

$e_c$-Filters and $e_c$-ideals in the functionally countable subalgebra of $C^*(X)$

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ABSTRACT

The purpose of this article is to study and investigate $e_c$-filters on $X$ and $e_c$-ideals in $C^*_c(X)$ in which they are in fact the counterparts of $z_c$-filters on $X$ and $z_c$-ideals in $C_c(X)$ respectively. We show that the maximal ideals of $C^*_c(X)$ are in one-to-one correspondence with the $e_c$-ultrafilters on $X$. In addition, the sets of $e_c$-ultrafilters and $z_c$-ultrafilters are in one-to-one correspondence. It is also shown that the sets of maximal ideals of $C_c(X)$ and $C^*_c(X)$ have the same cardinality. As another application of the new concepts, we characterized maximal ideals of $C^*_c(X)$. Finally, we show that whether the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

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1. INTRODUCTION

All topological spaces are completely regular Hausdorff spaces and we shall assume that the reader is familiar with the terminology and basic results of [6]. Given a topological space $X$, we let $C(X)$ denote the ring of all real-valued continuous functions defined on $X$. $C_c(X)$ is the subalgebra of $C(X)$ consisting of functions with countable image and $C^*_c(X)$ is its subalgebra consisting of bounded functions. In fact, $C^*_c(X) = C_c(X) \cap C^*(X)$, where elements of $C^*(X)$
are bounded functions of $C(X)$. Recall that for $f \in C(X)$, $Z(f)$ denotes its zero-set:

$$Z(f) = \{x \in X : f(x) = 0\}.$$

The set-theoretic complement of a zero-set is known as a cozero-set and we denote this set by $\text{coz}(f)$. Let us put $Z_c(X) = \{Z(f) : f \in C_c(X)\}$ and $Z^*_c(X) = \{Z(g) : g \in C^*_c(X)\}$. These two latter sets are in fact equal, since $Z(f) = Z(1 + |f|)$, where $f \in C_c(X)$. A nonempty subfamily $\mathcal{F}$ of $Z_c(X)$ is called a $z_c$-filter if it is a filter on $X$. If $I$ is an ideal in $C_c(X)$ and $\mathcal{F}$ is a $z_c$-filter on $X$ then, we denote $Z_c[I] = \{Z(f) : f \in I\}$, $\cap Z_c[I] = \cap \{Z(f) : f \in I\}$ and $Z^{-1}_c[\mathcal{F}] = \{f : Z(f) \in \mathcal{F}\}$. We see that $Z_c[I]$ is a $z_c$-filter and $Z^{-1}_c[\mathcal{F}] \supseteq I$. If the equality holds, then $I$ is called a $z_c$-ideal. Moreover, $Z^{-1}_c[\mathcal{F}]$ is a $z_c$-ideal and we always have $Z_c[Z^{-1}_c[\mathcal{F}]] = \mathcal{F}$. So maximal ideals in $C_c(X)$ are $z_c$-ideals. In [5], a Hausdorff space $X$ is called countably completely regular (briefly, c-completely regular) if whenever $F$ is a closed subset of $X$ and $x \notin F$, there exists $f \in C_c(X)$ such that $f(x) = 0$ and $f(F) = 1$. In addition, two closed sets $A$ and $B$ of $X$ are also called countably separated (in brief, c-separated) if there exists $f \in C_c(X)$ with $f(A) = 0$ and $f(B) = 1$. c-completely regular and zero-dimensional spaces are the same, see Theorem 1.1.

If we let $M^c_p = \{f \in C_c(X) : f(p) = 0\} (p \in X)$, then the ring isomorphism

$$\frac{C_c(X)}{M^c_p} \cong \mathbb{R}$$

gives that $M^c_p$ is a maximal ideal, in fact, $M^c_p$ is a fixed maximal ideal. Moreover, $\cap Z_c[M^c_p] = \{p\}$.

Our concentration is on the zero-dimensional spaces since in [5] the authors proved that for any space $X$ there is a zero-dimensional Hausdorff space $Y$ such that $C_c(X)$ and $C_c(Y)$ are isomorphic as rings, see Theorem 1.2.

In section 2, we study and investigate the $e_c$-filters on $X$ and $e_c$-ideals in $C^*_c(X)$ which they are in fact the counterpart of [6, 2L]. We show that the maximal ideals of $C^*_c(X)$ are in one-to-one correspondence with the $e_c$-ultrafilters on $X$. Moreover, the sets of $e_c$-ultrafilters and $z_c$-ultrafilters are in one-to-one correspondence. By using the latter facts, it is shown that the sets of maximal ideals of $C_c(X)$ and $C^*_c(X)$ have the same cardinality. Finally, maximal ideals of $C^*_c(X)$ are characterized based on these concepts. In Section 3, our concentration is on the uniform norm topology on $C^*_c(X)$ which is the restriction of the uniform norm topology on $C^*(X)$. It is shown that whenever the space $X$ is compact, a proper ideal $I$ of $C_c(X)$ is an $e_c$-ideal if and only if it is a closed ideal in $C_c(X)$ if and only if it is an intersection of maximal ideals of $C_c(X)$.

