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Master Thesis

Innermost Termination of Context-Sensitive Rewriting

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Abstract

Innermost context-sensitive rewriting (CSR) has been proved useful for modeling the computational behavior of programs of algebraic languages like Maude, OBJ, etc, which incorporate an innermost strategy which is used to break down the nondeterminism which is inherent to reduction relations. Furthermore, innermost termination of rewriting is often easier to prove than termination. Thus, under appropriate conditions, a useful strategy for proving termination of rewriting is trying to prove termination of innermost rewriting. This phenomenon has also been investigated for context-sensitive rewriting. Up to now, only few transformation-based methods have been proposed and used to (specifically) prove termination of innermost *CSR*. Powerful and efficient techniques for proving (innermost) termination of (unrestricted) rewriting like the dependency pairs framework have not been considered yet. In this work, we investigate the adaptation of the Dependency Pairs Framework to innermost *CSR*. We provide a suitable notion of innermost context-sensitive dependency pair and show how to extend and adapt the main notions which conform the framework (chain, termination problem, processor, etc.). We show how to mechanize proofs with the dependency pair framework for proving (innermost) termination of *CSR*. Thanks to the innermost context-sensitive dependency pairs, we can now use powerful techniques for proving termination of innermost *CSR*. This is made clear by means of some benchmarks showing that our techniques dramatically improve over previously existing transformational techniques, thus establishing the new state-of-the-art in the area. We have implemented them as part of the termination tool MU-TERM.

1
Introduction

Most computational systems whose operational principle is based on reducing expressions can be described and analyzed by using notions and techniques coming from the abstract model of Term Rewriting Systems (TRSs [BN98, TeR03]). Such computational systems (e.g., functional, algebraic, and equational programming languages as well as theorem provers based on rewriting techniques) often incorporate a predefined reduction strategy which is used to break down the nondeterminism which is inherent to reduction relations. Eventually, this can rise problems, as each kind of strategy only behaves properly (i.e., it is normalizing, optimal, etc.) for particular classes of programs. One of the most commonly used strategy is the *innermost* one, in which only innermost redexes are reduced. Here, by an innermost redex we mean a redex containing no other redex. The innermost strategy corresponds to call by value or *eager* computation, that is, the computational mechanism of several programming languages where the arguments of a function are always evaluated before the application of the function which use them. It is well-known, however, that programs written in eager programming languages frequently run into a nonterminating behavior if the programs have not carefully been written to avoid such problems. For this reason, the designers of such eager programming languages have also developed some features and language constructs aimed at giving the user more flexible control of the program execution. For instance, *syntactic annotations* (which are associated to arguments of symbols) have been used in programming languages such as Clean [NSEP92], Haskell [HPW92], Lisp [McC60], Maude [CDEL$^+$07], OBJ2 [FGJM85], OBJ3 [GWM+00], CafeOBJ [FN97], etc., to improve the termination and efficiency of computations. Lazy languages (e.g., Haskell, Clean) interpret them as *strictness annotations* in order to become 'more eager' and efficient. Eager languages (e.g., Lisp, Maude, OBJ2, OBJ3, CafeOBJ) use them as *replacement restrictions* to become 'more lazy' thus (hopefully) avoiding nontermination.

Context-sensitive rewriting (*CSR* [Luc98, Luc02]) is a restriction of rewriting that forbids reductions on some subexpressions and that has proved useful to model and analyze such programming language features at different levels, see, e.g., [BM06, DLMM$^+$04, DLM$^+$08, GM04, Luc01b, LM08a]. Such a restriction of the rewriting computations is formalized at a very simple syntactic level: that of the arguments of function symbols f in the signature $\mathcal{F}$. As usual, by a *signature* we mean a set of function symbols $f_1, \ldots, f_n, \ldots$ together with an *arity* function $ar : \mathcal{F} \to \mathbb{N}$ which establishes the number of 'arguments' associated to each symbol. A *replacement map* is a mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ satisfying $\mu(f) \subseteq \{1, \ldots, k\}$, for each k-ary symbol f in the signature $\mathcal{F}$ [Luc98]. We use them to discriminate the argument positions on which the rewriting steps are allowed. In *CSR* we only rewrite μ-replacing subterms: every term t (as a whole) is μ-replacing by definition; and t_i (as well as all its μ-replacing subterms) is a μ-replacing subterm of $f(t_1, \ldots, t_k)$ if $i \in \mu(f)$.

Example 1 Consider the following orthogonal TRS $\mathcal{R}$ which is a variant of an example in [Bor03]:

$$
\begin{array}{rcl}
\mathtt{from}(x) & \to & \mathtt{cons}(x, \mathtt{from}(\mathtt{s}(x))) \\
\mathtt{sel}(\mathtt{0}, \mathtt{cons}(x, xs)) & \to & x \\
\mathtt{sel}(\mathtt{s}(y), \mathtt{cons}(x, xs)) & \to & \mathtt{sel}(y, xs) \\
\mathtt{minus}(x, \mathtt{0}) & \to & x \\
\mathtt{minus}(\mathtt{s}(x), \mathtt{s}(y)) & \to & \mathtt{minus}(x, y) \\
\mathtt{quot}(\mathtt{0}, \mathtt{s}(y)) & \to & \mathtt{0} \\
\mathtt{quot}(\mathtt{s}(x), \mathtt{s}(y)) & \to & \mathtt{s}(\mathtt{quot}(\mathtt{minus}(x, y), \mathtt{s}(y))) \\
\mathtt{zWquot}(\mathtt{nil}, \mathtt{nil}) & \to & \mathtt{nil} \\
\mathtt{zWquot}(\mathtt{cons}(x, xs), \mathtt{nil}) & \to & \mathtt{nil} \\
\mathtt{zWquot}(\mathtt{nil}, \mathtt{cons}(x, xs)) & \to & \mathtt{nil} \\
\mathtt{zWquot}(\mathtt{cons}(x, xs), \mathtt{cons}(y, ys)) & \to & \mathtt{cons}(\mathtt{quot}(x, y), \mathtt{zWquot}(xs, ys))
\end{array}
$$

together with $\mu(\mathtt{cons}) = \{1\}$ and $\mu(f) = \{1, \ldots, ar(f)\}$ for all other symbols f. According to [GM02a], innermost μ-termination of $\mathcal{R}$ implies its μ-termination as well. We will show how $\mathcal{R}$ can easily be proved innermost μ-terminating (and hence μ-terminating) by using the results in this paper. ∎

The replacement map in Example 1 exemplifies one of the most typical applications of context-sensitive rewriting as a computational mechanism. The declaration $\mu(\mathtt{cons}) = \{1\}$ disallows reductions on the *list* part of the list constructor `cons`, thus making possible a kind of *lazy evaluation* of lists. We can still use projection operators as `sel` to continue the evaluation when needed. The other typical application is the declaration $\mu(\mathtt{if}) = \{1\}$ which allows us to forbid reductions on the two *alternatives* s and t of *if-then-else* expressions $\mathtt{if}(b, s, t)$ whereas it is still possible to perform reductions on the *boolean* part b, as required to implement the usual semantics of the operator.

Termination is one of the most interesting practical problems in computation and software engineering. A program or computational system is said to be *terminating* if it does not lead to any infinite computation for any possible call or input data. Ensuring termination is often a prerequisite for essential program properties like correctness. Termination is also one of the most interesting problems when dealing with *CSR*. With *CSR* we can *achieve* a terminating behavior with nonterminating TRSs by pruning (all) infinite rewrite sequences.

Our focus is on termination of *innermost context-sensitive rewriting* (i.e., the variant of *CSR* where only the deepest μ-replacing redexes are contracted). Termination of innermost context-sensitive rewriting has been proved useful for proving termination of programs in programming languages like Maude and OBJ* which permit to control the program execution by means of such context-sensitive annotations [Luc01a, Luc01b]. Techniques for proving termination of innermost *CSR* were first investigated in [GM02b, Luc01a]. These papers, though, only consider *transformational* techniques, where the original CS-TRS $(\mathcal{R}, \mu)$ is transformed into a TRS $\mathcal{R}^{\mu}_{\Theta}$ (where Θ represents the transformation which has been used) whose *innermost* termination implies the innermost termination of *CSR* for $(\mathcal{R}, \mu)$. The *dependency pairs method* [AG00, GAO02, GTS04, GTSF06, HM04, HM05], one of the most powerful techniques for proving termination of rewriting, had not been investigated in connection with proofs of termination of *CSR* until [AGL06]. As shown in [AGL07], proofs of termination using context-sensitive dependency pairs (CSDPs) are much more powerful and faster than any other technique for proving termination of *CSR*. As we show here, dealing with innermost *CSR*, we have a similar situation.

Proving innermost termination of rewriting is often easier than proving termination of rewriting [AG00] and, for some relevant classes of TRSs, innermost termination of rewriting is even equivalent to termination of rewriting [Gra95, Gra96]. In [GM02b, GL02] it is proved that the equivalence between termination of innermost *CSR* and termination of *CSR* holds in some interesting cases (e.g., for *orthogonal* CS-TRSs).

Plan of the paper

After some preliminaries in Chapter 2, we develop the material in the paper in three main parts:

1. We investigate the structure of infinite innermost context-sensitive rewrite sequences. This analysis is essential to provide an appropriate definition of innermost context-sensitive dependency pair, and the related notions of innermost chains, graph, etc. Chapter 3 provides appropriate notions of *minimal* innermost non-μ-terminating terms and introduces the main properties of such terms. Chapter 4 recalls the notion of *hidden term* in a CS-TRS. This notion turns to be essential for the appropriate treatment of our dependency pairs. Chapter 5 investigates the structure of infinite innermost context-sensitive rewrite sequences starting from minimal innermost non-μ-terminating terms.

2. We define the notions of *innermost context-sensitive dependency pair* and *innermost context-sensitive chain of pairs* and show how to use them to *characterize* innermost termination of *CSR*. Chapters 6 and 7 introduce the general framework to compute and use innermost context-sensitive dependency pairs

for proving (innermost) termination of *CSR*. The introduction of a new kind of dependency pairs (the *collapsing* dependency pairs) leads to a notion of innermost context-sensitive dependency *chain*, which is quite different from the standard one. In Chapter 8 we prove that our *innermost context-sensitive dependency pairs approach* fully characterizes termination of innermost *CSR*.

3. We describe a suitable *framework* for dealing with proofs of termination of (innermost) *CSR* by using the previous results. Chapter 9 provides an adaptation of the *dependency pair framework* [GTS04, GTSF06] to (innermost) *CSR* by defining appropriate notions of *CS-termination problem* and *CS-processor* which rely in the notions and results investigated in the second part of the paper. Chapter 10 relates innermost termination of *CSR* and μ-termination as well as with full rewriting. Chapter 11 introduces the notion of *innermost context-sensitive (dependency) graph* and the associated CS-processor which formalizes the usual practice of analyzing the absence of infinite (minimal) innermost chains by considering the (maximal) cycles in the dependency graph. As in the standard case, the ICS-dependency graph is not computable, so we show how to obtain the *estimated* ICS-dependency graph which is a computable overestimation of it. Chapter 12 adapts the notion of usable rules to deal with proofs of innermost *CSR* by using term orderings. We introduce the notion of *μ-reduction pair*, which is the straightforward adaptation of reduction pairs used for dealing with dependency pairs in the standard case. Chapter 13 adapts *narrowing transformation* of pairs in [GTSF06] to (innermost) *CSR* and the new framework. Chapter 14 adapts to the context-sensitive setting, the notion of *usable argument* introduced by Férnandez [Fer05] to prove innermost termination of rewriting by proving termination of *CSR*. In this way, we can prove innermost termination of *CSR* by proving innermost termination of *CSR* using a *more restrictive* replacement map. We also include this criterion as a processor in the (innermost) context-sensitive dependency pairs framework.

The paper ends with an experimental evaluation of our techniques in Chapter 15. Chapter 16 concludes.

2

Preliminaries

This chapter collects a number of definitions and notations about term rewriting. More details and missing notions can be found in [BN98, Ohl02, TeR03].

Let A be a set and $\mathsf{R} \subseteq \mathsf{A} \times \mathsf{A}$ be a binary relation on A. We denote the transitive closure of R by R^+ and its reflexive and transitive closure by R^*. We say that R is *terminating* (*strongly normalizing*) if there is no infinite sequence $a_1 \mathrel{\mathsf{R}} a_2 \mathrel{\mathsf{R}} a_3 \cdots$. A reflexive and transitive relation R is a quasi-ordering.

Signatures, Terms, and Positions

Throughout the paper, $\mathcal{X}$ denotes a countable set of variables and $\mathcal{F}$ denotes a signature, i.e., a set of function symbols $\{\mathtt{f}, \mathtt{g}, \ldots\}$, each having a fixed arity given by a mapping $ar : \mathcal{F} \to \mathbb{N}$. The set of terms built from $\mathcal{F}$ and $\mathcal{X}$ is $\mathcal{T}(\mathcal{F}, \mathcal{X})$. A term is *ground* if it contains no variable. A term is said to be *linear* if it has no multiple occurrences of a single variable.

Terms are viewed as labelled trees in the usual way. Positions $p, q, \ldots$ are represented by chains of positive natural numbers used to address subterms of t. We denote the empty chain by Λ. Given positions p, q, we denote their concatenation as $p.q$. Positions are ordered by the standard prefix ordering: $p \leq q$ if $\exists q'$ such that $q = p.q'$ If p is a position, and Q is a set of positions, $p.Q = \{p.q \mid q \in Q\}$. The set of positions of a term t is $\mathcal{P}os(t)$. Positions of nonvariable symbols in t are denoted as $\mathcal{P}os_{\mathcal{F}}(t)$, and $\mathcal{P}os_{\mathcal{X}}(t)$ are the positions of variables. The subterm at position p of t is denoted as $t|_p$ and $t[s]_p$ is the term t with the subterm at position p replaced by s.

We write $t \trianglerighteq s$, read *s is a subterm of t*, if $s = t|_p$ for some $p \in \mathcal{P}os(t)$ and $t \triangleright s$ if $t \trianglerighteq s$ and $t \neq s$. We write $t \ntrianglerighteq s$ and $t \ntriangleright s$ for the negation of the corresponding properties. The symbol labeling the root of t is denoted as $root(t)$. A *context* is a term $C \in \mathcal{T}(\mathcal{F} \cup \{\Box\}, \mathcal{X})$ with a 'hole' $\Box$ (a fresh constant symbol). We write $C[\,]_p$ to denote that there is a (usually single) hole $\Box$ at position p of C. Generally, we write $C[\,]$ to denote an arbitrary context and make explicit the position of the hole only if necessary. $C[\,] = \Box$ is called the *empty* context.

Substitutions

A substitution is a mapping $\sigma : \mathcal{X} \to \mathcal{T}(\mathcal{F}, \mathcal{X})$. Denote as ε the 'identity' substitution: $\varepsilon(x) = x$ for all $x \in \mathcal{X}$. The set $\mathcal{D}om(\sigma) = \{x \in \mathcal{X} \mid \sigma(x) \neq x\}$ is called the *domain* of σ.

Remark 2 In this paper, we do *not* impose that the domain of the substitutions is finite. This is usual practice in the dependency pairs approach, where a single substitution is used to instantiate an infinite number of variables coming from renamed versions of the dependency pairs (see below). ∎

Whenever $\mathcal{D}om(\sigma) \cap \mathcal{D}om(\sigma') = \varnothing$, for substitutions σ, σ', we denote by $\sigma \cup \sigma'$, a substitution such that $(\sigma \cup \sigma')(x) = \sigma(x)$ if $x \in \mathcal{D}om(\sigma)$ and $(\sigma \cup \sigma')(x) = \sigma'(x)$ if $x \in \mathcal{D}om(\sigma')$.

Renamings and unifiers

A *renaming* is an injective substitution ρ such that $\rho(x) \in \mathcal{X}$ for all $x \in \mathcal{X}$. For renamings, we assume that $\mathcal{V}ar(\rho)$ is finite (which is the usual practice) and also idempotency, i.e., $\rho(\rho(x)) = \rho(x)$ for all $x \in \mathcal{X}$.

The quasi-ordering of subsumption $\leq$ over $\mathcal{T}(\mathcal{F}, \mathcal{X})$ is $t \leq t' \Leftrightarrow \exists\sigma.\ t' = \sigma(t)$. We denote as $\sigma \leq \sigma'$ the fact that $\sigma(x) \leq \sigma'(x)$ for all $x \in \mathcal{X}$, thus extending the quasi-ordering to substitutions.

A substitution σ such that $\sigma(s) = \sigma(t)$ for two terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is called a *unifier* of s and t; we also say that s and t unify (with substitution σ). If two terms s and t unify, then there is a unique (up to renaming of variables) *most general unifier (mgu)* θ which is *minimal* (w.r.t. the subsumption quasi-ordering $\leq$) among all other unifiers of s and t.

A relation $\mathsf{R} \subseteq \mathcal{T}(\mathcal{F}, \mathcal{X}) \times \mathcal{T}(\mathcal{F}, \mathcal{X})$ on terms is *stable* if for all terms $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, and substitutions σ, we have $\sigma(s)\,\mathsf{R}\,\sigma(t)$ whenever $s\,\mathsf{R}\,t$.

Rewrite Systems and Term Rewriting

A rewrite rule is an ordered pair (l, r), written $l \to r$, with $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $l \notin \mathcal{X}$ and $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$. The left-hand side (*lhs*) of the rule is l and r is the right-hand side (*rhs*). A rewrite rule $l \to r$ is said to be *collapsing* if $r \in \mathcal{X}$. A *Term Rewriting System* (TRS) is a pair $\mathcal{R} = (\mathcal{F}, R)$, where R is a set of rewrite rules. Given TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{R}' = (\mathcal{F}', R')$, we let $\mathcal{R} \cup \mathcal{R}'$ be the TRS $(\mathcal{F} \cup \mathcal{F}', R \cup R')$. An instance $\sigma(l)$ of a *lhs* l of a rule is called a *redex*. Given $\mathcal{R} = (\mathcal{F}, R)$, we consider $\mathcal{F}$ as the disjoint union $\mathcal{F} = \mathcal{C} \uplus \mathcal{D}$ of symbols $c \in \mathcal{C}$, called *constructors* and symbols $f \in \mathcal{D}$, called *defined functions*, where $\mathcal{D} = \{root(l) \mid l \to r \in R\}$ and $\mathcal{C} = \mathcal{F} - \mathcal{D}$.

Example 3 Consider again the TRS in Example 1. The symbols `from`, `sel`, `minus`, `quot` and `zWquot` are defined, and `s`, `0`, `cons`, and `nil` are constructors. ■

For simplicity, we often write $l \to r \in \mathcal{R}$ instead of $l \to r \in R$ to express that the rule $l \to r$ is a rule of $\mathcal{R}$.

A term $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ rewrites to s (at position p), written $t \xrightarrow{p}_{\mathcal{R}} s$ (or just $t \to s$, or $t \to_{\mathcal{R}} s$), if $t|_p = \sigma(l)$ and $s = t[\sigma(r)]_p$, for some rule $l \to r \in R$, $p \in \mathcal{P}os(t)$ and substitution σ. We write $t \xrightarrow{>p}_{\mathcal{R}} s$ if $t \xrightarrow{q}_{\mathcal{R}} s$ for some $q > p$. A TRS $\mathcal{R}$ is terminating if its one step rewrite relation $\to_{\mathcal{R}}$ is terminating.

Innermost rewriting

A term is a normal form if it contains no redex. A substitution σ is normalized if $\sigma(x)$ is a normal form for all $x \in \mathcal{D}om(\sigma)$. A term $f(t_1, \ldots, t_k)$ is argument normalized if t_i is a normal form for all $1 \leq i \leq n$. An innermost redex is an argument normalized redex. A term s rewrites innermost to t, written $s \to_i t$, if $s \to t$ at position p and $s|_p$ is an innermost redex. Let $\mathcal{R}$ be a TRS. For any symbol f let *Rules*$(\mathcal{R}, f)$ be the set of rules $l \to r$ defining f and such that the left-hand sides l are argument normalized. For any term t the set of usable rules $\mathbf{U}(\mathcal{R}, t)$ is as follows:

$$\begin{array}{lcl} \mathbf{U}(\mathcal{R}, x) & = & \varnothing \\ \mathbf{U}(\mathcal{R}, f(t_1, \ldots, t_n)) & = & \mathit{Rules}(\mathcal{R}, f) \cup \bigcup\limits_{i \in ar(f)} \mathbf{U}(\mathcal{R}', t_i) \cup \bigcup\limits_{l \to r \in \mathit{Rules}(\mathcal{R}, f)} \mathbf{U}(\mathcal{R}', r) \end{array}$$

where $\mathcal{R}' = \mathcal{R} - \mathit{Rules}(\mathcal{R}, f)$.

(Innermost) Context-Sensitive Rewriting

A mapping $\mu : \mathcal{F} \to \wp(\mathbb{N})$ is a *replacement map* (or $\mathcal{F}$-map) if $\forall f \in \mathcal{F},\ \mu(f) \subseteq \{1, \ldots, ar(f)\}$ [Luc98]. Let $M_{\mathcal{F}}$ be the set of all $\mathcal{F}$-maps (or $M_{\mathcal{R}}$ for the $\mathcal{F}$-maps of a TRS $(\mathcal{F}, R)$). Let $\mu_{\top}$ be the replacement map given by $\mu_{\top}(f) = \{1, \ldots, ar(f)\}$ for all $f \in \mathcal{F}$ (i.e., no replacement restrictions are specified).

A binary relation R on terms is μ-monotonic if whenever $t \mathrel{\mathsf{R}} s$ we have that $f(t_1, \ldots, t_{i-1}, t, \ldots, t_k) \mathrel{\mathsf{R}} f(t_1, \ldots, t_{i-1}, s, \ldots, t_k)$ for all $f \in \mathcal{F}$, $i \in \mu(f)$, and t, s, $t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If R is $\mu_{\top}$-monotonic, we just say that R is *monotonic.*

The set of *μ-replacing positions* $\mathcal{P}os^{\mu}(t)$ of $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ is: $\mathcal{P}os^{\mu}(t) = \{\Lambda\}$, if $t \in \mathcal{X}$ and $\mathcal{P}os^{\mu}(t) = \{\Lambda\} \cup \bigcup_{i \in \mu(root(t))} i.\mathcal{P}os^{\mu}(t|_i)$, if $t \notin \mathcal{X}$. When no replacement map is made explicit, the μ-replacing positions are often called *active*; and the non-μ-replacing ones are often called *frozen.* The following result about *CSR* is often used without any explicit mention.

Proposition 4 [Luc98] *Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $p = q.q' \in \mathcal{P}os(t)$. Then $p \in \mathcal{P}os^{\mu}(t)$ iff $q \in \mathcal{P}os^{\mu}(t) \wedge q' \in \mathcal{P}os^{\mu}(t|_q)$*

The μ-replacing subterm relation $\unrhd_\mu$ is given by $t \unrhd_\mu s$ if there is $p \in \mathcal{P}os^\mu(t)$ such that $s = t|_p$. We write $t \rhd_\mu s$ if $t \unrhd_\mu s$ and $t \neq s$. We write $t \rhd_{\not\mu} s$ to denote that s is a non-μ-replacing (hence strict) subterm of t: $t \rhd_{\not\mu} s$ if there is $p \in \mathcal{P}os(t) - \mathcal{P}os^\mu(t)$ such that $s = t|_p$. The set of *μ-replacing* variables of a term t, i.e., variables occurring at some μ-replacing position in t, is $\mathcal{V}ar^\mu(t) = \{x \in \mathcal{V}ar(t) \mid t \unrhd_\mu x\}$. The set of non-$\mu$-replacing variables of t, i.e., variables occurring at some non-μ-replacing position in t, is $\mathcal{V}ar^{\not\mu}(t) = \{x \in \mathcal{V}ar(t) \mid t \rhd_{\not\mu} x\}$. Note that $\mathcal{V}ar^\mu(t)$ and $\mathcal{V}ar^{\not\mu}(t)$ do not need to be disjoint.

A pair $(\mathcal{R}, \mu)$ where $\mathcal{R}$ is a TRS and $\mu \in M_\mathcal{R}$ is often called a CS-TRS. In *context-sensitive rewriting*, we (only) contract *μ-replacing* redexes: t μ-rewrites to s, written $t \hookrightarrow_\mu s$ (or $t \hookrightarrow_{\mathcal{R},\mu} s$ and even $t \hookrightarrow s$), if $t \xrightarrow{p}_\mathcal{R} s$ and $p \in \mathcal{P}os^\mu(t)$.

