LOCAL COMPACTNESS IN RIGHT BOUNDED ASYMMETRIC NORMED SPACES

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Abstract. We characterize the finite dimensional asymmetric normed spaces which are right bounded and the relation of this property with the natural compactness properties of the unit ball, such as compactness and strong compactness. In contrast with some results found in the existing literature, we show that not all right bounded asymmetric norms have compact closed balls. We also prove that there are finite dimensional asymmetric normed spaces that satisfy that the closed unit ball is compact, but not strongly compact, closing in this way an open question on the topology of finite dimensional asymmetric normed spaces. In the positive direction, we will prove that a finite dimensional asymmetric normed space is strongly locally compact if and only if it is right bounded.

1. Introduction

It is well known that a normed vector space is locally compact if and only if it is finite dimensional. However, in the asymmetric case this is no longer true. In the context of asymmetric normed spaces, this matter is related to the relevant notion of right boundedness that has been widely used (see e.g. [2, 9]; see Section §2.1 for the definition). Right bounded asymmetric normed spaces were introduced in [14, Definition 16]. In that same paper it was stated that the closed unit ball of a right bounded asymmetric normed space is always compact ([14, Proposition 17]). However, we will show in Example 4.1 that —except if the constant r in the definition of right boundedness is 1— this result is not true (the proof given in [14, Proposition 17] is correct for $r = 1$). Therefore, it is natural to ask if it is possible to give a characterization of right boundedness for finite dimensional asymmetric normed spaces in terms of a weaker compactness-type property: local compactness. Recall that a topological space X is locally compact iff every point $x \in X$ has a local base consisting of compact neighborhoods.

The aim of this work is to solve that problem. Another relevant compactness property that can also be found in the literature —strong compactness–,

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and its local version will be considered. As part of our main result, we will prove that right boundedness and strong local compactness are equivalent notions in the class of asymmetric normed spaces. However, we will also show that there are finite dimensional asymmetric normed spaces of dimension 3 such that the closed unit ball is compact and right bounded, but not strongly compact (see Example 5.1). We will also prove other equivalences related with the geometry and the topology of the asymmetric closed unit balls.

2. Preliminaries

Consider a real linear space X and let $\mathbb{R}^+$ be the set of non-negative real numbers. An *asymmetric norm* q on X is a function $q : X \to \mathbb{R}^+$ such that

(1) $q(ax) = aq(x)$ for every $x \in X$ and $a \in \mathbb{R}^+$,
(2) $q(x + y) \leq q(x) + q(y)$, $x, y \in X$, and
(3) for every $x \in X$, if $q(x) = q(-x) = 0$, then $x = 0$.

The pair (X, q) is called an *asymmetric normed linear space*. An asymmetric norm defines a non-symmetric topology on X that is given by the open balls $B^q_\varepsilon(x) := \{y \in X : q(y-x) < \epsilon\}$. This topology is denoted by τ_q.

We will denote by θ_q the set of all $x \in X$ such that $q(x) = 0$. The set θ_q is a convex cone; this means that $\mu x \in \theta_q$ and $x + y \in \theta_q$ for every $x, y \in \theta_q$ and $\mu \geq 0$. This set plays a fundamental role in many topological, geometric and analytic results about asymmetric normed linear spaces. In particular, (X, q) is T_1 if and only if $\theta_q = \{0\}$.

The following are well known results concerning the set θ_q (see [14] for the proofs) that will be needed in the paper.

(O1) For any open set $U \subset X$, we always have that $U = U + \theta_q$.
(O2) $K \subset X$ is compact if and only if $K + \theta_q$ is compact.

For every asymmetric normed space (X, q) the map $q^s : X \to \mathbb{R}^+$ defined by the rule

$$q^s(x) := \max\{q(x), q(-x)\}, \quad x \in X,$$

is a norm that generates a topology stronger than the one generated by q. We will use the symbols B^q_ε and $B^{q^s}_\varepsilon$ to distinguish the balls of (X, q) and (X, q^s), respectively. More precisely, for every $x \in X$ and $\varepsilon > 0$ we will use the following notations

$$B^q_\varepsilon(x) = \{y \in X \mid q(y - x) < \varepsilon\},$$
$$B^q_\varepsilon[x] = \{y \in X \mid q(y - x) \leq \varepsilon\},$$
$$B^{q^s}_\varepsilon(x) = \{y \in X \mid q^s(y - x) < \varepsilon\},$$
$$B^{q^s}_\varepsilon[x] = \{y \in X \mid q^s(y - x) \leq \varepsilon\}.$$

The set $B^q_\varepsilon[x]$ is called the *closed ball* of radius ε centered at x. However, in general this set is not closed with respect to τ_q.