We recite the following results from [5].

**Theorem 1.1** ([5, Proposition 4.4]). Let $X$ be a topological space. Then, $X$ is a zero-dimensional space (i.e., a $T_1$-space with a base consisting of clopen sets) if and only if $X$ is c-completely regular space.

**Theorem 1.2** ([5, Theorem 4.6]). Let $X$ be any topological space (not necessarily completely regular). Then, there is a zero-dimensional space $Y$ which is a continuous image of $X$ with $C_c(X) \cong C_c(Y)$ and $C^F(X) \cong C^F(Y)$.
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**Remark 1.3** ([5, Remark 7.5]). There is a topological space $X$, such that there is no space $Y$ with $C_c(X) \cong C(Y)$.

The following results are the known facts about $C_c(X)$ and we are seeking to get similar results for $C^*_c(X)$.

**Proposition 1.4.** Let $I$ be a proper ideal in $C_c(X)$ and $F$ a $z_c$-filter on $X$. Then:

(i) $Z_c[I]$ is a $z_c$-filter and $Z_c^{-1}[F]$ is a $z_c$-ideal of $C_c(X)$.

(ii) If $I$ is maximal then $Z_c[I]$ is a $z_c$-ultrafilter, and the converse holds if $I$ is a $z_c$-ideal.

(iii) $F$ is a $z_c$-ultrafilter if and only if $Z_c^{-1}[F]$ is a maximal ideal.

(iv) If $F$ is a $z_c$-ultrafilter and $Z \in Z_c(X)$ meets each element of $F$, then $Z \in F$.

**Corollary 1.5.** There is a one-to-one correspondence $\psi$ between the sets of $z_c$-ideals of $C_c(X)$ and $z_c$-filters on $X$, defined by $\psi(I) = Z_c[I]$. In particular, the restriction of $\psi$ to the set of maximal ideals is a one-to-one correspondence between the sets of maximal ideals of $C_c(X)$ and $z_c$-ultrafilters on $X$.

2. $e_c$-FILTERS ON $X$ AND $e_c$-IDEALS IN $C^*_c(X)$

For $f \in C^*_c(X)$ and $\epsilon > 0$, we define

$$E^*_c(f) = f^{-1}([-\epsilon, \epsilon]) = \{x \in X : |f(x)| \leq \epsilon\}.$$ 

Each such set is a zero set, since it is equal to $Z(\langle|f| - \epsilon\rangle \cup 0)$. Conversely, every zero set is also of this form, since for $g \in C^*_c(X)$ we have $Z(g) = E^*_c(|g| + \epsilon)$.

For a nonempty subset $I$ of $C^*_c(X)$ we denote $E^*_c[I] = \{E^*_c(f) : f \in I\}$, and $E_c(I) = \bigcup \{E^*_c[f] : f \in I\}$. Moreover, if $F$ is a nonempty subfamily of $Z_c(X)$, then we define $E^{-1}_c[F] = \{f \in C^*_c(X) : E^*_c(f) \in F\}$ and $E^{-1}_c(F) = \bigcap \{E^{-1}_{c}[f] : f \in F\}$. So we have $E_c(I) = \{E^*_c(f) : f \in I \text{ and } \epsilon > 0\}$, and $E^{-1}_c(F) = \{f \in C^*_c(X) : E^*_c(f) \in F \text{, for all } \epsilon\}$. Moreover, $E_c^{-1}(E_c(I)) = \{g \in C^*_c(X) : E^*_c(g) \in E_c(I), \text{for all } \delta > 0\}$ and $E_c(E_c^{-1}(F)) = \{E^*_c(f) : E^*_c(f) \in F, \text{for all } \delta > 0\}$.

The next result is now immediate.

**Corollary 2.1.** The following statements hold.

(i) $I \subseteq E^{-1}_c(E_c(I))$ and $E_c(E^{-1}_c(F)) \subseteq F$.

(ii) The mappings $E_c$ and $E^{-1}_c$ preserve the inclusion.

(iii) If $f \in I$ then for each positive integer $n$, $E^*_c(f) = E^*_c(fn)$.

(iv) If $I$ is an ideal, then $E_c(I)$ is a $z_c$-filter.

**Proof.** The proofs of (i), (ii) and (iii) are clear. (iv) This is presented in the proof of Proposition 2.5. $\square$

Examples 2.2 and 2.3 below show that the inclusions in (i) of the above corollary may be strict even when $I$ is an ideal and $F$ is a $z_c$-filter.
Example 2.2. Let $X$ be the discrete space $\mathbb{N} \times \mathbb{N}$, $f(m, n) = \frac{1}{mn}$ and $I$ the ideal in $C^*_c(X)(= C^*(X))$ generated by $f^2$. Obviously $f \notin I$. Since $\{x \in X : f(x) \leq \epsilon\} = \{x \in X : f^2(x) \leq \epsilon^2\}$, we have $E^*_c(f) = E_c(I)$. So $I \subseteq E^*_c(I)$.

