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Extension of b_f-continuous functions and b_f-groups

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Abstract

Let X be a b_f-space and let G be a b_f-group. By means of the exponential mapping we characterize when a b_f-continuous function on $X \times G$ with values in a topologically complete sapce Z has a b_f-continuous extension to $\beta(X) \times G$. As a consequence we show that the product of a pseudocompact space and a b_f-group is a b_f-group. This result generalizes the fact that the product of a pseudocompact space and a pseudocompact group is pseudocompact.

Keywords: *b_f-space; b_f-group; b_f-continuous function; Stone-Čech compactification; Dieudonné topological completion; topologically complete space.*
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1. Introduction

Throughout, all spaces are by default Tychonoff and all topological groups are Hausdorff. A subset B of a space X is said to be *bounded* (in X) if each real-valued continuous function on X is bounded on B. Boundedness generalizes the notion of pseudocompactness introduced by Hewitt [9]: in fact, a space X is pseudocompact if and only if it is bounded in itself. This concept was implicit in the well-known theorem of Nachbin-Shirota which characterizes when the space of all real-valued continuous functions on a space X endowed with the compact-open topology is barrelled. The foregoing definition appears in a paper by Isiwata [10] (who called these subsets relatively pseudocompact). The denomination *bounded* is due to Buchwalter [3]. This concept also appears in Noble [12] with a different (equivalent) definition: a subset B of a space X is bounded (in X) if and only if for each locally finite family $\mathcal{U}$ of mutually disjoint, non-empty open sets in X, only finitely many members of $\mathcal{U}$ meet B. These subsets were denominated *relatively pseudocompact* in [2], [11] [12] and [14], and *functionally bounded* in [7] and [18].

Given a space X, the family of all bounded subsets of X is denoted by b. A function f from a space X into a space Y is said to be b_f-continuous if the restriction of f to each member of b can be extended to a continuous function on the whole X. A space X is called a *b_f-space* if every real-valued b_f-continuous function on X is continuous (equivalently, if every b_f-continuous function from X into a Tychonoff space Y is continuous).

It is apparent that locally pseudocompact spaces and k_r-spaces (spaces X where a real-valued function is continuous whenever its restriction to each compact subset of X is continuous) are examples of b_f-spaces. Thus, locally compact spaces, first countable spaces (in particular, metrizable spaces) are b_f-spaces too. The theory of z-closed projections [12], the distribution of the functor of the Dieudonné topological completion [4, 14], compactness of function spaces in the topology of the pointwise convergence [1], and locally pseudocompact groups [15] are some of the frameworks where b_f-spaces arise in a natural way. We encourage the reader unfamiliar with the techniques of the theory of bounded subsets to consult [16].

Let $F(X, Z)$ denote the set of all functions from a set X into a set Z. We denote by τ_b the topology of uniform convergence on members of b. It is a well-known fact that $((F(X, Z), \tau_b)$ is a Tychonoff space.

By a topological group it is understood an abstract group G equipped with a topology τ making the functions $\phi\colon G \times G \to G$ and $\varphi\colon G \to G$ defined as

$$\phi(g, h) = g \cdot h \quad \text{and} \quad \varphi(g) = g^{-1} \quad g, h \in G$$

continuous. (As usual, here $g \cdot h$ –respectively, g^{-1}– stands for the operation on G –respectively for the inverse of g–)

A b_f-group is a topological group whose underlying space is a b_f-space. Examples of b_f-groups which are neither locally pseudocompact nor first countable can be found in [15].

The aim of this note is to characterize when a b_f-continuous function on a product space $X \times G$ whit X a b_f-space and G a b_f-group has a b_f-continuous extension to $\beta(X) \times G$. The key tool is the exponential mapping. The characterization states here allows us to generalize the fact that the product of a pseudocompact space and a pseudocompact topological group is a pseudocompact space ([17]).

