# On Generating Continuous-Curvature Paths for Line Following Problem with Curvature and Sharpness Constrains

Vicent Girbés and Leopoldo Armesto and Josep Tornero

*Abstract*— **The main contribution of the paper is to provide a method to generate continuous curvature paths in order to converge to a line, based on combinations of clothoids with line segments and circular arcs. Different kinds of continuous curvature paths have been defined in order to solve problems with different complexity. The type of paths that we have generated get the benefits of higher comfortability and safety. Generated paths take into account lower and upper bounds of sharpness and curvature simultaneously, while it is not the case of other continuous curvature paths such as Elementary [1] and BiElementary [2] paths. Wheeled mobile robots following a path with continuous curvature may also get benefit on wheels slippage reduction and low odometry errors, since transitions are softer with constant curvature rates.**

## I. Introduction

It is well known that comfortability and safety increase when generating continuous curvature paths. This aspects become crucial in transporting people or dangerous goods. Wheeled mobile robots following a path with continuous curvature may also get benefit on wheels slippage reduction and low odometry errors, since transitions are softer with constant curvature rates. However, all these aspects have shown little attention and most of well known path planners do not take continuous curvature into account.

In order to generate continuous curvature paths, most of researchers have used clothoids as transitions curves between lines segments and circular segments and their combinations [1], [2], [3], [4], [5]. Clothoids are convenient because they provide better comfort (by increase gradually the centrifugal), desirable arragement for superelevation and satisfactory road appearance. The Standards usually stablish upper bounds on clothoid sharpness based on these three criteria. In addition to this, some authors, see [6] and the references therein, have shown the inconvenience of using too large clothoid segments since they can have a potentially negative effect on driver's curve perception and safety. As a consequence, some authors suggest lower bound on the clothoid sharpness. Moreover, mechanical constrains on orientable wheels must be also taking into account when generating a path tracking real vehicle which implies bounded curvature.

This work was supported by Generalitat Valenciana (VALI+D Program), PISALA Project (Vicerectorado de Investigación Desarrollo e Innovación, Universidad Politécnica de Valencia) and PROMETEO Program funded by Conselleria Educació, Generalitat Valenciana)

Vicent Girbés, Leopoldo Armesto and Josep Tornero are with Research Institute of Design and Manufacturing at Universidad Politécnica de Valencia, C/Camino de Vera, 46022, Valencia, Spain {vigirjua@upvnet.upv.es, leoaran@isa.upv.es, jtornero@upvnet.upv.es}

Line following problems have been studied intensively in the past and can be applied on different approaches, covering a wide spectrum of applications such as vision-based line following, path generation for overtakes, lateral tracking, parking, etc... The goal is to generate a path that converges to the line within a given maximum time or distance. However, to the authors knowledge, none technique can guarranty continuity on the curvature in order to line following problems.

The main contribution of the paper is to provide a method to generate continuous curvature paths in order to converge to a line, based on combinations of clothoids with line segments and circular arcs. Different kinds of continuous curvature paths have been defined in order to solve problems with different complexity. The simplest one provides continuous curvatures profiles that cannot change from positive to negative or viceversa. More complex solutions provides a general curvature profile that can cope with changes on curvature sign. The type of solutions that we provide get the benefits of higher comfortability and safety. In addition to this, lower and upper bound on sharpness and curvature are simultaneously taken into account, so every path satisfies those constrains. It is interesting to remark, that those constrains have not been considered simultaneously in other continuous curvature paths such as Elementary [1] and BiElementary [2] paths.

### A. Related Work

In recent years many researchers have used clothoids because of their interesting geometric properties and their benefits in comfort and safety. In mobile robotics, clothoids have been used to generate trajectories in navigation problems such as: obstacle avoidance [7], overtaking and lane changing [8], [9], [10], parking [11], [12], [13], among others. In addition to this, clothoids are commonly used in highway design [14] and coasters [15]. Road identification and modeling based on vision systems can be also carried out with clothoids [16], [17].

One of the simplest approximations for trajectory generation using clothoids is to use RS-paths [18], concatenating line segments and circles. The drawback of this technique is that the resulting curvature is discontinuous and clothoids must be introduce to guarantee a constant rate of curvature. In [4] clothoids were initially introduced as transition curves for these kind of trajectories, while [5] proposed using clothoids segments to interpolate and connect two points. In [19] a single continuous curvature path is created with a sequence of way points. [20] and [21] introduce the concept of anti-clothoid (the inverse of the clothoid) so that the trajectory changes from actual curvature of the vehicle to zero curvature (straight line).

In [1], Elementary paths were first introduced, a combination of two symmetrical clothoids with the same homotecial factor. These ideas were extended in [2], by introducing the concept of BiElementary paths, combinations of two Elementary paths. In BiElementary paths the initial and final configurations are not necessary symmetric, but the loci of the intermediate configuration is restricted to a circle with specific orientations to ensure that each Elementary path contain symmetrical clothoids. Obviously, the solution space is significantly limited in those cases and Elementary and BiElementary paths might not be appropiate to solve specific problems, specially the obstacle avoidance problem or the line following problem with bounded sharpness and curvature.

Dubin's curves were the inspiration in [3] to create the SCC-paths (simple continuous-curvature paths) and thus simplify the problem of finding optimal path. Each path is defined as the combination of a maximum of 7 parts between clothoids, circular arcs and line segments. In [22] a non-holonomic robot without curvature constrains was used to design a generic planner by combining clothoids and anti-clothoids segments. In [23] RS paths were introduced, and in [24] were used to create the CC paths that ensure continuity, by replacing circular arcs to the called CC-turns. In [25] generically global planner continuous curvature paths for vehicles is described. It combines existing systems based on collision avoidance introducing clothoids, lines and circles.