Example 2.3. Let $X$ be the zero-dimensional space $\mathbb{Q} \times \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers, and $\mathcal{F} = \{Z \in Z_c(X) : (0,0) \in Z\}$. Then $\mathcal{F}$ is a $z_c$-filter on $X$. Now, if we define $f(x,y) = \frac{|x|+|y|}{1+|x|+|y|}$ then $f \in C^*_c(X)(= C^*(X))$ and $Z(f) = \{(0,0)\}$. Given $\epsilon > 0$ and $g = f + \epsilon$, we have $E^*_c(g) = \{(0,0)\}$. If we take $0 < \delta < \epsilon$ then $E^*_c(g) = \varnothing$. Hence $E^*_c(g)$ is not contained in $E_c(E^*_c(F))$. Therefore the latter set is contained in $\mathcal{F}$ properly, which gives the result.

Definition 2.4. A $z_c$-filter $\mathcal{F}$ is called an $e_c$-filter if $\mathcal{F} = E_c(E^{-1}_c(\mathcal{F}))$, or equivalently, whenever $Z \in \mathcal{F}$ then there exist $f \in C^*_c(X)$ and $\epsilon > 0$ such that $Z = E^*_c(f)$ and $E^*_c(g) \in \mathcal{F}$, for each $\delta > 0$.

Proposition 2.5. If $I$ is a proper ideal in $C^*_c(X)$, then $E_c(I)$ is an $e_c$-filter.

Proof. First, we show that $E_c(I)$ is a $z_c$-filter, i.e., it satisfies the following conditions.

(i) $\emptyset \notin E_c(I)$.

(ii) $E^*_c(f), E^*_c(g) \in E_c(I)$, then $E^*_c(f) \cap E^*_c(g) \in E_c(I)$.

(iii) $E^*_c(f) \in E_c(I)$, $Z \in Z_c(X)$ with $Z \supseteq E^*_c(f)$, then $Z \in E_c(I)$.

(i). Suppose that for some $\epsilon > 0$ and $f \in I$, $E^*_c(f) = \emptyset$. So $\emptyset < |f|$, which yields $f$ is a bounded away from zero. Hence $I$ contains the unit $f$, which is impossible. (ii). This is equivalent to say that if $E^*_c(f), E^*_c(g) \in E_c(I)$, then $E^*_c(f) \cap E^*_c(g) \subseteq E_c(I)$. Suppose that $f', g' \in I$ and $\epsilon', \delta' > 0$ such that $E^*_c(f) = E^*_c(f')$ and $E^*_c(g) = E^*_c(g')$. Without loss of generality, we may suppose that $\delta' < \epsilon' < 1$. Hence $f'^2 + g'^2 \in I$ and $E^*_c(f'^2 + g'^2) \subseteq E^*_c(f) \cap E^*_c(g)$, which gives the result. (iii). Assume that $E^*_c(f) \subseteq Z(f')$, where $f \in I$ and $f' \in C^*_c(X)$. Since $E^*_c(f') = E^*_c(f^2)$ and $Z(f') = Z(|f'|)$, we can suppose that $f \geq 0$ and $f' \geq 0$. Now, define

$$g(x) = \begin{cases} 1, & \text{if } x \in E^*_c(f) \\ (f' + \frac{\epsilon}{|x|})^2, & \text{if } x \in X \setminus \text{int}E^*_c(f). \end{cases}$$

So $g$ is continuous, since it is continuous on two closed sets whose union is $X$, in fact, $g \in C^*_c(X)$. Note that $fg \in I$ and

$$(fg)(x) = \begin{cases} f(x), & \text{if } x \in E^*_c(f) \\ (ff') + \epsilon, & \text{if } x \in X \setminus \text{int}E^*_c(f). \end{cases}$$

It is easily seen that $Z(f') = E^*_c(fg)$. So $Z(f') \in E_c(I)$. This shows that $E_c(I)$ is a $z_c$-filter. Now, apply (i) and (ii) of Corollary 2.1 for the ideal $I$ and the $z_c$-filter $E_c(I)$, to get the inclusions $E_c(I) \subseteq E_c(E^{-1}_c(E_c(I)))$ and $E_c(E^{-1}_c(E_c(I))) \subseteq E_c(I)$, which yields $E_c(I)$ is an $e_c$-filter. \hfill \Box

Definition 2.6. An ideal $I$ in $C^*_c(X)$ is called $e_c$-ideal if $I = E^{-1}_c(E_c(I))$, or equivalently, if $f \in C^*_c(X)$ and $E^*_c(f) \in E_c(I)$ for all $\epsilon$, then $f \in I$.  

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Corollary 2.8. Let \( E_\epsilon \) be an \( e_\epsilon \)-ideal containing \( \emptyset \) and \( \epsilon > 0 \). Then \( E_\epsilon^{-1}(f) \) and \( E_\epsilon^{-1}(g) \) belong to \( F \). Hence \( E_\epsilon^{-1}(f) \cap E_\epsilon^{-1}(g) \) is an ideal in \( F \).\( \square \)

Proof. Let \( f, g \in C_\epsilon^{-1}(X) \), \( h \in C_\epsilon(X) \) and let \( u \) be an upper bound for \( h \) and \( \epsilon > 0 \). Then \( E_\epsilon^{-1}(f) \) and \( E_\epsilon^{-1}(g) \) belong to \( F \). Hence \( E_\epsilon^{-1}(f) \cap E_\epsilon^{-1}(g) \) implies that \( E_\epsilon^{-1}(f + g) \) is an ideal in \( F \), or equivalently, \( f + g \in E_\epsilon^{-1}(1) \). Moreover, \( E_\epsilon^{-1}(f) \cap E_\epsilon^{-1}(g) \) implies \( f \) and \( g \) are \( e_\epsilon \)-ideals.

