x)) + d(x, T(x))] = 2k\, d(x, T(x)),$$

so $d(x, T(x)) = 0$ for all $x \in X$. □

4.3. **2nd generalization of expansive mappings.** Theorem 3.5 of [12] asserts the following.

> Let (X, d, κ) be a complete digital metric space and let $T : X \to X$ be onto and continuous. Let
>
> $$d(T(x), T(y)) \geq k\mu(x, y)$$
>
> for all $x, y \in X$, where $k > 1$ and
>
> $$\mu(x, y) \in \left\{ d(x, y), \frac{d(x, T(x)) + d(y, T(y))}{2}, \frac{d(x, T(y)) + d(y, T(x))}{2} \right\}.$$
>
> Then T has a fixed point.

The argument given as proof for this assertion has flaws, including the use in its proof of an assumption that $k < 2$, not stated in the hypotheses; and an incorrect application of the triangle inequality (where we need the reverse of the inequality to proceed as the authors have done) in the attempt to reduce Case 3 to Case 2. Thus, the assertion as stated must be regarded as unproven. Also, the argument given for proof clarifies that the continuity assumption is of the $\varepsilon - \delta$ type, not digital. In the following, we obtain a version of this assertion with no continuity assumption, but with an additional assumption about X or d and with greater restriction on the possible values of $\mu(x, y)$.

Theorem 4.6. *Let (X, d, κ) be a digital metric space, such that X is finite or d is an ℓ_p metric. Let $T : X \to X$ be onto. Suppose*

$$d(T(x), T(y)) \geq k\mu(x, y)$$

for all $x, y \in X$, where $1 < k < 2$ and

$$\mu(x, y) \in \left\{ d(x, y), \frac{d(x, T(x)) + d(y, T(y))}{2} \right\}.$$

Then T has a fixed point.

Proof. A proof can be given via suitable modification of its analog in [12]. However, a simpler argument is as follows.

Without loss of generality, $|X| > 1$. Since X is finite or d is an ℓ_p metric, there exist $x_0, y_0 \in X$ such that

$$m = d(x_0, y_0) = \min\{d(x, y) \mid x, y \in X, x \neq y\} > 0.$$

Since T is onto, there exist $x', y' \in X$ such that $T(x') = x_0$ and $T(y') = y_0$.

Suppose T has no fixed point. Then for all $x, y \in X$, $d(x, T(x)) \geq m$ and $d(y, T(y)) \geq m$; hence $\mu(x, y) \geq m$. Therefore,

$$m = d(x_0, y_0) = d(T(x'), T(y')) \geq k\mu(x', y') \geq km,$$

a contradiction. Therefore, T must have a fixed point. $\square$

4.4. **3rd generalization of expansive mappings.** The next assertion of [12] is flawed in ways similar to the assertion discussed in section 4.3. Asserted as Theorem 3.6 of [12] is the following.

> Let (X, d, κ) be a complete digital metric space. Let T be an onto self-map of X that is continuous. Let $k > 1$ and suppose T satisfies
> $$d(T(x), T(y)) \geq k\mu(x, y) \text{ for all } x, y \in X,$$
> where $\mu(x, y)$ belongs to
> $$\left\{ d(x, y), \frac{d(x, T(x)) + d(y, T(y))}{2}, d(x, T(y)), d(y, T(x)) \right\}.$$
> Then T has a fixed point.

Observe the following.

- As above, the continuity used for the proof of this assertion is of the $\varepsilon - \delta$ kind, not digital continuity.
- As above, the argument given in proof for this assertion requires $1 < k < 2$.
- As above, the authors attempt to establish a Cauchy sequence, and in doing so, they incorrectly reverse the triangle inequality in order to reduce the 3rd case considered to the 2nd case.

Thus, as stated, the assertion presented as Theorem 3.6 of [12] must be regarded as unproven. Note that Theorem 4.6 above is a reasonable correct modification of this assertion.

