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A version of Stone-Weierstrass theorem in Fuzzy Analysis

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Abstract

Let $C(K, \mathbb{E}^1)$ be the space of continuous functions defined between a compact Hausdorff space K and the space of fuzzy numbers $\mathbb{E}^1$ endowed with the supremum metric. We provide a sufficient set of conditions on a subspace of $C(K, \mathbb{E}^1)$ in order that it be dense. We also obtain a similar result for interpolating families of $C(K, \mathbb{E}^1)$.

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MSC: *54E35; 54E40.*

1. Introduction

Fuzzy numbers provide formalized tools to deal with non-precise quantities. They are indeed fuzzy sets in the real line and were introduced in 1978 by Dubois and Prade ([3]), who also defined their basic operations. Since then, Fuzzy Analysis

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has developed based on the notion of fuzzy number just as much as classical Real Analysis did based on the concept of real number. Such development was eased by a characterization of fuzzy numbers provided in 1986 by Goetschel and Voxman ([5]) leaning on their level sets.

As real-valued functions do in the classical setting, fuzzy-number-valued functions, that is, functions defined on a topological space taking values in the space of fuzzy numbers, play a central role in Fuzzy Analysis. Namely, fuzzy-number-valued functions have become the main tool in several fuzzy contexts, such as fuzzy differential equations ([1]), fuzzy integrals ([12]) or fuzzy optimization ([6]). However the main difficulty of dealing with these functions is the fact that the space they form is not a linear space; indeed it is not a group with respect to addition.

In this paper we focus on the conditions under which continuous (with respect to the supremum metric) fuzzy-number-valued functions defined on a compact Hausdorff space can be (uniformly) approximated to any degree of accuracy. More precisely and based on ideas of R. I. Jewett ([8]) and J. B. Prolla ([11]), we provide a sufficient set of conditions on a subspace of the space of fuzzy-number-valued functions in order that it be dense, which is to say a Stone-Weierstrass type result. The celebrated Stone-Weierstrass theorem is one of the most important results in classical Analysis, plays a key role in the development of General Approximation Theory and, particularly, is in the essence of the approximation capabilities of neural networks. We also obtain a similar result for interpolating families of continuous fuzzy-number-valued functions in the sense that the uniform approximation can also demand exact agreement at any finite number of points.

2. Preliminaries

Let $F(\mathbb{R})$ denote the family of all fuzzy subsets on the real numbers $\mathbb{R}$. For $u \in F(\mathbb{R})$ and $\lambda \in [0,1]$, the λ-level set of u is defined by

$$[u]^\lambda := \{ x \in \mathbb{R} : u(x) \geq \lambda \}, \quad \lambda \in]0,1],$$
$$[u]^0 := \mathrm{cl}_{\mathbb{R}} \{ x \in \mathbb{R} : u(x) > 0 \}.$$

The fuzzy number space $\mathbb{E}^1$ is the set of elements u of $F(\mathbb{R})$ satisfying the following properties:

(1) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ with $u(x_0) = 1$;
(2) u is convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}, \lambda \in [0,1]$;
(3) u is upper-semicontinuous;
(4) $[u]^0$ is a compact set in $\mathbb{R}$.

Notice that if $u \in \mathbb{E}^1$, then the λ-level set $[u]^\lambda$ of u is a compact interval for each $\lambda \in [0,1]$. We denote $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$. Every real number r can be considered a fuzzy number since r can be identified with the fuzzy number $\tilde{r}$ defined as

$$\tilde{r}(t) := \begin{cases} 1 & \text{if } t = r, \text{mean} \\ 0 & \quad \text{if } t \neq r. \end{cases}$$

We can now state the characterization of fuzzy numbers provided by Goetschel and Voxman ([5]):

Theorem 1. *Let $u \in \mathbb{E}^1$ and $[u]^\lambda = [u^-(\lambda), u^+(\lambda)]$, $\lambda \in [0,1]$. Then the pair of functions $u^-(\lambda)$ and $u^+(\lambda)$ has the following properties:*

- *$u^-(\lambda)$ is a bounded left continuous nondecreasing function on $(0,1]$;*
- *$u^+(\lambda)$ is a bounded left continuous nonincreasing function on $(0,1]$;*
- *$u^-(\lambda)$ and $u^+(\lambda)$ are right continuous at $\lambda = 0$;*
- *$u^-(1) \leq u^+(1)$.*

Conversely, if a pair of functions $\alpha(\lambda)$ and $\beta(\lambda)$ satisfy the above conditions (i)-(iv), then there exists a unique $u \in \mathbb{E}^1$ such that $[u]^\lambda = [\alpha(\lambda), \beta(\lambda)]$ for each $\lambda \in [0,1]$.