Corollary 2.9. The correspondence \( I \mapsto E_\epsilon(I) \) is one-one from the set of \( e_\epsilon \)-ideals in \( C_\epsilon(X) \) onto the set of \( e_\epsilon \)-ideal filters on \( X \).

Proof. Let \( M \) be a maximal ideal of \( C_\epsilon(X) \). If \( E_\epsilon^{-1}(E_\epsilon(M)) \) is not a proper \( e_\epsilon \)-ideal, then it contains the constant function 1 and \( E_\epsilon^{-1}(1) = \emptyset \in E_\epsilon(M) \) \((0 < \epsilon < 1)\) which is impossible, see Propositions 2.5 and 2.7. Hence \( M = E_\epsilon^{-1}(E_\epsilon(M)) \), i.e., \( M \) is an \( e_\epsilon \)-ideal. The second part is obtained by this fact, the fact that the intersection of a family of maximal ideals is an ideal contained in each of them and (ii) of Corollary 2.1.\( \square \)

The next corollary is an immediate result of Propositions 2.5 and 2.7.

Corollary 2.10. Let \( I \) and \( J \) be ideals in \( C_\epsilon(X) \) and \( \epsilon > 0 \) an \( e_\epsilon \)-ideal. Then \( I \subseteq J \) if and only if \( E_\epsilon(I) \subseteq E_\epsilon(J) \).

Lemma 2.11. (i) Let \( F_1 \) and \( F_2 \) be \( e_\epsilon \)-filters on \( X \) and \( F \) an \( e_\epsilon \)-filter. Then \( F_1 \subseteq F_2 \) if and only if \( E_\epsilon^{-1}(F) \subseteq E_\epsilon^{-1}(F_2) \).

Proof. It is straightforward.\( \square \)

Proposition 2.11. Let \( I \) be an ideal in \( C_\epsilon(X) \) and \( \epsilon > 0 \) an \( e_\epsilon \)-filter on \( X \). Then:

(i) \( E_\epsilon^{-1}(E_\epsilon(I)) \) is the smallest \( e_\epsilon \)-ideal containing \( I \).

(ii) \( E_\epsilon(E_\epsilon^{-1}(F)) \) is the largest \( e_\epsilon \)-filter contained in \( F \).

Proof. (i). Propositions 2.5 and 2.7 respectively show that \( E_\epsilon(I) \) is an \( e_\epsilon \)-filter and \( E_\epsilon^{-1}(E_\epsilon(I)) \) is an \( e_\epsilon \)-ideal. Now, suppose that \( K \) is an \( e_\epsilon \)-ideal containing \( I \). So \( E_\epsilon^{-1}(E_\epsilon(I)) \subseteq E_\epsilon^{-1}(E_\epsilon(K)) \). Hence we are done. (ii). This is proved similarly.\( \square \)

The next theorem plays an important role in many of the following results.

Theorem 2.12. Let \( \mathcal{A} \) be a \( z_\epsilon \)-ultrafilter. Then a zero set \( Z \) meets every element of \( E_\epsilon(E_\epsilon^{-1}(\mathcal{A})) \) if and only if \( Z \notin \mathcal{A} \).

Proof. Since \( \mathcal{A} \) is a filter and \( E_\epsilon(E_\epsilon^{-1}(\mathcal{A})) \subseteq \mathcal{A} \), the sufficient condition is evident. For the necessary condition, it is recalled at first that if \( Z \) meets every element of \( \mathcal{A} \) then \( Z \notin \mathcal{A} \), see (iv) of Proposition 1.4. Now, we claim that if \( Z \) meets every element of \( E_\epsilon(E_\epsilon^{-1}(\mathcal{A})) \) as a particular subfamily of \( \mathcal{A} \), then also \( Z \notin \mathcal{A} \). Otherwise, for some \( Z' \notin \mathcal{A} \), \( Z \cap Z' = \emptyset \). Since the closed sets \( Z \) and
$Z'$ are $c$-completely separated, there is $f \in C_c^*(X)$ (in fact $0 \leq f \leq 1$) such that $f(Z) = 1$ and $f(Z') = 0$. Notice that $Z' \subseteq Z(f) \subseteq E_c^c(f)$, for all $c$, and; $E_c^c(f) \in A$, since $Z' \in A$. So $E_c^c(f) \in E_c(E_c^c(A))$. Now, if $c$ is taken less than 1, then $Z \cap E_c^c(f) = \emptyset$ which contradicts with our assumption of $Z$. So $Z \in A$ and the proof is complete. 