Our terminology and notation are standard. For instance, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{R}$ for the real numbers and $f|_A$ means the restriction of a function f to a subset A. $\beta(X)$ denotes the Stone-Čech compactification of a space X. We say that a space X is topologically complete if X is homeomorphic to a closed subspace of a product of metrizable spaces. It is known that for every space X there exists a unique topologically complete space γX, up to homeomorphisms which leave X pointwise fixed, in which X is dense and every continuous function f from X into a topologically complete space Z can be extended to a continuous function on γY. This space is called the Dieudonné topological completion of X. For notions which are not explicitly defined here, the reader might consult [6].

2. The results

One easily sees that the formula

$$\mu(f)(x)(y) = f(x, y)$$

where f is a function on $X \times Y$ into a set Z, defines a one-to-one correspondence μ between the set of all (not necessarily continuous) functions from $X \times Y$ into Z and the set of all functions from X into the set of all functions from Y into Z; this correspondence is called the *exponential mapping.* The restriction of this map to subspaces will also denoted by μ. The following theorem follows from [15, Theorem 3.2] and [8, Theorem 4.7]. It provides a useful tool for analysing b_f-extensions of b_f-continuous functions. The symbol $b_f C(X, Z)$ stands for the set of all b_f-continuous functions from a space X into a space Z. We write $b_f C(X)$ when $Z = \mathbb{R}$ endowed with its usual topology.

Theorem 1. *Let G be a b_f-group. For each space X and each topologically complete space Z, the equality*

$$\mu(b_f C(X \times G, Z)) = b_f C(X, C_b(G, Z)) \qquad (*)$$

holds.

Our basic result on extensions of b_f-continuous functions is the following

Theorem 2. *Let X be a b_f-space and let G be a b_f-group. If Z is a topologically complete space and $f \in b_f C(X \times G, Z)$, then the following conditions are equivalent:*

(i) *f has a b_f-continuous extension to $\beta(X) \times G$;*
(ii) *the closure of $\mu(f)(X)$ in $C_b(G, Z)$ is compact.*

Proof. (i)$\Longrightarrow$(ii) By Theorem 1, $\mu(f)$ belongs to $b_f C(\beta(X), C_b(G, Z))$. Being $\beta(X)$ a compact space, it is a b_f-space. Thus, $\mu(f)$ is a continuous function. The result now follows from the fact that $\mu(f)(\beta(X))$ is a compact subset of $C_b(G, Z)$.

(ii)$\Longrightarrow$(i) Since X is a b_f-space, the equality $(*)$ tells us that $\mu(f)$ is continuous. Being the closure of $\mu(f)(X)$ in $C_b(G, Z)$ compact, there exists a continuous extension, say $\widehat{\mu(f)}$, of $\mu(f)$ to $\beta(X)$. To finish the proof it suffices to apply the equality $(*)$. □

Remark 3. It is worth noting that the compact subsets of $C_b(G, Z)$ are characterized by Ascoli's theorem. Indeed, a subset K of $C_b(G, Z)$ is compact if, and only if, K is closed, pointwise bounded and evenly continuous (see [12] for details).

It is a well-known fact that the product of a compact space and a b_f-space is a b_f-space. Moreover, it follows from [14, Corollary 4.8] that, for every space Y, the equality $\gamma(K \times Y) = K \times \gamma(Y)$ holds whenever K is a compact space. Therefore we can rephrase the above result as

Theorem 4. *Let X be a b_f-space and let G be a b_f-group. If $f \in b_f C(X \times G, Z)$ with Z a topologically complete space, then the following conditions are equivalent:*

(i) *f has a continuous extension to $\beta(X) \times G$;*
(ii) *f has a continuous extension to $\beta(X) \times \gamma(G)$;*
(iii) *the closure of $\mu(f)(X)$ in $C_b(G, Z)$ is compact.*

Corollary 5. *Let X be a b_f-space and let G be a b_f-group. If $f \in b_f C(X \times G)$, the following conditions are equivalent:*

(i) *f has a continuous extension to $\beta(X) \times G$;*
(ii) *f has a continuous extension to $\beta(X) \times \gamma(G)$;*
(iii) *the closure of $\mu(f)(X)$ in $C_b(G)$ is compact.*

In particular, $X \times G$ is a b_f-space.