Example 5 Consider $\mathcal{R}$ and μ as in Example 1. Then, we have:

$$\underline{\texttt{from(0)}} \hookrightarrow_\mu \texttt{cons(0,}\underline{\texttt{from(s(0))}}\texttt{)} \not\hookrightarrow_\mu \texttt{cons(0,cons(s(0),from(s(s(0)))))}$$

Since the second argument of $\texttt{cons}$ is not μ-replacing, we have that $2 \notin \mathcal{P}os^\mu(\texttt{cons(0,from(s(0)))})$, and the redex $\texttt{from(s(0))}$ cannot be μ-rewritten. $\blacksquare$

A term t is μ-terminating (or $(\mathcal{R}, \mu)$-terminating, if we want an explicit reference to the involved TRS $\mathcal{R}$) if there is no infinite μ-rewrite sequence $t = t_1 \hookrightarrow_\mu t_2 \hookrightarrow_\mu \cdots \hookrightarrow_\mu t_n \hookrightarrow_\mu \cdots$ starting from t. A TRS $\mathcal{R}$ is μ-terminating if $\hookrightarrow_\mu$ is terminating.

A μ-normal form is a term which cannot be μ-rewritten. Let $\mathsf{NF}_\mu(\mathcal{R})$ (or just NF_μ if no confusion arises) be the set of μ-normal forms of a TRS $\mathcal{R}$.

A substitution σ is μ-normalized if $\sigma(x)$ is a μ-normal form for all $x \in \mathcal{D}om(\sigma)$. A term $t = f(t_1, \ldots, t_k)$ is argument μ-normalized if t_i is a μ-normal form for all $i \in \mu(f)$. A μ-innermost redex is an argument μ-normalized redex, i.e., $t = \sigma(l)$ for some substitution σ and rule $l \to r \in \mathcal{R}$ and for all $p \in \mathcal{P}os^\mu(t - \Lambda)$, $t|_p \in \mathsf{NF}_\mu$. A term s innermost μ-rewrites to t, written $s \hookrightarrow_\mathsf{i} t$, if $s \xrightarrow{p}_\mathcal{R} t$, $p \in \mathcal{P}os^\mu(s)$, and $s|_p$ is a μ-innermost redex. Let innermost μ-rewriting below the root be $\overset{>\Lambda}{\hookrightarrow}_\mathsf{i} = (\overset{>\Lambda}{\hookrightarrow} \cap \hookrightarrow_\mathsf{i})$. Termination of *CSR* is fully captured by the so-called μ-reduction orderings, i.e., well-founded, stable orderings $\sqsupset$ which are μ-monotonic. A TRS $\mathcal{R}$ is innermost μ-terminating if $\hookrightarrow_{\mu,\mathsf{i}}$ is terminating. We write $s \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} t$ if $s \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} t$ and $t \in \mathsf{NF}_\mu$.

A term t *μ-narrows* to a term s (written $t \rightsquigarrow_{\mathcal{R},\mu,\theta} s$), if there is a nonvariable μ-replacing position $p \in \mathcal{P}os^\mu_\mathcal{F}(t)$ and a rule $l \to r$ in $\mathcal{R}$ (sharing no variable with t) such that $t|_p$ and l unify with most general unifier θ and $s = \theta(t[r]_p)$.

3

Minimal innermost non-μ-terminating terms and infinite innermost μ-rewrite sequences

Given a TRS $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$, the *minimal* nonterminating terms associated to $\mathcal{R}$ are nonterminating terms t whose proper subterms u (i.e., $t \rhd u$) are terminating; $\mathcal{T}_\infty$ is the set of minimal nonterminating terms associated to $\mathcal{R}$ [HM04, HM07]. Minimal nonterminating terms have two important properties:

1. Every nonterminating term s contains a minimal nonterminating term $t \in \mathcal{T}_\infty$ (i.e., $s \unrhd t$), and
2. minimal nonterminating terms t are always rooted by a *defined* symbol $f \in \mathcal{D}$: $\forall t \in \mathcal{T}_\infty, root(t) \in \mathcal{D}$.

Considering the structure of the infinite rewrite sequences starting from a minimal nonterminating term $t = f(t_1, \ldots, t_k) \in \mathcal{T}_\infty$ is helpful to come to the notion of dependency pair.

Proposition 6 [HM04, Lemma 1] *Let $\mathcal{R} = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS. For all $t \in \mathcal{T}_\infty$, there exist $l \to r \in R$, a substitution σ and a term $u \in \mathcal{T}_\infty$ such that $root(u) \in \mathcal{D}$, $t \xrightarrow{>\Lambda}{}^* \sigma(l) \xrightarrow{\Lambda} \sigma(r) \unrhd u$ and there is a nonvariable subterm v of r, $r \unrhd v$, such that $u = \sigma(v)$.*

In the following, we show how to adapt this notion to investigate infinite innermost μ-rewrite sequences.

3.1 Minimal innermost non-μ-terminating terms

Before starting our discussion about (minimal) innermost non-μ-terminating terms, we provide an obvious auxiliary result about innermost μ-terminating terms.

Lemma 7 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $s, t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If t is innermost μ-terminating, then:*

1. *If $t \trianglerighteq_\mu s$, then s is innermost μ-terminating.*
2. *If $t \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} s$, then s is innermost μ-terminating.*

Given a TRS $\mathcal{R} = (\mathcal{F}, R)$ and a replacement map $\mu \in M_{\mathcal{F}}$, maybe the most straightforward definition of minimal innermost non-μ-terminating terms is the following: let $i\mathcal{T}_{\infty,\mu}$ be a set of minimal innermost non-μ-terminating terms in the following sense: t belongs to $i\mathcal{T}_{\infty,\mu}$ if t is innermost non-μ-terminating and every strict subterm u (i.e., $t \vartriangleright u$) is innermost μ-terminating. It is obvious that $root(t) \in \mathcal{D}$ for all $t \in i\mathcal{T}_{\infty,\mu}$. We also have:

Lemma 8 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If s is not innermost μ-terminating, then there is a subterm t of s ($s \trianglerighteq t$) such that $t \in i\mathcal{T}_{\infty,\mu}$.*

Proof.

By structural induction. If s is a constant symbol, it is obvious: take $t = s$. If $s = f(s_1, \ldots, s_k)$, then we proceed by contradiction. If there is no subterm t of s such that $t \in i\mathcal{T}_{\infty,\mu}$, then in particular $s \notin i\mathcal{T}_{\infty,\mu}$, i.e., (since s is not innermost μ-terminating) there is a strict subterm t of s ($s \vartriangleright t$) which is not innermost μ-terminating. By the Induction Hypothesis, there is $t' \in i\mathcal{T}_{\infty,\mu}$ such that $t \trianglerighteq t'$. Then, we have $s \vartriangleright t'$, thus leading to a contradiction. ■

Unfortunately, there can be innermost non-μ-terminating terms having no μ-*replacing* subterm in $i\mathcal{T}_{\infty,\mu}$.

Example 9 Consider the following TRS $\mathcal{R}$:

$$\mathtt{a} \;\to\; \mathtt{c}(\mathtt{f}(\mathtt{a})) \qquad\qquad \mathtt{f}(\mathtt{c}(x)) \;\to\; x$$

together with $\mu(\mathtt{c}) = \varnothing$ and $\mu(\mathtt{f}) = \{1\}$ and $s = \mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a})))$. Note that s is not innermost μ-terminating, but $s \notin i\mathcal{T}_{\infty,\mu}$ because $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a}))) \vartriangleright \mathtt{f}(\mathtt{a})$ and $\mathtt{f}(\mathtt{a})$ is not innermost μ-terminating. Note that $\mathtt{f}(\mathtt{c}(\mathtt{f}(\mathtt{a}))) \vartriangleright_{\not\mu} \mathtt{f}(\mathtt{a})$. The only μ-replacing strict subterm of s is $\mathtt{c}(\mathtt{f}(\mathtt{a}))$, which is innermost μ-terminating, i.e., $\mathtt{c}(\mathtt{f}(\mathtt{a})) \notin i\mathcal{T}_{\infty,\mu}$. ■

Therefore, this kind of minimal innermost non-μ-terminating terms are not the most natural ones because they could occur at non-μ-replacing positions, where no innermost μ-rewriting step is possible. So, this simple notion would not lead to an appropriate generalization of Proposition 6 to *CSR*. Still, we use them advantageously below; for this reason we pay them some attention here.

There is a suitable generalization of Proposition 6 to *CSR* (see Proposition 18 below) based on the following notion.

Definition 10 (Minimal innermost non-μ-terminating term) *Let $i\mathcal{M}_{\infty,\mu}$ be a set of minimal innermost non-μ-terminating terms in the following sense: t belongs to $i\mathcal{M}_{\infty,\mu}$ if t is not innermost μ-terminating and every strict* μ-replacing *subterm s of t (i.e., $t \vartriangleright_\mu s$) is innermost μ-terminating.*

Note that $i\mathcal{T}_{\infty,\mu} \subseteq i\mathcal{M}_{\infty,\mu}$. In the following we often say that terms in $i\mathcal{T}_{\infty,\mu}$ are *strongly minimal* innermost non-μ-terminating terms. Now we have the following.

Lemma 11 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. If s is not innermost μ-terminating, then there is a μ-replacing subterm t of s such that $t \in i\mathcal{M}_{\infty,\mu}$.*

Proof.

By structural induction. If s is a constant symbol, it is obvious: take $t = s$. If $s = f(s_1, \ldots, s_k)$, then we proceed by contradiction. If there is no μ-replacing subterm t of s such that $t \in i\mathcal{M}_{\infty,\mu}$, then in particular $s \notin i\mathcal{M}_{\infty,\mu}$, i.e., there is a strict μ-replacing subterm t of s which is not innermost μ-terminating. By the Induction Hypothesis, t contains a μ-replacing subterm t' which belongs to $i\mathcal{M}_{\infty,\mu}$. But, since t is a μ-replacing subterm of s (i.e., $t = s|_p$ for some $p \in \mathcal{P}os^{\mu}(s)$), t' itself is also a μ-replacing subterm of s (because $t' = t|_q$ for some $q \in \mathcal{P}os^{\mu}(t)$ and $p.q \in \mathcal{P}os^{\mu}(s)$ by Proposition 4) which belongs to $i\mathcal{M}_{\infty,\mu}$, thus leading to a contradiction. ■

Obviously, if $t \in i\mathcal{M}_{\infty,\mu}$, then $root(t)$ is a defined symbol. Since innermost μ-terminating terms are preserved under innermost μ-rewriting (Lemma 7), it follows that $i\mathcal{M}_{\infty,\mu}$ is preserved under innermost μ-rewritings below the root in the following sense.

Lemma 12 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS, $\mu \in M_{\mathcal{F}}$, and $t \in i\mathcal{M}_{\infty,\mu}$. If $t \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}}\, u$ and u is not innermost μ-terminating, then $u \in i\mathcal{M}_{\infty,\mu}$.*

Proof.

All innermost μ-rewritings below the root are issued on μ-replacing and innermost μ-terminating terms which remain innermost μ-terminating by Lemma 7. Then, if u is not innermost μ-terminating, all its μ-replacing subterms (which are the ones which can be originated or transformed by innermost μ-rewritings from t to u) have to be innermost μ-terminating as well. Hence, $u \in i\mathcal{M}_{\infty,\mu}$. ■

Lemma 12 does *not* hold for $i\mathcal{T}_{\infty,\mu}$: consider the CS-TRS $(\mathcal{R}, \mu)$ in Example 9. We have that $\mathtt{f(a)} \in i\mathcal{T}_{\infty,\mu}$. Now, $\mathtt{f(a)} \hookrightarrow_{\mu,\mathsf{i}} \mathtt{f(c(f(a)))}$ and $\mathtt{f(c(f(a)))}$ is not innermost μ-terminating. However, $\mathtt{f(c(f(a)))} \notin i\mathcal{T}_{\infty,\mu}$ as shown in Example 9.

123. **Minimal innermost non-μ-terminating terms and infinite innermost μ-rewrite sequences**

4

Hidden terms in minimal innermost μ-rewrite sequences

Given a CS-TRS $(\mathcal{R}, \mu)$ the *hidden terms* are nonvariable terms occurring on some frozen position in the right-hand side of some rule of $\mathcal{R}$. As we show in the next chapter they play an important role in infinite minimal innermost μ-rewrite sequences associated to $\mathcal{R}$.

Definition 13 (Hidden symbols and terms [AGL08]) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. We say that $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}$ is a* hidden term *if there is a rule $l \to r \in R$ such that $r \rhd_{\not\mu} t$. Let $\mathcal{HT}(\mathcal{R}, \mu)$ (or just $\mathcal{HT}$, if $\mathcal{R}$ and μ are clear for the context) be the set of all hidden terms in $(\mathcal{R}, \mu)$. We say that $f \in \mathcal{F}$ is a* hidden symbol *if it occurs in a hidden term. Let $\mathcal{H}(\mathcal{R}, \mu)$ (or just $\mathcal{H}$) be the set of all hidden symbols in $(\mathcal{R}, \mu)$.*

Example 14 For $\mathcal{R}$ and μ as in Example 1, the maximal hidden terms are $\mathtt{from}(\mathtt{s}(x))$, and $\mathtt{zWquot}(zs, ys)$. The hidden symbols are `from`, `s` and `zWquot`. ■

In the following, we also use $\mathcal{DHT} = \{t \in \mathcal{HT} \mid root(t) \in \mathcal{D}\}$ for the set of hidden terms which are rooted by a *defined* symbol.

The following lemma says that frozen subterms t in the contractum $\sigma(r)$ of a redex $\sigma(l)$ which do *not* contain t, are (at least partly) 'introduced' by a hidden term in the right-hand side r of the involved rule $l \to r$.

Lemma 15 ([AGL08]) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution. If there is a rule $l \to r \in R$ such that $\sigma(l) \not\rhd_{\not\mu} t$ and $\sigma(r) \rhd_{\not\mu} t$, then there is no $x \in \mathcal{V}ar(r)$ such that $\sigma(x) \unrhd t$. Furthermore, there is a term $t' \in \mathcal{HT}$ such that $r \rhd_{\not\mu} t'$ and $\sigma(t') = t$.*

The following lemma establishes that minimal not innermost μ-terminating and non-μ-replacing subterms occurring in a innermost μ-rewrite sequence involving only minimal terms directly come from the first term in the sequence or are instances of a hidden term.

Lemma 16 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let A be an innermost μ-rewrite sequence $t_1 \hookrightarrow_{\mathsf{i}} t_2 \hookrightarrow_{\mathsf{i}} \cdots \hookrightarrow_{\mathsf{i}} t_n$ with $t_i \in i\mathcal{M}_{\infty,\mu}$ for all i, $1 \leq i \leq n$ and $n \geq 1$. If there is a term $t \in i\mathcal{M}_{\infty,\mu}$ such that $t_1 \not\triangleright t$ and $t_n \triangleright_{\not\mu} t$, then $t = \sigma(s)$ for some $s \in \mathcal{DHT}$ and substitution σ.*

Proof. By induction on n:

1. If $n = 1$, then it is vacuously true because $t_1 \not\triangleright t$ and $t_1 \triangleright_{\not\mu} t$ do not simultaneously hold.
2. If $n > 1$, then we assume that $t_1 \not\triangleright t$ and $t_n \triangleright_{\not\mu} t$. Let $l \to r \in R$ be such that $t_{n-1} = C[\sigma(l)]$ and $t_n = C[\sigma(r)]$ for some context $C[\,]$ and substitution σ. We consider two cases: either $t_{n-1} \triangleright_{\not\mu} t$ holds or not.
   (a) If $t_{n-1} \triangleright_{\not\mu} t$, then by the induction hypothesis the conclusion follows.
   (b) If $t_{n-1} \triangleright_{\not\mu} t$ does not hold, then, since assuming $t_{n-1} \triangleright_{\mu} t$ leads to a contradiction (because $t_{n-1} \in i\mathcal{M}_{\infty,\mu}$ in the hypothesis implies that $t \notin i\mathcal{M}_{\infty,\mu}$), we have that $t_{n-1} \not\triangleright t$. In particular, $\sigma(l) \not\triangleright t$; then, since $t_n \triangleright_{\not\mu} t$ there must be $\sigma(r) \triangleright_{\not\mu} t$. Thus, by Lemma 15 we conclude that $t = \sigma(s)$ for some $s \in \mathcal{HT}$ and substitution σ. Since $t \in i\mathcal{M}_{\infty,\mu}$, it follows that $root(t) = root(s) \in \mathcal{D}$. Thus, $s \in \mathcal{DHT}$.

■

We use the previous results to investigate infinite sequences that combine innermost μ-rewriting steps on minimal innermost non-μ-terminating terms and the extraction of such subterms as μ-replacing subterms of (instances of) right-hand sides of the rules.

Proposition 17 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let A be a finite or infinite sequence of the form $t_1 \stackrel{\Lambda}{\hookrightarrow}_{\mathsf{i}} s_1 \trianglerighteq_{\mu} t'_2 \stackrel{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} t_2 \stackrel{\Lambda}{\hookrightarrow}_{\mathsf{i}} s_2 \trianglerighteq_{\mu} t'_3 \stackrel{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} t_3 \cdots$ with t_i, $t'_i \in i\mathcal{M}_{\infty,\mu}$ for all $i \geq 1$. If there is a term $t \in i\mathcal{M}_{\infty,\mu}$ such that $t_i \triangleright_{\not\mu} t$ for some $i \geq 1$, then $t_1 \triangleright_{\not\mu} t$ or $t = \sigma(s)$ for some $s \in \mathcal{DHT}$ and substitution σ.*

Proof. By induction on i:

1. If $i = 1$, it is trivial.
2. If $i > 1$ and $t_i \triangleright_{\not\mu} t$, then we consider two cases: either $t_{i-1} \triangleright_{\not\mu} t$ holds or not.
   (a) If $t_{i-1} \triangleright_{\not\mu} t$, then by the induction hypothesis the conclusion follows.
   (b) If $t_{i-1} \triangleright_{\not\mu} t$ does not hold, then let $l \to r \in R$ and σ be such that $t_{i-1} = \sigma(l)$ and $s_{i-1} = \sigma(r) \trianglerighteq_{\mu} t'_i$. Since $t_{i-1} \triangleright_{\mu} t$ leads to a contradiction (because $t_{i-1} \in i\mathcal{M}_{\infty,\mu}$ implies that $t \notin i\mathcal{M}_{\infty,\mu}$), we have that $t_{i-1} \not\triangleright t$. Then we consider two cases: either $t'_i \triangleright t$ or $t'_i \not\triangleright t$.

(A) If $t'_i \rhd t$, since $t'_i, t \in i\mathcal{M}_{\infty,\mu}$ the case $t'_i \rhd_\mu t$ is excluded and the only possibility is that $t'_i \rhd_{\not\mu} t$. Then, since $\sigma(l) = t_{i-1} \not\trianglerighteq t$ and $\sigma(r) \trianglerighteq_\mu t'_i \rhd_{\not\mu} t$, i.e. $\sigma(r) \rhd_{\not\mu} t$, by Lemma 15 we conclude that $t = \sigma(s)$ for some $s \in \mathcal{HT}$ and substitution σ. Since $t \in i\mathcal{M}_{\infty,\mu}$, it follows that $root(t) = root(s) \in \mathcal{D}$. Thus, $s \in \mathcal{DHT}$.

(B) If $t'_i \not\trianglerighteq t$, then, by applying Lemma 12 and Lemma 16 to the innermost μ-rewrite sequence $t'_i \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} t_i$ the conclusion follows.

∎

5

Infinite innermost μ-rewrite sequences starting from minimal terms

The following proposition establishes that, given a minimal not innermost μ-terminating term $t \in i\mathcal{M}_{\infty,\mu}$, there are only two ways for an infinite innermost μ-rewrite sequence to proceed. The first one is by using 'visible' parts of the rules which correspond to μ-replacing nonvariable subterms in the right-hand sides which are rooted by a defined symbol. The second one is by showing up 'hidden' not innermost μ-terminating subterms which are activated by *migrating* variables in a rule $l \to r$, i.e., variables $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$ which are *not* μ-replacing in the left-hand side l but become μ-replacing in the right-hand side r.

Proposition 18 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Then for all $t \in i\mathcal{M}_{\infty,\mu}$, there exist $l \to r \in R$, a substitution σ such that $\sigma(l)$ is argument μ-normalized and a term $u \in i\mathcal{M}_{\infty,\mu}$ such that $t \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} \sigma(l) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma(r) \trianglerighteq_{\mu} u$ and either*

1. *there is a μ-replacing subterm s of r, $r \trianglerighteq_{\mu} s$, such that $u = \sigma(s)$, or*
2. *there is $x \in \mathcal{V}ar^{\mu}(r) - \mathcal{V}ar^{\mu}(l)$ such that $\sigma(x) \trianglerighteq_{\mu} u$.*

Proof.

Consider an infinite innermost μ-rewrite sequence starting from t. By definition of $i\mathcal{M}_{\infty,\mu}$, all proper μ-replacing subterms of t are innermost μ-terminating. Therefore, t has an inner reduction (of innermost μ-rewriting steps) to an instance $\sigma(l)$ of the left-hand side of a rule $l \to r$ of $\mathcal{R}$, such that no strict μ-replacing subterm of $\sigma(l)$ is a redex, i.e. $\sigma(l)$ is argument μ-normalized. Then we have $t \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} \sigma(l) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma(r)$ and $\sigma(r)$ is not innermost μ-terminating. Note that, $\sigma(l)$ must be argument μ-normalized; otherwise, the last step would not be an innermost μ-rewriting step. Thus, we can write $t = f(t_1, \ldots, t_k)$ and $\sigma(l) = f(l_1, \ldots, l_k)$ for some k-ary defined symbol f, and $t_i \hookrightarrow^{*}_{\mathsf{i}} \sigma(l_i)$ for all i, $1 \leq i \leq k$. Since all t_i are innermost μ-terminating for $i \in \mu(f)$, by Lemma 7, $\sigma(l_i)$ and all its μ-replacing subterms also are. In particular, $\sigma(x)$ is innermost μ-terminating for all μ-replacing variables x in l: $x \in \mathcal{V}ar^\mu(l)$ (in fact $\sigma(x) \in \mathsf{NF}_\mu$). Since $\sigma(r)$ is not innermost μ-terminating, by Lemma 11 it contains a μ-replacing subterm $u \in i\mathcal{M}_{\infty,\mu}$: $\sigma(r) \trianglerighteq_\mu u$, i.e., there is a position $p \in \mathcal{P}os^\mu(\sigma(r))$ such that $\sigma(r)|_p = u$. We consider two cases:

1. If $p \in \mathcal{P}os_\mathcal{F}(r)$ is a nonvariable position of r, then there is a μ-replacing subterm s of r, such that $u = \sigma(s)$.

2. If $p \notin \mathcal{P}os_\mathcal{F}(r)$, then there is a μ-replacing variable position $q \in \mathcal{P}os^\mu(r) \cap \mathcal{P}os_\mathcal{X}(r)$ such that $q \leq p$. Let $x \in \mathcal{V}ar^\mu(r)$ be such that $r|_q = x$. Then, $\sigma(x) \trianglerighteq_\mu u$ and $\sigma(x)$ is not innermost μ-terminating (by assumption, $u \in i\mathcal{M}_{\infty,\mu}$ is not innermost μ-terminating: by Lemma 7, $\sigma(x)$ cannot be innermost μ-terminating either). Since $\sigma(l_i)$ is innermost μ-terminating for all $i \in \mu(f)$, and $\sigma(x)$ is also innermost μ-terminating (in fact $\sigma(x) \in \mathsf{NF}_\mu$) for all μ-replacing variables in l , we conclude that $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$.

■

Proposition 18 entails the following result, which establishes some properties of infinite sequences starting from minimal innermost non-μ-terminating terms.

Corollary 19 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_\mathcal{F}$. For all $t \in i\mathcal{M}_{\infty,\mu}$, there is an infinite sequence*

$$t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_1(r_1) \trianglerighteq_\mu t_1 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_2(r_2) \trianglerighteq_\mu t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$ are rewrite rules, σ_i are substitutions, $\sigma_i(l_i)$ is argument μ-normalized, and terms $t_i \in i\mathcal{M}_{\infty,\mu}$ are minimal innermost non-μ-terminating terms such that either

1. *$t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$, or*

2. *$\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$.*

Remark 20 The $(\hookrightarrow_{\mu,\mathsf{i}} \cup \trianglerighteq_\mu)$-sequence in Corollary 19 can be easily viewed as an infinite innermost μ-rewrite sequence by just introducing appropriate contexts $C_i[\,]_{p_i}$ with μ-replacing holes: since $\sigma_i(r_i) \trianglerighteq_\mu t_i$, there is $p_i \in \mathcal{P}os^\mu(\sigma_i(r_i))$ such that $\sigma_i(r_i) = \sigma_i(r_i)[t_i]_{p_i}$; just take $C_i[\,]_{p_i} = \sigma(r_i)[\Box]_{p_i}$. Then we get:

$$t \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} \sigma_1(l_1) \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} C_1[t_1]_{p_1} \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} C_1[\sigma_2(l_2)]_{p_1} \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} C_1[C_2[t_2]_{p_2}]_{p_1} \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} \cdots$$

Note that, e.g., $p_1.p_2 \in \mathcal{P}os^\mu(C_1[C_2[t_2]_{p_2}]_{p_1})$ (use Proposition 4). ■

5.1 Infinite innermost μ-rewrite sequences starting from strongly minimal terms

In the following, we consider a function $\textsc{Ren}^{\mu}$ which *independently* renames all *occurrences* of μ-replacing variables within a term t by using new fresh variables which are not in $\mathcal{V}ar(t)$:

- $\textsc{Ren}^{\mu}(x) = y$ if x is a variable, where y is intended to be a fresh new variable which has not yet been used (we could think of y as the 'next' variable in an infinite list of variables); and
- $\textsc{Ren}^{\mu}(f(t_1, \ldots, t_k)) = f([t_1]^f_1, \ldots, [t_k]^f_k)$ for evey k-ary symbol f, where given a term $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, $[s]^f_i = \textsc{Ren}^{\mu}(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$.