In order to avoid confusion, when necessary, we will say that a set is *q-compact* (q^s-compact) if it is compact in the topology generated by q (q^s). We will use the expressions *q-open* and *q-closed* sets (*q^s-open* and *q^s-closed* sets) in the same way.

A subset $K \subset X$ in an asymmetric normed space (X, q) is *strongly q-compact* (or simply, *strongly compact*) if and only if there exists a q^s-compact set $S \subset X$ such that

$$S \subset K \subset S + \theta_q.$$

Every strongly compact set is q-compact, but the converse is not true (see e.g. [2, 8, 15]). If each point of the asymmetric normed space X has a local base consisting of strongly compact sets we will say that X is *strongly locally compact*. Evidently, if X is strongly locally compact then it is also locally compact.

Since $q(x) \leq q^s(x)$ for every $x \in X$, we always have the following inclusions

$$B_\varepsilon^{q^s}(x) + \theta_q \subset B_\varepsilon^q(x) \quad \text{and} \quad B_\varepsilon^{q^s}[x] + \theta_q \subset B_\varepsilon^q[x]. \tag{1}$$

Addition in asymmetric normed spaces is always continuous but scalar multiplication is not (see e.g. [7]). However it is well-known that multiplication by non negative scalars is continuous. For the convenience of the reader, we include the proof of this result that will be used often in the paper.

Lemma 2.1. *Let (X, q) be an asymmetric normed space. The map $\mathbb{R}^+ \times X \to X$ given by $(\mu, x) \to \mu x$ is continuous if we endow $\mathbb{R}^+$ with the Euclidean topology.*

Proof. Let $\varepsilon > 0$ and $(\mu, x) \in \mathbb{R}^+ \times X$. Then there exists $\delta_1 > 0$ such that $\delta_1 q^s(x) < \varepsilon/2$. Since $\mu \geq 0$, if we define $\delta_2 := \varepsilon/2(\mu + \delta_1)$ we have that $\delta_2 > 0$. Now take $\lambda \geq 0$ such that $|\lambda - \mu| < \delta_1$ and $y \in B_{\delta_2}^q(x)$. Thus

$$\begin{aligned} q(\lambda y - \mu x) &= q(\lambda y - \lambda x + \lambda x - \mu x) \\ &\leq q(\lambda y - \lambda x) + q(\lambda x - \mu x) \\ &\leq \lambda q(y - x) + q^s(\lambda x - \mu x) \\ &< (\mu + \delta_1)\frac{\varepsilon}{2(\mu + \delta_1)} + |\lambda - \mu| q^s(x) \\ &< \frac{\varepsilon}{2} + \delta_1 q^s(x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves the lemma. □

Let A be a convex and absorbing subset of a vector space X. Let us recall that the gauge functional g_A of A is defined as

$$g_A(x) = \inf\{t > 0 \mid x \in tA\}, \quad x \in X.$$

It is well known that g_A satisfies the following properties

(a) $g_A(x + y) \leq g_A(x) + g_A(x)$ for every $x, y \in X$.

(b) $g_A(tx) = tg_A(x)$ for every $t \geq 0$ and $x \in X$.
(c) $\{x \in X \mid g_A(x) < 1\} \subset A \subset \{x \in X \mid g_A(x) \leq 1\}$.
(d) If $X = \mathbb{R}^n$ and A is closed with respect to the euclidean topology of $\mathbb{R}^n$, then $A = \{x \in X \mid g_A(x) \leq 1\}$.

(see, e.g., [18, Theorem 1.35] and [19, §5.6 p. 236]).

Furthermore, if A does not contain any line, then g_A also satisfies

(e) $g_A(x) = 0 = g_A(-x) \Longrightarrow x = 0$.

Since every n-dimensional normed space is linear and topologically isomorphic to the euclidean space $\mathbb{R}^n$, all previous properties directly imply the following lemma.

Lemma 2.2. *Let (X, q) be a finite dimensional asymmetric normed space. Let us assume that $A \subset X$ is an absorbing q^s-closed convex subset of X such that A does not contain any line. Then the gauge functional g_A of A is an asymmetric norm on X for which its unit closed ball coincides with the set A.*

2.1. **Equivalent and right bounded asymmetric norms.** Let us show some basic facts on the relation among equivalent asymmetric norms and the right boundedness property for the corresponding spaces that will be used later on. Some of the proofs are straightforward, so we will write only some hints for getting them.