The following proposition shows that, as $Z_c^{-1}(A)$ is a maximal ideal in $C_c(X)$, $E_c^{-1}(A)$ is also a maximal ideal in $C_c^*(X)$, where $A$ is a $z_c$-ultrafilter on $X$.

**Proposition 2.13.** Let $A$ be a $z_c$-ultrafilter on $X$. Then:

(i) $E_c^{-1}(A)$ is a maximal ideal.

(ii) $E_c^{-1}(A)$ is an $e_c$-ideal.

(iii) $E_c^{-1}(A) = E_c^{-1}(E_c(E_c^{-1}(A)))$.

**Proof.** (i). Let $M$ be a maximal ideal of $C_c^*(X)$ containing $E^{-1}_c(A)$. Hence $E_c(E_c^{-1}(A)) \subseteq E_c(M)$. Since every element of $E_c(M)$ meets every element of $E_c(E_c^{-1}(A))$, Theorem 2.12 gives $E_c(M) \subseteq A$. So $M = E_c^{-1}(E_c(M)) \subseteq E_c^{-1}(A)$ and hence $M = E_c^{-1}(A)$. (ii). It follows by (i). (iii). Since the maximal ideal $E_c^{-1}(A)$ is contained in the proper ideal $E_c^{-1}(E_c(E_c^{-1}(A)))$, the result now holds. □

An $e_c$-ultrafilter on $X$ is meant a maximal $e_c$-filter, i.e., one not contained in any other $e_c$-filter. As usual, every $e_c$-filter $F$ is contained in an $e_c$-ultrafilter. This is obtained by considering the collection of all $e_c$-filters containing $F$ and the use of the Zorn’s lemma, where the partially ordered relation on $F$ is inclusion.

**Proposition 2.14.** Let $M$ be an ideal in $C_c^*(X)$ and $F$ a $z_c$-filter on $X$. Then:

(i) If $M$ is a maximal ideal then $E_c(M)$ is an $e_c$-ultrafilter.

(ii) If $F$ is an $e_c$-ultrafilter then $E_c^{-1}(F)$ is a maximal ideal.

(iii) If $M$ is an $e_c$-ideal, then $M$ is maximal if and only if $E_c(M)$ is an $e_c$-ultrafilter.

(iv) If $F$ is an $e_c$-filter, then $F$ is an ultrafilter if and only if $E_c^{-1}(F)$ is a maximal ideal.

**Proof.** (i). Note that $M = E_c^{-1}(E_c(M))$. Let $F'$ be an $e_c$-ultrafilter containing $E_c(M)$, then $M \subseteq E_c^{-1}(F')$ and hence $M = E_c^{-1}(F')$. Therefore $E_c(M) = E_c(E_c^{-1}(F')) = F'$, which yields the result. (ii). Let $M$ be a maximal ideal of $C_c^*(X)$ containing $E_c^{-1}(F)$. Then $F \subseteq E_c(M)$. Hence $F = E_c(M)$ and so $E_c^{-1}(F) = M$. The proofs of (iii) and (iv) are similarly done and further details are omitted. □

**Corollary 2.15.** There is a one-to-one correspondence $\psi$ between the sets of maximal ideals of $C_c^*(X)$ and $e_c$-ultrafilters on $X$, defined by $\psi(M) = E_c(M)$.

**Proposition 2.16.** Let $A$ be a $z_c$-ultrafilter. Then it is the unique $z_c$-ultrafilter containing $E_c(E_c^{-1}(A))$, and also $E_c(E_c^{-1}(A))$ is the unique $e_c$-ultrafilter contained in $A$. Hence every $e_c$-ultrafilter is contained in unique $z_c$-ultrafilter.
Proof. Let \( \mathcal{B} \) be a \( z_c \)-ultrafilter containing \( E_c(E_c^{-1}(A)) \) and \( Z \in \mathcal{B} \). Since \( Z \) meets every element of \( E_c(E_c^{-1}(A)) \), Theorem 2.12 gives \( \mathcal{B} \subseteq \mathcal{A} \) and hence \( \mathcal{B} = \mathcal{A} \). So the first part of the proposition holds. Now, let \( \mathcal{K} \) be an \( e_c \)-ultrafilter contained in \( \mathcal{A} \). Then \( \mathcal{K} = E_c(E_c^{-1}(\mathcal{K})) \subseteq E_c(E_c^{-1}(A)) \). Since the latter set is an \( e_c \)-filter, the inclusion cannot be proper, i.e., \( \mathcal{K} = E_c(E_c^{-1}(A)) \). Hence the result is obtained. \( \square \)

Corollary 2.17. The \( z_c \)-ultrafilters are in one-to-one correspondence with the \( e_c \)-ultrafilters.