The product of two pseudocompact spaces need not be pseudocompact (see [13]). However, an outstanding result by Comfort and Ross [5] states that pseudocompactness is preserved by the product of two pseudocompact groups. This outcome was generalizes by Tkachenko [17] who shows that the product of a pseudocompact topological group and a pseudocompact space is pseudocompact. The following result extends Tkachenko's theorem.

Theorem 6. *The product of a pseudocompact space X and a b_f-group G is a b_f-space. In addition, the equality $\gamma(X \times G) = \beta(X) \times \gamma(G)$ holds.*

Proof. Let f be a b_f-continuous function on $X \times G$. An argument similar to the one used in (2)$\Longrightarrow$(1) of Theorem 1 shows that $\mu(f)$ is continuous. Therefore $\mu(f)(X)$ is a pseudocompact subset of $C_b(G)$. Being $C_b(G)$ a topologically complete space ([15, Lemma 3.1]), the closure of $\mu(f)(X)$ in $C_b(G)$ is compact. The result now easily follows from Theorem 2. $\square$

References

[1] A. V. Arkhangels'kii, Function spaces in the topology of pointwise convergence and compact sets, Russian Math. Surveys 39 (1984), 9–56.

[2] R. L. Blair and M. A. Swardson, Spaces with an Oz Stone-Čech compactification, Topology Appl. 36(1) (1990), 73–92.

[3] H. Buchwalter, Topologies et compactologies, Publ. Dép. Math. (Lyon) 6(2) (1969), 1–74.

[4] H. Buchwalter, Produit topologique, produit tensoriel et c-repletion, Actes du Colloque d'Analyse Fonctionnelle de Bordeaux (Univ. Bordeaux, 1971), pp. 5171. Bull. Soc. Math. France, Mem. No. 3132, Soc. Math. France, Paris, 1972 (in French).

[5] W. W. Comfort and K. A. Ross, Pseudocompactness and uniform continuity in topological groups, Pacific J. Math. 16 (1966), 483–496.

[6] R. Engelking, General Topology, Sigma Series in Pure Mathematics 6, Heldermann Verlag Berlin, 1989.

[7] S. Hernández and M. Sanchis, G_δ-open functionally bounded subsets in topological groups, Topology Appl. 53 (3) (1993), 289–299.

[8] S. Hernández, M. Sanchis and M. Tkačenko, Bounded sets in spaces and topological groups, Topology Appl. 101 (1) (2000), 21–43.

[9] E. Hewitt, Rings of real-valued continuous functions. I, Trans. Amer. Math. Soc. 64 (1948), 45–99.

[10] T. Isiwata, Mappings and spaces, Pacific J. Math. 20 (1967), 455–480; correction, ibid., 23 (1967), 630–631.

[11] M. Mandelker, Supports of continuous functions, Trans. Amer. Math. Soc. 156 (1971), 73–83.

[12] N. Noble, Ascoli theorems and the exponential map, Trans. Amer. Math. Soc. 143 (1969), 393–411.

[13] J. Novák, On the Cartesian product of two compact spaces, Fund. Math. 40 (1953), 106–112.

[14] R. Pupier, Topological completion of a product, Rev. Roumaine Math. Pures Appl. 19 (1974), 925–933.

[15] M. Sanchis, Continuous functions on locally pseudocompact groups, Topology Appl. 86 (1) (1998), 5–23.

[16] M. Sanchis, Problems on bounded subsets, Questions Answers Gen. Topology 28 (1) (2010), 65–79.

[17] M. G. Tkačenko, Compactness type properties in topological groups, Czechoslovak Math. J. 38 (113) (1988), 324–341.

[18] F. J. Trigos-Arrieta, Continuity, boundedness, connectedness and the Lindelöf property for topological groups, in Proceedings of the Conference on Locales and Topological Groups (Curaçao, 1989), volume 70, pages 199–210, 1991.