Note that $\textsc{Ren}^{\mu}(t)$ renames all μ-replacing positions of variables in t by new fresh variables y but keeps variables at non-μ-replacing positions untouched. Note that, in contrast to a *renaming* substitution (often denoted by ρ), $\textsc{Ren}^{\mu}$ is *not* a substitution: it will replace different μ-replacing occurrences of the same variable by different variables.

Proposition 21 ([AGL08]) *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}$ be a nonvariable term and σ be a substitution. If $\sigma(t) \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} \sigma(l)$ for some (probably renamed) rule $l \to r \in R$, then $\textsc{Ren}^{\mu}(t)$ is μ-narrowable.*

Corollary 22 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. Let $t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}$ be a nonvariable term and σ be a substitution such that $\sigma(t) \in i\mathcal{M}_{\infty,\mu}$. Then, $\textsc{Ren}^{\mu}(t)$ is μ-narrowable.*

Proof.

By Proposition 18, there is a rule $l \to r$ and a substitution σ s.t. $\sigma(t) \overset{>\Lambda}{\hookrightarrow}{}^{*}_{\mathsf{i}} \sigma(l)$ (since we can assume that variables in l and variables in t are disjoint we can apply the same substitution σ to t and l without any problem). By Proposition 21, the conclusion follows. ■

In the following, we write $\textsc{Narr}^{\mu}(t)$ to indicate that t is μ-narrowable (w.r.t. the intended TRS $\mathcal{R}$). We also let

$$\mathcal{NHT}(\mathcal{R}, \mu) = \{t \in \mathcal{DHT} \mid \textsc{Narr}^{\mu}(\textsc{Ren}^{\mu}(t))\}$$

be the set of *hidden terms* which are rooted by a *defined* symbol, and that, after applying $\textsc{Ren}^{\mu}$, become *μ-narrowable*. As a consequence of the previous results, we have the following main result which we will use later.

Theorem 23 *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. For all $t \in i\mathcal{T}_{\infty,\mu}$, there is an infinite sequence*

$$t = t_0 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_1(r_1) \trianglerighteq_\mu t_1 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_2(r_2) \trianglerighteq_\mu t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$, σ_i is a substitution, $\sigma_i(l_i)$ is argument μ-normalized, and $t_i \in i\mathcal{M}_{\infty,\mu}$ is a minimal innermost non-μ-terminating term such that either

1. *$t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$, or*
2. *$\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$ and $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{NHT}$ and substitution θ_i.*

Proof.

Since $i\mathcal{T}_{\infty,\mu} \subseteq i\mathcal{M}_{\infty,\mu}$, by Corollary 19, we have a sequence

$$t = t_0 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_1(r_1) \trianglerighteq_\mu t_1 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_2(r_2) \trianglerighteq_\mu t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \cdots$$

where, for all $i \geq 1$, $l_i \to r_i \in R$, σ_i is a substitution such that $\sigma(l_i)$ is argument μ-normalized, $t_i \in i\mathcal{M}_{\infty,\mu}$, and either (1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$ or (2) $\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$ (and hence $\sigma(l_i) \rhd_{\not\mu} t_i$ and $\sigma(r_i) \trianglerighteq_\mu t_i$ as well). We only need to prove that terms t_i are instances of hidden terms in $\mathcal{NHT}$ whenever the second condition holds. By Proposition 17, for all such terms t_i, we have that either (A) $\sigma_1(l_1) \rhd_{\not\mu} t_i$ or (B) $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{DHT}$ and substitution θ_i. In the second case (B), we are done by just considering Corollary 22, which ensures that $t'_i \in \mathcal{NHT}$. In the first one (A), since $t \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_1(l_1)$ and $\sigma_1(l_1)$ is not innermost μ-terminating, by Lemma 12 all terms u_j in the μ-rewrite sequence

$$t = u_1 \overset{>\Lambda}{\hookrightarrow}_{\mathsf{i}} u_2 \overset{>\Lambda}{\hookrightarrow}_{\mathsf{i}} \cdots \overset{>\Lambda}{\hookrightarrow}_{\mathsf{i}} u_m = \sigma_1(l_1)$$

for $m \geq 1$, belong to $i\mathcal{M}_{\infty,\mu}$: $u_j \in i\mathcal{M}_{\infty,\mu}$ for all j, $1 \leq j \leq m$. Since $t \in i\mathcal{T}_{\infty,\mu}$, all its strict subterms (disregarding their μ-replacing character) are innermost μ-terminating. Therefore, $t \not\rhd t_i$ (because t_i is not μ-terminating) and by Lemma 16, $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{DHT}$ and substitution θ_i. Again, by Corollary 22 we have $t'_i \in \mathcal{NHT}$. ∎

6

Innermost Context-Sensitive Dependency Pairs

Lemma 8 and Theorem 23 are the basis for our definition of *Innermost Context-Sensitive Dependency Pairs* (ICS-DPs) and the corresponding *chains*. Together, they show that every innermost non-μ-terminating term s has an associated infinite innermost μ-rewrite sequence starting from a strongly minimal subterm $t \in i\mathcal{T}_{\infty,\mu}$ (i.e., $s \trianglerighteq t$). Such a sequence proceeds by first performing some innermost μ-rewriting steps *below the root* of t to obtain a term t' (i.e., $t \xhookrightarrow{>\Lambda}{}^*_{\mathsf{i}}\, t'$) and then applying a rule $l \to r$ at the *topmost* position of t' (i.e., $t' = \sigma(l)$ for some matching substitution σ such that $\sigma(l)$ is argument μ-normalized). According to Proposition 18, the application of such a rule either

1. *introduces* a new minimal innermost non-μ-terminating subterm u having a prefix s which is a μ-replacing subterm of r (i.e., $r \trianglerighteq_\mu s$ and $u = \sigma(s)$). Furthermore, by Corollary 22, $\textsc{Ren}^\mu(s)$ must be μ-narrowable; or else
2. *takes* a minimal innermost non-μ-terminating and non-μ-replacing subterm u of t' (i.e., $t' \triangleright_{\not\mu} u$) and
   (a) brings it up to an *active* position by means of the binding $\sigma(x)$ (i.e., $\sigma(x) \trianglerighteq_\mu u$) for some *migrating variable* x in $l \to r$ (i.e., $x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)$).
   (b) At this point, we know that u, which is rooted by a defined symbol due to $u \in i\mathcal{M}_{\infty,\mu}$, is an instance of a hidden term $u' \in \mathcal{NHT}$: $u = \theta(u')$ for some substitution θ.
   (c) Afterwards, further *inner* μ-rewritings on u lead to match the left-hand-side l' of a new rule $l' \to r'$ and everything starts again.

This process is abstracted in the following definition of *innermost context-sensitive dependency pairs*.

Given a signature $\mathcal{F}$ and $f \in \mathcal{F}$, we let $f^\sharp$ be a new fresh symbol (often called *tuple* symbol or DP-symbol) associated to a symbol f [AG00]. Let $\mathcal{F}^\sharp$ be the set of tuple symbols associated to symbols in $\mathcal{F}$. As usual, for $t = f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^\sharp$ to denote the *marked* term $f^\sharp(t_1, \ldots, t_k)$. Conversely, given a marked term $t = f^\sharp(t_1, \ldots, t_k)$, where $t_1, \ldots, t_k \in \mathcal{T}(\mathcal{F}, \mathcal{X})$, we write $t^\natural$ to denote the term $f(t_1, \ldots, t_k) \in \mathcal{T}(\mathcal{F}, \mathcal{X})$. Let $\mathcal{T}^\sharp(\mathcal{F}, \mathcal{X}) = \{t^\sharp \mid t \in \mathcal{T}(\mathcal{F}, \mathcal{X}) - \mathcal{X}\}$ be the set of marked terms.

Definition 24 (Innermost Context-Sensitive Dependency Pairs) *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_\mathcal{F}$. We define $\mathsf{iDP}(\mathcal{R}, \mu) = \mathsf{iDP}_\mathcal{F}(\mathcal{R}, \mu) \cup \mathsf{iDP}_\mathcal{X}(\mathcal{R}, \mu)$ to be the set of* innermost context-sensitive *dependency pairs (ICS-DPs) where:*

$$\begin{array}{rcl} \mathsf{iDP}_\mathcal{F}(\mathcal{R}, \mu) & = & \{l^\sharp \to s^\sharp \mid l \to r \in R, l^\sharp \in \mathsf{NF}_\mu(\mathcal{R}), r \trianglerighteq_\mu s, root(s) \in \mathcal{D}, l \not\triangleright_\mu s, \textsc{Narr}^\mu(\textsc{Ren}^\mu(s))\} \\ \mathsf{iDP}_\mathcal{X}(\mathcal{R}, \mu) & = & \{l^\sharp \to x \mid l \to r \in R, l^\sharp \in \mathsf{NF}_\mu(\mathcal{R}), x \in \mathcal{V}ar^\mu(r) - \mathcal{V}ar^\mu(l)\} \end{array}$$

We extend $\mu \in M_\mathcal{F}$ into $\mu^\sharp \in M_{\mathcal{F} \cup \mathcal{D}^\sharp}$ by $\mu^\sharp(f) = \mu(f)$ if $f \in \mathcal{F}$, and $\mu^\sharp(f^\sharp) = \mu(f)$ if $f \in \mathcal{D}$.

The ICS-DPs $u \to v \in \mathsf{iDP}_\mathcal{X}(\mathcal{R}, \mu)$ in Definition 24, consisting of collapsing rules only, are called the *collapsing* ICS-DPs.

A rule $l \to r$ of a TRS $\mathcal{R}$ is μ-conservative if $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$, i.e., it does not contain migrating variables; $\mathcal{R}$ is μ-conservative if all its rules are (see [Luc96, Luc06]). The following fact is obvious from Definition 24.

Proposition 25 *If $\mathcal{R}$ is a μ-conservative TRS, then $\mathsf{iDP}(\mathcal{R}, \mu) = \mathsf{iDP}_\mathcal{F}(\mathcal{R}, \mu)$.*

Therefore, in order to deal with μ-conservative TRSs $\mathcal{R}$ we only need to consider the 'classical' dependency pairs in $\mathsf{iDP}_\mathcal{F}(\mathcal{R}, \mu)$.

Example 26 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rclcrcl} \mathtt{g}(x) & \to & \mathtt{h}(x) & \qquad & \mathtt{h}(\mathtt{d}) & \to & \mathtt{g}(\mathtt{c}) \\ \mathtt{c} & \to & \mathtt{d} & & & & \end{array}$$

together with $\mu(\mathtt{g}) = \mu(\mathtt{h}) = \varnothing$ [Zan97, Example 1]. Note that $\mathcal{R}$ is μ-conservative. $\mathsf{iDP}(\mathcal{R}, \mu)$ consists of the following (noncollapsing) ICS-DPs:

$$\begin{array}{rclcrcl} \mathtt{G}(x) & \to & \mathtt{H}(x) & \qquad & \mathtt{H}(\mathtt{d}) & \to & \mathtt{G}(\mathtt{c}) \end{array}$$

with $\mu^\sharp(\mathtt{G}) = \mu^\sharp(\mathtt{H}) = \varnothing$. ■

If the TRS $\mathcal{R}$ contains non-μ-conservative rules, then we also need to consider dependency pairs with variables in the right-hand side.

7

Innermost chains of ICS-DPs

An essential property of the dependency pairs method is that it provides a *characterization* of termination of TRSs $\mathcal{R}$ as the absence of infinite (minimal) *chains of dependency pairs* [AG00, GTSF06]. As we prove in Chapter 8, this is also true for innermost *CSR* when ICS-DPs are considered. First, we have to introduce a suitable notion of innermost chain which can be used with ICS-DPs. As in the DP-framework [GTS04, GTSF06], where the procedence of *pairs* does not matter, we rather think of another TRS $\mathcal{P}$ which is used together with $\mathcal{R}$ to build the chains. Once this more abstract notion of chain is introduced, it can be particularized to be used with ICS-DPs, by just taking $\mathcal{P} = \mathsf{iDP}(\mathcal{R}, \mu)$.

In innermost *CSR*, we only perform reduction steps on *innermost μ-replacing redexes*. Therefore, we have to restrict the definition of chains in order to obtain an appropriate notion corresponding to innermost *CSR*. Regarding innermost reductions, arguments of a redex should be in *normal form* before the redex is contracted and, regarding *CSR*, the redex to be contracted has to be in a *μ-replacing* position.

Definition 27 (Innermost chain of pairs - Minimal chain) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. An innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain is a finite or infinite sequence of pairs $u_i \to v_i \in \mathcal{P}$, together with a substitution σ satisfying that, for all $i \geq 1$, $\sigma(u_i) \in \mathsf{NF}_\mu(\mathcal{R})$ and :*

1. *if $v_i \notin \mathcal{V}ar(u_i) - \mathcal{V}ar^\mu(u_i)$, then $\sigma(v_i) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_{i+1})$, and*
2. *if $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^\mu(u_i)$, then there is $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(v_i) \trianglerighteq_\mu s_i$ and $s_i^\sharp \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_{i+1})$.*

As usual, we assume that different occurrences of dependency pairs do not share any variable (renaming substitutions are used if necessary). An innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain is called minimal *if for all $i \geq 1$,*

1. *if $v_i \notin \mathcal{V}ar(u_i) - \mathcal{V}ar^\mu(u_i)$, then $\sigma(v_i)$ is innermost $(\mathcal{R}, \mu)$-terminating, and*
2. *if $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^\mu(u_i)$, then $s_i^\sharp$ is innermost $(\mathcal{R}, \mu)$-terminating and $\exists \bar{s}_i \in \mathcal{NHT}(\mathcal{R}, \mu)$ such that $s_i = \sigma(\bar{s}_i)$.*

Note that the condition $v_i \in \mathcal{V}ar(u_i) - \mathcal{V}ar^{\mu}(u_i)$ in Definition 27 implies that v_i is a variable. Furthermore, since each $u_i \to v_i \in \mathcal{P}$ is a rewrite rule (i.e., $\mathcal{V}ar(v_i) \subseteq \mathcal{V}ar(u_i)$), v_i is a *migrating variable* in the rule $u_i \to v_i$.

Remark 28 (Conventions about $\mathcal{P}$) The following conventions about the component $\mathcal{P} = (\mathcal{G}, P)$ of our chains will be observed during our development:

1. According to the usual terminology [GTSF06], we often call *pairs* to the rules $u \to v \in \mathcal{P}$.
2. Marking is part of the definition of chain: we have to mark terms $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ before *connecting* them to the instance $\sigma(u_{i+1})$ of the left-hand side of the next pair. Since marked symbols $f^\sharp$ are *fresh* (w.r.t. the signature $\mathcal{F}$ of the TRS $\mathcal{R}$), we also assume that $\mathcal{D}^\sharp \cap \mathcal{F} = \varnothing$ and $\mathcal{D}^\sharp \subseteq \mathcal{G}$ (since we only mark defined symbols, we do not need to extend the marking to $\mathcal{F}$).
3. We also silently assume that $\mathcal{P}$ contains a *finite* set of rules. This is essential in many proofs.

■

In the following, the pairs in a CS-TRS $(\mathcal{P}, \mu)$, where $\mathcal{P} = (\mathcal{G}, P)$, are partitioned according to its role in Definition 27 as follows:

$$P_{\mathcal{X}} = \{u \to v \in P \mid v \in \mathcal{V}ar(u) - \mathcal{V}ar^{\mu}(u)\} \text{ and } P_{\mathcal{G}} = P - P_{\mathcal{X}}$$

Remark 29 (Collapsing pairs) Note that all pairs in $\mathcal{P}_{\mathcal{X}} = (\mathcal{G}, P_{\mathcal{X}})$ are *collapsing.* The rules in $\mathcal{P}_{\mathcal{G}} = (\mathcal{G}, P_{\mathcal{G}})$ *can be collapsing* as well: a rewrite rule $f(x) \to x \in \mathcal{P}$ with $\mu(f) = \{1\}$ does *not* belong to $\mathcal{P}_{\mathcal{X}}$ but rather to $\mathcal{P}_{\mathcal{G}}$ because x is not a migrating variable.

Despite this fact, we refer to $\mathcal{P}_{\mathcal{X}}$ as the set of *collapsing* pairs in $\mathcal{P}$ because its intended role in Definition 27 is capturing the computational behavior of collapsing ICS-DPs in $\mathsf{iDP}_{\mathcal{X}}(\mathcal{R}, \mu)$. ■

Remark 30 (Notation for chains) In general, an innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain can be written as follows:

$$\sigma(u_1) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu t_1 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_2) \hookrightarrow_{\mathcal{P},\mu} \circ \trianglerighteq^\sharp_\mu t_2 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \cdots$$

where, for all $i \geq 1$ and $u_i \to v_i \in \mathcal{P}$,

1. if $u_i \to v_i \notin \mathcal{P}_{\mathcal{X}}$, then $t_i = \sigma(v_i)$,
2. if $u_i \to v_i \in \mathcal{P}_{\mathcal{X}}$, then $t_i = s_i^\sharp$ for some term s_i such that $\sigma(v_i) \trianglerighteq_\mu s_i$.

The relation $\trianglerighteq^\sharp_\mu$ is defined as follows:

- $s \trianglerighteq^\sharp_\mu t$ is equivalent to $s \trianglerighteq_\mu t^\natural$ if $s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and $t \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$, and
- $s \trianglerighteq^\sharp_\mu t$ is equivalent to $s = t$ for $s, t \in \mathcal{T}^\sharp(\mathcal{F}, \mathcal{X})$.

■

7.1 Properties of some particular chains

In the following, we let $\mathcal{NHT}_{i,\mathcal{P}}(\mathcal{R},\mu) \subseteq \mathcal{NHT}(\mathcal{R},\mu)$ (or just $\mathcal{NHT}_{i,\mathcal{P}}$ if $\mathcal{R}$ and μ are clear from the context) be as follows:

$$\mathcal{NHT}_{i,\mathcal{P}}(\mathcal{R},\mu) = \{t \in \mathcal{NHT}(\mathcal{R},\mu) \mid \exists u \to v \in \mathcal{P}, \exists \theta, \theta', \theta(t^\sharp) \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathrm{i}} \theta'(u)\}$$

This sets contains the narrowable hidden terms which innermost 'connect' with some pair in $\mathcal{P}$.

Remark 31 Note that $\mathcal{NHT}_{i,\mathcal{P}}(\mathcal{R},\mu)$ is not computable, in general, due to the need of checking the reachability of $\theta'(u)$ from $\theta(t^\sharp)$ using innermost *CSR*. Suitable (over) *approximations* are discussed below. ■

We let $\mathcal{P}^1_{\mathcal{X}}$ denote the subTRS of $\mathcal{P}_{\mathcal{X}}$ containing the rules whose migrating variables occur on non-μ-replacing immediate subterms in the left-hand side:

$$\mathcal{P}^1_{\mathcal{X}} = \{f(u_1,\ldots,u_k) \to x \in \mathcal{P}_{\mathcal{X}} \mid \exists i, 1 \leq i \leq k, i \notin \mu(f), x \in \mathcal{V}ar(u_i)\}$$

Proposition 32 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

1. *If $\mathcal{NHT}_{i,\mathcal{P}} = \varnothing$, then every infinite minimal innermost $(\mathcal{P},\mathcal{R},\mu)$-chain is an infinite minimal innermost $(\mathcal{P}_{\mathcal{G}},\mathcal{R},\mu)$-chain and there is no infinite minimal innermost $(\mathcal{P}_{\mathcal{X}},\mathcal{R},\mu)$-chain.*

2. *If $\mathcal{P} = \mathcal{P}^1_{\mathcal{X}}$, then there is no infinite innermost $(\mathcal{P},\mathcal{R},\mu)$-chain.*

Proof.

1. By contradiction. Assume that there is an infinite minimal innermost $(\mathcal{P},\mathcal{R},\mu)$-chain containing any $u_i \to v_i \in \mathcal{P}_{\mathcal{X}}$. By Definition 27, such a pair must be followed by a pair $u_{i+1} \to v_{i+1} \in \mathcal{P}$ such that $\theta_i(\bar{s}_i^\sharp) \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathrm{i}} \sigma(u_{i+1})$ for some $\bar{s}_i \in \mathcal{NHT}$ and substitution θ_i. Therefore, $t'_i \in \mathcal{NHT}_{i,\mathcal{P}}$, but $\mathcal{NHT}_{i,\mathcal{P}} = \varnothing$, leading to a contradiction.

2. By contradiction. Assume that there is an infinite innermost chain which only uses dependency pairs $u_i \to x_i \in \mathcal{P}^1_{\mathcal{X}}$ for all $i \geq 1$. Let $f_i = root(u_i)$ for $i \geq 1$. Then, by definition of $\mathcal{P}^1_{\mathcal{X}}$, for all $i \geq 1$ there is $j_i \in \{1,\ldots,ar(f_i)\} - \mu(f_i)$ such that $u_i|_{j_i} \trianglerighteq x_i$. According to Definition 27, we have that $\sigma(u_i)|_{j_i} \trianglerighteq \sigma(x_i) \trianglerighteq_\mu s_i$ for some term s_i such that $s_i^\sharp \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathrm{i}} \sigma(u_{i+1})$, with $root(s_i^\sharp) = root(u_{i+1}) = f_{i+1}$ and $j_{i+1} \notin \mu(f_{i+1})$. Since no innermost μ-rewriting step is possible on the j_{i+1}-th immediate subterm $s_i|_{j_{i+1}}$ of s_i, it follows that $s_i|_{j_{i+1}} = \sigma(u_{i+1})|_{j_{i+1}} \trianglerighteq \sigma(x_{i+1})$, i.e., $\sigma(x_i) \triangleright \sigma(x_{i+1})$ for all $i \geq 1$. We get an infinite sequence $\sigma(x_1) \triangleright \sigma(x_2) \triangleright \cdots$ which contradicts well-foundedness of $\triangleright$.

■

The following proposition establishes some important 'basic' cases of (absence of) infinite context-sensitive chains of pairs which are used later. Note that in the innermost case, obviously, also holds.

Proposition 33 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

1. *If $P = \varnothing$, then there is no (innermost) $(\mathcal{P}, \mathcal{R}, \mu)$-chain.*
2. *If $R = \varnothing$, then there is no infinite (innermost) $(\mathcal{P}_{\mathcal{X}}, \mathcal{R}, \mu)$-chain.*
3. *Let $u \to v \in \mathcal{P}_{\mathcal{G}}$ be such that $v' = \theta(u)$ for some substitution θ such that $\theta(u) \in \mathsf{NF}_\mu(\mathcal{R})$ and renamed version v' of v. Then, there is an infinite innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain.*

Proof.

1. Trivial.
2. By contradiction. If there is an infinite $(\mathcal{P}_{\mathcal{X}}, \mathcal{R}, \mu)$-chain, then, since there is no rule in $\mathcal{R}$, there is a substitution σ such that

   $$\sigma(u_1) \hookrightarrow_{\mathcal{P},\mu} \sigma(x_1) \trianglerighteq^\sharp_\mu t_1 = \sigma(u_2) \hookrightarrow_{\mathcal{P},\mu} \sigma(x_2) \trianglerighteq^\sharp_\mu t_2 = \sigma(u_3) \cdots$$

   where $t_i = s_i^\sharp$ for some terms $s_i \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ for $i \geq 1$. Since $x_i \in \mathcal{V}ar(u_i)$ and u_i is not a variable, we have $u_i \triangleright x_i$, hence $\sigma(u_i) \triangleright \sigma(x_i)$ (by stability of $\triangleright$), and also $\sigma(u_i) \triangleright s_i$ for all $i \geq 1$. Since s_i and $\sigma(u_{i+1})$ only differ in the root symbol, we can actually say that $s_i \triangleright s_{i+1}$ for all $i \geq 1$. Thus, we obtain an infinite sequence $s_1 \triangleright s_2 \triangleright \cdots$ which contradicts the well-foundedness of $\triangleright$.
3. Since we always deal with *renamed* versions $u_i \to v_i$ of the pair $u \to v \in \mathcal{P}$, for each $x \in \mathcal{V}ar(u)$, we write x_i to denote the 'name' of the variable x in $u_i \to v_i$. According to our hypothesis, we can assume the existence of substitutions θ_{i+1} such that $v_i = \theta_{i+1}(u_{i+1})$. Note that, for all $x \in \mathcal{V}ar(u)$ and $i \geq 1$, $\mathcal{V}ar(\theta_{i+1}(u_{i+1})) \subseteq \mathcal{V}ar(v_i) \subseteq \mathcal{V}ar(u_i)$ and $\theta(u) \in \mathsf{NF}_\mu(\mathcal{R})$ is needed to deal only with innermost μ-chains.

