WATS'17

Workshop on Applied Topological Structures
Valencia, 11-12 July 2017
pp. 95 – 102

An overview on transformations on generalized metrics

Juan José Miñana

Departament de Ciències Matemàtiques i Informàtica, Universitat de les Illes Balears, Carretera de Valldemossa km 7.5, 07122 Palma, Spain (jj.minana@uib.es)

Abstract

We will present an overview on the results appeared in the literature about the study of those functions that preserve or transform a generalized metric.

Keywords: *Generalized metric; preserving function; symmetrization.*
MSC: *54E35; 54E40; 68T27.*

1. Introduction

In 1981, J. Borsík and J. Doboš studied the problem of characterizing the class of functions that preserve metrics, i.e., those functions whose composition with each metric provide a metric (see [2]). Later in [3], the same authors continued their study approaching the problem of merging a family of metric spaces into a single one (we can find a whole study related to these topics in [4]). Both cases can be seen as the study of functions that transform metrics (a single one or a family) in metrics. This idea opened a via of research, which is extending the study of transformations on the different notions of generalized metrics. For instance, quasi-metric spaces (see Definition 7), a generalization of metric space in which the axiom of symmetry is non-demanded, or partial metric spaces (see Definition 9), known also as non-zero distances.

The aforementioned topic, has been tackled in two different senses. On the one hand, we can find in the literature some studies about functions that transform a class of generalized metric in the same class. For instance, in [8] it was characterized those functions that transform each quasi-metric (single one or family) into a quasi-metric, and in [5] it was provided a respective characterization to the partial metric case. On the other hand, a natural problem, related to the last one, is to study the functions that convert a class of generalized metrics in a distinct one. In this last line, we can find in [7] a characterization of those functions that "symmetrize" quasi-metrics.

In this paper, we have collected some results appeared in the literature about the topics exposed in the last two paragraphs. In addition, we propose some observations about open topics of research related to the presented results.

Along the paper we will denote the interval $[0, \infty[$ by $\mathbb{R}_+$.

2. Metric Preserving Functions

In this section, we present the main results about the study of those functions whose composition with each metric provide a metric. We begin by the following concept introduced by Doboš.

Definition 1. We will say that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a *metric preserving function* if for each metric space (X, d) the function d_f is a metric on X, where $d_f(x, y) = f(d(x, y))$ for each $x, y \in X$.

From now on, we will denote by $\mathcal{M}$ the class of all metric preserving functions.

An example of metric preserving function is the following one:

$$f(x) = \frac{x}{1 + x}, \text{ for each } x \in \mathbb{R}_+.$$

Accordig to the notation used by Doboš in [4] we have the next definition.

Definition 2. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. Then, we will say that

(i) f is *amenable* if $f^{-1}(0) = \{0\}$.
(ii) f is *subadditive* if for each $a, b \in \mathbb{R}_+$ it is hold:

$$f(a+b) \leq f(a) + f(b).$$

In the rest of the paper we will denote by $\mathcal{O}$ the class of all amenable functions.

On the one hand, each metric preserving function is amenable and subadditve. However, in [4] it was introduced the next example to show an amenable and subbaditive function which is not included in $\mathcal{M}$.

Example 3. Define $f : \mathbb{R}_+ \to \mathbb{R}_+$ as follows:

$$f(x) = \begin{cases} \frac{x}{1+x}, & \text{if } x \in \mathbb{Q} \cap \mathbb{R}_+ (\mathbb{Q} \text{ denotes the set of rational numbers}); \\ 1, & \text{elsewhere.} \end{cases}$$

On the other hand, every amenable, subadditive and non-decreasing function preserves metrics. Nevertheless, there exists functions in $\mathcal{M}$ which are not non-decreasing such as shows the following instance based on Example 8 in [8].

Example 4. Consider the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by:

$$f(x) = \begin{cases} 0, & \text{if } x = 0; \\ 2, & \text{if } x \in]0,1[; \\ 1, & \text{if } x \in [1, \infty[. \end{cases}$$

It is clear that $f(1/2) > f(1)$, but $1/2 < 1$.

We continue recalling a notion used in the Doboš' characterization, which was introduced by F. Terpe in [9] and it will be crucial for a subsequent discussion.

Definition 5. Let $a, b, c \in \mathbb{R}_+$. We will say that (a, b, c) is a *triangle triplet* if

$$a \leq b + c; \quad b \leq a + c \quad \text{and} \quad c \leq a + b.$$

A metric provides an easy way to construct triangle triplets. Indeed, if we consider a metric space (X, d) and we take $x, y, z \in X$, then the triangle inequality ensures that (a, b, c) is a triangle triplet, where $a = d(x, z)$, $b = d(x, y)$ and $c = d(y, z)$.