Let X be a real vector space. Two asymmetric norms in X, q and p, are said to be equivalent if and only if there exist $\kappa > 0$ and $\lambda > 0$ such that

$$\kappa q(x) \leq p(x) \leq \lambda q(x) \quad \text{for all} \;\; x \in X,$$

(or equivalently, if and only if $\lambda B_1^q[0] \subset B_1^p[0] \subset \kappa B_1^q[0]$). Obviously, two equivalent asymmetric norms in X generate the same topology. For the proof of the next result it is enough to take into account that for equivalent q and p, $q(x) = 0$ if and only if $p(x) = 0$.

Lemma 2.3. *Let q and p be equivalent asymmetric norms in a vector space X. Then q^s and p^s are equivalent and $\theta_p = \theta_q$.*

An asymmetric norm q in a vector space X is said to be *right bounded* if and only if there exists $r > 0$ such that

$$rB_1^q[0] \subset B_1^{q^s}[0] + \theta_q.$$

In this case we also say that $B_1^q[0]$ is right bounded. In the particular case when $r = 1$, we will simply say that q is 1-bounded. Using (1) we infer the following

Remark 2.4. *Let (X, q) be an asymmetric normed space. The norm q is 1-bounded if and only if $B_1^q[0] = B_1^{q^s}[0] + \theta_p$.*

Lemma 2.5. *An asymmetric normed space (X, q) is right bounded if and only if there exists a q^s-bounded set K such that*

$$B_1^q[0] \subset K + \theta_q$$

Proof. If (X, q) is right bounded, we can find $r > 0$ such that $rB_1^q[0] \subset B_1^{q^s}[0] + \theta_q$. Then the set $K := \frac{1}{r}B_1^{q^s}[0]$ is q^s-bounded and

$$B_1^q[0] \subset K + 1/r\theta_q = K + \theta_q.$$

For the second implication, let $h > 0$ be such that $K \subset hB_1^{q^s}[0]$. Then

$$B_1^q[0] \subset K + \theta_q \subset hB_1^{q^s}[0] + \theta_q = hB_1^{q^s}[0] + h\theta_q.$$

Thus $1/hB_1^q[0] \subset B^{q^s} + \theta_q$, which proves that q is right bounded. □

2.2. **The canonical 1-bounded equivalent asymmetric norm.** In [9] Conradie introduced a new asymmetric norm associated to an asymmetric normed space (X, p) in the following way. For every $x \in X$ let us define

$$q_p(x) = \inf\{p^s(x - y) \mid y \in \theta_p\}.$$

It was proven in [9] that $q_p : X \to [0, \infty)$ is an asymmetric norm satisfying the following properties:

(p1) $p(x) \leq q_p(x)$ for all $x \in X$,
(p2) $B_1^{q_p}[0] \subset B_1^p[0]$,
(p3) $B_1^{q_p}[0] = B_1^{p^s}[0] + \theta_p$,
(p4) p is equivalent to q_p if and only if p is right bounded.

In the next lemma we prove some other fundamental properties about the norm q_p that will be used later on.

Lemma 2.6. *Let (X, p) be an asymmetric normed space. Then the following statements hold*

(p5) $\theta_p = \theta_{q_p}$ *(equivalently, $p(x) = 0$ if and only if $q_p(x) = 0$).*
(p6) $p^s = q_p^s$ *(in particular $B_1^{p^s}[0] = B_1^{q_p^s}[0]$).*
(p7) q_p *is 1-bounded.*
(p8) $B_1^{q_p}[0] = B_1^p[0]$ *if and only if p is 1-bounded.*

Proof. (p5) The inclusion $\theta_{q_p} \subset \theta_p$ follows from property (p1). Now, if $x \in \theta_p$, we can use the definition of q_p to infer that

$$0 \leq q_p(x) \leq p^s(x - x) = p^s(0) = 0.$$

Then $q_p(x) = 0$ and therefore $x \in \theta_{q_p}$.

(p6) Let $x \in X$. By property (p1), we have that $q_p(x) \geq p(x)$ and $q_p(-x) \geq p(-x)$. Thus

$$q_p^s(x) = \max\{q_p(x), q_p(-x)\} \geq \max\{p(x), p(-x)\} = p^s(x).$$

On the other hand, the definition of q_p guarantees that $p^s(x) = p^s(x - 0) \geq q_p(x)$ and $p^s(-x) = p^s(-x-0) \geq q_p(-x)$. Since p^s is a norm, $p^s(x) = p^s(-x)$ and therefore

$$p^s(x) \geq \max\{q_p(x), q_p(-x)\} = q_p^s(x).$$

Then we get $p^s(x) = q_p^s(x)$, as desired.