Given $u, v \in \mathbb{E}^1$ and $k \in \mathbb{R}$, we can define $u + v := [u^-(\lambda), u^+(\lambda)] + [v^-(\lambda), v^+(\lambda)]$ and $ku := k[u^-(\lambda), u^+(\lambda)]$. It is well-known that $\mathbb{E}^1$ endowed with this two natural operations is not a vector space. Indeed $(\mathbb{E}^1, +)$ is not a group.

On the other hand, we can endow $\mathbb{E}^1$ with the following metric:

Definition 2 ([5, 2])**.** For $u, v \in \mathbb{E}^1$, we can define

$$d_\infty(u,v) := \sup_{\lambda \in [0,1]} \max \left\{ |u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)| \right\}.$$

It is called the supremum metric on $\mathbb{E}^1$, and $(\mathbb{E}^1, d_\infty)$ is well-known to be a complete metric space. Notice that, by the definition of d_∞, $\mathbb{R}$ endowed with the euclidean topology can be topologically identified with the closed subspace $\tilde{R} = \{ \tilde{x} : x \in \mathbb{R} \}$ of $(\mathbb{E}^1, d_\infty)$ where $\tilde{x}^+(\lambda) = \tilde{x}^-(\lambda) = x$ for all $\lambda \in [0,1]$. As a metric space, we shall always consider $\mathbb{E}^1$ equipped with the metric d_∞.

Proposition 3. *The metric d_∞ satisfies the following properties:*

(1) $d_\infty(\sum_{i=1}^m u_i, \sum_{i=1}^m v_i) \leq \sum_{i=1}^m d_\infty(u_i, v_i)$ *where* $u_i, v_i \in \mathbb{E}^1$ *for* $i = 1, ..., m$.
(2) $d_\infty(ku, kv) = k d_\infty(u,v)$ *where* $u, v \in \mathbb{E}^1$ *and* $k > 0$.
(3) $d_\infty(ku, \mu u) = \mid k - \mu \mid d_\infty(u, 0)$, *where* $u \in \mathbb{E}^1$, $k \geq 0$ *and* $\mu \geq 0$.
(4) $d_\infty(ku, \mu v) \leq \mid k - \mu \mid d_\infty(u, 0) + \mu d_\infty(u, v)$, *where* $u, v \in \mathbb{E}^1$, $k \geq 0$ *and* $\mu \geq 0$.

We shall denote by $C(K, \mathbb{E}^1)$ the space of continuous functions defined between the compact Hausdorff space K and the metric space $(\mathbb{E}^1, d_\infty)$. In $C(K, \mathbb{E}^1)$ we shall consider the following metric:

$$D(f,g) = \sup_{t \in K} d_\infty(f(t), g(t)),$$

which induces the uniform convergence topology on $C(K, \mathbb{E}^1)$.

Proposition 4. *Let $\phi \in C(K, \mathbb{R}^+)$ and $f \in C(K, \mathbb{E}^1)$. Then the function $k \mapsto \phi(k)f(k)$, $k \in K$, belongs to $C(K, \mathbb{E}^1)$.*

3. A version of the Stone-Weierstrass theorem in Fuzzy Analysis.

Let us first introduce a basic tool to obtain our main theorem (Theorem 11).

Definition 5. Let W be a nonempty subset of $C(K, \mathbb{E}^1)$. We define

$$Conv(W) = \{\varphi \in C(K, [0,1]) : \varphi f + (1 - \varphi) g \in W \quad for \;\; all \;\; f, g \in W\}.$$

Proposition 6. *Let W be a nonempty subset of $C(K, \mathbb{E}^1)$. Then we have:*

(1) $\phi \in Conv(W)$ implies that $1 - \phi \in Conv(W)$.
(2) If $\phi, \varphi \in Conv(W)$, then $\phi \cdot \varphi \in Conv(W)$.
(3) If ϕ belongs to the uniform closure of $Conv(W)$, then so does $1 - \phi$.
(4) If ϕ, φ belong to the uniform closure of $Conv(W)$, then so does $\phi \cdot \varphi$.
(5) Uniform closure

Definition 7. It is said that $M \subset C(K, [0, 1])$ separates the points of K if given $s, t \in K$, there exists $\phi \in M$ such that $\phi(s) \neq \phi(t)$.