   We can define an infinite innermost $(\{u \to v\}, \varnothing, \mu)$-chain (hence an innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain) by using the renamed versions $u_i \to v_i$ of $u \to v$ for $i \geq 1$ together with σ given (inductively) as follows: for all $x \in \mathcal{V}ar(u)$, $\sigma(x_1) = x_1$ and $\sigma(x_i) = \sigma(\theta_i(x_i))$ for all $i > 1$. Note that $\sigma(v_i) = \sigma(\theta_{i+1}(u_{i+1})) = \sigma(u_{i+1})$ for all $i \geq 1$.

■

The following example shows that Proposition 33(2) does not hold for TRSs $\mathcal{P}$ with arbitrary rules.

Example 34 Consider $\mathcal{P} = \{\mathtt{F}(x) \to x, \mathtt{G}(x) \to \mathtt{F}(\mathtt{g}(x))\}$ together with a TRS $\mathcal{R}$ with an emtpy set of rules: $\mathcal{R} = (\mathcal{F}, \varnothing)$. Let μ be given by $\mu(f) = \varnothing$ for all $f \in \mathcal{F} \cup \mathcal{G}$. Note that $\mathcal{P}_{\mathcal{X}}$ consists of the pair $\mathtt{F}(x) \to x$ because $x \in \mathcal{V}ar(\mathtt{F}(x)) - \mathcal{V}ar^{\mu}(\mathtt{F}(x))$. Then, we have an infinite chain

$$\mathtt{F}(\mathtt{g}(x)) \hookrightarrow_{\mathcal{P},\mu} \mathtt{g}(x) \trianglerighteq^{\sharp}_{\mu} \mathtt{G}(x) \hookrightarrow_{\mathcal{P},\mu} \mathtt{F}(\mathtt{g}(x)) \hookrightarrow_{\mathcal{R},\mu} \cdots$$

Note that this chain is *not* minimal because $\mathcal{NHT} = \varnothing$, hence $\mathtt{g}(x)$ is not an instance of any term in $\mathcal{NHT}$. ■

8

Characterizing innermost termination of CSR using chains of ICS-DPs

The following result establishes the soundness of the innermost context-sensitive dependency pairs approach. As usual, in order to fit the requirement of variable-disjointness among two arbitrary pairs in a chain of pairs, we assume that appropriately renamed ICS-DPs are available when necessary.

Theorem 35 (Soundness) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If there is no infinite minimal innermost* $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$*-chain, then $\mathcal{R}$ is innermost μ-terminating.*

Proof.

By contradiction. If $\mathcal{R}$ is not innermost μ-terminating, then by Lemma 8 there is $t \in i\mathcal{T}_{\infty,\mu}$. By Theorem 23, there are rules $l_i \to r_i \in \mathcal{R}$, matching substitutions σ_i, such that $\sigma_i(l_i)$ is argument μ-normalized and terms $t_i \in i\mathcal{M}_{\infty,\mu}$, for $i \geq 1$ such that

$$t = t_0 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}}\ \sigma_1(l_1) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_1(r_1)\ \trianglerighteq_\mu\ t_1 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}}\ \sigma_2(l_2) \overset{\Lambda}{\hookrightarrow}_{\mathsf{i}} \sigma_2(r_2)\ \trianglerighteq_\mu\ t_2 \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \cdots$$

where either (D1) $t_i = \sigma_i(s_i)$ for some s_i such that $r_i \trianglerighteq_\mu s_i$ or (D2) $\sigma_i(x_i) \trianglerighteq_\mu t_i$ for some $x_i \in \mathcal{V}ar^\mu(r_i) - \mathcal{V}ar^\mu(l_i)$ and $t_i = \theta_i(t'_i)$ for some $t'_i \in \mathcal{NHT}$. Furthermore, since $t_{i-1} \overset{>\Lambda}{\hookrightarrow}{}^*_{\mathsf{i}} \sigma_i(l_i)$ and $t_{i-1} \in i\mathcal{M}_{\infty,\mu}$ (in particular, $t_0 = t \in i\mathcal{T}_{\infty,\mu} \subseteq i\mathcal{M}_{\infty,\mu}$), by Lemma 12, $\sigma_i(l_i) \in i\mathcal{M}_{\infty,\mu}$ for all $i \geq 1$. Note that, since $t_i \in i\mathcal{M}_{\infty,\mu}$, we have that $t_i^\sharp$ is innermost μ-terminating (with respect to $\mathcal{R}$), because all μ-replacing subterms of t_i (hence of $t_i^\sharp$ as well) are innermost μ-terminating and $root(t^\sharp)$ is not a defined symbol of $\mathcal{R}$.

First, note that $\mathsf{iDP}(\mathcal{R},\mu)$ is a TRS $\mathcal{P}$ over the signature $\mathcal{G} = \mathcal{F} \cup \mathcal{D}^\sharp$ and $\mu^\sharp \in M_{\mathcal{F}\cup\mathcal{G}}$ as required by Definition 27. Furthermore, $\mathcal{P}_\mathcal{G} = \mathsf{iDP}_\mathcal{F}(\mathcal{R},\mu)$ and $\mathcal{P}_\mathcal{X} = \mathsf{iDP}_\mathcal{X}(\mathcal{R},\mu)$. We can define an infinite minimal innermost $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain using ICS-DPs $u_i \to v_i$ for $i \geq 1$, where $u_i = l_i^\sharp$, and

1. $v_i = s_i^\sharp$ if (D1) holds. Since $t_i \in i\mathcal{M}_{\infty,\mu}$, we have that $root(s_i) \in \mathcal{D}$ and, because $t_i = \sigma_i(s_i)$, by Corollary 22 $\textsc{Ren}^\mu(s_i)$ is μ-narrowable. Furthermore, if we

assume that s_i is a μ-replacing subterm of l_i (i.e., $l_i \rhd_\mu s_i$), then $\sigma_i(l_i) \rhd_\mu \sigma_i(s_i)$ which (since $\sigma_i(s_i) = t_i \in i\mathcal{M}_{\infty,\mu}$) contradicts that $\sigma_i(l_i) \in i\mathcal{M}_{\infty,\mu}$. Thus, $l_i \not\rhd_\mu s_i$. Moreover, since $\sigma_i(l_i)$ is argument μ-normalized, it implies that $\sigma_i(l_i^\sharp)$ also, which means that $\sigma_i(l_i^\sharp) \in \mathsf{NF}_\mu(\mathcal{R})$ (since $root(l_i^\sharp)$ is not a defined symbol of $\mathcal{R}$) and trivially also is $l_i^\sharp$. Hence, $u_i \to v_i \in \mathsf{iDP}_\mathcal{F}(\mathcal{R},\mu)$. Furthermore, $t_i^\sharp = \sigma_i(v_i)$ is innermost μ-terminating. Finally, since $t_i = \sigma_i(s_i) \stackrel{>\Lambda}{\hookrightarrow}{}_{\mathsf{i}}^{*} \sigma_{i+1}(l_{i+1})$ and $\mu^\sharp$ extends μ to $\mathcal{F} \cup \mathcal{D}^\sharp$ by $\mu^\sharp(f^\sharp) = \mu(f)$ for all $f \in \mathcal{D}$, we also have that $\sigma_i(v_i) = \sigma_i(s_i^\sharp) \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma_{i+1}(u_{i+1})$.

2. $v_i = x_i$ if (D2) holds. Again, since $\sigma_i(l_i)$ is argument μ-normalized, it implies that $\sigma_i(l_i^\sharp)$ also, which means that $\sigma_i(l_i^\sharp) \in \mathsf{NF}_\mu(\mathcal{R})$ (since $root(l_i^\sharp)$ is not a defined symbol of $\mathcal{R}$) and trivially also is $l_i^\sharp$. Clearly, $u_i \to v_i \in \mathsf{iDP}_\mathcal{X}(\mathcal{R},\mu)$. As discussed above, $t_i^\sharp$ is innermost μ-terminating. Since $\sigma_i(x_i) \unrhd_\mu t_i$, we have that $\sigma_i(v_i) \unrhd_\mu t_i$. Finally, since $t_i \stackrel{>\Lambda}{\hookrightarrow}{}_{\mathsf{i}}^{*} \sigma_{i+1}(l_{i+1})$, again we have that $t_i^\sharp \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma_{i+1}(u_{i+1})$. Furthermore, $t_i = \theta_i(t_i')$ for some $t_i' \in \mathcal{NHT}$.

Regarding σ, w.l.o.g. we can assume that $\mathcal{V}ar(l_i) \cap \mathcal{V}ar(l_j) = \varnothing$ for all $i \neq j$, and therefore $\mathcal{V}ar(u_i) \cap \mathcal{V}ar(u_j) = \varnothing$ as well. Then, σ is given by $\sigma(x) = \sigma_i(x)$ whenever $x \in \mathcal{V}ar(u_i)$ for $i \geq 1$. From the discussion in points (1) and (2) above, we conclude that the CSDPs $u_i \to v_i$ for $i \geq 1$ together with σ define an infinite minimal innermost $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain which contradicts our initial assumption. ∎

As for arbitrary pairs, we use $\mathsf{iDP}_\mathcal{X}^1$ to denote the subset of dependency pairs in $\mathsf{iDP}_\mathcal{X}(\mathcal{R},\mu)$ whose migrating variables occur on non-μ-replacing immediate subterms in the left-hand side.

As an immediate consequence of Theorem 35 and Propositions 32 and 33, we have the following.

Corollary 36 (Basic innermost μ-termination criteria) *Let $\mathcal{R}$ be a TRS and $\mu \in M_\mathcal{R}$.*

1. *If $\mathsf{iDP}(\mathcal{R},\mu) = \varnothing$, then $\mathcal{R}$ is innermost μ-terminating.*

2. *If $\mathcal{NHT}_{i,\mathsf{iDP}(\mathcal{R},\mu)}(\mathcal{R},\mu) = \varnothing$ and $\mathsf{iDP}_\mathcal{F}(\mathcal{R},\mu) = \varnothing$, then $\mathcal{R}$ is innermost μ-terminating.*

3. *If $\mathsf{iDP}(\mathcal{R},\mu) = \mathsf{iDP}_\mathcal{X}^1(\mathcal{R},\mu)$, then $\mathcal{R}$ is innermost μ-terminating.*

Example 37 Consider the following TRS $\mathcal{R}$ [Luc98, Example 15]:

$$
\begin{array}{rclrcl}
\mathtt{and}(\mathtt{true},x) & \to & x & \mathtt{add}(\mathtt{0},x) & \to & x\\
\mathtt{and}(\mathtt{false},y) & \to & \mathtt{false} & \mathtt{add}(\mathtt{s}(x),y) & \to & \mathtt{s}(\mathtt{add}(x,y))\\
\mathtt{if}(\mathtt{true},x,y) & \to & x & \mathtt{from}(x) & \to & \mathtt{cons}(x,\mathtt{from}(\mathtt{s}(x)))\\
\mathtt{if}(\mathtt{false},x,y) & \to & y & \mathtt{first}(\mathtt{0},x) & \to & \mathtt{nil}\\
\mathtt{first}(\mathtt{s}(x),\mathtt{cons}(y,z)) & \to & \mathtt{cons}(y,\mathtt{first}(x,z)) & & &
\end{array}
$$

with $\mu(\mathtt{cons}) = \mu(\mathtt{s}) = \mu(\mathtt{from}) = \varnothing$, $\mu(\mathtt{add}) = \mu(\mathtt{and}) = \mu(\mathtt{if}) = \{1\}$, and $\mu(\mathtt{first}) = \{1,2\}$. Then, $\mathsf{iDP}(\mathcal{R},\mu) = \mathsf{iDP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ is:

$$
\begin{array}{rclrcl}
\mathtt{AND}(\mathtt{true},x) & \to & x & \mathtt{IF}(\mathtt{true},x,y) & \to & x\\
\mathtt{ADD}(\mathtt{0},x) & \to & x & \mathtt{IF}(\mathtt{false},x,y) & \to & y
\end{array}
$$

Note also that $\mathcal{NHT}_{i,\mathsf{iDP}(\mathcal{R},\mu)} = \varnothing$. Thus, by any of the last two statements of Corollary 36, we conclude the innermost μ-termination of $\mathcal{R}$. ■

The following example shows that Corollary 36(3) does *not* hold for chains consisting of *arbitrary* collapsing ICS-DPs.

Example 38 Consider the CS-TRS $(\mathcal{R},\mu)$ in Example 9. Note that $\mathsf{iDP}(\mathcal{R},\mu) = \mathsf{iDP}_{\mathcal{X}}(\mathcal{R},\mu)$ (both $\mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$ and $\mathsf{iDP}^1_{\mathcal{X}}(\mathcal{R},\mu)$ are empty!). We have the following infinite $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain:

$$\mathtt{F}(\underline{\mathtt{a}}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \underline{\mathtt{F}(\mathtt{c}(\mathtt{f}(\mathtt{a})))} \hookrightarrow_{\mathsf{iDP}(\mathcal{R},\mu),\mu^\sharp} \mathtt{F}(\underline{\mathtt{a}}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \cdots$$

■

Now we prove that the previous ICS-dependency pairs approach is not only correct but also complete for proving innermost termination of *CSR*.

Theorem 39 (Completeness) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. If $\mathcal{R}$ is innermost μ-terminating, then there is no infinite minimal innermost $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain.*

Proof.

By contradiction. If there is an infinite minimal innermost $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain, then there is a substitution σ and dependency pairs $u_i \to v_i \in \mathsf{iDP}(\mathcal{R},\mu)$ such that $\sigma(u_i) \in \mathsf{NF}_\mu(\mathcal{R})$ and

1. $\sigma(v_i) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma(u_{i+1})$, if $u_i \to v_i \in \mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$, and
2. if $u_i \to v_i = u_i \to x_i \in \mathsf{iDP}_{\mathcal{X}}(\mathcal{R},\mu)$, then there is $s_i \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that $\sigma(x_i) \trianglerighteq_\mu s_i$ and $s_i^\sharp \overset{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma(u_{i+1})$.

for $i \geq 1$. Now, consider the first dependency pair $u_1 \to v_1$ in the sequence:

1. If $u_1 \to v_1 \in \mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$, then $v_1^\natural$ is a μ-replacing subterm of the right-hand-side r_1 of a rule $l_1 \to r_1$ in $\mathcal{R}$. Therefore, $r_1 = C_1[v_1^\natural]_{p_1}$ for some $p_1 \in \mathcal{P}os^\mu(r_1)$ and, since $\sigma(u_1) \in \mathsf{NF}_\mu(\mathcal{R})$, we can perform the innermost μ-rewriting step $t_1 = \sigma(u_1^\natural) \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} \sigma(r_1) = \sigma(C_1)[\sigma(v_1^\natural)]_{p_1} = s_1$, where $\sigma(v_1^\natural)^\sharp = \sigma(v_1) \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma(u_2)$ and $\sigma(u_2)$ also initiates an infinite minimal innermost $(\mathcal{R},\mathsf{iDP}(\mathcal{R},\mu),\mu^\sharp)$-chain. Note that $p_1 \in \mathcal{P}os^\mu(s_1)$.
2. If $u_1 \to x \in \mathsf{iDP}_{\mathcal{X}}(\mathcal{R},\mu)$, then there is a rule $l_1 \to r_1$ in $\mathcal{R}$ such that $u_1 = l_1^\sharp$, and $x \in \mathcal{V}ar^\mu(r_1) - \mathcal{V}ar^\mu(l_1)$, i.e., $r_1 = C_1[x]_{q_1}$ for some $q_1 \in \mathcal{P}os^\mu(r_1)$. Furthermore, since there is a subterm s such that $\sigma(x) \trianglerighteq_\mu s$ and $s^\sharp \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma(u_2)$, we can write $\sigma(x) = C_1'[s]_{p_1'}$ for some $p_1' \in \mathcal{P}os^\mu(\sigma(x))$ and context $C_1'[\,]_{p_1'}$. Therefore, since $\sigma(u_1) = \sigma(l_1)^\sharp \in \mathsf{NF}_\mu(\mathcal{R})$, we can perform the innermost μ-rewriting step $t_1 = \sigma(l_1) \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} \sigma(r_1) = \sigma(C_1)[C_1'[s]_{p_1'}]_{q_1} = s_1$ where $s^\sharp \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu^\sharp,\mathsf{i}} \sigma(u_2)$ (hence $s \stackrel{!}{\hookrightarrow}_{\mathsf{i}} u_2^\natural$) and $\sigma(u_2)$ initiates an infinite minimal innermost $(\mathcal{R},\mathsf{iDP}(\mathcal{R},\mu),\mu^\sharp)$-chain. Note that $p_1 = q_1.p_1' \in \mathcal{P}os^\mu(s_1)$.

Since $\mu^\sharp(f^\sharp) = \mu(f)$, and $p_1 \in \mathcal{P}os^\mu(s_1)$, we have that $s_1 \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} t_2[\sigma(u_2)]_{p_1} = t_2$ and $p_1 \in \mathcal{P}os^\mu(t_2)$. Therefore, we can build in that way an infinite μ-rewrite sequence

$$t_1 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_1 \stackrel{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} t_2 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} \cdots$$

which contradicts the innermost μ-termination of $\mathcal{R}$.

■

According to this, Proposition 33(3) suggests a simple checking of *innermost non-μ-termination.*

Corollary 40 (Innermost non-μ-termination criterion) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. If there is $u \to v \in \mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$ such that $v' = \theta(u)$ for some substitution θ and renamed version v' of v, then $\mathcal{R}$ is not innermost μ-terminating.*

As a corollary of Theorems 35 and 39, we have.

Corollary 41 (Characterization of innermost μ-termination) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Then, $\mathcal{R}$ is innermost μ-terminating if and only if there is no infinite minimal innermost $(\mathsf{iDP}(\mathcal{R},\mu),\mathcal{R},\mu^\sharp)$-chain.*

Example 42 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{b} & \to & \mathtt{c}(\mathtt{b}) \\ \mathtt{f}(\mathtt{c}(x),x) & \to & \mathtt{f}(x,x) \end{array}$$

together with $\mu(\mathtt{f}) = \{1,2\}$ and $\mu(\mathtt{c}) = \varnothing$.
There is only one iCS-DP:

$$\mathtt{F}(\mathtt{c}(x), x) \;\rightarrow\; \mathtt{F}(x, x)$$

Since $\mu^\sharp(\mathtt{F}) = \{1, 2\}$, if a substitution σ satisfies $\sigma(\mathtt{F}(\mathtt{c}(x), x)) \in \mathsf{NF}_\mu(\mathcal{R})$, then $\sigma(x) = s$ is in μ-normal form. Assume that the dependency pair is part of an innermost CS-DP-chain. Since there is no way to μ-rewrite $F(s, s)$, there must be $F(s, s) = F(c(t), t)$ for some term t, which means that $s = t$ and $c(t) = s$, i.e., $t = c(t)$ which is not possible. Thus, there is no infinite innermost chain of iCS-DPs for $\mathcal{R}$, which is proved innermost terminating by Theorem 35. ■

Of course, ad-hoc reasonings like in Example 42 do not lead to automation. In following chapters we discuss how to prove termination of innermost *CSR* by giving *constraints* on terms that can be solved by using standard methods.

9

Mechanizing proofs of innermost μ-termination using ICS-DPs

During the last ten years, the dependency pairs method has evolved to a powerful technique for proving termination of TRSs in practice. From the already classical Arts and Giesl's article [AG00] to the last developments corresponding to the so-called *dependency pair framework* [GTS04, GTSF06, Thi07] many new improvements have been introduced.

In the DP-approach [AG00], the starting point is a TRS $\mathcal{R}$ from which a set of dependency pairs $\mathsf{DP}(\mathcal{R})$ is obtained. Then, such dependency pairs are organized in a dependency graph $\mathsf{DG}(\mathcal{R})$ and the cycles of the graph are analyzed to show that no infinite chains of DPs can be obtained from them. The dependency pairs approach emphasizes (at least theoretically) a 'linear' (although somehow *modular*, see [GAO02]) procedure for proving termination. In this sense, the treatment of strongly connected components of the graph (SCCs) instead of cycles, as suggested by Hirokawa and Middeldorp [HM04, HM05], brought an important improvement in its practical use because it provides a way to make the proofs more incremental without running out of the basic DP-approach. In the DP-approach, dependency pairs are considered as components of the chains (or cycles). Since they only make sense when an underlying TRS is given as the source of the dependency pairs, transforming DPs is possible (the *narrowing* transformation is already described in [AG00]) but only as a final step because, afterwards, they are not dependency pairs of the original TRS anymore.

The dependency pair framework solves these problems in a clean way, leading to a more powerful mechanization of termination proofs.

9.1 Mechanizing termination proofs with the dependency pair framework

An appealing aspect of the DP-framework [GTS04, GTSF06] is that the procedence of *pairs* does not matter; they can be *independent* from the considered TRS. The notion of chain is parametric on both a TRS $\mathcal{R}$ and a set of pairs $\mathcal{P}$ (a TRS, actually) which are connected by using $\mathcal{R}$-rewrite sequences. Regarding termination proofs, the central notion now is that of *DP-termination problem*: given a TRS $\mathcal{R}$ and a set of pairs $\mathcal{P}$, the goal is checking the absence (or presence) of infinite (minimal) chains. Termination of a TRS $\mathcal{R}$ is addressed as a DP-termination problem where $\mathcal{P} = \mathsf{DP}(\mathcal{R})$. The most important notion regarding mechanization of the proofs is that of *processor*. A (correct) processor basically transforms DP-termination problems into (hopefully) *simpler* ones, in such a way that the existence of an infinite chain in the original DP-termination problem implies the existence of an infinite chain in the transformed one. Here 'simpler' usually means that fewer pairs are involved. Often, processors are not only correct but also *complete*, i.e., there is an infinite minimal chain in the original DP-termination problem if and only if there is an infinite minimal chain in the transformed problem. This is essential if we are interested in *disproving* termination.

Processors are used in a pipe (more precissely, a *tree*) to incrementally simplify the original DP-termination problem as much as possible, possibly decomposing it into smaller pieces which are then independently treated in the very same way. The trivial case of this *iterative* process comes when the set of pairs $\mathcal{P}$ becomes empty. Then, no infinite chain is possible and we can provide a *positive* answer yes to the DP-termination problem which is propagated upwards to the original problem in the root of the tree. In some cases it is also possible to witness the existence of infinite chains for a given DP-termination problem; then a *negative* answer no can be provided and propagated upwards. Of course, DP-termination problems are undecidable (in general), thus don't know answers can also be generated (for instance by a time-out system which interrupts the usually complex search processes which are involved in the proofs). When all answers are collected, a final conclusion about the whole DP-termination problem can be given:

1. if we have positive answers (yes) for all problems in the leaves of the tree, then we conclude yes as well;

2. if a negative answer (no) was raised somewhere and the DP-processors which were used in the path from the root to the node producing the negative answer were *complete*, then we conclude no as well;

3. Otherwise, the conclusion is don't know.

The notions of graph, cycles, SCCs, etc., are also part of the framework but (1) they are incorporated as *processors* now, and (2) they do *not* refer to dependency pairs anymore, but rather to the pairs in the (different) sets of pairs which are obtained through the process sketched above. In this way, we obtain a much more flexible framework to mechanize termination proofs and also to benefit from new future developments which could lead to the introduction of new processors.

In the following, we adapt these ideas to *CSR* to provide a suitable framework for mechanizing proofs of (innermost) termination of *CSR*.

9.2 CS-termination problems and processors

The following definition adapts the notion of (DP-)termination problem in [GTSF06] to (innermost) *CSR*. In our definition, we prefer to avoid '*DP*' because, as discussed above, dependency pairs (as such) are relevant in the theoretical framework only for investigating a particular problem (termination of TRSs), whereas some transformations can yield sets of pairs which are not dependency pairs of the underlying TRS anymore.

Definition 43 (CS-termination problems) *A CS-termination problem τ is a tuple $\tau = (\mathcal{P}, \mathcal{R}, \mu, e)$, where $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ are TRSs, $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$ and $e \in \{\mathbf{t}, \mathbf{i}\}$ is a flag that stands for* **t**ermination *or* **i**nnermost termination *of* CSR.

A CS-termination problem is finite if there is no infinite minimal (innermost) $(\mathcal{P}, \mathcal{R}, \mu)$-chain.

Finite CS-termination problems correspond to those generating a positive answer yes in the full proof process sketched above. Accordingly, CS-termination problems which are *not* finite correspond to a negative answer no.

Remark 44 According to Corollary 41, we can say now that a TRS $\mathcal{R}$ is innermost μ-terminating if and only if the CS-termination problem $(\mathsf{iDP}(\mathcal{R}, \mu), \mathcal{R}, \mu, i)$ is finite. ■

According to our previous results (Propositions 32 and 33), for some specific CS-termination problems it is easy to say whether they are finite or not.