In [26], a Chebyshev model is used to approximate Fresnel integrals and in [27] is developed a rational approximation, both using the Taylor series expansion. In [28], Fresnel integral are approximated using a recursive method with an error less than $6 \cdot 10^{-10}$, while in [29] clothoids in the range $[0 - \frac{\pi}{2}]$ are approximated using Bezier curves and B-splines. In [30] made the estimation of the curve by a pair of adjacent clothoids. In [31] try to show that third order polynomial approximation to estimate the curve is a poor method, because it is very sensitive to noise of the sensors or the lateral offset of vehicle on the road. A method for performing a polynomial clothoid approximation by s-power series can be found in [32] and in [33] the approach is based on arc splines. Recently, in [9], [34], clothoids were approximated using Rational Bezier curves, providing a real-time planner using clothoidal trajectories.

## II. Clothoid Properties Summary

**Definition** Cornu's Spiral or Clothoid is defined by the Fresnel integrals in $\Re^2$ as follows:

$$\mathcal{C}(\gamma) = \begin{bmatrix} C_x(\gamma) \\ C_y(\gamma) \end{bmatrix} = K \begin{bmatrix} \int_0^\gamma \cos \frac{\pi}{2}\xi^2 d\xi \\ \int_0^\gamma \sin \frac{\pi}{2}\xi^2 d\xi \end{bmatrix} \tag{1}$$

where $K$ is the Homotetical factor, i.e.: the scale of the spiral, and $\gamma$ is comprises the integration interval. Unfortunately, there is no closed-form solution to compute Fresnel integrals, however some interesting geometric properties of clothoids can be analytically computed.

**Properties** Let $\mathcal{C}(\gamma)$ be a clothoid curve, the so called clothoid parameter $A$ and its homotetical factor are related by $K = \sqrt{\pi}A$. The tangent angle $\tau$ with respect to the abscissa axis $\mathcal{X}^+$ of $\mathcal{C}(\gamma)$ is $\tau = \frac{\pi}{2}\gamma^2$. The curvature $\kappa$ and length $L$ of the clothoid $\mathcal{C}(\gamma)$ increase proportionally with $\gamma$ for a given Homotetical factor, being the expression $\kappa = \pi\frac{\gamma}{K}$ for curvature and $L = K\gamma$ for the length. It is straight forward to see that both, curvature and length are related by the clothoid parameter as $\kappa = \frac{L}{A^2}$, which implies that constant changes on the curvature are proportional to changes on the length of the curve.

**Properties** Let $\mathcal{C}(\gamma)$ be a clothoid curve with a constant velocity $v$ and the sharpness $\sigma \equiv A^{-2}$, the curvature derivative of $\mathcal{C}(\gamma)$ is constant and given by $\dot{\kappa} = v\sigma$. Clothoid derivatives can be analytically computed[35] The tangential and normal components of acceleration are $a_t = 0$ and $a_n = v^2\kappa$, respectively, while the tangential and normal components of rate of acceleration (jerk) are $j_t = -v^3\kappa^2$ and $j_n = v^2\dot{\kappa} = v^3\sigma$ respectively.

## III. Line Following with Continuous Curvature Paths

### A. Problem Statement

**Definition** Let $\mathcal{R}$ a non-holonomic wheeled robot moving on a 2D plane with extended state space $\mathbf{q} = (x, y, \theta, \kappa)^T \in \Re^2 \times \mathcal{S} \times \Re$ containing the robot Cartesian positions $x$ and $y$, the robot orientation $\theta$ and the curvature $\kappa$. The kinematic model for $\mathcal{R}$ is:

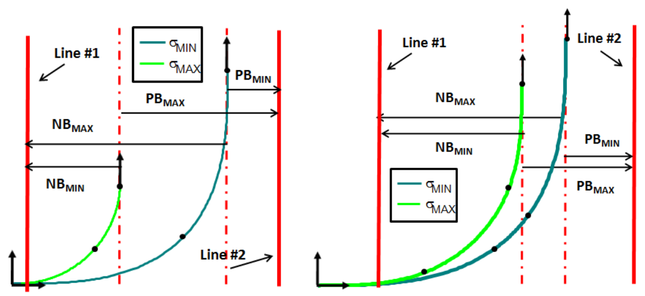
$$\dot{\mathbf{q}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\theta}(t) \\ \dot{\kappa}(t) \end{pmatrix} = \begin{pmatrix} v\cos(\theta) \\ v\sin(\theta) \\ \kappa(t)v \\ v\sigma \end{pmatrix} \tag{2}$$

being $v$, $\sigma$ the velocity and sharpness to describe a path respectively, both assumed constant for simplicity. Without loss of generality, the robot configuration is located at the origin $\mathbf{q}_A = (x_A, x_B, \theta_A, \kappa_A)^T$ with $x_A = 0$, $y_A = 0$, initial null curvature $\kappa_A = 0$ and any arbitrary orientation $\theta_A$.

**Definition** Let $\mathcal{R}$ a non-holonomic wheeled robot moving on a 2D plane with bounded curvature $\kappa \in [-\kappa_{max}, \kappa_{max}]$ and sharpness $\sigma \in [\sigma_{min}, \sigma_{max}]$. The curvature bounds are due to mechanical constrains of orientable wheels, while sharpness bound are introduces to satisfy comfortability, desirable arragement for superelevation and road appereance and increase safety as discussed on section I.

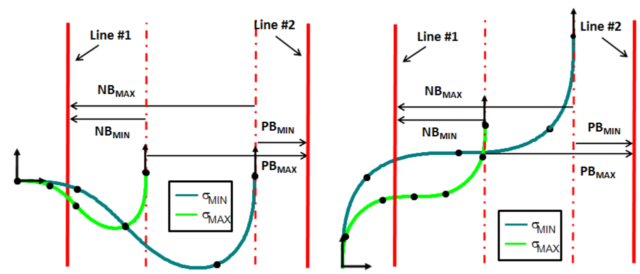
**Remark** Relaxation of the lower bound $\sigma_{min} = 0$ is very common and in most of the cases feasible even when generating paths for a mobile robot, although we include it here in order to provide a general formulation.