4.5. **Examples of [12].** In Examples 3.8, 3.9, 3.16, and 3.17 of [12], the authors lose track of the standard assumption that a digital image X is a subset of $\mathbb{Z}^n$. In each of these examples, they write of an unspecified X using functions that clearly place X in $\mathbb{R}$, but not clearly in $\mathbb{Z}$.

4.6. $\alpha - \psi$ **expansive maps.** In the following, we let Ψ be the set of functions [14] $\psi : [0, \infty) \to [0, \infty)$ such that

- $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ ([12] misquotes this requirement as $\psi^n(t) < \infty$ for each $t > 0$), and
- ψ is non-decreasing.

Definition 4.7 ([12])**.** Let (X, d, κ) be a digital metric space. Let $T : X \to X$. T is a *digital* $\alpha - \psi$ *expansive mapping* if $\alpha : X \times X \to [0, \infty)$, $\psi \in \Psi$, and for all $x, y \in X$,
$$\psi(d(T(x), T(y))) \geq \alpha(x, y)d(x, y).$$

Definition 4.8 ([12])**.** Let $T : X \to X$. Let $\alpha : X \times X \to [0, \infty)$. T is α*-admissible* if $\alpha(x, y) \geq 1$ implies $\alpha(T(x), T(y)) \geq 1$

Theorem 4.9 ([12])**.** *Let (X, d, κ) be a complete digital metric space. Let $T : X \to X$ be a bijective, digital $\alpha - \psi$ expansion mapping such that*

- *T^{-1} is α-admissible;*

- *There exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$; and*
- *T is digitally continuous.*

Then T has a fixed point.

Despite the use of "digitally continuous" in the statement of Theorem 4.9, the continuity assumption used in its proof is of the $\varepsilon - \delta$ variety. In fact, the assumption is unnecessary if we assume additional common conditions, as in the following.

Theorem 4.10. *Let (X, d, κ) be a digital metric space, where X is finite or d is an ℓ_p metric. Let $T : X \to X$ be a bijective, digital $\alpha - \psi$ expansion mapping such that*

- *T^{-1} is α-admissible; and*
- *there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$.*

Then T has a fixed point.

Proof. Our argument borrows from its analog in [12].

By hypothesis, there exists $x_0 \in X$ such that $\alpha(x_0, T^{-1}(x_0)) \geq 1$. By induction, we obtain $S = \{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} = T^{-1}(x_n)$ for $n > 0$.

Since T^{-1} is α-admissible, by induction we have

$$\alpha(x_n, x_{n+1}) = \alpha(T^{-1}(x_{n-1}), T^{-1}(x_n)) \geq 1$$

for all n. Since T is a *digital $\alpha - \psi$ expansive mapping*, for all n we have

$$d(x_n, x_{n+1}) \leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) \leq \psi(d(T(x_n), T(x_{n+1})) = \psi(d(x_{n-1}, x_n)).$$

By induction, it follows that $d(x_n, x_{n+1}) \leq \psi^n(d(x_0, x_1))$. Since $\psi \in \Psi$, it follows that S is a Cauchy sequence. By Theorem 2.6, S is eventually constant, so there exists m such that $T(x_{m+1}) = x_m = x_{m+1}$; thus, x_{m+1} is a fixed point of T. $\square$

5. Weakly commuting mappings

The paper [11] presents a fixed point assertion for "weakly commuting mappings," defined as follows.

Definition 5.1 ([15])**.** Let (X, d) be a metric space and let $f, g : X \to X$. Then f and g are *weakly commuting* if for all $x \in X$, $d(f(g(x)), g(f(x))) \leq d(f(x), g(x))$.

Presented as Theorem 3(A) of [11] is the following.