Proof. Consider the mapping \( \psi \) from the set of \( z_c \)-ultrafilters into the set of \( e_c \)-ultrafilters defined by \( \psi(A) = E_c(E_c^{-1}(A)) \). If \( \psi(A) = \psi(B) \), then we have that \( E_c(E_c^{-1}(A)) = E_c(E_c^{-1}(B)) \) and it is contained in both \( \mathcal{A} \) and \( \mathcal{B} \). So each element of \( \mathcal{B} \) meets each element of \( E_c(E_c^{-1}(A)) \). Now, Theorem 2.12 gives \( \mathcal{B} \subseteq \mathcal{A} \). Similarly, \( \mathcal{A} \subseteq \mathcal{B} \). Therefore \( \psi \) is one-one. Let \( \mathcal{K} \) be an \( e_c \)-ultrafilter and \( \mathcal{A} \) the unique \( z_c \)-ultrafilter containing it (Proposition 2.16). Then \( \mathcal{K} = E_c(E_c^{-1}(A)) \) and hence \( \psi(A) = \mathcal{K} \). Therefore \( \psi \) is onto. \( \square \)

Our next two theorems are applications that are based on the concepts of \( e_c \)-filters and \( e_c \)-ideals. In the first result (Theorem 2.18) we show that the maximal ideals of \( C_c(X) \) are in one-to-one correspondence with those ones of \( C^*_c(X) \) and the second result (Theorem 2.20) involves characterization of maximal ideals of \( C^*_c(X) \).

Theorem 2.18. Let \( \mathcal{M} \) (resp. \( \mathcal{M}^* \)) be the set of maximal ideals of \( C_c(X) \) (resp. \( C^*_c(X) \)). Then \( \mathcal{M} \) and \( \mathcal{M}^* \) have the same cardinality.

Proof. If \( M \in \mathcal{M} \) then \( Z_c[M] \) is a \( z_c \)-ultrafilter and hence \( E_c^{-1}(Z_c[M]) \in \mathcal{M}^* \), see Propositions 1.4 and 2.13. Define

\[ \varphi : \mathcal{M} \to \mathcal{M}^* \text{ which } M \mapsto E_c^{-1}(Z_c[M]). \]

If \( \varphi(M) = \varphi(M') \) then \( E_c(E_c^{-1}(Z_c[M])) = E_c(E_c^{-1}(Z_c[M'])) \) and it is contained in both \( Z_c[M] \) and \( Z_c[M'] \). Since each element of \( Z_c[M'] \) meets each element of \( E_c(E_c^{-1}(Z_c[M])) \), Theorem 2.12 yields \( Z_c[M'] \subseteq Z_c[M] \). Similarly, \( Z_c[M] \subseteq Z_c[M'] \). Therefore \( M = M' \). This verifies \( \varphi \) is one-one. To show that \( \varphi \) is onto, suppose that \( M^* \in \mathcal{M}^* \). Hence \( E_c(M^*) \) is an \( e_c \)-ultrafilter (Proposition 2.14). Now, let \( \mathcal{A} \) be the unique \( z_c \)-ultrafilter containing \( E_c(M^*) \) (Proposition 2.16), then \( Z_c^{-1}[A] \) is a maximal ideal in \( C_c(X) \) (Proposition 1.4) and \( M^* = E_c^{-1}(A) \). Recall that if \( F \) is a \( z_c \)-filter, then we always have \( Z_c[Z_c^{-1}[F]] = F \) and \( E_c(E_c^{-1}(F)) \subseteq F \), but the equality occurs if \( F \) is an \( e_c \)-filter. Now, if we let \( M = Z_c^{-1}[A] \) then \( \varphi(M) = E_c^{-1}(Z_c[M]) = E_c^{-1}(A) = M^* \). Hence \( \varphi \) is onto, which it completes the proof. \( \square \)

Remark 2.19. Combining Corollaries 1.5, 2.15 and 2.17 gives another proof of the above theorem.

Theorem 2.20. Let \( M \) be an ideal in \( C^*_c(X) \). Then \( M \) is maximal if and only if whenever \( f \in C^*_c(X) \) and each \( E_c(f) \) meets every element of \( E_c(M) \), then \( f \in M \).
Proof. Necessity: Suppose that \( f \notin M \). So \((M, f) = C_c^*(X)\). Hence \( h + fg = 1 \), for some \( h \in M \) and \( g \in C_c^*(X) \). Let \( u \) be an upper bound for \( g \) and \( 0 < \epsilon < 1 \). Then
\[
\emptyset = E_{\epsilon}^c(1) \supseteq E_{\epsilon}^c(h + fg) \supseteq E_{\epsilon}^c(h) \cap E_{\epsilon}^c(fg) \supseteq E_{\epsilon}^c(h) \cap E_{\epsilon}^c(f),
\]
which contradicts with our assumption, since \( E_{\epsilon}^c(h) \cap E_{\epsilon}^c(f) \neq \emptyset \). So we are done.