The goal is to generate a continuous curvature path $\mathcal{P}$ connecting the robot pose $\mathbf{q}$ (with initial null curvature) to a target configuration $\mathbf{q}_B = (x_B, y_B, \theta_B, \kappa_B)^T$ with



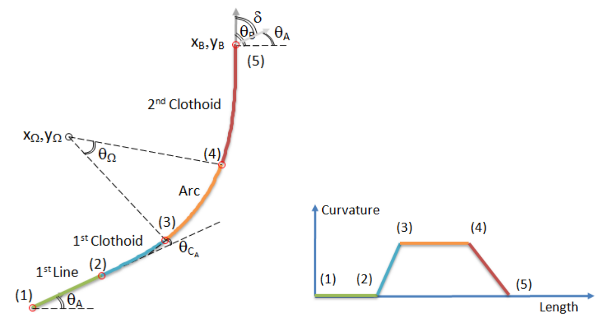

(a) Two clothoids (b) Two clothoids and circular arc

Fig. 1. Solution space with two and four clothoids and optionally circular arcs and line segments to compensate bias.




(a) Four clothoids (b) Four clothoids and circular arcs

Fig. 2. Solution space with four clothoids and optionally circular arcs and line segments to compensate bias.




(a) Variables definition (b) Curvature Profile

Fig. 3. Representative example of a SCC path.

separation distance $x_B = d_h$, final null curvature $\kappa_B = 0$, orientation $\theta_B$ and a given vertical separation $y_B$. Without loss of generality $\theta_B = \frac{\pi}{2}$ and $d_h > 0$. It should be remarked that $\mathcal{P}$ may contain not only Cartesian points on the 2D plane, but also orientations (tangential directions), curvatures and their derivatives, which can be obtained using clothoid properties of section II.

Let's first study how to solve the line following problem with continuous curvature paths with bounded sharpness and curvature. Figure 1(a) shows the type of solutions that can be obtained using two symmetric clothoids. Depending on the angle between the line and the initial configuration (deflection angle), clothoid sharpness and curvature can be adjusted to converge parallel to the line, but in general we can not guarantee convergence without bias and therefore an initial line segment is nedded to compensate such as bias. When deflection angles are too high or bounds are too tight, two symmetric clothoids can not provide the desired final orientation and therefore a circular arc segment is needed to compensate deflection angle in addition to a line segment to compensate the bias, as shown in Figure 1(b). The figures show the cases where the bias to be compensated with the line segments should be positive (PB) of negative (NB) for their minimum and maximum values. In addition to this, there might be situations where a change on the curvature sign is mandatory, as shown in Figure 2(b). In that particular case, the initial and final orientations are the same, but in order to compensate the bias, we need to increase and decrease the curvature, requiring four clothoids to perform the complete path. Like in the previous examples, two circular arcs might be needed in case of large deflection angles or too tight bounds, each of them connecting the clothoids are their maximum curvature points. Similary, two line segments might be required to compensate the bias, one at the beginning of the path as shown in Figure 2(a) or in between the intermediate clothoids as shown in Figure 2(b).

### *B. Single Continuous Curvature Path*

**Definition** A single continuous curvature path (SCC) is composed by a line segment, a first clothoid, a circle segment (arc) and a second clothoid, as shown in Figure 3(a) with a curvature profile like the one shown in Figure 3(b). These paths are similar to the ones defined in [3], but clothoids are not neccesarily symmetric. As particular cases the length of line segment and arc can be zero and so only two clothoids are necessary, known as Elementary path [1], which requires that clothoids to be symmetric (with the same sharpness and curvature).

In SCC paths, the arc angle $\theta_\Omega$ must satisfy $\theta_\Omega = \theta_B - \theta_A - \theta_{C_A} - \theta_{C_B}$, being $\theta_{C_A} \in ]0, \kappa_{max}^2 \sigma_{min}^{-1}/2]$ the tangent angle of the first clothoid, with sharpness $\sigma_{C_A}$ and maximum curvature $\kappa_{max}$ and $\theta_{C_B} \in ]0, \kappa_{max}^2 \sigma_{min}^{-1}/2]$ the tangent angle of the second clothoid, with sharpness $\sigma_{C_B}$, where both tangent angles must satisfy $\theta_{C_A} + \theta_{C_B} < \delta$. Therefore, $\delta_{max} \equiv \kappa_{max}^2 \sigma_{min}^{-1}$ is the maximum deflection angle that can be covered only with two clothoids.

Let's assume that clothoids sharpness are known and given by:

$$\sigma_{C_A} = \alpha_A(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{3}$$

$$\sigma_{C_B} = \alpha_B(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{4}$$

being $\alpha_A \in [0, 1]$ and $\alpha_B \in [0, 1]$ two design parameters. Note that the clothoids do not need to be symmetric.

Therefore, the curvature $\kappa_{C_A} = \kappa_{C_B}$ of the circle joining the clothoids must satisfy:

$$\kappa_{C_A} = \min\{\sqrt{\sigma_{C_A}\delta}, \sqrt{\sigma_{C_B}\delta}, \kappa_{max}\} \tag{5}$$

obtaining a family of curvature curves for different values of $\sigma$ depending on the deflection angle as shown in Figure 4.