Proposition 45 (Basic innermost CS-termination problems) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

1. *If $P = \varnothing$, or $\mathcal{P} = \mathcal{P}^1_{\mathcal{X}}$, or $R = \varnothing$ and $\mathcal{P} = \mathcal{P}_{\mathcal{X}}$, then the CS-termination problem $(\mathcal{P}, \mathcal{R}, \mu, i)$ is finite.*

2. *If there is $u \to v \in \mathcal{P}_{\mathcal{G}}$ such that $v' = \theta(u)$ for some substitution θ such that $\sigma(u) \in \mathsf{NF}_\mu(\mathcal{R})$ and renamed version v' of v, then the CS-termination problem $(\mathcal{P}, \mathcal{R}, \mu, i)$ is not finite.*

The CS-termination problems in Proposition 45 provide the necessary *base* cases for our proofs of innermost termination of *CSR*. The following definition adapts the notion of *processor* [GTSF06] to *CSR*.

Definition 46 (CS-processor) *A CS-processor* Proc *is a mapping from CS-termination problems into sets of CS-termination problems. A CS-processor* Proc *is*

- sound *if for all CS-termination problems* τ*,* τ *is finite whenever* τ' *is finite for all* $\tau' \in \mathsf{Proc}(\tau)$*.*
- complete *if for all CS-termination problems* τ*, whenever* τ *is finite, then* τ' *is finite for all* $\tau' \in \mathsf{Proc}(\tau)$*.*

In following chapters we describe a number of sound and (most of them) complete CS-processors.

10

Innermost termination and termination of CSR

In this chapter we are going to comment some interesting points that relate innermost termination of *CSR* to μ-termination. We show that our definition of ICS-DPs coincides (in the most important points) with the standard one for proving termination of innermost rewriting if no replacement restrictions are considered (equivalently, if the top replacement map $\mu_\top(f) = \{1, \ldots, ar(f)\}$ for all $f \in \mathcal{F}$ is used). Of course, when proving μ-termination of a TRS we are also proving innermost μ-termination (the converse does not hold). For this reason, although many standard techniques that we have developed for proving termination of *CSR* are not mentioned here (see [AGL08]) they can also be used for proving innermost μ-termination (as in full rewriting). However, it is most interesting the other side, since proving innermost termination always offers simpler proofs. Something similar happens with *CSR*.

10.1 Switching to innermost termination of *CSR*

Proving innermost termination of rewriting is often easier than proving termination of rewriting [AG00] and, for some relevant classes of TRSs (locally confluent overlay systems), innermost termination of rewriting is equivalent to termination of rewriting [Gra95, Gra96]. In [GM02b] it is proved that the equivalence between termination of innermost *CSR* and termination of *CSR* holds in some interesting cases (e.g., for *orthogonal*). The pair $\langle \sigma(l)[\sigma(r')]_p, \sigma(r) \rangle$ is called a critical pair and is also called an overlay if $p = \Lambda$. A critical pair $\langle t, s \rangle$ is trivial if $t = s$. The critical pairs of a TRS $\mathcal{R}$ are the critical pairs between any two of its (renamed) rules; this includes overlaps of a rule with a renamed variant of itself, except at the root, i.e., if $p = \Lambda$. A TRS $\mathcal{R}$ is left-linear if for all $l \to r \in R$, l is a linear term. A left-linear TRS without critical pairs is called orthogonal. This fact was noticed by Lucas in a personal communication to the authors showing the following example:

Example 47 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{f}(x,x) & \to & \mathtt{b} \\ \mathtt{f}(x,g(x)) & \to & \mathtt{f}(x,x) \\ \mathtt{c} & \to & \mathtt{g}(c) \end{array}$$

together with $\mu(\mathtt{f}) = \{1,2\}$ and $\mu(\mathtt{g}) = \varnothing$. This system is nonoverlapping and innermost μ-terminating, but not μ-terminating since $\mathtt{f}(\mathtt{c},\mathtt{c})) \hookrightarrow_\mu \underline{\mathtt{f}(\mathtt{c},\mathtt{g}(\mathtt{c})))} \hookrightarrow_\mu \mathtt{f}(\mathtt{c},\mathtt{c}) \hookrightarrow_\mu \cdots$ ■

Thanks to this, the following result was formulated:

Theorem 48 ([GM02b]) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a orthogonal TRSs and $\mu \in M_{\mathcal{F}}$. $\mathcal{R}$ is μ-terminating if and only if it is innermost μ-terminating.*

So, whenever it is possible, we switch to innermost μ-termination since, proofs often are easier due to the fact that when we consider an innermost rewriting step, we know that every possible subterm of our redex is in normal form with respect to our rewriting relation. For instance, in the next chapter we show that this is an advantage when estimating the graph.

Now we show that our framework coincide with the standard one for proving full (innermost) termination when no replacement map is considered.

10.2 ICS-DPs and IDPs

Given a TRS $\mathcal{R}$ and a replacement map μ, if no replacement restrictions are imposed, i.e., $\mu(f) = \{1,\ldots,ar(f)\}$ for all $f \in \mathcal{F}$, then no collapsing pair is possible, and we would have $\trianglerighteq_\mu = \trianglerighteq$, and $\mathsf{iDP}(\mathcal{R},\mu) = \mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$.

Regarding the ICS-DPs in $\mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$, Definition 24 differs from the standard definition of dependency pair (e.g., [AG00, GTSF06]) in two additional requirements:

1. As in [HM04], which follows Dershowitz's proposal in [Der04], we require that subterms s of the right-hand sides r of the rules $l \to r$ which are considered to build the dependency pairs $l^\sharp \to s^\sharp$ are not subterms of the left-hand side (i.e., $l \ntrianglerighteq_\mu s$).

2. As in [LM08b], we require 'narrowability' of the (appropriately renamed) term s: $\textsc{Narr}^\mu(\textsc{Ren}^\mu(s))$.

3. We explicitly require that the left-hand side l of a rule $l \to r$ is argument μ-normalized when considering the dependency pair $l^\sharp \to s^\sharp$ (or $l^\sharp \to x$), that is, $l^\sharp \in \mathsf{NF}_\mu(\mathcal{R})$. In the standard definition, this is exploited when building the graph but, in our definition, we avoid the introduction of spurious pairs from the beginning.

Except for these provisos, we could say that Definition 24 boils down to the definition of dependency pair when no replacement restrictions are imposed.

Regarding the definition of (minimal) chain (Definition 27), the correspondence is exact: if μ imposes no replacement restriction, then $\rightarrow_{\mathcal{R},\mathsf{i}} = \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}}$ and our definition coincides with the standard one (see, e.g., [GTSF06, Definition 3]): again, since all variables are μ-replacing now, item (2) in Definition 27 does not apply. Due to the absence of replacement restrictions, we have $\mathcal{V}ar^{\mu}(u) = \mathcal{V}ar(u)$, hence $\mathcal{V}ar(u) - \mathcal{V}ar^{\mu}(u) = \varnothing$ for all $u \rightarrow v \in \mathcal{P}$. Then, the condition $v \notin \mathcal{V}ar(u) - \mathcal{V}ar^{\mu}(u)$ vacuously holds and all pairs in $\mathcal{P}$ satisfy item (1) of Definition 27.

In the following chapters we are going to show some powerful techniques adapted from standard rewriting to deal with proofs of innermost termination of *CSR*.

11

Innermost Context-Sensitive Dependency Graph

CS-termination problems draw our attention to the analysis of *infinite minimal (innermost)* $(\mathcal{P}, \mathcal{R}, \mu)$*-chains*. In general, an infinite sequence $S = a_1, a_2, \ldots, a_n, \ldots$ of objects a_i belonging to a set A can be represented as a path in a *graph* G whose nodes are the objects in A, and whose arcs among them are appropriately established to represent S (in particular, an arc from a_i to a_{i+1} should be established if we want to be able to capture the sequence above). Actually, if A is *finite*, then the infinite sequence S defines at least one *cycle* in G: since there is a finite number of different objects $a_i \in A$ in S, there is an infinite tail $S' = a_m, a_{m+1}, \ldots$ of S where *all objects* a_i *occur infinitely many times* for all $i \geq m$. This clearly corresponds to a cycle in G.

In the dependency pairs approach [AG00], a *dependency graph* $\mathsf{DG}(\mathcal{R})$ is associated to the TRS $\mathcal{R}$. The nodes of the dependency graph are the dependency pairs in $\mathsf{DP}(\mathcal{R})$; there is an arc from a dependency pair $u \to v$ to a dependency pair $u' \to v'$ if there are substitutions θ and θ' such that $\theta(v) \to^*_{\mathcal{R}} \theta'(u')$.

In more recent approaches, the analysis of infinite chains of dependency pairs as such is just a starting point. Many often, chains of dependency pairs are *transformed* into chains of more general *pairs* which cannot be considered dependency pairs anymore. This is the case for the *narrowing* or *instantiation* transformations, among others, see [GTSF06] for instance. Still, the analysis of the cycles in the graph build out from such pairs is useful to investigate the existence of infinite (minimal) chains of pairs. Thus, a more general notion of *graph of pairs* $\mathsf{DG}(\mathcal{P}, \mathcal{R})$ associated to a set of pairs $\mathcal{P}$ and a TRS $\mathcal{R}$ is considered; the pairs in $\mathcal{P}$ are used now as the nodes of the graph but they are connected by $\mathcal{R}$-rewriting in the same way [GTSF06, Definition 7].

In the following section we consider these points to provide an appropriate definition of the innermost context-sensitive (dependency) graph which takes into account the ideas developed in previous chapters.

11.1 Definition of the innermost context-sensitive dependency graph

According to the discussion above, our starting point are two TRSs $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ together with a replacement map $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Our aim is obtaining a notion of graph which is able to represent all infinite *minimal* innermost chains of pairs as given in Definition 27.

When considering pairs $u \to v \in \mathcal{P}_\mathcal{G}$, we can proceed as in the standard case to define appropriate connections to other pairs $u' \to v' \in \mathcal{P}$: there is an arc from $u \to v$ to $u' \to v'$ if $\theta(v) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ for some substitutions θ and θ' such that $\theta(u)$ and $\theta'(u') \in \mathsf{NF}_\mu(\mathcal{R})$. When considering collapsing pairs $u \to v \in \mathcal{P}_\mathcal{X}$, we know that such pairs can only be followed by a pair $u' \to v' \in \mathcal{P}$ such that $\theta(t^\sharp) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ for some $t \in \mathcal{NHT}$ and substitutions θ and θ' such that $\theta'(u') \in \mathsf{NF}_\mu(\mathcal{R})$ (see Definition 27).

Definition 49 (Innermost Context-Sensitive Graph of Pairs) *Let* $\mathcal{R} = (\mathcal{F}, R)$ *and* $\mathcal{P} = (\mathcal{G}, P)$ *be TRSs and* $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$*. The* innermost context-sensitive (ICS-)graph *associated to* $\mathcal{R}$ *and* $\mathcal{P}$ *(denoted* $\mathsf{IG}(\mathcal{P}, \mathcal{R}, \mu)$*) has* $\mathcal{P}$ *as the set of nodes and arcs which connect them as follows:*

1. *There is an arc from* $u \to v \in \mathcal{P}_\mathcal{G}$ *to* $u' \to v' \in \mathcal{P}$ *if there are substitutions* θ *and* θ' *such that* $\theta(v) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ *and* $\theta(u), \theta'(u') \in \mathsf{NF}_\mu(\mathcal{R})$.
2. *There is an arc from* $u \to v \in \mathcal{P}_\mathcal{X}$ *to* $u' \to v' \in \mathcal{P}$ *if there is* $t \in \mathcal{NHT}(\mathcal{R}, \mu)$ *and substitutions* θ *and* θ' *such that* $\theta(t^\sharp) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ *and* $\theta'(u') \in \mathsf{NF}_\mu(\mathcal{R})$

In termination proofs, we are concerned with the so-called *strongly connected components* (SCCs) of the dependency graph, rather than with the cycles themselves (which are exponentially many) [HM05]. A strongly connected component in a graph is a *maximal cycle*, i.e., a cycle which is not contained in any other cycle. The following result justifies the use of SCCs for proving the absence of infinite minimal (innermost) $(\mathcal{P}, \mathcal{R}, \mu)$-chains.

Theorem 50 (SCC processor) *Let* $\mathcal{R} = (\mathcal{F}, R)$ *and* $\mathcal{P} = (\mathcal{G}, P)$ *be TRSs and* $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$*. Then, the processor* Proc_{SCC} *given by*

$$\mathsf{Proc}_{SCC}(\mathcal{P}, \mathcal{R}, \mu, i) = \{(\mathcal{Q}, \mathcal{R}, \mu, i) \mid \mathcal{Q} \text{ contains the pairs of an SCC in } \mathsf{IG}(\mathcal{P}, \mathcal{R}, \mu)\}$$

is sound and complete.

Proof.

We prove soundness by contradiction. Assume that Proc_{SCC} is not sound. Then, there is an innermost CS-termination problem $\tau = (\mathcal{P}, \mathcal{R}, \mu, i)$ such that, for all

$\tau' \in \mathsf{Proc}_{SCC}(\tau)$, τ' is finite but τ is not finite. Thus, there is an infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain A. Since $\mathcal{P}$ contains a finite number of pairs, there is $\mathcal{P}' \subseteq \mathcal{P}$ and a tail B of A which is an infinite minimal innermost $(\mathcal{P}', \mathcal{R}, \mu)$-chain where all pairs in $\mathcal{P}'$ are infinitely often used. According to Definition 49, this means that $\mathcal{P}'$ is a cycle in $\mathsf{IG}(\mathcal{P}, \mathcal{R}, \mu)$, hence it belongs to some SCC with nodes in $\mathcal{Q}$, i.e., $\mathcal{P}' \subseteq \mathcal{Q}$. Hence B is an infinite minimal innermost $(\mathcal{Q}, \mathcal{R}, \mu)$-chain, i.e., $\tau' = (\mathcal{Q}, \mathcal{R}, \mu, i)$ is not finite. Since $\tau' \in \mathsf{Proc}_{SCC}(\tau)$, we obtain a contradiction.

Regarding completeness, since $\mathcal{Q} \subseteq \mathcal{P}$ for some SCC in $\mathsf{IG}(\mathcal{P}, \mathcal{R}, \mu)$ with nodes in $\mathcal{Q}$, every infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu, e)$-chain is an infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain, hence the processor is complete as well. ■

As a consequence of this theorem, we can *separately* work with the strongly connected components of $\mathsf{IG}(\mathcal{P}, \mathcal{R}, \mu)$, disregarding other parts of the graph.

Now we can use these notions to introduce the innermost context-sensitive dependency graph.

Definition 51 (Innermost Context-Sensitive Dependency Graph (ICS-DG)) *Let $\mathcal{R} = (\mathcal{F}, R)$ be a TRS and $\mu \in M_{\mathcal{F}}$. The* Innermost Context-Sensitive Dependency Graph *associated to $\mathcal{R}$ and μ is $\mathsf{IDG}(\mathcal{R}, \mu) = \mathsf{IG}(\mathsf{iDP}(\mathcal{R}, \mu), \mathcal{R}, \mu^\sharp)$.*

Example 52 Consider the following CS-TRS $\mathcal{R}$ in [GM02a]:

$$\begin{array}{rcl} \mathtt{f(g(b))} & \to & \mathtt{f(g(a))} \\ \mathtt{f(a)} & \to & \mathtt{f(a)} \\ \mathtt{a} & \to & \mathtt{b} \end{array}$$

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \varnothing$. Then the set of dependency pairs in [AL07a] for proving innermost μ-termination of $\mathcal{R}$ was:

$$\begin{array}{rcl} \mathtt{F(g(b))} & \to & \mathtt{F(g(a))} \\ \mathtt{F(a)} & \to & \mathtt{F(a)} \\ \mathtt{F(a)} & \to & \mathtt{A} \end{array}$$

and $\mu^\sharp(\mathtt{F}) = \{1\}$. Now, with the new definition of $\mathsf{iDP}(\mathcal{R}, \mu)$ we do not obtain any of these pairs since the first one does not have the μ-narrowability condition and the rest have the left-hand side not argument μ-normalized. So, now, $\mathsf{iDP}(\mathcal{R}, \mu) = \varnothing$. ■

11.2 Estimating the ICS-dependency graph

In general, the (innermost context-sensitive) dependency graph of a TRS is *not* computable: it involves reachability of $\theta'(u')$ from $\theta(v)$ (for $u \to v \in \mathcal{P}_{\mathcal{G}}$) or $\theta(t^\sharp)$

(for $t \in \mathcal{NHT}_{i,\mathcal{P}}$) using innermost *CSR*; as in the unrestricted case, the reachability problem for innermost *CSR* is undecidable.

So, we need to use some approximation of it. Following [AG00], we describe how to approximate the ICS-dependency graph of a CS-TRS.

Given a set Δ of 'defined' symbols, we let $\textsc{Cap}^{\mu}_{\Delta}$ be as follows (if Δ is clear from the context, sometimes we omit it in our examples):

$$\begin{array}{rcl} \textsc{Cap}^{\mu}_{\Delta}(x) & = & x \text{ if } x \text{ is a variable} \\ \textsc{Cap}^{\mu}_{\Delta}(f(t_1,\ldots,t_k)) & = & \begin{cases} y & \text{if } f \in \Delta \\ f([t_1]^f_1,\ldots,[t_k]^f_1) & \text{otherwise} \end{cases} \end{array}$$

where y is intended to be a new, fresh variable which has not yet been used and given a term s, $[s]^f_i = \textsc{Cap}^{\mu}_{\Delta}(s)$ if $i \in \mu(f)$ and $[s]^f_i = s$ if $i \notin \mu(f)$. Function $\textsc{Cap}^{\mu}_{\Delta}$ is intended to provide a suitable approximation of reachability problems $\theta(s) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(t)$ by means of unification. The idea is obtaining the maximal prefix context $C[\,]$ of s (i.e., $s = C[s_1,\ldots,s_n]$ for some terms $s_1,\ldots,s_n$) which we know (without any 'look-ahead' for applicable rules) that cannot be changed by any reduction starting from s. Therefore, $C[\,]$ is a *constructor* context, i.e., only function symbols which are not in Δ occur in $C[\,]$. Furthermore, terms $s_1,\ldots,s_n$ above must be rooted by defined symbols (i.e., $root(s_i) \in \Delta$ for $i \in \{1,\ldots,n\}$). Now, we replace those subterms s_i which are at μ-replacing positions (i.e., $s_i = s|_{p_i}$ for some $p_i \in \mathcal{P}os^{\mu}(s)$) by some variable x, and we leave untouched the non-μ-replacing ones. In this way, we are able to capture any possible 'evolution' of $\theta(s_i)$ by μ-rewriting.

The following result whose proof is similar to that of [AG00, Theorem 21] (we only need to take into account the replacement restrictions indicated by the replacement map μ) formalizes the correctness of this approach.

Proposition 53 *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $s, t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ be such that $\mathcal{V}ar(s) \cap \mathcal{V}ar(t) = \varnothing$ and θ, θ' be substitutions. If $\theta(s) \hookrightarrow^*_{\mathcal{R},\mu} \theta'(t)$, then $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\Delta}(s))$ and t unify.*

According to Proposition 53, given terms $s, t \in \mathcal{T}(\mathcal{F},\mathcal{X})$ and substitutions θ, θ', the reachability of $\theta'(t)$ from $\theta(s)$ by μ-rewriting can be *approximated* as unification of $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\Delta}(s))$ and t. However, since we are dealing with *innermost* μ-termination, we can take advantage of this and go further in a more accurate approximation. In order to automatically build the Innermost Context-Sensitive Dependency Graph it is necessary to approximate it since for two noncollapsing pairs $u \to v$ and $u' \to v'$ it is undecidable to know if there exist two substitutions θ and θ' such that (v) μ-reduces innermost to $\theta(u')$ and $\theta(u)$ and $\theta'(u')$ are instantiated to μ-normal forms. As we have commented, in the context-sensitive setting, we have adapted functions $\textsc{Cap}$ and $\textsc{Ren}$ to be applied only on *μ-replacing* subterms [AGL06]. In the innermost setting it is not necessary to use $\textsc{Ren}$ since all variables are always instantiated to normal forms and cannot be reduced and

$\textsc{Cap}_u(v)$ substitutes every subterm with a defined root symbol by fresh variables only if the term is not equal to subterms of u. To approximate the ICS-dependency graph, however, we have to combine both of them: we use $\textsc{Cap}^\mu_{\Delta,u}(v)$ to replace all μ-replacing subterm rooted with a defined symbol whenever the term was not equal to a μ-replacing subterm of the left-hand side of the dependency pair u. We use $\textsc{Ren}^\mu_u(v)$ to replace by fresh variables those ones that are replacing in v but not in u since they are not μ-normalized. The differences between the above definition are: for the $\textsc{Cap}$ function, now, $[s]^f_i = \textsc{Cap}^\mu_{\Delta,u}(s)$ if $i \in \mu(f)$ and s is not equal to a μ-replacing subterm of u and $[s]^f_i = s$ otherwise and for the function $\textsc{Ren}$ (see Section 5.1) we have that now, $[s]^f_i = \textsc{Ren}^\mu_u(s)$ if $i \in \mu(f)$ and the variable is not μ-replacing in u and $[s]^f_i = s$ otherwise. So, we have the following:

Definition 54 (Estimated Innermost Context-Sensitive Graph of Pairs) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. The* estimated *ICS-graph associated to $\mathcal{R}$ and $\mathcal{P}$ (denoted $\mathsf{EIG}(\mathcal{P}, \mathcal{R}, \mu)$) has $\mathcal{P}$ as the set of nodes and arcs which connect them as follows:*

1. *There is an arc from $u \to v \in \mathcal{P}_\mathcal{G}$ to $u' \to v' \in \mathcal{P}$ if $\textsc{Ren}^\mu_u(\textsc{Cap}^\mu_{\Delta,u}(v))$ and u' unify by some mgu σ such that $\sigma(u), \sigma(u') \in \mathsf{NF}_\mu(\mathcal{R})$.*

2. *There is an arc from $u \to v \in \mathcal{P}_\mathcal{X}$ to $u' \to v' \in \mathcal{P}$ if there is $t \in \mathcal{NHT}_{i,\mathcal{P}}(\mathcal{R}, \mu)$ such that $\textsc{Ren}^\mu(\textsc{Cap}^\mu_\Delta(t^\sharp))$ and u' unify by some mgu σ such that $\sigma(u') \in \mathsf{NF}_\mu(\mathcal{R})$.*

Proposition 55 (Correctness of the Estimated ICS-Graph of Pairs) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$.*

1. *If there are pairs $u \to v \in \mathcal{P}_\mathcal{G}$ and $u' \to v' \in \mathcal{P}$ and substitutions θ and θ' such that $\theta(v) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ and $\theta(u), \theta'(u') \in \mathsf{NF}_\mu(\mathcal{R})$, then $\textsc{Ren}^\mu_u(\textsc{Cap}^\mu_{\Delta,u}(v))$ and u' unify by some mgu σ such that $\sigma(u), \sigma(u') \in \mathsf{NF}_\mu(\mathcal{R})$.*

2. *If there are pairs $u \to v \in \mathcal{P}_\mathcal{X}$ and $u' \to v' \in \mathcal{P}$ and there is $t \in \mathcal{NHT}(\mathcal{R}, \mu)$ and substitutions θ and θ' such that $\theta(t^\sharp) \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \theta'(u')$ and $\theta'(u') \in \mathsf{NF}_\mu(\mathcal{R},$ then there is $t \in \mathcal{NHT}_{i,\mathcal{P}}(\mathcal{R}, \mu)$ such that $\textsc{Ren}^\mu(\textsc{Cap}^\mu_\Delta(t^\sharp))$ and u' unify by some mgu σ such that $\sigma(u') \in \mathsf{NF}_\mu(\mathcal{R})$;.*

According to Definition 51, we would have the corresponding one for the *estimated* ICS-DG: $\mathsf{EIDG}(\mathcal{R}, \mu) = \mathsf{EIG}(\mathsf{iDP}(\mathcal{R}, \mu), \mathcal{R}, \mu^\sharp)$.