Sufficiency: Let \( M' \) be a maximal ideal of \( C_c^*(X) \) containing \( M \) and \( f \in M' \). Then \( E_c(M) \subseteq E_c(M') \) and \( E_c^c(f) \subseteq E_c(M') \), for all \( \epsilon \). Since \( E_c(M') \) is an \( e_c \)-filter, \( E_c^c(f) \) meets every element of \( E_c(M') \). Hence it also meets each element of \( E_c(M) \). Now, by hypothesis \( f \in M \). Therefore \( M = M' \), which gives the result. \( \square \)

3. **Uniform norm topology on \( C_c^*(X) \) and related closed ideals**

Consider the supremum-norm on \( C_c^*(X) \), i.e., \( \|f\| = \sup_{x \in X} |f(x)| \), where \( f \in C_c^*(X) \). So its restriction on \( C_c^*(X) \) is also the supremum-norm. This defines a metric \( d \) as usual, \( d(f, g) = \|f - g\| \). The resulting metric topology is called the uniform norm topology on \( C_c^*(X) \). Convergence in this topology is uniform convergence of the functions. A base for the neighborhood system at \( g \) consists of all sets of the form
\[
\{ f : \|f - g\| \leq \epsilon \} \quad (\epsilon > 0).
\]
Equivalently, a base at \( g \) is given by all sets
\[
\{ f : |f(x) - g(x)| \leq u(x) \text{ for every } x \in X \},
\]
where \( u \) is a positive unit of \( C_c^*(X) \).

If \( I \) is an ideal in \( C_c^*(X) \) then its closure in \( C_c^*(X) \) is denoted by \( cl(I) \).

**Proposition 3.1.** Let \( I \) be an ideal in \( C_c^*(X) \). Then:

(i) \( clI \) is ideal.

(ii) If \( I \) is a proper ideal then \( clI \) is also a proper ideal.

(iii) If \( I \) is an \( e_c \)-ideal then it is a closed ideal.

Proof. (i). Let \( f, g \in clI, h \in C_c^*(X) \) and let \( u \) be an upper bound for \( h \) and \( \epsilon > 0 \) is fixed. Then for some \( f' \in N_{\epsilon/2}(f) \cap I \) and \( g' \in N_{\epsilon/2}(g) \cap I \) we have \( f' + g' \in N_{\epsilon/2}(f + g) \cap I \). Moreover, there exists \( f_1 \in N_{\epsilon/2}(f) \cap I \) and hence \( f_1h \in N_{\epsilon/2}(fh) \cap I \). So \( clI \) contains \( f + g \) and \( fh \). Hence \( clI \) is ideal. (ii). If \( clI \) is not a proper ideal then \( \exists 1 \in clI \) and hence \( N\epsilon(1) \cap I \) contains a unit element of \( C_c^*(X) \) such as \( f \), since \( 1 - \epsilon < f < 1 + \epsilon \) gives \( f \) is bounded away from zero (of course, when \( 0 < \epsilon < 1 \)). But this is impossible since \( f \in I \). Thus \( clI \) is a proper ideal.

(iii). Let \( g \in clI \) and \( \epsilon > 0 \) arbitrary. Then for some \( f \in N_{\epsilon/2}(g) \cap I \) and all \( x \in E_{\epsilon/2}^c(f) \), we have
\[
|g(x)| = |g(x) - f(x) + f(x)| \leq |g(x) - f(x)| + |f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]
Hence $E^c_\epsilon(f) \subseteq E^c_\epsilon(g)$. Since the $z_\epsilon$-filter $E_\epsilon(I)$ contains $E^c_\epsilon(f)$, it also contains $E^c_\epsilon(g)$, for all $\epsilon$. So $g \in E^{-1}_c(E_\epsilon(I)) = I$ and therefore $cI \subseteq I$. This proves that $I$ is closed and hence the proof is complete. \hfill \square

Immediately, we find there is no proper dense ideal in $C^*_c(X)$, and further maximal ideals of $C^*_c(X)$ and hence every intersection of them are closed, see Corollary 2.8 and (iii) of the above proposition.

We recall that $[6, 1D(1)]$ plays a useful role in the context of $C(X)$. The following is the counterpart for $C^*_c(X)$.

Proposition 3.2. If $f, g \in C_c(X)$ and $Z(f)$ is a neighborhood of $Z(g)$, then $f = gh$ for some $h \in C_c(X)$.

In the remainder of this section, the zero-dimensional topological space $X$ will be assumed to be compact. Hence it is $c$-pseudo-compact, i.e, $C_c(X) = C^*_c(X)$.

Lemma 3.3. Let $X$ be a compact space, $I$ an ideal in $C_c(X)$, $f \in C_c(X)$ and $Z(f)$ a neighborhood of $\cap Z_c[I]$. Then $f \in I$.

Proof. First, we recall that $X$ is compact if and only if the intersection of members of any collection consisting of nonempty closed subsets of $X$ with the finite intersection property (i.e., the intersection of each of a finite number of them is nonempty) is nonempty. The lemma is obvious when $I = C_c(X)$. Now, if $I$ is a proper ideal in $C_c(X)$ then $Z_c[I]$ satisfies the finite intersection property and hence $\cap Z_c[I] \neq \emptyset$. By assumption $\cap Z_c[I] \subseteq \text{int} Z(f)$. Hence $X \setminus \text{int} Z(f) \subseteq \bigcup_{g \in I} \text{coz}(g)$ and so $X = \bigcup_{g \in I} \text{coz}(g) \cup \text{int} Z(f)$. By compactness of $X$, there are a finite number of elements of $I$, say $g_1, g_2, \ldots, g_n$, such that

$$X = \bigcup_{i=1}^n \text{coz}(g_i) \cup \text{int} Z(f).$$

Now, if we let $g = \sum_{i=1}^n g_i^2$ then $g \in I$ and $\emptyset \neq Z(g) = \bigcap_{i=1}^n Z(g_i) \subseteq \text{int} Z(f)$. So the proof is complete. \hfill \square

Proposition 3.4. If $g \in C_c(X)$ and $\epsilon > 0$ is fixed, then there exists $f \in C_c(X)$ such that $\|g - f\| \leq \epsilon$ and $Z(f)$ is a neighborhood of $Z(g)$.