> Let (X, d, κ) be a complete digital metric space, $X \neq \varnothing$. Let $S, T : X \to X$ such that
>
> (3.1) $T(X) \subset S(X)$;
>
> (3.2) S is κ-continuous;
>
> (3.3) For some α such that $0 < \alpha < 1$ and all $x, y \in X$, $d(T(x), T(y)) \leq \alpha d(S(x), S(y))$.
>
> If S and T are weakly commuting, then they have a unique common fixed point.

The argument given in proof of this assertion is flawed by the unjustified statement (rephrased slightly), "From (3.2) the κ-continuity of S implies the κ-continuity of T." This reasoning is incorrect, as shown in the following.

Example 5.2. Let $X = \{p_0 = (0,0,0), p_1 = (1,1,1), p_2 = (2,0,0)\} \subset \mathbb{Z}^3$. Let $S = 1_X : X \to X$ and let $T : X \to X$ be defined by $T(p_0) = T(p_2) = p_2$, $T(p_1) = p_0$. Let κ be the c_3-adjacency. Clearly, (3.1) and (3.2) of the above are satisfied. Let d be the ℓ_1 metric. Then (3.3) above is satisfied with $\alpha = 2/3$. However, T is not c_3-continuous, since $p_0 \leftrightarrow_{c_3} p_1$ but $f(p_0)$ and $f(p_1)$ are not c_3-adjacent.

Therefore, the assertion stated as Theorem 3(A) of [11] must be regarded as unproven. However, we see below that replacing the assumptions of completeness and (3.2) by assumptions that are commonly realized yields a valid statement.

Theorem 5.3. *Let (X, d, κ) be a digital metric space, $X \neq \varnothing$, with X finite or d an ℓ_p metric. Let $S, T : X \to X$ such that*

(1) *$T(X) \subset S(X)$;*
(2) *For some α such that $0 < \alpha < 1$ and all $x, y \in X$,*
$d(T(x), T(y)) \leq \alpha d(S(x), S(y))$.

If S and T are weakly commuting, then they have a unique common fixed point.

Proof. We use ideas from the analogous argument of [11].

Let $x_0 \in X$. By assumption 1, there exists $x_1 \in X$ such that $S(x_1) = T(x_0)$. By induction we have a sequence $\{x_n\}_{n=0}^{\infty}$ such that for all n, $S(x_{n+1}) = T(x_n)$, and we have

$$d(S(x_n), S(x_{n+1})) = d(T(x_{n-1}), T(x_n)) \leq \alpha d(S(x_{n-1}), S(x_n)).$$

By a simple induction, this yields

$$d(S(x_n), S(x_{n+1})) \leq \alpha^n d(S(x_0), S(x_1)).$$

Thus, $\{S(x_n)\}_{n=0}^{\infty}$ is a Cauchy sequence, hence by Proposition 2.6 is eventually constant, i.e., there exists $z \in X$ such that for sufficiently large n,

$$S(x_n) = z.$$

By our definition of the sequence $\{x_n\}$, we also have, for sufficiently large n,

$$T(x_n) = z.$$

So for n sufficiently large, and since S and T are weakly commuting,

$$d(S(z), T(z)) = d(S(T(x_n)), T(S(x_n))) \leq d(S(x_n), T(x_n)) = d(z, z) = 0,$$

i.e., $S(z) = T(z)$ and therefore, by the weakly commuting property,

$$d(S(T(z)), T(S(z))) \leq d(S(z), T(z)) = 0,$$

i.e., $S(T(z)) = T(S(z))$. So

$$d(T(z), T(T(z))) \leq \alpha d(S(z), S(T(z))) = \alpha d(T(z), T(S(z))) =$$
$$\alpha d(T(z), T(T(z))).$$

Thus $d(T(z), T(T(z))) = 0$, i.e., $T(z)$ is a fixed point of T. Further, substituting from the above gives

$$d(S(T(z)), T(z)) = d(T(S(z)), T(z)) \leq \alpha d(S(S(z)), S(z)) = \alpha d(S(T(z)), T(z));$$

since $\alpha > 0$, it follows that $d(S(T(z)), T(z)) = 0$. Thus, $T(z)$ is a common fixed point of S and T.