Example 56 Consider the following TRS $\mathcal{R}$ [Zan97, Example 4]:

$$\begin{array}{rcl} \mathtt{f}(x) & \to & \mathtt{cons}(x, \mathtt{f}(\mathtt{g}(x))) \\ \mathtt{g}(\mathtt{0}) & \to & \mathtt{s}(\mathtt{0}) \end{array}$$

$$\begin{array}{rcl}
\mathtt{g}(\mathtt{s}(x)) & \to & \mathtt{s}(\mathtt{s}(\mathtt{g}(x))) \\
\mathtt{sel}(\mathtt{0},\mathtt{cons}(x,y)) & \to & x \\
\mathtt{sel}(\mathtt{s}(x),\mathtt{cons}(y,z)) & \to & \mathtt{sel}(x,z)
\end{array}$$

with $\mu(\mathtt{0}) = \varnothing$, $\mu(\mathtt{f}) = \mu(\mathtt{g}) = \mu(\mathtt{s}) = \mu(\mathtt{cons}) = \{1\}$, and $\mu(\mathtt{sel}) = \{1,2\}$. Then, $\mathsf{iDP}(\mathcal{R},\mu)$ is:

$$\begin{array}{rcll}
\mathtt{G}(\mathtt{s}(x)) & \to & \mathtt{G}(x) & (11.1) \\
\mathtt{SEL}(\mathtt{s}(x),\mathtt{cons}(y,z)) & \to & \mathtt{SEL}(x,z) & (11.2) \\
\mathtt{SEL}(\mathtt{s}(x),\mathtt{cons}(y,z)) & \to & z & (11.3)
\end{array}$$

and $\mathcal{NHT} = \{\mathtt{f}(\mathtt{g}(x)), \mathtt{g}(x)\}$. Regarding pairs (11.1) and (11.2) in $\mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$, there is an arc from (11.1) to itself and another one from (11.2) to itself. Regarding the only collapsing pair (11.3), we have $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}(\mathtt{F}(\mathtt{g}(x)))) = \mathtt{F}(y)$ and $\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}(\mathtt{G}(x))) = \mathtt{G}(y)$. Since $\mathtt{F}(y)$ does not unify with the left-hand side of any pair, and $\mathtt{G}(y)$ unifies with the left-hand side $\mathtt{G}(\mathtt{s}(\mathrm{x}))$ of (11.1) and $\mathtt{G}(\mathtt{s}(\mathrm{x}))$ is in μ-normal form, there is an arc from (11.3) to (11.1), see Figure 11.1. Thus, there are two cycles: $\{(11.1)\}$ and $\{(11.2)\}$. ■

The following example shows that using $\textsc{Ren}^{\mu}_{u}$ provides a better approximation of the ICS-DG than using $\textsc{Ren}^{\mu}$ for noncollapsing pairs.

Example 57 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl}
\mathtt{f}(\mathsf{a},\mathsf{b},x) & \to & \mathtt{f}(x,x,x) \\
\mathsf{c} & \to & \mathsf{a} \\
\mathsf{c} & \to & \mathsf{b}
\end{array}$$

together with $\mu(\mathtt{f}) = \{1,2\}$. There are two ICS-dependency pairs:

$$\begin{array}{rcl}
\mathtt{F}(\mathsf{a},\mathsf{b},x) & \to & \mathtt{F}(x,x,x) \\
\mathtt{F}(\mathsf{a},\mathsf{b},x) & \to & x
\end{array}$$

$\mathcal{R}$ is not innermost μ-terminating:

$$\mathtt{F}(\underline{\mathsf{c}},\mathsf{c},\mathsf{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \mathtt{F}(\mathsf{a},\underline{\mathsf{c}},\mathsf{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \underline{\mathtt{F}(\mathsf{a},\mathsf{b},\mathsf{c})} \hookrightarrow_{\mathsf{iDP}(\mathcal{R},\mu,\mathsf{i}),\mu^\sharp} \mathtt{F}(\underline{\mathsf{c}},\mathsf{c},\mathsf{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \cdots$$

In order to build the ICS-DG, since there are not hidden terms, we have to check if $\textsc{Ren}^{\mu}_{u}(\textsc{Cap}^{\mu}_{u}(\mathtt{F}(x,x,x))) = \textsc{Ren}^{\mu}_{\mathtt{f}(\mathsf{a},\mathsf{b},x)}(\textsc{Cap}^{\mu}_{\mathtt{f}(\mathsf{a},\mathsf{b},x)}(\mathtt{F}(x,x,x))) = \mathtt{F}(x''',x'',x)$ unifies with $\mathtt{F}(\mathsf{a},\mathsf{b},y)$ so, we get a cycle. However, if we use $\mu(\mathtt{f}) = \{1,3\}$, the system now is innermost μ-terminating but if we use the $\textsc{Ren}$ version for *CSR* we get

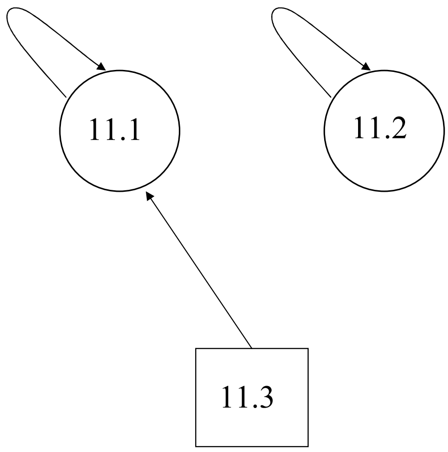

Figure 11.1: Estimated ICS-DG for the CS-TRS $(\mathcal{R}, \mu)$ in Example 56

$\textsc{Ren}^{\mu}(\textsc{Cap}^{\mu}_{\mathtt{f(a,b,}x\mathtt{)}}(\mathtt{F}(x,x,x))) = \mathtt{F}(x''',x'',x)$ again unifies with $\mathtt{F}(\mathtt{a},\mathtt{b},x)$ and we obtain a spurious cycle. By using $\textsc{Ren}^{\mu}_{u}$, we obtain $\textsc{Ren}^{\mu}_{\mathtt{f(a,b,}x\mathtt{)}}(\textsc{Cap}^{\mu}_{\mathtt{f(a,b,}x\mathtt{)}}(\mathtt{F}(x,x,x))) = \mathtt{F}(x,x,x)$ (since there are not migrating variables now) which does not unify with $\mathtt{F}(\mathtt{a},\mathtt{b},y)$. Now, innermost μ-termination can be easily proved since there are no cycles in the ICS-DG.

■

After showing that $\textsc{Ren}^{\mu}_{u}$ provides a better approximation of the ICS-DG for noncollapsing pairs, we are going to show that for the collapsing pairs this is not true since we can lead into and underestimation of the graph and conclude a false result.

Example 58 Consider the following TRS $\mathcal{R}$ which is a variant of Example 57:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{a},\mathtt{b},x) & \to & \mathtt{g}(\mathtt{f}((x,x,x)) \\ \mathtt{g}(x) & \to & x \\ \mathtt{c} & \to & \mathtt{a} \\ \mathtt{c} & \to & \mathtt{b} \end{array}$$

together with $\mu(\mathtt{f}) = \{1,2\}$ and $\mu(\mathtt{g}) = \varnothing$. There are two ICS-dependency pairs:

$$\mathtt{F}(\mathtt{a},\mathtt{b},x) \to \mathtt{G}(\mathtt{f}(x,x,x)) \tag{11.4}$$

$$\mathtt{G}(x)) \to x \tag{11.5}$$

$\mathcal{R}$ is not innermost μ-terminating:

$$\mathtt{F}(\underline{\mathtt{c}},\mathtt{c},\mathtt{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \mathtt{F}(\mathtt{a},\underline{\mathtt{c}},\mathtt{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \underline{\mathtt{F}(\mathtt{a},\mathtt{b},\mathtt{c})} \hookrightarrow_{\mathsf{iDP}(\mathcal{R},\mu),\mu^\sharp,\mathsf{i}} \underline{\mathtt{G}(\mathtt{f}(\mathtt{a},\mathtt{b},\mathtt{c}))} \hookrightarrow_{\mathsf{iDP}(\mathcal{R},\mu),\mu^\sharp,\mathsf{i}} \mathtt{F}(\underline{\mathtt{c}},\mathtt{c},\mathtt{c}) \hookrightarrow_{\mathcal{R},\mu^\sharp,\mathsf{i}} \cdots$$

We have $\mathcal{NHT} = \{\mathtt{f}(x,x,x)\}$. Regarding the pair (11.4) $\in \mathsf{iDP}_{\mathcal{F}}(\mathcal{R},\mu)$, there is an obvious arc from (11.4) to (11.5). Regarding the only collapsing pair (11.5), since we do not have any information in the hidden terms about migrating variables, we have to use $\textsc{Ren}^\mu$. In this way, we have that $\textsc{Ren}^\mu(\textsc{Cap}^\mu(\mathtt{F}(x,x,x))) = \mathtt{F}(x'',x',x)$ unifies with $\mathtt{F}(\mathtt{a},\mathtt{b},y)$ and we obtain an arc from (11.5) to (11.4), thus obtaining the existing cycle $\{$(11.5)-(11.4)$\}$

∎

Note that Proposition 53 also provides a way to *estimate* the set $\mathcal{NHT}_{i,\mathcal{P}}$: if $t \in \mathcal{NHT}_{i,\mathcal{P}}$, then $\textsc{Ren}^\mu(\textsc{Cap}^\mu(t^\sharp))$ and u unify for some $u \to v \in \mathcal{P}$. In the following, our presentations of $\mathcal{NHT}_{i,\mathcal{P}}$ in the examples are computed in this way.

Example 59 (Continuing Example 42) Since $\textsc{Ren}^\mu_u(\textsc{Cap}^\mu_u(\mathtt{F(x,x)})) = \mathtt{F(x,x)}$ and $\mathtt{F(c(y),y)}$ do not unify we conclude (and this can easily be implemented) that the ICS-dependency graph for the CS-TRS $(\mathcal{R},\mu)$ in Example 42 contains no cycles.

∎

12

Usable Rules

An interesting feature in the treatment of innermost termination problems using the dependency pairs approach is that, since the variables in the right-hand side of the dependency pairs are in normal form, the rules which can be used to connect contiguous dependency pairs are usually a proper subset of the rules in the TRS. This leads to the notion of *usable rules* [AG00, Definition 32] which simplifies the proofs of innermost termination of rewriting. We adapt this notion to the context-sensitive setting.

Definition 60 (Basic usable CS-rules) *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. For any symbol f let $Rules(\mathcal{R}, f)$ be the set of rules of $\mathcal{R}$ defining f and such that the left-hand side l has no proper μ-replacing $\mathcal{R}$-redex. For any term t, the set of* basic usable rules *$\mathbf{U}_0(\mathcal{R}, \mu, t)$ is as follows:*

$$\begin{array}{lcl}\mathbf{U}_0(\mathcal{R}, \mu, x) & = & \varnothing \\ \mathbf{U}_0(\mathcal{R}, \mu, f(t_1, \ldots, t_n)) & = & Rules(\mathcal{R}, f) \cup \bigcup_{i \in \mu(f)} \mathbf{U}_0(\mathcal{R}', \mu, t_i) \cup \bigcup_{l \to r \in Rules(\mathcal{R}, f)} \mathbf{U}_0(\mathcal{R}', \mu, r)\end{array}$$

where $\mathcal{R}' = \mathcal{R} - Rules(\mathcal{R}, f)$. Consider now a TRS $\mathcal{P}$. Then, $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}) = \bigcup_{l \to r \in \mathcal{P}} \mathbf{U}_0(\mathcal{R}, \mu, r)$. Obviously, $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}) \subseteq \mathcal{R}$ for all TRSs $\mathcal{P}$ and $\mathcal{R}$.

Interestingly, although our definition is a straightforward extension of the classical one (which just takes into account that μ-rewritings are possible only on μ-replacing subterms), some subtleties arise due to the presence of *non-conservative* rules.

Basic usable rules $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ in Definition 60 can be used instead of $\mathcal{R}$ when dealing with innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chains associated to μ-conservative TRSs $\mathcal{P}$ provided that $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ is also μ-conservative. This is proved in Theorem 64 below. First, we need some auxiliary results.

Proposition 61 *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^{\mu}(t)$, $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. If $s \hookrightarrow_{\mathsf{i}} s'$ by applying a rule $l \to r \in \mathcal{R}$, then there is a substitution σ' such that $s' = \sigma'(t')$ for $t' = t[r]_p$ and $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$.*

Proof. Let $p \in \mathcal{P}os^{\mu}(s)$ be the position of an innermost redex $s|_p = \theta(l)$ for some substitution θ. Since $s = \sigma(t)$ and for all replacing variables in t, we have $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$, it follows that p is a non-variable (replacing) position of t. Therefore, $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$. Since $s = \sigma(t)$, we have that $s' = \sigma(t)[\theta(r)]_p$ and since $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$, by defining $\sigma'(x) = \sigma(x)$ for all $x \in Var(t)$ and $\sigma(x) = \theta(x)$ for all $x \in \mathcal{V}ar(r)$ (as usual, we assume $\mathcal{V}ar(t) \cap \mathcal{V}ar(r) = \varnothing$), we have $s' = \sigma'(t[r]_p)$. ∎

The following proposition states that an innermost μ-rewrite step by applying a conservative rule makes the set of μ-replacing variables of the contractum will be instantiated to μ-normal forms.

Proposition 62 *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^{\mu}(t)$, $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. If $s \hookrightarrow_{\mathsf{i}} s'$ by applying a conservative rule $l \to r \in \mathcal{R}$, then there is a substitution σ' such that $s' = \sigma'(t')$ for $t' = t[r]_p$, $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$ and $\forall x \in \mathcal{V}ar^{\mu}(t')$, $\sigma'(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$.*

Proof. By Proposition 61, we know that σ', as in Proposition 61, satisfies $s' = \sigma'(t')$ for θ as in Proposition 61 and some $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$. Since $s|_p$ is an innermost μ-replacing redex, we have that $\forall y \in \mathcal{V}ar^{\mu}(l)$, $\theta(y) \in \mathsf{NF}_{\mu}(\mathcal{R})$. Since the rule $l \to r$ is conservative, $\mathcal{V}ar^{\mu}(r) \subseteq \mathcal{V}ar^{\mu}(l)$, hence $\forall z \in \mathcal{V}ar^{\mu}(r)$, $\sigma'(z) \in \mathsf{NF}_{\mu}(\mathcal{R})$. Since $\mathcal{V}ar^{\mu}(t[r]_p) \subseteq \mathcal{V}ar^{\mu}(t) \cup \mathcal{V}ar^{\mu}(r)$, we have that $\forall x \in \mathcal{V}ar^{\mu}(t')$, $\sigma'(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. ∎

Now, we prove that in an innermost μ-rewrite sequence starting from a term instantiated with a μ-normalized substitution, the only rules that can be applied are the usable rules (if they are μ-conservative).

Proposition 63 *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. Let $t, s \in \mathcal{T}(\mathcal{F}, \mathcal{X})$ and σ be a substitution such that $s = \sigma(t)$ and $\forall x \in \mathcal{V}ar^{\mu}(t)$, $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. If $\mathbf{U}_0(\mathcal{R}, \mu, t)$ is conservative and $s = s_1 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_2 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} \cdots \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_n \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_{n+1} = u$ for some $n \geq 0$ then $s_i \hookrightarrow_{\mathbf{U}_0(\mathcal{R},\mu,t),\mu,\mathsf{i}} s_{i+1}$ for all i, $1 \leq i \leq n$.*

Proof. By induction on n. If $n = 0$, then $s = \sigma(t) = u$, it is trivial. Otherwise, if $s_1 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_2 \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} u$, we first prove that the result also holds in $s_1 \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} s_2$. By Proposition 61, $s_1 = \sigma(t)$, and $s_2 = \sigma'(t')$ for $t' = t[r]_p$ is such that $s_1|_p = \theta(l)$ and $s_2|_p = \theta(r)$ for some $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(t)$. Thus, $root(l) = root(t|_p)$ and by Definition 60, we can conclude that $l \to r \in \mathbf{U}_0(\mathcal{R}, \mu, t)$. By hypothesis, $\mathbf{U}_0(\mathcal{R}, \mu, t)$ is conservative. Thus, $l \to r$ is conservative and by Proposition 62, $s_2 = \sigma'(t')$ and $\forall x \in \mathcal{V}ar^{\mu}(t')$, $\sigma'(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$. Since $t' = t[r]_p$ and $root(t|_p) = root(l)$, we have that $\mathbf{U}_0(\mathcal{R}, \mu, t') \subseteq \mathbf{U}_0(\mathcal{R}, \mu, t)$ and (since $\mathbf{U}_0(\mathcal{R}, \mu, t)$ is conservative) $\mathbf{U}_0(\mathcal{R}, \mu, t')$ is conservative as well. By the induction hypothesis we know that $s_i \hookrightarrow_{\mathbf{U}_0(\mathcal{R},\mu,t'),\mu,\mathsf{i}} s_{i+1}$ for all i, $2 \leq i \leq n$. Thus we have $s_i \hookrightarrow_{\mathbf{U}_0(\mathcal{R},\mu,t),\mu,\mathsf{i}} s_{i+1}$ for all i, $1 \leq i \leq n$ as desired.

∎

The following theorem formalizes a processor to remove pairs from $\mathcal{P}$ by using the previous result and μ-reduction pairs.

Theorem 64 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $(\gtrsim, \sqsupset)$ be a μ-reduction pair such that*

1. *$\mathcal{P}$ and $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ are conservative,*
2. *$\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}) \subseteq \gtrsim$ and $\mathcal{P} \subseteq \gtrsim \cup \sqsupset$,*

Let $\mathcal{P}_{\sqsupset} = \{u \to v \in \mathcal{P} \mid u \sqsupset v\}$. Then, the processor Proc_{UR} given by

$$\mathsf{Proc}_{UR}(\mathcal{P}, \mathcal{R}, \mu, i) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_{\sqsupset}, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu, i)\} & \textit{if (1) and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu, i)\} & \textit{otherwise} \end{cases}$$

is sound.

Proof. We proceed by contradiction. Assume that there is an infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain A, but that there is no infinite minimal innermost $(\mathcal{P} - \mathcal{P}_{\sqsupset}, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu)$-chain. Due to the finiteness of $\mathcal{P}$, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B

$$\sigma(u_1) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_1 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_2) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_2 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_3) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} \cdots$$

for some substitution σ, where all pairs in $\mathcal{Q}$ are infinitely often used, and, for all $i \geq 1$, since all $u_i \to v_i \in \mathcal{P}$ are conservative $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$, then $t_i = \sigma(v_i)$ and $\sigma(u_i) \in \mathsf{NF}_{\mu}(\mathcal{R})$, this implies that $\forall x \in \mathcal{V}ar^{\mu}(v_i)$, $\sigma(x) \in \mathsf{NF}_{\mu}(\mathcal{R})$ and by Proposition 63 the sequence can be seen as:

$$\sigma(u_1) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_1 \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu,\mathsf{i}} \sigma(u_2) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} t_2 \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu,\mathsf{i}} \sigma(u_3) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^{\sharp}_{\mu} \cdots$$

Furthermore, by minimality, $\sigma(v_i)$ is innermost $(\mathcal{R}, \mu)$-terminating for all $i \geq 1$. Since $u_i\ (\gtrsim \cup \sqsupset)\ v_i$ for all $u_i \to v_i \in \mathcal{Q} \subseteq \mathcal{P}$, by stability of $\gtrsim$ and $\sqsupset$, we have $\sigma(u_i)\,(\gtrsim \cup \sqsupset)\,\sigma(v_i)$ for all $i \geq 1$. No pair $u \to v \in \mathcal{Q}$ satisfies that $u \sqsupset v$. Otherwise, we get a contradiction by considering that since all pairs in $\mathcal{P}$ are conservative, we have that $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$. Then, $t_i = \sigma(v_i) \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu,\mathsf{i}} \sigma(u_{i+1})$ and $t_i \gtrsim \sigma(u_{i+1})$. Since we have $\sigma(u_i)\ (\gtrsim \cup \sqsupset)\ \sigma(v_i) = \sigma(v_i) = t_i$, by using transitivity of $\gtrsim$ and compatibility between $\gtrsim$ and $\sqsupset$, we conclude that $\sigma(u_i)\ (\gtrsim \cup \sqsupset)\ \sigma(u_{i+1})$. Since $u \to v$ occurs infinitely often in B, there is an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that $\sigma(u_i) \sqsupset \sigma(u_{i+1})$ for all $i \in \mathcal{I}$. And we have $\sigma(u_i)\ (\gtrsim \cup \sqsupset)\ \sigma(u_{i+1})$ for all other $u_i \to v_i \in \mathcal{Q}$. Thus, by using the compatibility conditions of the μ-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$. Therefore, $\mathcal{Q} \subseteq (\mathcal{P} - \mathcal{P}_{\sqsupset})$. Since $\mathsf{NF}_{\mu}(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})) \supseteq \mathsf{NF}_{\mu}(\mathcal{R})$, we have that $\sigma(u_i) \in \mathsf{NF}_{\mu}(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}))$. By Proposition 63, innermost $(\mathcal{R}, \mu)$-termination of $\sigma(v_i)$ implies innermost $(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu)$-termination of $\sigma(v_i)$ for all $i \geq 1$. Hence, B is an infinite minimal innermost $(\mathcal{P} - \mathcal{P}_{\sqsupset}, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu)$-chain, thus leading to a contradiction.

∎

Unfortunately, dealing with nonconservative pairs, considering the basic usable CS-rules does *not* ensure a correct approach.

Example 65 Consider again the TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{b} & \to & \mathtt{c}(\mathtt{b}) \\ \mathtt{f}(\mathtt{c}(x), x) & \to & \mathtt{f}(x, x) \end{array}$$

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{c}) = \varnothing$. There are *two* non-conservative ICS-DPs (note that $\mu^\sharp(\mathtt{F}) = \mu(\mathtt{f}) = \{1\}$):

$$\begin{array}{rcl} \mathtt{F}(\mathtt{c}(x), x) & \to & \mathtt{F}(x, x) \\ \mathtt{F}(\mathtt{c}(x), x) & \to & x \end{array}$$

and only one cycle in the ICS-DG:

$$\{\mathtt{F}(\mathtt{c}(x), x) \to \mathtt{F}(x, x)\}$$

Note that $\mathbf{U}_0(\mathcal{R}, \mu, \mathtt{F(x,x)}) = \varnothing$. Since this ICS-DP is strictly compatible with, e.g., an LPO, we would conclude the innermost μ-termination of $\mathcal{R}$. However, this system is *not* innermost μ-terminating:

$$\mathtt{f(\underline{b},b)} \hookrightarrow_{\mathsf{i}} \underline{\mathtt{f(c(b),b)}} \hookrightarrow_{\mathsf{i}} \mathtt{f(\underline{b},b)} \hookrightarrow_{\mathsf{i}} \cdots$$

■

The problem is that we have to take into account the special status of variables in the right-hand side of a nonconservative ICS-DP. Instances of such variables are *not* guaranteed to be μ-normal forms. For this reason, when a cycle contains at least one nonconservative pair, we have to consider the whole set of rules of the system.

Furthermore, conservativeness of $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ cannot be dropped either since we could infer an incorrect result as shown by the following example.

Example 66 Consider the TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{b} & \to & \mathtt{c}(\mathtt{b}) \\ \mathtt{f}(\mathtt{c}(x), x) & \to & \mathtt{f}(\mathtt{g}(x), x) \\ \mathtt{g}(x) & \to & x \end{array}$$

together with $\mu(\mathtt{f}) = \{1\}$ and $\mu(\mathtt{g}) = \mu(\mathtt{c}) = \varnothing$. There is only one conservative cycle:

$$\{\mathtt{F}(\mathtt{c}(x), x) \to \mathtt{F}(\mathtt{g}(x), x)\}$$

having only one usable (but non-conservative!) rule $\mathtt{g}(x) \to x$ This is compatible with the μ-reduction pair induced by the following polynomial interpretation:

$$[\mathtt{f}](x,y) = 0 \qquad [\mathtt{c}](x) = x+1 \qquad [\mathtt{g}](x) = x \qquad [\mathtt{F}](x,y) = x$$

However the system is not innermost μ-terminating:

$$\underline{\mathtt{f(c(b),b)}} \hookrightarrow_{\mathsf{i}} \mathtt{f(}\underline{\mathtt{g(b)}}\mathtt{,b)} \hookrightarrow_{\mathsf{i}} \mathtt{f(}\underline{\mathtt{b}}\mathtt{,b)} \hookrightarrow_{\mathsf{i}} \underline{\mathtt{f(c(b),b)}} \hookrightarrow_{\mathsf{i}} \cdots$$

■

Nevertheless, Theorem 64 is useful to improve the proofs of termination of innermost *CSR* as the following example shows.

Example 67 Consider again the TRS $\mathcal{R}$ in example 1. The system contains three cycles in the ICS-DG:

$$\begin{array}{rcl}
\{\mathtt{SEL}(\mathtt{s}(y), \mathtt{cons}(x, xs)) & \to & \mathtt{SEL}(y, xs)\} \\
\{\mathtt{MINUS}(\mathtt{s}(x), \mathtt{s}(y)) & \to & \mathtt{MINUS}(x, y)\} \\
\{\mathtt{QUOT}(\mathtt{s}(x), s(y)) & \to & \mathtt{QUOT}(\mathtt{minus}(x, y), \mathtt{s}(y))\}
\end{array}$$

The first two cycles can be solved by using the subterm processor (see [AGL08]). However, without the notion of usable rules, the last one is difficult to solve. The cycle is μ-conservative and the obtained usable rules are also μ-conservative:

$$\mathtt{minus}(x, 0) \to x$$

and

$$\mathtt{minus}(\mathtt{s}(x), \mathtt{s}(x)) \to \mathtt{minus}(x, y)$$

According to Theorem 64, the cycle can be easily solved by using a polynomial interpretation:

$$\begin{array}{rclrcl}
[\mathtt{minus}](x,y) & = & x & [\mathtt{0}] & = & 0 \\
[\mathtt{s}](x) & = & x+1 & [\mathtt{QUOT}](x,y) & = & x
\end{array}$$

■

13

Narrowing Transformation

The starting point of a proof of termination is the computation of the *estimated* dependency graph followed by the use of the SCC processor (Theorem 50). The estimation of the graph can lead to *overestimate* the arcs that connect two dependency pairs.