(p7) Using Remark 2.4 and properties (p3), (p5) and (p6) we get that

$$B_1^{q_p}[0] = B_1^{p^s}[0] + \theta_p = B_1^{q_p^s}[0] + \theta_{q_p}.$$

Then q_p is 1-bounded.

(p8) This is a direct consequence of Remark 2.4 and property (p3). □

Lemma 2.7. *Let q and p be equivalent asymmetric norms in a vector space X. Then q is right bounded if and only if p is right bounded.*

Proof. If q is equivalent to p, then by Lemma 2.3 q^s and p^s are also equivalent and $\theta_q = \theta_p$. It follows from this that q_p and q_q are equivalent. If q is right bounded, by property (p4) q is equivalent to q_q which is equivalent to q_p, therefore p is equivalent to q_p and using property (p4) again we infer that p is right bounded, as desired. □

3. Convexity and compactness

In this section we will show some results relating to compactness properties of subsets of asymmetric normed spaces, mainly for convex sets. The notion of local compactness arises in a natural way in the characterizations, which are given in terms of an equivalent asymmetric norm satisfying some special properties.

It is well known that in finite dimensional normed spaces, the convex hull of a compact set is compact. In the following lemma we show the asymmetric version of this result.

Lemma 3.1. *Let $A \subset X$ be a q-compact set in the finite dimensional asymmetric normed space (X, q). Then the convex hull $\operatorname{conv}(A)$ is also q-compact.*

Proof. Let n be the (algebraic) dimension of X. We apply Carathéodory's Theorem ([19, Theorem 2.2.4]) to infer that

$$\operatorname{conv}(A) = \left\{ \sum_{i=0}^{n} \lambda_i a_i : \ a_i \in A, \, \lambda_i \geq 0 \ \text{ and } \ \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

On the other hand, the topological product A^{n+1} is also compact since A is compact. Now, let us consider the n-dimensional simplex

$$\Delta = \left\{ (\lambda_0, \dots, \lambda_n) \in \mathbb{R}^{n+1} \mid \lambda_i \geq 0 \text{ and } \sum_{i=0}^{n} \lambda_i = 1 \right\}.$$

If we equip Δ with the Euclidean topology, it becomes a compact space. Now, let $\varphi : A^{n+1} \times \Delta \to \operatorname{conv}(A)$ be the natural map defined by

$$\varphi(a_0, \dots, a_n, \lambda_0, \dots, \lambda_n) = \sum_{i=0}^{n} \lambda_i a_i.$$

Clearly φ is an onto function. Now observe that φ is continuous since addition is continuous in X ([7, Proposition 1.1.40]) and scalar multiplication by positive numbers is also continuous (Lemma 2.1). Using the fact that $A^{n+1} \times \Delta$ is compact we can then conclude that $\operatorname{conv}(A)$ is also compact. □

Lemma 3.2. *Let A be a strongly compact set in the finite dimensional asymmetric normed space (X, q). Then $\operatorname{conv}(A)$ is also strongly compact.*

Proof. Let $K \subset A$ be a q^s-compact set such that $K \subset A \subset K + \theta_q$. By [19, Theorem 2.2.6], $\operatorname{conv}(K)$ is q^s-compact. Furthermore, we have that

$$\operatorname{conv}(K) \subset \operatorname{conv}(A) \subset \operatorname{conv}(K + \theta_q).$$

Since $\operatorname{conv}(K) + \theta_q$ is a convex set containing $K + \theta_q$, we conclude that $\operatorname{conv}(K + \theta_q) \subset \operatorname{conv}(K) + \theta_q$ and therefore

$$\operatorname{conv}(K) \subset \operatorname{conv}(A) \subset \operatorname{conv}(K) + \theta_q.$$

This proves that $\operatorname{conv}(A)$ is strongly compact. □

Lemma 3.3. *Let A be a strongly compact set in an asymmetric normed space (X, q). Then the q^s-closure of A is also strongly compact.*

Proof. Let $K \subset X$ be a q^s-compact set such that $K \subset A \subset K + \theta_q$. For any $B \subset X$, let us denote by $\overline{B}^s$ the q^s-closure of B. Then we have that

$$\overline{K}^s \subset \overline{A}^s \subset \overline{K + \theta_q}^s.$$

Since K is q^s-closed we have that $K = \overline{K}^s$. On the other hand, since θ_q is q^s-closed (see e.g. [16, Lemma 3.6]) and the sum of a compact set with a closed set is closed in normed spaces ([19, Theorem 1.8.10-(ii)]) we infer that $K + \theta_q$ is q^s-closed. Therefore $K + \theta_q = \overline{K + \theta_q}^s$ and then $K \subset \overline{A}^s \subset K + \theta_q$ which proves that $\overline{A}^s$ is strongly compact, as desired. □