It can be shown that, the arc has zero length if $\sqrt{\sigma_{C_A}\delta} = \sqrt{\sigma_{C_B}\delta} < \kappa_{max}$, otherwise, the center of the

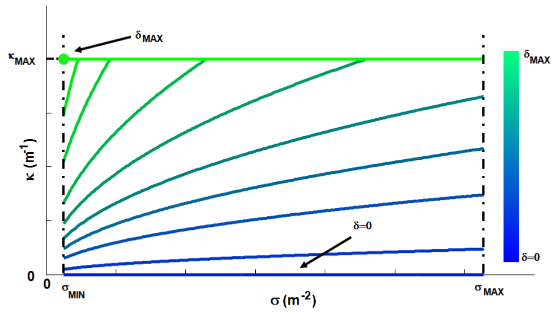



Fig. 4. Curvature curve selection for different values of sigma.

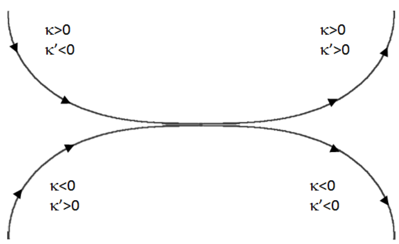



Fig. 5. Clothoid paths defined in four quadrants.

circle with radius $r_\Omega = \kappa_{C_A}^{-1}$ whose frame is located at the origin of the first clothoid is:

$$x_\Omega = x_{C_A} - r_\Omega \sin\theta_{C_A} \tag{6}$$

$$y_\Omega = y_{C_A} + r_\Omega \cos\theta_{C_A} \tag{7}$$

being $\mathbf{q}_{C_A} = (x_{C_A}, y_{C_A}, \theta_{C_A}, \kappa_{C_A})^T$ the final point of the first clothoid given by Fresnel integrals restated here for convenience with $\gamma_{C_A} = \sqrt{\frac{\kappa_{C_A}^2}{\pi\sigma_{C_A}}}$:

$$x_{C_A} = \pm\sqrt{\frac{\pi}{\sigma_{C_A}}}\int_0^{\gamma_{C_A}} \cos\frac{\pi}{2}\xi^2 d\xi \tag{8}$$

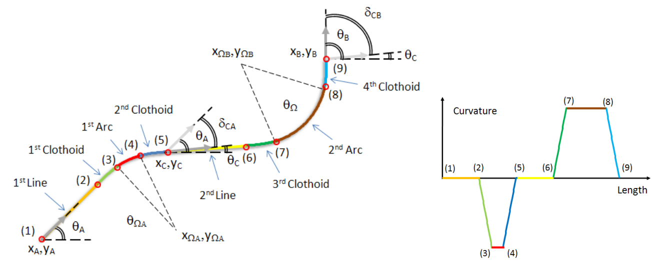
$$y_{C_A} = \pm\sqrt{\frac{\pi}{\sigma_{C_A}}}\int_0^{\gamma_{C_A}} \sin\frac{\pi}{2}\xi^2 d\xi \tag{9}$$

$$\theta_{C_A} = \pm\frac{\kappa_{C_A}^2}{2\sigma_{C_A}} \tag{10}$$

where $\mathbf{q}_{C_B} = (x_{C_B}, y_{C_B}, \theta_{C_B}, \kappa_{C_B})^T$ is similarly defined. Eqs. (8) and (9) indicate that any clothoidal combination is possible (on each quadrant) and its sign depends on the conditions that we need positive/negative curvature and positive/negative curvature rate, as shown in Figure 5.

The final configuration $\mathbf{q}_B$ depends on the selected values $\alpha_A$ and $\alpha_B$, determining the curvatures $\kappa_{C_A}$ and $\kappa_{C_B}$ and sharpness $\sigma_{C_A}$ and $\sigma_{C_B}$ of each clothoid:

$$\begin{aligned}\mathbf{q}_B = &\underbrace{\begin{pmatrix} x_A \\ y_A \\ \theta_A \\ \kappa_A \end{pmatrix}}_{\text{origin}} + \underbrace{\begin{pmatrix} l_A\cos\theta_A \\ l_A\sin\theta_A \\ 0 \\ 0 \end{pmatrix}}_{\text{1st line segment}} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_A)\begin{bmatrix} x_{C_A} \\ y_{C_A} \end{bmatrix} \\ \theta_{C_A} \\ \kappa_{C_A} \end{pmatrix}}_{\text{1st clothoid}} + \\ &+ \underbrace{\begin{pmatrix} r_\Omega \mathbf{R}(\theta_A + \theta_{C_A})\begin{bmatrix} \sin\theta_\Omega \\ 1-\cos\theta_\Omega \end{bmatrix} \\ \theta_\Omega \\ 0 \end{pmatrix}}_{arc} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_B)\begin{bmatrix} x_{C_B} \\ -y_{C_B} \end{bmatrix} \\ \theta_{C_B} \\ -\kappa_{C_B} \end{pmatrix}}_{\text{2nd clothoid}}\end{aligned} \tag{11}$$




(a) Variables definition (b) Curvature Profile

Fig. 6. Representative examples of DCC paths.

where $\mathbf{R}(\bullet)$ is a rotation matrix and $l_A$ is the length of the line segment. Also note that the curvature of the circle is $\kappa_{C_A}$ but doesn't change, so its contribution to the final curvature is zero.

For the line following case, It is quite straight forward to compute the length of segments from Eq. (11) to converge to the line given $\alpha_A$ and $\alpha_B$:

$$l_A = \frac{X}{\cos\theta_A} \tag{12}$$

with,

$$\begin{aligned} X = &x_B - x_A - y_{C_B} - x_{C_A}\cos\theta_A + y_{C_A}\sin\theta_A - \\ &- 2r_\Omega\cos(\theta_A + \theta_{C_A} + \theta_\Omega/2)\sin(\theta_\Omega/2) \end{aligned} \tag{13}$$

and the distance where the SCC path will reach the line is:

$$\begin{aligned} y_B = &x_{C_A}\sin\theta_A + y_{C_A}\cos\theta_A + x_{C_B} + l_A\sin\theta_A + \\ &+ 2r_\Omega\sin(\theta_A + \theta_{C_A} + \theta_\Omega/2)\sin(\theta_\Omega/2) \end{aligned} \tag{14}$$

**Remark** SCC paths have a singular situation when the initial configuration is aligned with the line without performing loops. It is quite obvious that with a curvature profile that can not change the sign it is not possible to perform the expected “s” shape to converge to the line.