Example 68 Consider the following example [Luc06, Proposition 7]:

$$\begin{array}{rcl} \mathtt{f(0)} & \to & \mathtt{cons(0,f(s(0)))} \\ \mathtt{f(s(0))} & \to & \mathtt{f(p(s(0)))} \\ \mathtt{p(s(}x\mathtt{))} & \to & x \end{array}$$

together with $\mu(\mathtt{f}) = \mu(\mathtt{p}) = \mu(\mathtt{s}) = \mu(\mathtt{cons}) = \{1\}$ and $\mu(\mathtt{0}) = \varnothing$. Then, $\mathsf{DP}(\mathcal{R},\mu)$ (see [AGL06, AGL07, AGL08]) consists of the following pairs:

$$\mathtt{F(s(0))} \to \mathtt{F(p(s(0)))} \tag{13.1}$$

$$\mathtt{F(s(0))} \to \mathtt{P(s(0))} \tag{13.2}$$

The *estimated* CS-dependency graph [AGL06, AGL07, AGL08] contains one cycle: $\{(13.1)\}$. Note, however, that this cycle does *not* belong to the CS-dependency graph because there is no way to μ-rewrite $\mathtt{F(p(s(0)))}$ into $\mathtt{F(s(0))}$! ■

As already observed by Arts and Giesl for the standard case [AG00], in our case the overestimation comes when a (noncollapsing) pair $u_i \to v_i$ is followed in a chain by a second one $u_{i+1} \to v_{i+1}$ and v_i and u_{i+1} are not directly unifiable, i.e., at least one μ-rewriting step is needed to μ-reduce $\sigma(v_i)$ to $\sigma(u_{i+1})$. Then, the μ-reduction from $\sigma(v_i)$ to $\sigma(u_{i+1})$ requires at least one step, i.e., we always have $\sigma(v_i) \hookrightarrow_{\mathcal{R},\mu^\sharp} \sigma(v'_i) \hookrightarrow^*_{\mathcal{R},\mu^\sharp} \sigma(u_{i+1})$. Then, v'_i is a one-step μ-narrowing of v_i and we could require $u_i \sqsupset v'_i$ (which could be easier to prove) instead of $u_i \sqsupset v_i$. Furthermore, we could discover that v_i has *no μ-narrowings*. In this case, we know that no chain starts from $\sigma(v_i)$.

According to the discussion above, we can be more precise when connecting two pairs $u \to v$ and $u' \to v'$ in a chain, if we perform all possible one-step μ-narrowings on v in order to develop the possible reductions from $\sigma(v)$ to $\sigma(u')$. Then, we obtain new terms $v_1, \ldots, v_n$ which are one-step μ-narrowings of v using unifiers θ_i (i.e., $v \leadsto_{\mathcal{R},\mu,\theta_i} v_i$) for $i \in \{1, \ldots, n\}$, respectively. These unifiers are also applied to the left-hand side u of the pair $u \to v$. Therefore, we can replace a pair $u \to v$ by all its (one-step) *μ-narrowed pairs* $\theta_1(u) \to v_1, \ldots, \theta_n(u) \to v_n$.

As in [AG00, GTSF06], a pair $u \to v \in \mathcal{P}$ may only be replaced by its narrowings if the right-hand side v does not unify with any left-hand side u' of a (possibly renamed) pair $u' \to v' \in \mathcal{P}$ (note that this excludes pairs $u \to v$ with $v \in \mathcal{X}$). Moreover, the term v must be *linear*. We need to demand linearity instead of (the apparently more natural) μ-linearity (i.e., something like "no multiple *μ-replacing* occurrences of the same variable are allowed").

Example 69 Consider the following TRS which is used in [AG00] to motivate the requirement of linearity:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{s}(x)) & \to & \mathtt{f}(\mathtt{g}(x,x)) \\ \mathtt{g}(\mathtt{0},\mathtt{1})) & \to & \mathtt{s}(\mathtt{0}) \\ \mathtt{0} & \to & \mathtt{1} \end{array}$$

We make it a CS-TRS by adding a replacement map μ given by $\mu(\mathtt{f}) = \mu(\mathtt{s}) = \{1\}$, $\mu(\mathtt{g}) = \{2\}$. The only cycle in the CS-DG consists of the pair

$$\mathtt{F}(\mathtt{s}(x)) \to \mathtt{F}(\mathtt{g}(x,x)).$$

If linearity of the right-hand sides is not required for narrowing CSDPs, then it will be removed since $\mathtt{F}(\mathtt{g}(x,x))$ and the (renamed version of) the left-hand side $\mathtt{F}(\mathtt{s}(x'))$ do not unify, thus, there are no μ-narrowings. However the system is *not μ-terminating*:

$$\underline{\mathtt{f}(\mathtt{s}(\mathtt{0}))} \hookrightarrow \mathtt{f}(\mathtt{g}(\mathtt{0},\underline{\mathtt{0}})) \hookrightarrow \mathtt{f}(\underline{\mathtt{g}(\mathtt{0},\mathtt{1})}) \hookrightarrow \underline{\mathtt{f}(\mathtt{s}(\mathtt{0}))} \ldots$$

The problem is that the μ-reduction from $\sigma(\mathtt{F}(\mathtt{g}(x,x)))$ to $\sigma(\mathtt{F}(\mathtt{s}(x')))$ takes place 'in σ' and therefore it cannot be captured by μ-narrowing. Note that $\mathtt{F}(\mathtt{g}(x,x))$ is "μ-linear". ■

Another restriction to take into account when μ-narrowing a noncollapsing pair $u \to v$ is that the μ-replacing variables in v have to be μ-replacing in u as well (this corresponds with the notion of conservativeness). Furthermore, they cannot be both μ-replacing and non-μ-replacing at the same time. This corresponds to the following definition.

Definition 70 (Strongly Conservative [GLU08]**)** *Let $\mathcal{R}$ be a TRS and $\mu \in M_{\mathcal{R}}$. A rule $l \to r$ is strongly μ-conservative if it is μ-conservative and $\mathcal{V}ar^{\mu}(l) \cap \mathcal{V}ar^{\not\mu}(l) = \mathcal{V}ar^{\mu}(r) \cap \mathcal{V}ar^{\not\mu}(r) = \varnothing$.*

The following result shows that, under these conditions, the set of CSDPs can be safely replaced by their μ-narrowings.

Theorem 71 (Narrowing processor) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $u \to v \in \mathcal{P}$ be such that*

1. *$u \to v$ is strongly conservative,*
2. *v linear, and*
3. *for all $u' \to v' \in \mathcal{P}$ (with possibly renamed variables), v and u' do not unify.*

Let $\mathcal{Q} = (\mathcal{P} - \{u \to v\}) \cup \{u' \to v' \mid u' \to v'$ is a μ-narrowing of $u \to v\}$. Then, the processor Proc_{narr} given by

$$\mathsf{Proc}_{narr}(\mathcal{P}, \mathcal{R}, \mu, t) = \begin{cases} \{(\mathcal{Q}, \mathcal{R}, \mu, t)\} & \textit{if (1), (2), and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu, t)\} & \textit{otherwise} \end{cases}$$

is sound and complete.

Proof.

We have to prove that there is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain iff there is an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu, t)$-chain. The proof of this theorem is analogous to the proof of [GTSF06, Theorem 31], which we adapt here. For the first direction, we prove that given a minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$", there is a μ-narrowing v' of v with the mgu θ such that "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is also a minimal $(\mathcal{Q}, \mathcal{R}, \mu, t)$-chain. Hence, every infinite minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain yields an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu, t)$-chain.

If "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is a minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain, then there is an substitution σ such that for all pairs $s \to t$ in the chain,

1. if $s \to t \in \mathcal{P}_{\mathcal{G}}$, then $\sigma(t)$ is μ-terminating and it μ-reduces to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain
2. if $s \to t = s \to x \in \mathcal{P}_{\mathcal{X}}$ then, $\sigma(x)$ has a μ-replacing subterm s_0, $\sigma(x) \trianglerighteq_\mu s_0$ such that $s_0^\sharp$ is μ-terminating and it μ-reduces to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain; furthermore, there is $\bar{s}_0 \in \mathcal{NHT}(\mathcal{R}, \mu)$ such that $s_0 = \theta_0(\bar{s}_0)$ for some substitution θ_0.

Assume that σ is a substitution satisfying the above requirements and such that the length of the sequence $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$ is *minimal.*

Note that the length of this μ-reduction sequence cannot be zero because v and u_2 do not unify, that is, $\sigma(v) \neq \sigma(u_2)$. Hence, there is a term q such that $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$. We consider two possible cases:

1. The reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ takes place within a binding of σ, i.e., there is a term r, a μ-replacing variable position $p \in \mathcal{P}os^{\mu}_{\mathcal{X}}(v)$, and a μ-replacing variable $x \in \mathcal{V}ar^{\mu}(v)$ such that $v|_p = x$, $q = \sigma(v[r]_p)$ and $\sigma(x) \hookrightarrow_{\mathcal{R},\mu} r$. Since v is linear, x occurs only once in v. Thus, $q = \sigma'(v)$ for the substitution σ' with $\sigma'(x) = r$ and $\sigma'(y) = \sigma(y)$ for all variables $y \neq x$. As we assume that all occurrences of pairs in the chain are variable disjoint, $\sigma'(x)$ behaves like σ for all pairs except $u \to v$. We have $\sigma(z) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(z)$ for all $z \in \mathcal{X}$. Since $u \to v$ is strongly conservative we also have $\sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$ because all occurrences of x in u must be μ-replacing. Hence, if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}$ we have
$$\sigma'(v_1) = \sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$$
and if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$, then there is $s_1 \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that
$$\sigma'(v_1) = \sigma(v_1) \trianglerighteq_{\mu} s_1 \; s_1^{\sharp} \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u)$$
and, in both cases,
$$\sigma'(v) = q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2) = \sigma'(u_2).$$
Note that, by minimality and because $u \to v \in \mathcal{P}_{\mathcal{G}}$, $\sigma(v)$ is $(\mathcal{R},\mu)$-terminating and, since $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$, the term q is $(\mathcal{R},\mu)$-terminating as well. Therefore, $\sigma'(x) = q$ is $(\mathcal{R},\mu)$-terminating and σ' satisfies the two conditions above. Since the length of the sequence $\sigma'(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma'(u_2)$ is shorter than the sequence $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$, we obtain a contradiction and we conclude that the μ-reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ cannot take place in a binding of σ.

2. The reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu} q$ 'touches' v, i.e., there is a nonvariable position $p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(v)$, and a rewrite rule $l \to r \in \mathcal{R}$ such that $\sigma(v|_p) = \rho(l)$, for some substitution ρ and
$$\sigma(v) = \sigma(v)[\sigma(v|_p)]_p = \sigma(v)[\rho(l)]_p \hookrightarrow_{\mathcal{R},\mu} \sigma(v)[\rho(r)]_p = q$$
Since we can assume that variables in l are fresh, we can extend σ to behave like ρ on variables in l. Thus, $\sigma(l) = \sigma(v|_p)$, i.e, l and $v|_p$ unify and there is a mgu θ and an substitution τ satisfying $\sigma(x) = \tau(\theta(x))$ for all variables x. We have that v μ-narrows to $\theta(v)[\theta(r)]_p = v'$ with unifier θ. Again, we can extend σ to behave like τ on the variables of $\theta(u)$ and v'. Therefore, if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}$ we have
$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) = \tau(\theta(u)) = \sigma(\theta(u))$$
and if $u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$, then there is $s_1 \in \mathcal{T}(\mathcal{F},\mathcal{X})$ such that
$$\sigma(v_1) = \sigma(x) \trianglerighteq_{\mu} s_1 \; s_1^{\sharp} \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) = \tau(\theta(u)) = \sigma(\theta(u))$$
and
$$\sigma(v') = \tau(v') = \tau(\theta(v))[\tau(\theta(r))]_p = \sigma(v)[\sigma(r)]_p = \sigma(v)[\rho(r)]_p = q \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$$
Hence, "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is also a minimal chain.

The other side is also analogous to the 'completeness' part of [GTSF06, Theorem 31]. If "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is an infinite minimal $(\mathcal{Q}, \mathcal{R}, \mu, t)$-chain where v' is a one-step μ-narrowing of v using the mgu θ, then "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is an infinite minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain. There is a substitution σ such that

$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(\theta(u)) \;\text{ if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}, \text{ and}$$

$$\sigma(v_1) = \sigma(x) \trianglerighteq_\mu s_1 \text{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(\theta(u)) \quad \text{if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$$

Finally, we also have

$$\sigma(v') \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2).$$

Since the variables in the pairs are pairwise disjoint, we may extend σ to behave like $\sigma(\theta(x))$ on $x \in \mathcal{V}ar(u)$ then $\sigma(u) = \sigma(\theta(u))$ and therefore

$$\sigma(v_1) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \;\text{ if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{G}}, \text{ and}$$

$$\sigma(v_1) \trianglerighteq_\mu s_1 \text{ and } s_1^\sharp \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u) \quad \text{if } u_1 \to v_1 \in \mathcal{P}_{\mathcal{X}}$$

Moreover, by definition of μ-narrowing, we have $\theta(v) \hookrightarrow_{\mathcal{R},\mu} v'$. This implies that $\sigma(\theta(v)) \hookrightarrow_{\mathcal{R},\mu} \sigma(v')$ and since $\sigma(v) = \sigma(\theta(v))$, we obtain

$$\sigma(v) \hookrightarrow_{\mathcal{R},\mu} \sigma(v') \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2).$$

Hence, "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is a minimal $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain as well. ■

Example 72 (Continuing Example 68) Since the right-hand side of pair (13.1) in Example 68 does not unify with any (renamed) left-hand side of a pair (including itself) and it can be μ-narrowed at position 1 (notice that $\mu(\mathtt{f})=\{1\}$) by using the rule $\mathtt{p}(\mathtt{s}(x)) \to x$, we can replace it by its μ-narrowed pair:

$$\mathtt{F}(\mathtt{s}(\mathtt{0})) \to \mathtt{F}(\mathtt{0})$$

The *μ-narrowed* pair does not form any cycle in the estimated narrowed ICS-dependency graph and μ-termination is easily proved now. ■

The following example shows that strongly conservativeness cannot be dropped for the pair $u \to v$ to be μ-narrowed. This requirement was not taken into account in [AGL07, Theorem 5.3].

Example 73 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{c}(\mathtt{e}(x)) & \to & \mathtt{d}(x, x) \\ \mathtt{a} & \to & \mathtt{e}(\mathtt{a}) \end{array}$$

and $\mathcal{P}$ consisting of the following pair:

$$\mathtt{F}(\mathtt{d}(x,x)) \to \mathtt{F}(\mathtt{c}(x))$$

together with $\mu(\mathtt{c}) = \mu(\mathtt{d}) = \mu(\mathtt{F}) = \{1\}$ and $\mu(\mathtt{e}) = \varnothing$. There is an infinite $(\mathcal{P}, \mathcal{R}, \mu, t)$-chain as follows:

$$\mathtt{F}(\mathtt{c}(\underline{\mathtt{a}})) \hookrightarrow_{\mathcal{R},\mu} \mathtt{F}(\underline{\mathtt{c}(\mathtt{e}(\mathtt{a}))}) \hookrightarrow_{\mathcal{R},\mu} \underline{\mathtt{F}(\mathtt{d}(\mathtt{a},\mathtt{a}))} \hookrightarrow_{\mathcal{P},\mu} \mathtt{F}(\mathtt{c}(\underline{\mathtt{a}})) \hookrightarrow_{\mathcal{R},\mu} \cdots$$

Since $\mathtt{F}(\mathtt{c}(x))$ does not unify with any left-hand side of another pair, we can μ-narrow the pair in $\mathcal{P}$. We obtain $\mathcal{P}'$ consisting of the μ-narrowed pair

$$\mathtt{F}(\mathtt{d}(\mathtt{e}(x),\mathtt{e}(x))) \to \mathtt{F}(\mathtt{d}(x,x))$$

No infinite $(\mathcal{P}', \mathcal{R}, \mu, t)$-chain is possible now.

Note that $\mathcal{P}$ is μ-conservative, but it is not strongly μ-conservative (the variable x is both μ-replacing and non-μ-replacing in $\mathtt{F}(\mathtt{d}(x,x))$). ■

Of course, μ-narrowing can also be used in proofs of innermost termination of *CSR*. In the standard setting, when using narrowing for proving innermost termination we do not require that the right-hand side of the dependency pair to be narrowed is linear since the involved substitution σ is normalized. However, in the context-sensitive setting, if the pair to be μ-narrowed is *not strongly μ-conservative*, we can not ensure that the variables on the right-hand side are μ-normalized so *we also have to demand linearity.* When dealing with innermost narrowing in context-sensitive rewriting we can drop the linearity condition if the pair to be μ-narrowed is strongly conservative since all μ-replacing variables in the right-hand side of a pair are instantiated to μ-normal form and μ-reductions can not take place on them.

Theorem 74 (Innermost Narrowing processor) *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $u \to v \in \mathcal{P}$ be such that*

1. *$u \to v$ is strongly conservative,*
2. *for all $u' \to v' \in \mathcal{P}$ (with possibly renamed variables), v and u' do not unify or they unify by some mgu θ such that one of the terms $\theta(u)$ or $\theta(u')$ is not a μ-normal form.*

Let $\mathcal{Q} = (\mathcal{P} - \{u \to v\}) \cup \{u' \to v' \mid u' \to v' \text{ is a } \mu\text{-narrowing of } u \to v\}$. Then, the processor Proc_{narr} given by

$$\mathsf{Proc}_{Inarr}(\mathcal{P}, \mathcal{R}, \mu, i) = \begin{cases} \{(\mathcal{Q}, \mathcal{R}, \mu, i)\} & \text{if (1), and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu, i)\} & \text{otherwise} \end{cases}$$

is sound and complete.

Proof.

We have to prove that there is an infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu, i)$-chain iff there is an infinite minimal innermost $(\mathcal{Q}, \mathcal{R}, \mu, i)$-chain. We prove that for every minimal innermost $(\mathcal{P}, \mathcal{R}, \mu, i)$-chain "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$", there is an innermost μ-narrowing v' of v with the mgu θ such that "$\ldots, u_1 \to v_1, \theta(u) \to v', u_2 \to v_2, \ldots$" is also a minimal innermost $(\mathcal{Q}, \mathcal{R}, \mu, i)$-chain.

If "$\ldots, u_1 \to v_1, u \to v, u_2 \to v_2, \ldots$" is a minimal innermost $(\mathcal{P}, \mathcal{R}, \mu, i)$-chain, then there is an substitution σ such that for all pairs $s \to t$ in the chain,

1. if $s \to t \in \mathcal{P}_{\mathcal{G}}$, then $\sigma(t)$ is μ-terminating and it μ-reduces innermost to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain

2. if $s \to t = s \to x \in \mathcal{P}_{\mathcal{X}}$ then, $\sigma(x)$ has a μ-replacing subterm s_0, $\sigma(x) \trianglerighteq_{\mu} s_0$ such that $s_0^{\sharp}$ is μ-terminating and it μ-reduces to the instantiated left-hand side $\sigma(s')$ of the next pair $s' \to t'$ in the chain; furthermore, there is $\bar{s}_0 \in \mathcal{NHT}(\mathcal{R}, \mu)$ such that $s_0 = \theta_0(\bar{s}_0)$ for some substitution θ_0.

3. all instantiated left-hand sides are μ-normal forms w.r.t. $(\mathcal{R}, \mu)$

Assume that σ is a substitution satisfying the above requirements and such that the length of the sequence $\sigma(v) \hookrightarrow^*_{\mathcal{R},\mu} \sigma(u_2)$ is *minimal*.

Note that $\sigma(v) \neq \sigma(u_2)$. Otherwise σ would unify v and u_2, where both, u and v_2 are μ-normal forms, hence, there is a term q such that $\sigma(v) \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} q \hookrightarrow^*_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_2)$.

The reduction $\sigma(v) \hookrightarrow_{\mathcal{R},\mu,\mathsf{i}} q$ cannot take place within a binding of σ because $u \to v$ is strongly conservative. Hence, $\sigma(u)$ would not be a μ-normal form which violates the last condition for σ. In the innermost case, we do not have to demand linearity since μ-replacing variables in v come from being replacing in u (strongly conservative) and they are instantiated to μ-normal forms and no one can be reduced in v. The remainder of the proof is completely analogous to the noninnermost case.

■

Example 75 Consider again the TRS $(\mathcal{R},\mu)$ in Example 69. The only CS-DP in the existing cycle can be removed since it does not unify with other (renamed) CS-DP and has no μ-narrowing. Since the CS-DP is strongly conservative we do not require that the right-hand side is linear. So the system is easily proved *innermost μ-terminating* thanks to the innermost μ-narrowing transformation.

■

14

Simplifying monotonicity requirements for innermost μ-termination

In the innermost setting, matching substitutions are always normalized. For this reason, in an innermost sequence $t_1 \xrightarrow{p_1}_i t_2 \xrightarrow{p_2}_i \cdots \xrightarrow{p_n}_i t_{n+1}$ starting at root position (i.e., $p_1 = \Lambda$), every redex $t_j|_{p_j}$ for $j > 1$ comes from a defined symbol introduced after applying a rule $l_k \to r_k$ in a previous step $k < j$. Hence the set of arguments which are reduced can be handled by looking for defined symbols in right-hand sides of the involved rules $l \to r$.

In [AG00], Arts and Giesl already noticed that in the treatment of innermost chains, monotonicity requirements for the reduction pairs can be weaker. In [Fer05] Fernández defines the notion of *usable arguments* for a function symbol when proving innermost termination. The idea is that, in innermost sequences, some arguments are not relevant for proving termination.

Example 76 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{s}(\mathtt{0}), \mathtt{s}(\mathtt{0})) & \to & \mathtt{f}(x, \mathtt{g}(x)) \\ \mathtt{g}(\mathtt{s}(x)) & \to & \mathtt{g}(x) \end{array}$$

Any innermost sequence starting at root position does not take into account the first argument of `f` nor the argument of `g`. The reason is that since an innermost redex is an argument normalized redex, that means that all variables (e.g. `x`) of the applied rule are normalized and cannot be reduced. Only the second argument `g(x)` of `f` in the right-hand side of the first rule could be innermost reduced after applying it.

■

Definition 77 (Usable arguments [Fer05], Definition 3) *Let $\mathcal{R} = (\mathcal{F}, R) = (\mathcal{C} \uplus \mathcal{D}, R)$ be a TRS and $\mathcal{P}$ a set of pairs of terms s.t. for all $u \to v \in \mathcal{P}$, u is argument normalized with respect to $\mathcal{R}$. The set of usable arguments for a function symbol $f \in \mathcal{F}$ with respect to $\mathcal{R}$ and $\mathcal{P}$ is defined as* $\mathbf{UA}(f, \mathcal{R}, \mathcal{P}) = \{1 \leq k \leq ar(f) \mid \exists u \to$

$v \in \mathcal{P} \cup \mathbf{U}(\mathcal{R},\mathcal{P})$, $\exists p, p' \in \mathcal{P}os(v)$ *s.t.* $root(v|_{p'}) = f$, $root(v|_p) \in \mathcal{D}$, $p'.k \leqslant p$, $u \not\trianglerighteq v|_p\}$.

Considering those usable arguments could be helpful in proofs of innermost termination since they impose weaker monotonicity requirements. For instance, when using polynomial orderings, we can use even negative or rational coefficients for interpreting the symbols that do not need to be monotonic.

As Fernández noticed, the set of usable arguments can be seen as a replacement map which specifies the arguments to be reduced. In her approach, proving the μ-termination of a TRS $\mathcal{R}$ implies the innermost termination of $\mathcal{R}$ if $\mu(f)$=$\mathbf{UA}(f,\mathcal{R},R)$ for all $f \in \mathcal{F}$ where R only contains rules such that all left-hand sides are argument normalized.

Corollary 78 ([Fer05], Corollary 11) *Let $\mathcal{R}$ be a TRS and $\mu(f) = \mathbf{UA}(f,\mathcal{R},R')$ for every $f \in \mathcal{F}$ where $R' \subseteq R$ contains all rules $l \to r \in R$ such that l is argument normalized. If $\mathcal{R}$ is μ-terminating, then $\mathcal{R}$ is innermost terminating.*

This observation is very useful since now, all techniques for proving termination of *CSR* can be used for proving innermost termination. Several methods and techniques for proving termination of *CSR* have been developed so far [GM04, AGL06, Luc06, AGL07].