Proposition 3.4. *Let (X, q) be a finite dimensional asymmetric normed space. If the origin has a compact neighbourhood U, then there exists an equivalent asymmetric norm p such that each closed ball $B^p_r[x]$ is compact. If additionally U is strongly compact, then the balls $B^p_r[x]$ are also strongly compact.*

Proof. Let U be a q-compact neighbourhood of the origin. Then there exist $\varepsilon > 0$ and $\kappa > 0$ such that

$$\varepsilon B^q_1[0] \subset U \subset \kappa B^q_1[0].$$

Let us denote by A the q^s-closure of the convex hull of U. By Lemma 3.1 and [16, Lemma 3.2], we have that A is compact. Since $\kappa B^q_1[0]$ is convex and q^s-closed, we have that

$$\varepsilon B^q_1[0] \subset U \subset A \subset \kappa B^q_1[0] \tag{2}$$

Now let p be defined as the gauge functional of A. Namely,

$$p(x) := \{\inf t > 0 \mid x \in tA\}.$$

Since A is absorbent, convex and does not contain any line, p defines an asymmetric norm such that $B^p_1[0] = A$ (Lemma 2.2). Furthermore, by (2)

p is equivalent to q. Since A is q-compact, it is also p-compact and all the closed balls $B^p_r[x] = rB^p_1[0] + x$ are also compact.

Now, if U is strongly compact we get by Lemma 3.3 that $A = B^p_1[0]$ is strongly compact and therefore all closed balls $B^p_r[x]$ are strongly compact, as desired. □

4. Strong local compactness

The aim of this section is to analyze the boundedness properties of strong locally compact asymmetric spaces. As we said in the introduction, in general right bounded spaces do not have compact unit balls. Indeed, let us start with the following example, which is a counterexample to [14, Proposition 17].

Example 4.1. *There exists a finite dimensional asymmetric normed space (X, q) such that the closed unit ball $B^q_1[0]$ is right bounded and not q-compact.*

Proof. Let $X = \mathbb{R}^2$. Define the set $A \subset X$ as follows

$$A = \left\{ (x, y) \in \mathbb{R}^2 \mid y \leq \frac{1}{x-2} + 2,\ x < 2 \right\}.$$

Let $q : X \to [0, \infty)$ be defined as the gauge functional of A. Namely

$$q((x, y)) = \inf\{t > 0 \mid (x, y) \in tA\}.$$

Clearly q is an asymmetric norm in X such that $A = B^q_1[(0, 0)]$. We also consider the asymmetric lattice norm $p : \mathbb{R}^2 \to \mathbb{R}$ given by

$$p((x, y)) = \max\{x^+, y^+\}$$

where $x^+ = \max\{x, 0\}$. Clearly $B^p_1[0] \subset B^q_1[0] \subset 2B^p_1[0]$ which implies that p and q are equivalent. Since p is right bounded, it follows from Lemma 2.7 that q is also right bounded.

Now let us observe that $B^q_1[(0, 0)]$ is non compact. Indeed, let us consider the family $\mathcal{U} = \{B^q_1((2, t))\}_{t<0}$. Clearly, for every $t < 0$,

$$B^q_1((2, t)) = \left\{ (x, y) \in \mathbb{R}^2, y < \frac{1}{x-4} + 2 + t, x < 4 \right\}$$

and $\mathcal{U}$ is an open cover for $B^q_1[(0, 0)]$. Furhtermore, if $t > s$, then $B^q_1((2, s)) \subset B^q_1((2, t))$ and therefore $\mathcal{U}$ is a nested family. A simple calculation shows that every point $(x, y) \in B^q_1[(0, 0)]$ such that $y = 1/(x - 2) + 2$ and $x \geq 3 - \sqrt{1 - 2/t}$ cannot belong to $B^q_1((2, t))$. Since $\mathcal{U}$ is nested we conclude that $B^q_1[(0, 0)]$ cannot be covered with a finite subfamily of $\mathcal{U}$ and therefore $B^q_1[(0, 0)]$ is not compact.

The fact that $B^q_1[0]$ is not compact, can also be proved using [8, Theorem 4.4] (c.f. [8, Examples 2.7 and 2.11-(a)]).

□

Remark 4.2. *Let p be the asymmetric lattice norm defined in Example 4.1. It is well known that $B_1^p[(0,0)]$ is p-compact and therefore (X,p) is locally compact (in fact, it is strongly locally compact). Since (X,p) is equivalent to (X,q) we infer that an asymmetric normed space can be locally compact, even if the closed unit ball is not compact.*

After Example 4.1 and Remark 4.2, it is natural to ask what is the relation between right boundedness and strong local compactness in finite dimensional asymmetric normed spaces. The aim of this section is to answer that question.