### *C. Double Continuous Curvature Paths*

**Definition** Double continuous curvature paths (DCC) use two single continuous curvature paths (SCC) to provide a set of general solutions. The first SCC path is noted with subindex $A$ and starts at point $\mathbf{q}_A$, while the second SCC path is noted with subindex $B$ finishing at point $\mathbf{q}_B$. The configuration joining both SCC paths is $\mathbf{q}_C = (x_C, y_C, \theta_C, \kappa_C)^T$, with $\kappa_C = 0$. Figure 6(a) shows an example of a DCC path together with the curvature profile in Figure 6(b). It can be appreciated that in that case, we need to generate four clothoids named as $A1$, $A2$, $B1$ and $B2$, two circular segments $\Omega_A$ and $\Omega_B$ and two line segments with length $l_A$ and $l_B$ to properly guarantee the appropiate changes on the curvature. Particular solutions can be derived using only four clothoids, obtaining similar solutions of BiElementary path [2]. However, it is interesting to remark the generalization of the formulation provided in this section will allow us to generate a wide spectrum of possible paths with clothoids with different sharpness that can be indistinctly combined.

Let's assume that the sharpness of the four clothoids is given by:

$$\sigma_{C_{A1}} = \alpha_{A1}(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{15}$$
$$\sigma_{C_{A2}} = \alpha_{A2}(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{16}$$
$$\sigma_{C_{B1}} = \alpha_{B1}(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{17}$$
$$\sigma_{C_{B2}} = \alpha_{B2}(\sigma_{max} - \sigma_{min}) + \sigma_{min} \tag{18}$$

being $\alpha_{A1}$, $\alpha_{A2}$, $\alpha_{B1}$ and $\alpha_{B2}$ design parameters to be determined.

The maximum curvature of each clothoid is:

$$\kappa_{A1} = \kappa_{A2} = \min\{\sqrt{\sigma_{C_{A1}}\delta_A}, \sqrt{\sigma_{C_{A2}}\delta_A}, \kappa_{max}\} \tag{19}$$
$$\kappa_{B1} = \kappa_{B2} = \min\{\sqrt{\sigma_{C_{B1}}\delta_B}, \sqrt{\sigma_{C_{B2}}\delta_B}, \kappa_{max}\} \tag{20}$$

being $\delta_A = |\theta_A - \theta_C|$ and $\delta_B = |\theta_B - \theta_C|$ the deflection angles between configurations $\angle(\mathbf{q}_A, \mathbf{q}_C)$ and $\angle(\mathbf{q}_C, \mathbf{q}_B)$.

With out loss of generality and for $x_A = 0$, $y_A = 0$, $x_B > 0$, $\theta_B = \pi/2$, if $\theta_A > \theta_B$ implies that $\delta_A > \delta_B$ and therefore $\theta_C$ should be bounded between $\theta_C \in [\theta_A - \pi/2, \theta_B]$. On the other hand, if $\theta_A \leq \theta_B$, implies that $\delta_A \leq \delta_B$ and consequently $\theta_C \in [\theta_B - \pi/2, \theta_A]$. The lower bound of $\theta_C$ is selected to avoid deflection angles higher than $\pi$ and the upper bound to is chosen to ensure that the continuous curvature paths will converge to the line, avoiding singular cases.

Let $\alpha_C$ an additional design parameter to be determined so that:

$$\theta_C = \begin{cases} \alpha_C(\theta_B - \theta_A + \pi/2) + \theta_A + \pi/2 & \text{if } \theta_A > \theta_B \\ \alpha_C(\theta_A - \theta_B + \pi/2) + \theta_B + \pi/2 & \text{if } \theta_A \leq \theta_B \end{cases} \tag{21}$$

The intermediate configuration $\mathbf{q}_C$ depends on the selected values $\alpha_{A1}$ and $\alpha_{A2}$, determining the curvatures $\kappa_{C_{A1}}$ and $\kappa_{C_{A2}}$ and sharpness $\sigma_{C_{A1}}$ and $\sigma_{C_{A2}}$ of each clothoid:

$$\begin{aligned}\mathbf{q}_C = &\underbrace{\begin{pmatrix} x_A \\ y_A \\ \theta_A \\ \kappa_A \end{pmatrix}}_{\text{origin}} + \underbrace{\begin{pmatrix} l_A \cos\theta_A \\ l_A \sin\theta_A \\ 0 \\ 0 \end{pmatrix}}_{1^{\text{st}}\text{ line segment}} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_A)\begin{bmatrix} x_{C_{A1}} \\ y_{C_{A1}} \end{bmatrix} \\ -\theta_{C_{A1}} \\ -\kappa_{C_{A1}} \end{pmatrix}}_{1^{\text{st}}\text{ clothoid}} + \\ &+ \underbrace{\begin{pmatrix} r_{\Omega_A}\mathbf{R}(\theta_A - \theta_{C_{A1}})\begin{bmatrix} \sin\theta_{\Omega_A} \\ \cos\theta_{\Omega_A} - 1 \end{bmatrix} \\ -\theta_{\Omega_A} \\ 0 \end{pmatrix}}_{arc} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_C)\begin{bmatrix} x_{C_{A2}} \\ y_{C_{A2}} \end{bmatrix} \\ -\theta_{C_{A2}} \\ \kappa_{C_{A2}} \end{pmatrix}}_{2^{\text{nd}}\text{ clothoid}}\end{aligned} \tag{22}$$