Proof. The trivial solution is $f = g$, of course when $Z(g)$ is open. In general, it suffices to define

$$f(x) = \begin{cases} 
  g(x) - \epsilon, & \text{if } x \in g^{-1}([\epsilon, +\infty)) \\
  0, & \text{if } x \in E^c_\epsilon(g) \\
  g(x) + \epsilon, & \text{if } x \in g^{-1}((-\infty, -\epsilon]).
\end{cases}$$

We note that $X$ is the union of three closed sets $g^{-1}([\epsilon, +\infty))$, $E^c_\epsilon(g)$ and $g^{-1}((-\infty, -\epsilon])$ and further $f$ is continuous on each of them. Therefore $f$ is continuous on $X$, i.e., $f \in C_c(X)$. Notice that the definition of $f$ makes the
cardinality of the range of \( f \) the same cardinality of the range of \( g \). Hence this leads us \( f \in C_c(X) \). Moreover, \( \|g - f\| \leq \epsilon \). Evidently, \( Z(g) \subseteq g^{-1}((-\epsilon, \epsilon)) \subseteq \text{int}Z(f) \) which yields \( Z(f) \) is a neighborhood of \( Z(g) \).

**Theorem 3.5.** Let \( I \) be a proper ideal in \( C_c(X) \), \( \mathcal{T} = \cap \{M_p^c : M_p^c \supseteq I\} \) and \( J = \{g \in C_c(X) : Z(g) \supseteq \cap Z_c[I]\} \). Then:

(i) \( \mathcal{T} = J \).

(ii) \( \cap Z_c[I] = \cap Z_c[\mathcal{T}] \).

*Proof.* (i). Let \( g \in J \) and \( M_p^c \) be a fixed maximal ideal of \( C_c(X) \) containing \( I \). Then \( Z(g) \supseteq \cap Z_c[I] \supseteq \cap Z_c[M_p^c] = \{p\} \). So \( g(p) = 0 \) and hence \( g \in M_p^c \). Therefore \( g \in \mathcal{T} \). For the reverse inclusion, we show that if \( g \notin J \) then \( g \notin \mathcal{T} \).

If \( g \notin J \) then there exists \( x \in \cap Z_c[I] \setminus Z(g) \). So \( I \subseteq M_x^c \) but \( g \notin M_x^c \). This means that \( g \notin \mathcal{T} \). The proof of (i) is now complete.

(ii). By (i), we have \( \cap Z_c[\mathcal{T}] = \cap Z_c[J] \supseteq \cap Z_c[I] \). On the other hand, \( I \subseteq \mathcal{T} \) implies \( Z_c[I] \subseteq Z_c[\mathcal{T}] \) and therefore \( \cap Z_c[I] \supseteq \cap Z_c[\mathcal{T}] \). So it gives the result.

**Corollary 3.6.** Let \( I \) be a proper ideal in \( C_c(X) \) and \( \mathcal{T} \) as defined in Theorem 3.5. Then \( \mathcal{T} = \text{cl}I \).

*Proof.* Since maximal ideals are closed, \( \cap_{I \subseteq M} M \) is also closed, where \( M \) is a maximal ideal in \( C_c(X) \). Therefore \( \text{cl}I \subseteq \cap_{I \subseteq M} M \subseteq \mathcal{T} \). Let \( g \in \mathcal{T} \) and \( N_c(g) \) is a neighborhood of \( g \). By Proposition 3.4, there is \( f \) such that \( Z(f) \) is a neighborhood of \( Z(g) \) and \( \|g - f\| \leq \epsilon \). Hence, by Theorem 3.5, \( \cap Z_c[I] \subseteq Z_c[I] \subseteq \text{int}Z(f) \) and therefore Lemma 3.3 implies \( f \in I \). Now, since \( f \in N_c(g) \cap I \), it gives \( g \in \text{cl}I \) and we are done.

We conclude the article with the following results for the proper ideals of \( C_c(X) \). Corollary 3.7 is a consequence of Corollary 2.8, Proposition 3.1 (iii) and Corollary 3.6; by the same results, plus Corollary 3.7 we obtain Corollary 3.8; finally Corollary 3.9 is the combination of Corollaries 3.7 and 3.8.

**Corollary 3.7.** An ideal \( I \) of \( C_c(X) \) is closed in \( C_c(X) \) if and only if it is an intersection of maximal ideals of \( C_c(X) \).

**Corollary 3.8.** An ideal \( I \) of \( C_c(X) \) is an \( e_c \)-ideal if and only if it is closed in \( C_c(X) \).

**Corollary 3.9.** An ideal \( I \) of \( C_c(X) \) is an \( e_c \)-ideal if and only if it is an intersection of maximal ideals of \( C_c(X) \).

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