Now, we present the enunciated characterization of the class of metric preserving functions as a modification of the one given by Doboš in [4].

Theorem 6. *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is a metric preserving function if and only if f satisfies the following properties:*

(1) $f \in \mathcal{O}$,
(2) if (a, b, c) is a triangle triplet, then so is $(f(a), f(b), f(c))$.

3. Quasi-metric and partial metric preserving functions

Borsík and Doboš continued the work exposed in Section 2 characterizing those functions that merge a family of metric spaces (see [3]). This study was extended to the context of quasi-metrics in [8] and partial metrics in [5]. In this paper we are just interested in functions that transform a generalized metric. For this reason, we have adapted the results of the aforementioned papers to the case that the family of metrics is formed by a unique element. As in Definition 1, we define a quasi-metric (or partial metric) preserving function as those functions whose composition with each quasi-metric (or partial metric) provide another one. We will denote the class of quasi-metric and partial metric preserving functions by $\mathcal{Q}$ and $\mathcal{P}$, respectively.

Next, let us recall the concept of quasi-metric space.

Definition 7. Let X be a non-empty set and let q be a non-negative real-valued function on $X \times X$. We will say that (X, q) is a *quasi-metric space* if for each $x, y, z \in X$ the following is hold:

(q1) $q(x, y) = q(y, x) = 0$ if and only if $x = y$;
(q2) $q(x, z) \leq q(x, y) + q(y, z)$.

Now, we present a characterization of those functions which preserve quasi-metrics. It is based on Theorem 1 in [8].

Theorem 8. *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is a quasi-metric preserving function if and only if f satisfies the following properties:*

(1) $f \in \mathcal{O}$,
(2) for each $a, b, c \in \mathbb{R}_+$, *with* $a \leq b + c$, *it is satisfied that* $f(a) \leq f(b) + f(c)$.

In [8] it was pointed out that $\mathcal{Q} \subsetneq \mathcal{M}$. Indeed, they provided (the above) Example 4 to show an instance of metric preserving function which is not a quasi-metric preserving one.

Analogously to the study for quasi-metrics, in [5] it was approached the problem of characterizing the functions that agregate partial metrics in a single one. In order to present such a characterization for the one-dimensional case, we will recall the notion of partial metric space introduced by S.G. Matthews in [6].

Definition 9. Let X be a non-empty set and let p a non-negative real-valued function on $X \times X$. We will say that (X, p) is a *partial metric space* if for each $x, y, z \in X$ the following is hold:

(p1) $p(x, x) = p(x, y) = p(y, y)$ if and only if $x = y$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

The next result is an adaptation, to the one-dimensional case, of Theorem 10 in [5].

Theorem 10. *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is a partial metric preserving function if and only if f satisfies the following properties:*

(1) $f(a) + f(b) \leq f(c) + f(d)$ *whenever* $a + b \leq c + d$ *and* $b \leq \min\{c, d\}$,
(2) If $\max\{b, c\} \leq a$ *and* $f(a) = f(b) = f(c)$, *then* $a = b = c$.

Attending to the preceding characterization, one can observe that the class $\mathcal{P}$ is not included in $\mathcal{Q}$, and consequently, it is not contained in $\mathcal{M}$ too. Indeed, a function $f \in \mathcal{P}$ is not necessarily included in $\mathcal{O}$. However, in [5] it was shown that if a partial metric preserving function f is included in $\mathcal{O}$, then $f \in \mathcal{Q}$.

Taking into account the studies presented in this section, it seems interesting to approach the problem of characterizing those functions that preserve another classes of generalized metrics as a future work. For instance, among others, the notion of metric like (or dislocated metric), introduced in [1].

4. Symmetrization of quasi-metrics

In the last two preceeding sections, we have presented some characterizations of functions that preserve (generalized) metrics. A natural problem to study, related to the aforementioned topic, is to characterize those functions that transform a generalized metric into a metric. In fact, it is well-known that each quasi-metric generates a metric: given a quasi-metric space (X, q), then the function d^s given by $d^s(x, y) = \max\{q(x, y), q(y, x)\}$, for each $x, y \in X$, is a metric on X. Thus, an interesting topic is to generalize the way of obtaining a metric deduced from a quasi-metric by means of tranformation functions. This problem was discussed in [7]. In this section, we will recall some results presented in the aforementioned paper. With this aim we introduce some pertinent notions.