While the final configuration $\mathbf{q}_B$ depends on the selected values $\alpha_{B1}$ and $\alpha_{B2}$, determining the curvatures $\kappa_{C_{B1}}$ and $\kappa_{C_{B2}}$ and sharpness $\sigma_{C_{B1}}$ and $\sigma_{C_{B2}}$ of each clothoid:

$$\begin{aligned}\mathbf{q}_B = &\underbrace{\begin{pmatrix} x_C \\ y_C \\ \theta_C \\ \kappa_C \end{pmatrix}}_{\text{origin}} + \underbrace{\begin{pmatrix} l_B \cos\theta_C \\ l_B \sin\theta_C \\ 0 \\ 0 \end{pmatrix}}_{1^{\text{st}}\text{ line segment}} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_C)\begin{bmatrix} x_{C_{B1}} \\ y_{C_{B1}} \end{bmatrix} \\ \theta_{C_{B1}} \\ \kappa_{C_{B1}} \end{pmatrix}}_{1^{\text{st}}\text{ clothoid}} + \\ &+ \underbrace{\begin{pmatrix} r_{\Omega_B}\mathbf{R}(\theta_C + \theta_{C_{B1}})\begin{bmatrix} \sin\theta_{\Omega_B} \\ 1 - \cos\theta_{\Omega_B} \end{bmatrix} \\ \theta_{\Omega_B} \\ 0 \end{pmatrix}}_{arc} + \underbrace{\begin{pmatrix} \mathbf{R}(\theta_B)\begin{bmatrix} x_{C_{B2}} \\ -y_{C_{B2}} \end{bmatrix} \\ \theta_{C_{B2}} \\ -\kappa_{C_{B2}} \end{pmatrix}}_{2^{\text{nd}}\text{ clothoid}}\end{aligned} \tag{23}$$

with $x_{C_{A1}}, y_{C_{A1}}, \theta_{C_{A1}}, x_{C_{A2}}, y_{C_{A2}}, \theta_{C_{A2}}, x_{C_{B1}}, y_{C_{B1}}, \theta_{C_{B1}}$, $x_{C_{B2}}$, $y_{C_{B2}}$ and $\theta_{C_{B1}}$ similar to those defined in Eqs. (8) to (10), being radius of arcs $r_{\Omega_A} = \kappa_{C_{A1}}^{-1}$, $r_{\Omega_B} = \kappa_{C_{B1}}^{-1}$ with angles $\theta_{\Omega_A} = \theta_C - \theta_A - \theta_{C_{A1}} - \theta_{C_{A2}}$ and $\theta_{\Omega_B} = \theta_B - \theta_C - \theta_{C_{B1}} - \theta_{C_{B2}}$, respectively.

**Remark** Equations (22) and (23) have been formulated for the case for curvature profiles like those shown in Figure 6(b). They can be generalized to cases where the sign of curvature doesn't change even with four clothoids.

In order to converge to the line, in Line Following problem, we must satisfy $X = l_A \cos\theta_A + l_B \cos\theta_C$. Therefore, we can find multiple solutions to satisfy this relation. We have followed the following criteria to select $l_A$ or $l_B$:

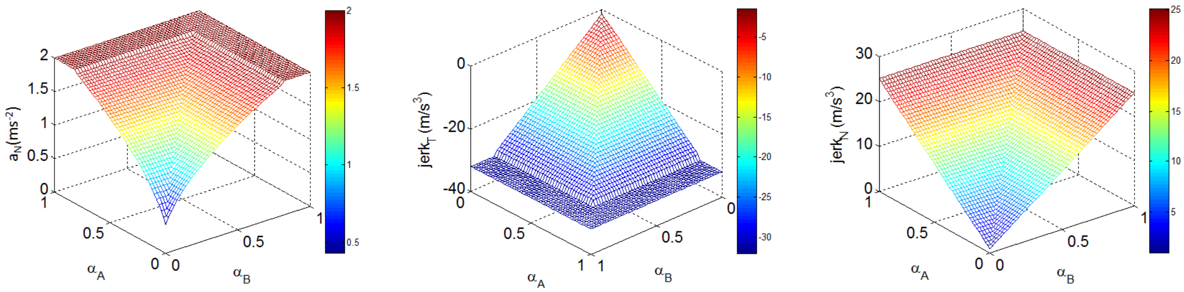
- Choose positive values (if possible) for $l_A$ or $l_B$ depending on the sign of $X$, $\cos\theta_A$ and $\cos\theta_C$ by forcing one of the length to be zero. This situation will happen whenever the signs of $X$, $\cos\theta_A$ and $\cos\theta_C$ are different or $\cos\theta_A = 0$ and $\cos\theta_C \neq 0$ or $\cos\theta_A \neq 0$ and $\cos\theta_C = 0$. So, for instance, if $X > 0$ and $\cos\theta_A > 0$, but $\cos\theta_C < 0$, then we force $l_B = 0$, being $l_A > 0$.
- There might be situations that $l_A > 0$ and $l_B > 0$ because signs of $X$, $\cos\theta_A$ and $\cos\theta_C$ are equal. In those cases, we force one of the values to zero, trying to converge to the line with the minimum $y_B$. Therefore, if $|\cos\theta_A| > |\cos\theta_C|$ then $l_A \neq 0$ and $l_B = 0$, while if $|\cos\theta_A| < |\cos\theta_C|$ then $l_A = 0$ and $l_B \neq 0$.

A singular situation can occur if both $\cos\theta_A = 0$ and $\cos\theta_C = 0$, which means that configurations $\mathbf{q}_A$, $\mathbf{q}_C$ and $\mathbf{q}_B$ have the same orientation. Therefore, an appropiate orientation for $\mathbf{q}_C$ should be selected to avoid this singularity.