To show the common fixed point is unique, suppose y and y' are common fixed points, i.e.,

$$S(y) = T(y) = y, \quad S(y') = T(y') = y'.$$

Then

$$d(y, y') = d(T(y), T(y')) \leq \alpha d(S(y), S(y')) = \alpha d(y, y'),$$

so $d(y, y') = 0$. Hence $y = y'$. □

Note the following limitation on Theorem 5.3 is applicable if X is finite, or if d is an ℓ_p metric.

Proposition 5.4. *Let $(X, d, \kappa), S, T, \alpha$ be as in Theorem 5.3, where*

$$d_0 = \min\{d(x, x') \mid x, x' \in X, x \leftrightarrow_\kappa x'\} > 0,$$

$$d_1 = \max\{d(x, x') \mid x, x' \in X, x \leftrightarrow_\kappa x'\}.$$

If X is κ-connected, S is κ-continuous, and $0 < \alpha < d_0/d_1$, then T is a constant function.

Proof. Let $x \leftrightarrow_\kappa x'$. Since S is κ-continuous we have $S(x) \leftrightarrows_\kappa S(x')$, and therefore $d(S(x), S(x')) \leq d_1$. Thus,

$$d(T(x), T(x')) \leq \alpha d(S(x), S(x')) < \frac{d_0}{d_1} d_1 = d_0.$$

Our choice of d_0 implies $T(x) = T(x')$. Since X is connected, the assertion follows. □

6. Weakly compatible maps

The paper [7] discusses "weakly compatible" or "coincidentally commuting" maps, defined as follows.

Definition 6.1. Let $S, T : X \to X$. Then S and T are *weakly compatible* or *coincidentally commuting* if, for every $x \in X$ such that $S(x) = T(x)$ we have $S(T(x)) = T(S(x))$.

The following assertion is stated as Theorem 3.1 of [7].

> Let $A, B, S, T : X \to X$, where (X, d, κ) is a complete digital metric space. Suppose the following are satisfied.
> - $S(X) \subset B(X)$ and $T(X) \subset A(X)$.
> - The pairs (A, S) and (B, T) are coincidentally commuting.
> - One of $S(X), T(X), A(X), B(X)$ is a complete subspace of X.

- For all $x, y \in X$, $d(S(x), T(y)) \leq$

$$\phi(\max\{d(A(x), B(y)), d(S(x), A(x)), d(S(x), B(y)), d(B(y), T(y))\})$$

where $\phi : [0, \infty) \to [0, \infty)$ is continuous, monotone increasing, and satisfies $\phi(t) < t$ for all $t > 0$.

Then A, B, S, and T have a unique common fixed point.

However, the argument offered as proof of this assertion is flawed as follows. A sequence $\{y_n\}_{n=0}^{\infty}$ is established and it is shown that $\lim_{n\to\infty} d(y_{2n}, y_{2n+1}) = 0$. From this, it is claimed that $\{y_n\}_{n=0}^{\infty}$ is a Cauchy sequence. But such reasoning is incorrect, as shown in the following.

Example 6.2. For $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 0 & \text{if } (n \mod 4) \in \{0, 1\}; \\ 1 & \text{if } (n \mod 4) \in \{2, 3\}, \end{cases}$$

For all $n \in \mathbb{N}$, $d(y_{2n}, y_{2n+1}) = 0$, yet $\{y_n\}_{n=0}^{\infty}$ is not a Cauchy sequence.

Thus, the assertion of [7] stated as Theorem 3.1, and its dependent assertion stated as Theorem 3.2, must be regarded as unproven.

7. Further remarks

We have discussed assertions that appeared in [4, 7, 11, 12]. We have discussed errors or corrections for some, shown some to be limited or trivial, and offered improvements for others.

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