The proof of the following result is given by the same argument used in [14, Proposition 17].

Lemma 4.3. *Let (X,q) be a finite dimensional asymmetric normed space. If q is 1-bounded, then $B_1^q[0]$ is strongly compact and therefore (X,q) is strongly locally compact.*

Proof. Observe that $B_1^{q^s}[0]$ is q^s-compact (because X is finite dimensional). Now, since q is 1-bounded, we have that

$$B_1^{q^s}[0] \subset B_1^q[0] = B_1^{q^s}[0] + \theta_q.$$

This directly implies that $B_1^q[0]$ is strongly compact.

□

Many important asymmetric normed spaces are 1-bounded. For example, every asymmetric norm defined by a Banach lattice norm is 1-bounded ([2, Lemma 1]). Furthermore, as we have already shown, the norm q_p associated to an asymmetric normed space (X,p) is 1-bounded. This gives us the following

Corollary 4.4. *If (X,p) is a finite dimensional asymmetric normed space, then $B_1^{q_p}[0]$ is strongly compact and therefore (X,q_p) is strongly locally compact.*

Lemma 4.5. *The asymmetric normed space (X,p) is right bounded if and only if there exists an equivalent norm q such that q is* 1*-bounded.*

Proof. For the first implication just consider $q = q_p$ and properties (p4) and (p7). The second implication follows directly from Lemma 2.7 □

Proposition 4.6. *Let (X,p) be a finite dimensional asymmetric normed space. If the origin has a strongly compact neighborhood, then p is right bounded.*

Proof. By Proposition 3.4, we can find an equivalent norm q such that $B_1^q[0]$ is strongly compact. Then there exists a q^s-compact set $K \subset X$ such that

$$K \subset B_1^q[0] \subset K + \theta_q.$$

Since X is finite dimensional we have that q^s and p^s are equivalent norms, then K is p^s compact and we can find $a > 0$ such that $K \subset aB_1^{p^s}[0]$. On the other hand, since q is equivalent to p, there exists $b > 0$ with $bB_1^p[0] \subset B_1^q[0]$. Using Lemma 2.3, we infer that

$$\begin{aligned} bB_1^p[0] \subset B_1^q[0] \subset K + \theta_q &\subset aB_1^{p^s}[0] + \theta_q \\ &= aB_1^{p^s}[0] + a\theta_p = a(B_1^{p^s}[0] + \theta_p). \end{aligned}$$

Therefore $rB_1^p[0] \subset B_1^{p^s}[0] + \theta_p$ with $r = b/a$, and then p is right bounded, as desired. □

Now we summarize all the previous work in the main result of this section.

Theorem 4.7. *Let (X, p) be a finite dimensional asymmetric normed space. The following statements are equivalent:*

(1) *p is right bounded.*
(2) *There exists an equivalent asymmetric norm q in X such that q is 1-bounded.*
(3) *There exists an equivalent asymmetric norm q in X such that $B_1^q[0]$ is strongly compact.*
(4) *(X, p) is strongly locally compact.*

Proof. The part (1) $\iff$ (2) was already proved in Lemma 4.5.

Implication (2) $\implies$ (3) follows from Lemma 4.3.

Finally, implication (3) $\implies$ (4) is obvious and (4) $\implies$ (1) was proved in Proposition 4.6. □

If the closed unit ball of an asymmetric normed space (X, p) is p-compact (or even strongly p-compact), then X need not be finite-dimensional. Consider for example $X = \ell_\infty$ and $p((x_n)) = \sup_{n\in\mathbb{N}} x_n^+$. It was shown in Example 5 of [2], that the closed unit ball of this space is p-precompact, and so by [8, Proposition 4.10] it is also strongly p-compact.

We finish this section by giving a useful characterization of equivalent right bounded asymmetric norms.

Theorem 4.8. *Let X be a real finite dimensional linear space. Two right bounded asymmetric norms p and p' are equivalent if and only if $\theta_p = \theta_{p'}$.*

Proof. The "only if" implication was already mentioned in Lemma 2.3. For the "if" implication, observe that right boundedness of p and p' implies that p is equivalent to q_p and p' is equivalent to $q_{p'}$. Since X is finite dimensional, p^s and q^s are equivalent, and if $\theta_p = \theta_{p'}$, it follows that q_p and $q_{p'}$ are equivalent. But then p and p' are also equivalent.

□

5. Right boundedness and the geometry of the unit ball

Let us finish the paper by exploring the relation among right boundedness, compactness and the geometry of the closed unit ball in a finite dimensional asymmetric normed space.