## IV. Analysis

The design parameters of DCC paths determine the clothoid sharpness of each clothoid. The selected combination of clothoid sharpness not only affects to the final configuration $\mathbf{q}_B$ but also to the maximum attained curvature, the overall length and path derivatives (normal acceleration, tangential jerk and normal jerk). Therefore, it is necessary to analyse the effect these design parameters on all these aspects. For the analysis, the deflection angle has been set to zero, the start and final configuration have the same orientation, but the intermediate configuration introduces a deflection angle of $\pi/2$, that is $\theta_C = 0$ for $\alpha_C = 0.5$. Although results may change depending on the starting and intermediate configurations, we consider that is case is one of the most representative ones. It is interesting to remark that the analysis has been performed using a constant velocity.

The normal accelerations, tangential jerk and normal jerk increase with increasing sharpness as shown in Figures 7(a), 7(b) and 7(c). The normal acceleration is a square root function with respect to the sharpness because $a_n = v^2\kappa^2 = v^2 \min\{\sqrt{\sigma_A\delta}, \sqrt{\sigma_B\delta}, \kappa_{max}\}$, the tangential jerk is linear $\jmath_t = -v^3\kappa^2 = -v^3 \min\{\sigma_A\delta, \sigma_B\delta, \kappa_{max}^2\}$ and the normal jerk is linear with respect to the sharpness $j_n = v^3 \max\{\sigma_A, \sigma_B\}$.




(a) Normal Acceleration (b) Tangential Jerk (c) Normal Jerk

Fig. 7. Analysis of maximum derivative values for different values of clothoid sharpness.

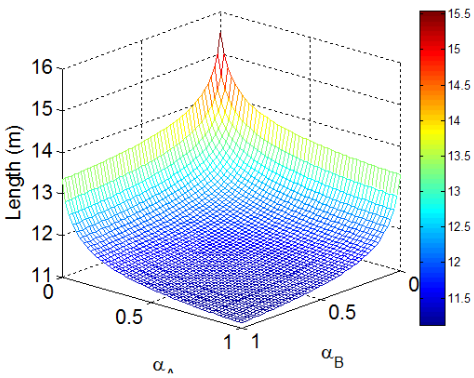
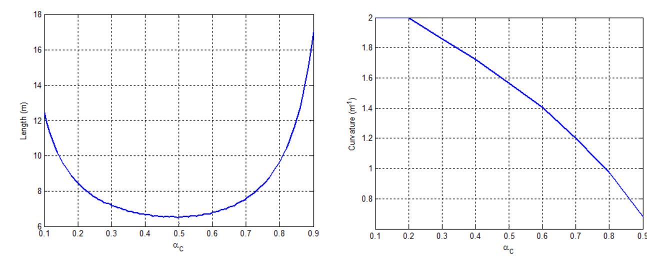



Fig. 8. Analysis of path length for different values of clothoid sharpness.




(a) Length (b) Curvature

Fig. 9. Analysis of the influence of $\alpha_C$ parameter on length and curvature of DCC paths.

On the contrary, in Figure 8, the overall length of the path can be appreciated against the design parameters $\alpha_A$ and $\alpha_B$, where in this case, we have restricted each pair of clothoids to be symmetric. As expected, the length of the path decreases with increasing sharpness since $L = \frac{\kappa}{\sigma} = \frac{\min\{\sqrt{\sigma\delta}, \kappa_{max}\}}{\sigma}$.

Now, we analyse the effect assigning different values to the design parameter $\alpha_C$. Again the initial and final configurations have the same orientation, but now the $\alpha_A$ and $\alpha_B$ parameters are kept constant. On the one hand, Figure 9(a) shows the variation of the length path with respect to the design parameter $\alpha_C$. It can be appreciated that minimum length case is for $\alpha_C = 0.5$ which means that intermediate configuration is perpendicular with respect to the line and initial configuration, which is the case of minimum deflection angle. On the other hand, Figure 9(b) shows the variation of the maximum curvature with respect to $\alpha_C$, which is again a square root function, since $\alpha_C$ affects linearly to the deflection angle.

## V. CONCLUSIONS

This paper describes a method for generating continuous curvature paths subject to constrains on curve sharpness and maximum allowable curvature. The paper first describes Single Continuous Curvature (SCC) paths , which consist of a line segment, a circular arc segment and two transition curves based on clothoids in order to guarantee a continuous curvature profile. SCC paths are extended to Double Continuous Curvature paths (DCC), which provide a complete solution to include curvature profiles with positive and negative curvatures. It has been shown that SCC and DCC paths can be applied to solve the line following problem, where the DCC paths can cope with any general configuration, while SCC provides a singular solution when the initial configuration is aligned with the line.

Wheeled mobile robots following a path with continuous curvature may also get benefit on wheels slippage reduction and low odometry errors, since transitions are softer with constant curvature rates.

Design parameters of DCC paths are the clothoid sharpness and the orientation of the intermediate configuration, which are quite intuitive to determine according to different criteria. The analysis on DCC has shown that, length, curvature and path derivatives are affected by design paramenters, where we can clearly see that configurations with symmetric SSCs provide the most balance situation for derivatives. All these aspects become crucial in transporting people or dangerous goods providing higher comfortability and safety.

## REFERENCES


[1] A. Scheuer and T. Fraichard, “Planning continuous-curvature paths for car-like robots,” in *Int. Conf. on Intelligent Robots and Systems*, 1996.

[2] ——, “Collision-free and continuous-curvature path planning for car-like robots,” in *Int. Conf. on Robotics and Automation*, 1997.

[3] ——, “Continuous-curvature path planning for multiple car-like vehicle,” in *Int. Conf. on Intelligent Robots and Systems*, 1997.

[4] J. Iijima, Y. Kanayama, and S.-i. Yuta, “A locomotion control system for mobile robots,” in *IJCAI'81: Proceedings of the 7th international joint conference on Artificial intelligence*. San Francisco, CA, USA: Morgan Kaufmann Publishers Inc., 1981, pp. 779–784.