Next we state two technical lemmas which will used in the sequel:

Lemma 8 ([8, Lemma 2])**.** *Let $0 < a < b < 1$ and $0 < \delta < \frac{1}{2}$. There exists a polynomial $p(x) = (1 - x^m)^n$ such that*

(1) $p(x) > 1 - \delta$ for all $0 \leq x \leq a$,
(2) $p(x) < \delta$ for all $b \leq x \leq 1$.

Lemma 9 ([8, Theorem 1])**.** *Let $W \subset C(K, \mathbb{E}^1)$. The maximum of two elements of $Conv(W)$ belongs to the uniform closure of $Conv(W)$.*

Lemma 10. *Let $W \subseteq C(K, \mathbb{E}^1)$. If $Conv(W)$ separates the points of K, then, given $x_0 \in K$ and a open neighborhood N of x_0, there exists a neighborhood U of x_0 such that for all $0 < \delta < \frac{1}{2}$, there is $\varphi \in Conv(W)$ such that*

(1) $\varphi(t) > 1 - \delta$, for all $t \in U$;
(2) $\varphi(t) < \delta$, for all $t \notin N$.

Gathering the information obtained so far, we can now state and prove a version of the Stone-Weierstrass theorem for fuzzy-number-valued continuous functions:

Theorem 11. *Let W be a nonempty subset of $C(K, \mathbb{E}^1)$ and assume that $Conv(W)$ separates points. If given $f \in C(K, \mathbb{E}^1)$ and $\varepsilon > 0$, there exists, for each $x \in K$, $g_x \in W$ such that $d_\infty(f(x), g_x(x)) < \varepsilon$, then W is dense in $(C(K, \mathbb{E}^1), D)$.*

4. Conclusion

We have proved that, under certain natural assumptions, continuous (with respect to the supremum metric) fuzzy-number-valued functions defined on a compact Hausdorff space can be (uniformly) approximated to any degree of accuracy, which yields a Stone-Weierstrass type result in this setting. A similar result for interpolating families of continuous fuzzy-number-valued functions in the sense that the uniform approximation can also demand exact agreement at any finite number of points.

References

[1] B. Bede and S. Gal, Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems 151 (2005) 581–599.

[2] P. Diamond and P. Kloeden, Metric Spaces of Fuzzy Sets: Theory and Applications, World Scientific, Singapore, 1994.

[3] D. Dubois and H. Prade, Operations on fuzzy numbers, Internat. J. of Systems Sci. 9 (1978) 613-626.

[4] J-X. Fang and Q-Y. Xue, Some properties of the space of fuzzy-valued continuous functions on a compact set, Fuzzy Sets and Systems 160 (2009) 1620–1631.

[5] R. Goetschel and W. Voxman, Elementary fuzzy calculus, Fuzzy Sets and Systems 18 (1986) 31–42.

[6] S. Hai, Z. Gong and H. Li, Generalized differentiability for n-dimensional fuzzy-number-valued functions and fuzzy optimization, Inf. Sci. 374 (2016) 151–163.

[7] H. Huang and C. Wu, Approximation of level fuzzy-valued functions by multilayer regular fuzzy neural networks, Math. Comp. Model. 49 (2009) 1311–1318.

[8] R. I. Jewett, A variation on the Stone-Weierstrass theorem, Proc. Amer. Math. Soc. 14 (1963) 690–693.

[9] M. Leshno, V. Y. Lin, A. Pinkus and S. Schocken, Multilayer feedforward networks with a nonpolynomial activation function can approximate any function, Neural Networks 6 (1993) 861–867.

[10] P. Y. Liu, Universal approximations of continuous fuzzy-valued function by multi-layer regular fuzzy neural networks, Fuzzy Sets and Systems 119 (2001) 313-320.

[11] J. B. Prolla, On the Weierstrass-Stone Theorem, J. Approx. Theory, 78 (1994), 299–313.

[12] C. Wu and Z. Gong, On Henstock integral of fuzzy-number-valued functions (I), Fuzzy Sets and Systems 120 (2001) 523–532.