In [16] it was proven that the convex hull of the extreme points of a compact convex set in an asymmetric normed space defines the topology of the set. In this section we will use the techniques used there to understand the relation among the geometry and the topology of the closed unit balls in finite dimensional asymmetric normed spaces.

Given a convex set K in an asymmetric normed space (X, q), let us denote by $E(K)$ the extreme points[1] of $K + \theta_q$, and by $S(K)$ the convex hull of $E(K)$.

First, it is interesting to note compactness of the set $E(K)$ is not a necessary condition to guarantee the compactness of a convex set K. For example, consider the the asymmetric lattice norm $(\mathbb{R}^2, p)$ defined in Example 4.1. Then the set

$$K = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0 \text{ and } y \leq -x^2\}$$

is a strongly compact convex set with an unbounded set $E(K)$.

On the other hand it was proven in [8] that in a finite dimensional asymmetric normed lattice (X, q), all q-compact and q^s-closed sets are strongly compact. If the dimension of X is 2, then it is also known (see [15]) that every q-compact and convex set is strongly compact (even if it is not q^s-closed). Nevertheless, not all compact convex sets are strongly compact (the reader can find an example of this in [15]). After these results, it is natural to ask if in general, a compact convex set of a finite dimensional asymmetric normed space (X, q) is strongly compact if it is q^s-closed. Even more, after Theorem 4.7 another natural question arises. On one hand, Theorem 4.7 shows that local compactness of an equivalent norm characterizes right boundedness and strong local compactness. Therefore, we can ask if in general compactness of the unit ball implies strong compactness. The next example solves these two questions negatively.

Example 5.1. *There is a finite dimensional asymmetric normed space such that $B_1^q[0]$ is compact, but not strongly compact.*

Proof. In order to see this, consider the cylinder $C = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 \leq 1 \text{ and } x \leq 1\}$ in $\mathbb{R}^3$, and let us define

$$R = \{(x, 0, 0) \in \mathbb{R}^3 \mid x \leq 0\}.$$

Let $p : \mathbb{R}^3 \to [0, \infty)$ be the gauge functional over the set C

$$p(u) = \inf\{t \geq 0 \mid u \in tC\}.$$

Then p is an asymmetric norm in $\mathbb{R}^3$, for whom the closed unit ball is the set C and such that $\theta_p = R$. We will define an equivalent norm $q : \mathbb{R}^3 \to [0, \infty)$ with the required characteristics.

To do that, let us start by defining, for each $n \in \mathbb{N}$, the values:

$$y_n = \cos\left(\frac{n\pi}{2(n+1)}\right) \quad \text{and} \quad z_n = \sin\left(\frac{n\pi}{2(n+1)}\right).$$

[1]A point $x \in A$ is an *extreme* point of A if $x = y = z$ whenever $y, z \in A$ and $x = \lambda y + (1 - \lambda)z$ for some $\lambda \in (0, 1)$.

And let us consider the sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^3$, where $u_n = (-n, y_n, z_n)$. Evidently, in the topology τ_p, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to the point $(1, 0, 1)$, and therefore the set $A_1 = \{u_n \mid n \in \mathbb{N}\} \cup \{(1, 0, 1)\}$ is p-compact.

On the other hand, the set $A_2 = \{(1, y, z) \in \mathbb{R}^3 \mid |y| + |z| \leq 1\}$ is clearly p^s-compact and therefore it is also p-compact. Thus $A_1 \cup A_2$ is compact. Let $A = \text{conv}(A_1 \cup A_2)$. By Lemma 3.1, A is p-compact and therefore $\overline{A}^s$ is also p-compact. This in combination with [16, Lemma 3.7] and property (O2) yields that $B := \overline{A}^s + \theta_p$ is p-compact. Furthermore, the set B satisfies

(1) $B = B + \theta_p$
(2) $\frac{1}{\sqrt{2}}C \subset B \subset C$

Define now $q : \mathbb{R}^3 \to [0, \infty)$ as the gauge functional over B. Namely

$$q(u) = \inf\{t \geq 0 \mid u \in tB\}.$$

Clearly B is the closed unit ball with respect to q and $\theta_q = R = \theta_p$. Property (2) implies that q is equivalent to p. In particular, since p is obviously right bounded, we also get that q is right bounded (by Lemma 2.7). By the same reason, since B is p-compact it is also q-compact, as desired.