[5] Y. Kanayama and N. Miyake, "Trajectory generation for mobile robots," in *Int. Symp. on Robotic Research*, 1985.

[6] M. A. and P. P., "A proposal to update the clothoid parameter limiting criteria of the italian standard," in *4th International Società Italiana Infrastrutture Viarie Congress*, 2007.

[7] N. Montes and J. Tornero, "Lane changing using s-series clothoidal approximation and dual-rate based on bezier points to controlling vehicle," in *Int. Conference on Systems Theory and Scientific Computation*. Stevens Point, Wisconsin, USA: World Scientific and Engineering Academy and Society (WSEAS), 2004, pp. 1–6.

[8] I. Papadimitriou and M. Tomizuka, "Fast lane changing computations using polinomials," in *American Control Conference*, 2003.

[9] N. Montes and J. Tornero, "Trajectory generation based on rational bezier curves as clothoids," in *Intelligent Vehicles Symposium*, 2007.

[10] D. K. Wilde, "Computing lothoid segments for trajectory generation," in *Int. Conference on Robotics and Automation*, 2009.

[11] J.-P. Laumond, P. Jacobs, M. Taix, and R. Murray, "A motion planner for nonholonomic mobile robots," *Robotics and Automation, IEEE Transactions on*, vol. 10, no. 5, pp. 577 –593, oct. 1994.

[12] K.Jiang, D.Z.Zhang, and L.D.Seneviratne, "A parallel parking system for a car-like robot with sensor guidance," *Proceedings of the Institution of Mechanical Engineers, Part C: Journal of Mechanical Engineering Science*, vol. 213, no. 6, pp. 591–600, 1999.

[13] L. Gracia and J. Tornero, "Geometric parallel parking planner for car-like vehicles," in *Int. Industrial Simulation Conference*, 2003.

[14] I. del Corral, *Topografía de obras*, Politext, Ed. Edicions UPC, 2001.

[15] D. L. Weiss, "Dynamic simulation and analysis of roller coasters," Ph.D. dissertation, University of California, Davis, 1998.

[16] C. Corridori and M. Zanin, "High curvature two-clothoid road model estimation," in *Intelllgenl Trsnsportation Systems Conference*, 2004.

[17] M. Manz, F. von Hundelshausen, and H. Wuensche, "A hybrid estimation approach for autonomous dirt road following using multiple clothoid segments," in *Int. Conference on Robotics and Automation*, 2010.

[18] P. Jacobs and J. Canny, "Planning smooth paths for mobile robots," in *Int. Conference on Robotics and Automation*, may. 1989, pp. 2 –7 vol.1.

[19] D. H. Shin and S. Singh, "Path generation for robot vehicles using composite clothoid segments," Robotics Institute, Pittsburgh, PA, Tech. Rep. CMU-RI-TR-90-31, December 1990.

[20] S. Fleury, P. Soueres, J.P.Laumond, and R. Chatila, "Primitives for smoothing paths of mobile robot," in *Int. Conference on Robotics and Automation*, 1993.

[21] S. Fleury, P. Souéres, J. Laumond, and R. Chatila, "Primitives for smoothing mobile robot trajectories," *IEEE Trans. Robotics and Automation*, vol. 11, pp. 441–448, 1995.

[22] A. Scheuer and Xie, "Continuous-curvature trajectory planning for maneuverable non-holonomic robots," in *Int. Conf. on Intelligent Robots and Systems*, 1999.

[23] J. Reeds and L. Shepp, "Optimal paths for a car that goes both forwards and backwards," *Pacific Journal of Mathematics*, vol. 145, pp. 367–393, 1990.

[24] S. Fraichard and Desvigne, "Form reeds and sepp's to continuous curvature paths," in *Int. Conference on Advanced Robotics*, 1999.

[25] Fraichard and Ahuactzin, "Smooth path planning for cars," in *Int. Conference on Robotics and Automation*, 2001.

[26] W. Cody, "Chevyshev approximation for fresnel integrals," *Mathematics of Computation*, vol. 22, pp. 450–453, 1968.

[27] M. Heald, "Rational approximations for the fresnel integrals," *Mathematics of Computation*, vol. 44, pp. 459–461, 1985.

[28] K. D. Mielenz, "Computation of fresnel integrals ii," *Journal of Research of the National Institute of Standards and Technology*, vol. 105, no. 4, pp. 589–590, 2000.

[29] L. Wang, K. Miura, E. Nakamae, T. Yamamoto, and T. Wang, "An approximation approach of the clothoid curve define in the interval $[0, \frac{\pi}{2}]$ and its offset by free-form curves," *Computer Aided Design*, vol. 33, pp. 1049–1058, 2001.

[30] D. Khosla, "Accurate estimation of forward path geometry using two-clothoid road model," in *Intelligent Vehicle Symposium*, 2002.

[31] D. Schwartz, "Clothoid road geometry unsuitable for sensor fusion," in *Intelligent Vehicle Symposium*, 2003.

[32] J. Sánchez-Reyes and J. Chac'on, "Polynomial approximation to clothoids via s-power series," *Computer Aided Design*, vol. 35, pp. 1305–1313, 2003.

[33] D. Meek and D. Walton, "An arc spline approximation to a clothoid," *Journal of Computational and Applied Mathematics*, vol. 170, pp. 59–77, 2004.

[34] N. Montes, A. Herraez, L. Armesto, and J. Tornero, "Real-time clothoid approximation by rational bezier curves," in *Int. Conference on Robotics and Automation*, may. 2008, pp. 2246 –2251.

[35] S. H. Schot, "Jerk: The time rate of chnge of acceleration," *American Journal of Physics*, vol. 46, pp. 1090–1094, 1978.