Definition 11. We will say that $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric generating function if $d_\Phi : X \times X \to \mathbb{R}_+$ is a metric on X for every quasi-metric space (X, q), where the function d_Φ is defined by

$$d_\Phi(x, y) = \Phi(q(x, y), q(y, x)), \text{ for each } x, y \in X.$$

As we have mentioned above, the function defined by $\Phi_{max}(a, b) = \max\{a, b\}$, for each $a, b \in \mathbb{R}_+$, is a metric generating function. Furthermore, it is easy to verify that the function defined by $\Phi_+(a, b) = a + b$, for each $a, b \in \mathbb{R}_+$, is a metric generating function too.

Note that a metric generating function is defined on $\mathbb{R}_+^2$ instead of $\mathbb{R}_+$ contrary to the case of metric preserving functions. On account of [3], we can extend the notions of monotonicity and subadditivity of a function to this context from the one-dimensional framework as follows.

Definition 12. Consider the set $\mathbb{R}_+^2$ ordered by the pointwise order relation $\preceq$, i.e. $(a,b) \preceq (c,d)$ if and only if $a \leq b$ and $c \leq d$, and let $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$. Then, we will say that:

(i) Φ is monotone if for each $(a,b),(c,d) \in \mathbb{R}_+^2$, with $(a,b) \preceq (c,d)$, it is hold: $\Phi(a,b) \leq \Phi(c,d)$.

(ii) We will say that Φ is subadditive if for each $(a,b),(c,d) \in \mathbb{R}_+^2$ it is hold:

$$\Phi\left((a,b)+(c,d)\right) \leq \Phi(a,b) + \Phi(c,d).$$

In a similar way to the one-dimensional case, we will denote by $\mathcal{O}^2$ the set of all functions $\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\Phi(a,b) = 0$ if and only if $(a,b) = (0,0)$.

The following is a crucial notion in order to symmetrize a quasi-metric.

Definition 13. Let $a,b,c,x,y,z \in \mathbb{R}_+$. We will say that the triplets (a,b,c) and (x,y,z) are mixed triplets if they satisfy the following inequalities:

$$\begin{array}{lll} a \leq b+c; & b \leq a+y; & c \leq a+z; \\ x \leq y+z; & y \leq x+b; & z \leq x+c. \end{array}$$

In [7] it was observed that this last concept is related to the notion of triangle triplet. In fact, it was pointed out that (a,b,c) forms a triangle triplet if and only if (a,b,c) and (a,c,b) are mixed ones.

Now, we can present the promiseed characterization of metric generating functions, provided in [7].

Theorem 14. *$\Phi : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a metric generating function if and only if it satisfies the following properties:*

(1) $\Phi \in \mathcal{O}^2$.

(2) Φ *is symmetric, i.e* $\Phi(a,b) = \Phi(b,a)$ *for each* $(a,b) \in \mathbb{R}_+^2$.

(3) $\Phi(a,x) \leq \Phi(b,y)+\Phi(c,z)$, *whenever* (a,b,c) *and* (x,y,z) *are mixed triplets.*

A natural way to continue the above study is motivated by the fact that each partial metric generates a metric and a quasi-metric as follows. Let p be a partial metric on a non-empty set X. Then, the functions d_p and q_p given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y), \text{ for each } x, y \in X, \text{ and}$$

$$q_p(x, y) = p(x, y) - p(x, x), \text{ for each } x, y \in X$$

are a metric and a quasi-metric on X, respectively.

Thus, an interesting item to approach in the future is to generalize, by means of transformation functions, the last two constructions.

References

[1] A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory and Applications 2012:204 (2012), 1–10.

[2] J. Borsík, J. Doboš, Functions whose composition with every metric is a metric, Math. Slovaca 31 (1) (1981), 3–12.

[3] J. Borsík, J. Doboš, On a product of metric spaces, Math. Slovaka 31 (1981), 193–205.

[4] J. Doboš, Metric preserving functions, Štroffek, Košice, 1998.

[5] S. Massanet, O. Valero, New results on metric aggregation, Proc. of the 17th Spanish Conference on Fuzzy Technology and Fuzzy Logic-Estylf, Valladolid (2012), 558–563.

[6] S. G. Matthews, Partial metric topology, Ann. New York Acad. Sci. 728 (1994), 183–197.

[7] J. Martín, G. Mayor, O. Valero, On the Symmetrization of Quasi-metrics: An Aggregation Perspective, in: H. Bustince, J. Fernandez, R. Mesiar, T. Calvo Sánchez (eds), Aggregation Functions in Theory and in Practise. Advances in Intelligent Systems and Computing, vol 228. Springer, Berlin, Heidelberg, 2013.

[8] G. Mayor, O. Valero, Aggregation of assymmetric distances in Computer Science, Inform. Sciences 180 (2010), 803–8012.

[9] F. Terpe, Metric preserving functions, Proc. Conf. Topology and Measure IV, Greifswald (1984), 189–197.