It just remains to prove that B is not strongly compact. For that purpose, suppose the opposite. Then we can find a q^s-compact set K such that $K \subset B \subset K + \theta_q$. Since B is convex, we can use Lemma 3.1 in order to assume without any loss of generality that the set K is convex. We can also observe that

$$K \subset B \subset K + \theta_q \subset B + \theta_q = B,$$

and therefore $B = K + \theta_q$. Now, by [16, Theorem 4.1] the set of extreme points $E(B)$ is contained in K. Since every point in the set A_1 is an extreme point of B, we infer that $A_1 \subset K$. In particular, we get that A_1 is q^s-bounded, which is a contradiction.

□

In particular this gives the existence of q^s-closed convex sets in finite dimensional asymmetric normed spaces that are compact but not strongly compact (c.f. [8]). Example 5.1 also shows that there is a q-compact closed unit ball of an asymmetric normed space of dimension 3 such that the set of its extreme points $E(B_1^q[0])$ is not q^s-bounded.

In what follows we show that some positive results can also be found on the relation among extreme points and right boundedness.

Other results on the behavior of asymmetric norms in finite dimensional spaces can be proven using the tools developed above.

Lemma 5.2. *For any finite dimensional asymmetric normed space (X, q), we always have $B_1^q[0] = S(B_1^q[0]) + \theta_q$.*

Proof. Using a well-known result of Klee [17], we know that $B_1^q[0]$ is the convex hull of its extreme points and its extreme rays[2]. This implies that $S(B_1^q[0]) \subset B_1^q[0]$ and therefore

$$S(B_1^q[0]) + \theta_q \subset B_1^q[0] + \theta_q = B_1^q[0].$$

In order to prove the other inclusion, let $x \in B_1^q[0]$. By [16, Lemma 3.6] every extreme ray of $B_1^q[0]$ is parallel to some ray contained in θ_q. Then we can find $a_1, \dots, a_n, a_{n+1}, \dots, a_p \in E(B_1^q[0])$, $b_1, \dots, b_n \in \theta_q$, and scalars $\lambda_1, \dots, \lambda_p, t_1, \dots, t_n \geq 0$ such that

$$x = \sum_{i=1}^{n} \lambda_i(a_i + t_i b_i) + \sum_{i=n+1}^{p} \lambda_i a_i \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1.$$

Since θ_q is a convex cone, the point $\sum_{i=1}^{n} \lambda_i t_i b_i$ lies in θ_q. On the other hand, we know that $\sum_{i=1}^{p} \lambda_i a_i \in S(B_1^q[0])$. Thus

$$\begin{aligned} x &= \sum_{i=1}^{n} \lambda_i(a_i + t_i b_i) + \sum_{i=n+1}^{p} \lambda_i a_i \\ &= \sum_{i=1}^{p} \lambda_i a_i + \sum_{i=1}^{n} \lambda_i t_i b_i \\ &\in S(B_1^q[0]) + \theta_q. \end{aligned}$$

□

Proposition 5.3. *Let (X, q) be an asymmetric normed space such that $E(B_1^q[0])$ is q^s-bounded. Then q is right bounded.*

Proof. If $E(B_1^q[0])$ is q^s-bounded, so is $S(B_1^q[0])$. Thus, by Lemmas 2.5 and 5.2 we conclude that q is right bounded. □

The converse of Proposition 5.3 is false. A counterexample of this situation was already shown in Example 4.1, where the norm is right bounded, but the set of extreme points of $B_1^q[0]$ is not q^s-bounded.

However, if the norm is 1-bounded, the situation is different as we can see in the following

Proposition 5.4. *Let (X, q) be a finite dimensional 1-bounded asymmetric normed space. Then $E(B_1^q[0]) \subset B_1^{q^s}[0]$ and therefore $E(B_1^q[0])$ is q^s-bounded.*

Proof. Since q is 1-bounded, $B_1^q[0] = B_1^{q^s}[0] + \theta_0$. Since $B_1^{q^s}[0]$ is q^s-compact (and q-compact), we can use [16, Theorem 4.1] to infer that $E(B_1^q[0]) \subset B_1^{q^s}[0]$ as desired. □

[2]An open half line $R = \{a + tb \mid a, b \in X,\ t > 0\}$ is called an *extreme ray* of A if $y, z \in R$ whenever $\lambda y + (1-\lambda)z \in R$, where $y, z \in A$ and $\lambda \in (0, 1)$. In this case, if R is an extreme ray and the extreme a lies in A, then a is an extreme point of A.

Let us finish the paper with the following open question, that is suggested by the strong relation between right boundedness and local compactness.

Question 5.5. *Let (X,q) be a finite dimensional asymmetric normed space. If X is locally compact, is q right bounded? In particular, if $B_1^q[0]$ is compact, is q right bounded?*

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