14.1 Usable arguments for *CSR*

Following Fernández's ideas, in the innermost context-sensitive setting (for a given replacement map μ) we could relax monotonicity requirements by taking into account that reductions only take place on μ-replacing positions of the right-hand sides of the rules which are rooted by a defined symbol. We adapt Fernández's ideas to *CSR*. In sharp contrast to the unrestricted case, we need to take into account that in innermost *CSR* a redex does not need to be argument normalized. Only argument μ-normalization can be assumed. Thus, non-μ-replacing subterms may contain redexes that can be reduced later on if they come to a replacing position.

Proposition 79 *A CS-TRS $(\mathcal{R},\mu)$ is innermost μ-terminating iff $\mathcal{R}'$ is innermost μ-terminating, where $\mathcal{R}' \subseteq \mathcal{R}$ contains all rules $l \to r \in \mathcal{R}$ such that l is argument μ-normalized.*

Proof. Trivial since the only rules that can be applied in innermost μ-reductions are those whose the left-hand sides are argument μ-normalized as we have shown in the definition 24 of ICS-DPs. ∎

In the following, we assume that all CS-TRS $(\mathcal{R}, \mu)$ are argument μ-normalized, i.e., for all rule $l \to r$ in $\mathcal{R}$, l is argument μ-normalized. Proposition 79 ensures that this entails no lack of generality regarding our research on innermost termination of *CSR*.

The straightforward adaptation of Fernandez's criterion to *CSR* yields the following definition: the usable CS-arguments for a function symbol $f \in \mathcal{F}$ are those arguments with a μ-replacing subterm rooted by a defined symbol in some right-hand side of a pair or usable rule.

Definition 80 (Basic usable CS-arguments) *Let $(\mathcal{R}, \mu) = ((\mathcal{C} \uplus \mathcal{D}, R), \mu)$ be a CS-TRS and $\mathcal{P}$ be a set of pairs of terms s.t. for all $u \to v \in \mathcal{P}$, u is argument μ-normalized. The* basic usable CS-arguments *for a function symbol $f \in \mathcal{F}$ are defined as $\mathbf{UA}_\mu(f, \mathcal{R}, \mathcal{P}) = \{i \in \mu(f) \mid \exists u \to v \in \mathcal{P} \cup \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \exists p, p' \in \mathcal{P}os^\mu(v)$ s.t. $root(v|_{p'}) = f$, $root(v|_p) \in \mathcal{D}$, $p'.i \leqslant p$, $u \not\trianglerighteq_\mu v|_p\}$.*

Note that the replacement map given by $\mu'(f) = \mathbf{UA}_\mu(f, \mathcal{R}, \mathcal{P})$ for all $f \in \mathcal{F}$ is more restrictive than μ: $\mu'(f) \subseteq \mu(f)$ for all $f \in \mathcal{F}$.

The following proposition is the context-sensitive version of [Fer05, Lemma 5].

Proposition 81 *Let $(\mathcal{R}, \mu)$ be a CS-TRS and $\mathcal{P}$ be a set of pairs of terms s.t. for all $u \to v \in \mathcal{P}$, u is argument μ-normalized and $\mathcal{P} \cup \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ is μ-conservative. Let $l \to r \in \mathcal{P} \cup \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ be such that $\sigma(r) \xhookrightarrow{>\Lambda}{}^{*}_{\mathsf{i}\ \mathbf{U}_0(\mathcal{R},\mu,\mathcal{P})} t$ for some term t and substitution σ s.t. $\sigma(l)$ is argument μ-normalized . If $t|_p$ is an innermost μ-redex, then for all $p'.k \leqslant p$, we have that $k \in \mathbf{UA}_\mu(root(t|_{p'}), \mathcal{R}, \mathcal{P})$.*

Proof. By induction on the length n of the rewriting sequence. If $n = 0$, then $\sigma(r) = t$. Then, since $\sigma(l)$ is argument μ-normalized, it follows that for all $x \in \mathcal{V}ar^\mu(l), \sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$. Since the rule $l \to r$ is conservative (that is $\mathcal{V}ar^\mu(r) \subseteq \mathcal{V}ar^\mu(l)$), we have that for all $x \in \mathcal{V}ar^\mu(r), \sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$. It follows that p is a nonvariable (μ-replacing) position of r, i.e. $p \in \mathcal{P}os^\mu_\mathcal{F}(r)$. Thus, root$(r|_p) \in \mathcal{D}$ and the result follows by Definition 80.

If $n > 0$, then there is a term s such that $\sigma(r) \xhookrightarrow{>\Lambda}{}^{*}_{\mathsf{i}} s$ and $s \xhookrightarrow{>\Lambda}_{\mathsf{i}} t$ at some μ-replacing position q. By the induction hypothesis, every μ-replacing position of the term t above, which equal or disjoint to q satisfies the result and we only have to prove it for innermost redexes $t|_p$ s.t. $q < p$, it is say, we have to prove that $k \in \mathbf{UA}_\mu(root(t|_{p'}), \mathcal{R}, \mathcal{P})$, for all $q < p'.k \leqslant p$. If $s \xhookrightarrow{>\Lambda}_{\mathsf{i}} t$, then $s|_q = \sigma'(l')$ and $t|_q = \sigma'(r')$, for some rule $l' \to r' \in \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ and substitution σ' s.t. $\sigma'(l')$ is argument μ-normalized. This implies that every innermost redex of $t|_q$ occurs at a position $p'' \in \mathcal{P}os^\mu(r')$ s.t. $root(r'|_{p''}) \in \mathcal{D}$ (since the rule $l' \to r'$ is conservative we have that for all $x \in \mathcal{V}ar^\mu(r'), \sigma(x) \in \mathsf{NF}_\mu(\mathcal{R})$) and $l' \not\trianglerighteq_\mu r'|_{p''}$(otherwise, $\sigma'(l')$ would not be an innermost redex of s. By definition, when $p'' > \Lambda$, $p'.k \leqslant p''$, $k \in \mathbf{UA}_\mu(root(t|_{q.p'}), \mathcal{R}, \mathcal{P})$ which is equivalent to what we needed to prove ($k \in \mathbf{UA}_\mu(root(t|_{p'}), \mathcal{R}, \mathcal{P})$, for all $q < p'.k \leqslant p$). ■

Corollary 78 suggests that innermost μ-termination could be proved by using a μ'-reduction ordering for μ' given by $\mu'(f) = \mathbf{UA}_\mu(f, \mathcal{R}, \mathcal{P})$ for all $f \in \mathcal{F}$. This is true for μ'-conservative CS-TRSs, as the following theorem shows.

Theorem 82 *A μ-conservative CS-TRS $(\mathcal{R}, \mu)$ is innermost μ-terminating if there is a μ'-reduction ordering $\succ$ s.t. $\mathcal{R} \subseteq \succ$, where for all symbol $f \in \mathcal{F}$, $\mu'(f) = \mathbf{UA}_\mu(f, \mathcal{R}, R)$.*

Proof. By contradiction. Assume that $\mathcal{R}$ is not innermost μ-terminating. By the argument of size minimality, there is a infinite innermost μ-rewrite sequence with the first step at position Λ: $s_1 \hookrightarrow_{\mathsf{i}} s_2 \hookrightarrow_{\mathsf{i}} s_3 \hookrightarrow_{\mathsf{i}} \cdots$ (without loss of generality). By Proposition 81 (where we let $\mathcal{P} = \mathcal{R}$), every step $s_j \stackrel{>\Lambda}{\hookrightarrow}_{\mathsf{i}} s_{j+1}$ at position p satisfies that $p'.k \leqslant p$, $k \in \mathbf{UA}_\mu(root(s_j|_{p'}), \mathcal{R}, \mathcal{P})$. Since $\mathcal{R} \subseteq \succ$ and $\succ$ is stable and μ'-monotonic, $s_j \succ s_{j+1}$ holds. Therefore, there is an infinite $\succ$-decreasing sequence of terms $s_1 \succ s_2 \succ \cdots \succ s_n \succ \cdots$ which contradicts the well-foundedness of $\succ$.

■

Since μ-reduction orderings characterize termination of *CSR* we have the following corollary.

Corollary 83 *Let $\mathcal{R}$ be a μ-conservative TRS for $\mu \in M_{\mathcal{R}}$. Let μ' be given by $\mu'(f) = \mathbf{UA}_\mu(f, \mathcal{R}, R)$ for every $f \in \mathcal{F}$. If $\mathcal{R}$ is innermost μ'-terminating, then $\mathcal{R}$ is innermost μ-terminating.*

Example 84 Consider the TRS $\mathcal{R}$:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{a}, \mathtt{b}, x) & \rightarrow & \mathtt{f}(x, x, x) \\ \mathtt{c} & \rightarrow & \mathtt{a} \\ \mathtt{c} & \rightarrow & \mathtt{b} \end{array}$$

together with $\mu(\mathtt{f}) = \{1, 3\}$. According to [AGL06, AGL07, AGL08] there is only one CS-DP:

$$\mathtt{F}(\mathtt{a}, \mathtt{b}, x) \rightarrow \mathtt{F}(x, x, x)$$

In the CS-DG (see [AGL06, AGL07, AGL08]), the context-sensitive dependency pair forms a cycle. However, by using $\mu'(f) = \mathbf{UA}_\mu(f, \mathcal{R}, \mathcal{P})$ for every $f \in \mathcal{F}$ we obtain $\mu'(\mathtt{f}) = \varnothing$. Now, the pair does not form a cycle thus easily concluding the μ'-termination of $\mathcal{R}$ and, by Corollary 83, the innermost μ-termination of $\mathcal{R}$. ■

This fact is important since now, all techniques for proving termination of *CSR* can be used to prove termination of innermost *CSR* for μ-conservative systems. The following example shows that μ-conservativeness cannot be dropped in Theorem 82 and Corollary 83.

Example 85 Consider again the TRS $\mathcal{R}$ in Example 84 but now together with $\mu(\mathtt{f}) = \{1,2\}$. If we try to apply Corollary 83 to prove innermost μ-termination of $\mathcal{R}$, we obtain $\mu'(\mathtt{f}) = \varnothing$ and (as discussed in Example 84) the CS-dependency graph has no cycle thus concluding the innermost μ-termination of $\mathcal{R}$. However,$\mathcal{R}$ is *not* innermost μ-terminating:

$$\underline{\mathtt{f(a,b,c)}} \hookrightarrow_{\mathsf{i}} \mathtt{f(\underline{c},c,c)} \hookrightarrow_{\mathsf{i}} \mathtt{f(a,\underline{c},c)} \hookrightarrow_{\mathsf{i}} \underline{\mathtt{f(a,b,c)}} \hookrightarrow_{\mathsf{i}} \cdots$$

Note that the first rule of $\mathcal{R}$ is not μ-conservative now. ■

14.2 Relaxing monotonicity with CS-DPs

Fernández's criterion was also adapted to deal with proofs of termination of rewriting using dependency pairs, what allows us using reduction pairs instead of reduction orderings in proofs of termination.

In previous chapters, we have shown how to prove innermost termination of *CSR* by using ICS-DPs. Now, we can adapt the use of CS-usable arguments to be applied in proofs of innermost μ-termination with ICS-DPs. We do that by providing a new processor for dealing with innermost μ-termination problems

Theorem 86 *Let $\mathcal{R} = (\mathcal{F}, R)$ and $\mathcal{P} = (\mathcal{G}, P)$ be TRSs and $\mu \in M_{\mathcal{F}\cup\mathcal{G}}$. Let $\mu_{\mathcal{P}}(f) = \mathbf{UA}_\mu(f, \mathcal{R}, \mathcal{P})$ for all $f \in \mathcal{F} \cup \mathcal{G}$ and $(\gtrsim, \sqsupset)$ be a $\mu_{\mathcal{P}}$-reduction pair such that*

1. *$\mathcal{P}$ and $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$ are μ-conservative,*
2. *$\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}) \subseteq \gtrsim$ and $\mathcal{P} \subseteq \gtrsim \cup \sqsupset$,*

Let $\mathcal{P}_\sqsupset = \{u \to v \in \mathcal{P} \mid u \sqsupset v\}$. Then, the processor Proc_{Fer} given by

$$\mathsf{Proc}_{Fer}(\mathcal{P}, \mathcal{R}, \mu, i) = \begin{cases} \{(\mathcal{P} - \mathcal{P}_\sqsupset, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}}, i)\} & \textit{if (1) and (2) hold} \\ \{(\mathcal{P}, \mathcal{R}, \mu, i)\} & \textit{otherwise} \end{cases}$$

is sound.

Proof. We have to prove that every infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain introduces an infinite minimal innermost $(\mathcal{P}-\mathcal{P}_\sqsupset, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}})$-chain. We proceed by contradiction. Assume that there is an infinite minimal innermost $(\mathcal{P}, \mathcal{R}, \mu)$-chain A, but that there is no infinite minimal innermost $(\mathcal{P} - \mathcal{P}_\sqsupset, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}})$-chain. Due to the finiteness of $\mathcal{P}$, we can assume that there is $\mathcal{Q} \subseteq \mathcal{P}$ such that A has a tail B

$$\sigma(u_1) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^\sharp_\mu t_1 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_2) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^\sharp_\mu t_2 \overset{!}{\hookrightarrow}_{\mathcal{R},\mu,\mathsf{i}} \sigma(u_3) \hookrightarrow_{\mathcal{Q},\mu} \circ \trianglerighteq^\sharp_\mu \cdots$$

for some substitution σ, where all pairs in $\mathcal{Q}$ are infinitely often used, and, for all $i \geq 1$, since all $u_i \to v_i \in \mathcal{P}$ are conservative $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$, then $t_i = \sigma(v_i)$ such that for all $i > 0$, $\sigma(u_i)$ is argument μ-normalized and $\sigma(v_i)$ is innermost $(\mathcal{R}, \mu)$-terminating. By Proposition 63 and 81, every innermost step in the sequence $t_i \overset{>\Lambda}{\hookrightarrow}^*_{\mathsf{i}} \sigma(u_{i+1})$ is performed at a $\mu_{\mathcal{P}}$-replacing position by means of a conservative rule in $\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})$:

$$\sigma(u_1) \hookrightarrow_{\mathcal{Q},\mu_{\mathcal{P}}} \circ \trianglerighteq^{\sharp}_{\mu_{\mathcal{P}}} t_1 \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu_{\mathcal{P}},\mathsf{i}} \sigma(u_2) \hookrightarrow_{\mathcal{Q},\mu_{\mathcal{P}}} \circ \trianglerighteq^{\sharp}_{\mu_{\mathcal{P}}} t_2 \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu_{\mathcal{P}},\mathsf{i}} \sigma(u_3) \hookrightarrow \cdots$$

Since $u_i \; (\gtrsim \cup \sqsupset) \; v_i$ for all $u_i \to v_i \in \mathcal{Q} \subseteq \mathcal{P}$, by stability of $\gtrsim$ and $\sqsupset$, we have $\sigma(u_i) \; (\gtrsim \cup \sqsupset) \; \sigma(v_i)$ for all $i \geq 1$.

No pair $u \to v \in \mathcal{Q}$ satisfies that $u \sqsupset v$. Otherwise, we get a contradiction by considering that since all pairs $\in \mathcal{P}$ are conservative $u_i \to v_i \in \mathcal{Q}_{\mathcal{G}}$, then $t_i = \sigma(v_i) \overset{!}{\hookrightarrow}_{\mathbf{U}_0(\mathcal{R},\mu,\mathcal{P}),\mu_{\mathcal{P}},\mathsf{i}} \sigma(u_{i+1})$ and $t_i \gtrsim \sigma(u_{i+1})$. Since we have $\sigma(u_i) \; (\gtrsim \cup \sqsupset \;) \; \sigma(v_i) = \sigma(v_i) = t_i$, by using transitivity of $\gtrsim$ and compatibility between $\gtrsim$ and $\sqsupset$, we conclude that $\sigma(u_i) \; (\gtrsim \cup \sqsupset) \; \sigma(u_{i+1})$. Since $u \to v$ occurs infinitely often in B, there is an infinite set $\mathcal{I} \subseteq \mathbb{N}$ such that $\sigma(u_i) \;\; \sqsupset \;\; \sigma(u_{i+1})$ for all $i \in \mathcal{I}$. And we have $\sigma(u_i) \; (\gtrsim \cup \sqsupset) \; \sigma(u_{i+1})$ for all other $u_i \to v_i \in \mathcal{Q}$. Thus, by using the compatibility conditions of the μ-reduction pair, we obtain an infinite decreasing $\sqsupset$-sequence which contradicts well-foundedness of $\sqsupset$. Therefore, $\mathcal{Q} \subseteq (\mathcal{P} - \mathcal{P}_{\sqsupset})$. Since $\mu_{\mathcal{P}} \subseteq \mu$ and $\mathsf{NF}_{\mu_{\mathcal{P}}}(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P})) \supseteq \mathsf{NF}_{\mu}(\mathcal{R})$, we have that $\sigma(u_i) \in \mathsf{NF}_{\mu_{\mathcal{P}}}(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}))$. By Proposition 63, innermost $(\mathcal{R}, \mu)$-termination of $\sigma(v_i)$ implies innermost $(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu)$- termination of $\sigma(v_i)$ for all $i \geq 1$ and by Proposition 81, innermost $(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu)$-termination of $\sigma(v_i)$ implies innermost $(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}})$-termination, so we get that innermost $(\mathcal{R}, \mu)$-termination of $\sigma(v_i)$ implies innermost $(\mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}})$-termination. Hence, B is an infinite minimal innermost $(\mathcal{P} - \mathcal{P}_{\sqsupset}, \mathbf{U}_0(\mathcal{R}, \mu, \mathcal{P}), \mu_{\mathcal{P}})$-chain, thus leading to a contradiction.

■

Corollary 83 can be generalized to (certain) non-μ-conservative CS-TRSs thanks to Theorem 86 and the results for proving innermost termination of *CSR* in [AL07a]. Now, for a given CS-TRS $(\mathcal{R}, \mu)$ that satisfies the conditions of Theorem 86, we can prove its innermost μ-termination by relaxing μ-monotonicity requirements for each cycle.

Example 87 Consider the following TRS $\mathcal{R}$:

$$\begin{array}{rcl}
\mathtt{f}(\mathtt{s}(x)) & \to & \mathtt{f}(x) \\
\mathtt{g}(\mathtt{cons}(\mathtt{0}, y)) & \to & \mathtt{g}(y) \\
\mathtt{f}(\mathtt{cons}(\mathtt{s}(x), y)) & \to & \mathtt{s}(y) \\
\mathtt{h}(\mathtt{cons}(x, y)) & \to & \mathtt{h}(\mathtt{g}(\mathtt{cons}(x, y)))
\end{array}$$

together with $\mu(\mathtt{f}) = \mu(\mathtt{s}) = \mu(\mathtt{cons}) = \mu(\mathtt{h}) = \{1\}$ and $\mu(\mathtt{g}) = \varnothing$. Note that $\mathcal{R}$ is not μ-conservative due to the third rule. The set of CS-DPs consists of the following ones:

$$\begin{array}{rcl} \mathtt{F}(\mathtt{s}(x)) & \to & \mathtt{F}(x) \\ \mathtt{G}(\mathtt{cons}(\mathtt{0}, y)) & \to & \mathtt{G}(y) \\ \mathtt{F}(\mathtt{cons}(\mathtt{s}(x), y)) & \to & y \\ \mathtt{H}(\mathtt{cons}(x, y)) & \to & \mathtt{H}(\mathtt{g}(\mathtt{cons}(x, y))) \\ \mathtt{H}(\mathtt{cons}(x, y)) & \to & \mathtt{G}(\mathtt{cons}(x, y))) \end{array}$$

The CSDP $\mathtt{H}(\mathtt{cons}(x, y)) \to \mathtt{H}(\mathtt{g}(\mathtt{cons}(x, y)))$ could be μ-narrowed and we obtain a new pair $\mathtt{H}(\mathtt{cons}(\mathtt{0}, y)) \to \mathtt{H}(\mathtt{g}(y))$ There are three cycles in the ICS-DG:

$$\begin{array}{rcl} \{\mathtt{F}(\mathtt{s}(x)) & \to & \mathtt{F}(x)\}, \\ \{\mathtt{G}(\mathtt{cons}(\mathtt{0}, y)) & \to & \mathtt{G}(y)\}, \\ \{\mathtt{H}(\mathtt{cons}(\mathtt{0}, y)) & \to & \mathtt{H}(\mathtt{g}(y))\} \end{array}$$

Although the system is not μ-conservative, all the obtained cycles (and the corresponding sets of basic CS-usable rules) are. The first cycle can be solved by applying subterm criterion (see [AGL08]). The second one can be oriented using a polynomial interpretation where $[\mathtt{G}](\mathrm{x}) = \mathrm{x}$, $[\mathtt{cons}](\mathrm{x,y}) = \mathrm{y} + 1$ and $[0] = 0$ (note that the set of basic CS-usable rules is empty for this cycle). For the last cycle, we have to take into account the (μ-conservative) basic CS-usable rule `g(cons(0,y))` $\to$ `g(y)`. However it can be solved by using the following interpretation:

$$[\mathtt{H}](x) = x \quad [\mathtt{cons}](x, y) = x \quad [\mathtt{g}](x) = 0 \quad [\mathtt{h}](x) = 0 \quad [\mathtt{0}] = 1$$

■

15

Experiments

We have implemented the techniques described in the previous chapters as part of the tool MU-TERM [Luc04]. In order to evaluate the techniques which are reported in this paper we have made some benchmarks. We have considered the examples in the Termination Problem Data Base (TPDB, version 3.2) available through the URL:

`http://www.lri.fr/~marche/tpdb/`

15.1 Proving termination of innermost *CSR*: Direct techniques vs. transformations

Although there is no special TPDB category for innermost termination of *CSR* (yet) we have used the TRS/CSR directory in order to test our techniques for proving termination of innermost *CSR*. The TPDB v3.2 contains 90 examples of CS-TRSs. In order to evaluate our direct techniques in comparison with the transformational approach of [GM02b, GM04, Luc01a], where termination of innermost *CSR* for a CS-TRS $(\mathcal{R}, \mu)$ is proved by proving innermost termination of a transformed TRS $\mathcal{R}^{\mu}_{\Theta}$, where Θ specifies a particular transformation (see [GM02a, GM02b] for a survey on this topic), we have transformed the set of examples by using the transformations that are correct for proving innermost termination of *CSR*: Giesl and Middeldorp's correct transformations for proving termination of innermost *CSR*, see [GM02b], although we use the 'authors-based' notation introduced in [Luc06]: GM and C for transformations 1 and 2 for proving termination of *CSR* introduced in [GM04], and iGM for the specific transformation for proving termination of innermost *CSR* introduced in [GM02b]. Then we have proved innermost termination of the set of examples with AProVE [GST06], which is able to prove innermost termination of standard rewriting. The results are summarized in Table 15.1 and 15.2. Further details can be found here:

`http://www.dsic.upv.es/~balarcon/MasterThesis/benchmarks/InnermostCSR.html`

These are the first known benchmarks comparing not only transformational techniques vs. direct (DP-based) techniques, but also the existing correct transformations for proving innermost termination of *CSR* among them. They show that, quite

	ICS-DPs	Transformations
YES score	70	44
YES average time	3.2 sec.	5 sec.

Table 15.1: Comparative in proofs of termination of innermost *CSR*

	C	GM	iGM
YES score	24	41	30

Table 15.2: Comparing transformations for proving termination of innermost *CSR*

surprisingly, the iGM transformation (which is in principle the more suitable one for proving innermost termination of *CSR*) obtains worse results than GM (in the average).

From the results in Table 15.1, it is clear that using transformations for proving termination of innermost *CSR* makes few sense after introducing the ICS-DP framework.

15.2 Proving innermost termination of rewriting as termination of *CSR*

We have implemented the use of Corollary 78 for proving innermost termination of rewriting as termination of *CSR* (this was one of the main results in Fernández's paper). The relevance of this result in practice had not been tested yet as no implementation of Fernández's results was available (to our knowledge). In order to evaluate it, we have considered the examples used in the *innermost category* of the 2006 termination competition[1], which are part of the TPDB. There are 69 examples, 66 of them are known to be innermost terminating. With Fernández's criterion (Corollary 78) MU-TERM succeeds in 32 examples (success rate of 48.5%). This is acceptable if we think that (except for AProVE, which succeeds in the 100% of the examples) the success rate for all other participants in this category is around 20%. However, we have also implemented the use of (standard) dependency pairs for proving innermost termination (according to [AG00, Theorem 37]) together with the narrowing refinement (we call this tool MU-TERM iDPs) and we are able to prove 39 examples, including all examples solved with Fernández's criterion. Moreover, we have included Fernández's criterion as a technique to be applied when trying to solve a cycle in the innermost termination proof (see [Fer05], Theorem 9). There are two news approximations: when MU-TERM has to solve a cycle, the first version

[1] http://www.lri.fr/~marche/termination-competition/2006

	MU-TERM Fernández	MU-TERM iDPs [AG00]
YES score	32	39
YES average time	0.03 sec.	0.03 sec.

Table 15.3: Summary of benchmarks for innermost termination of rewriting

	MU-TERM Fer. cycle-based	MU-TERM Fer. cycle-based (only)
YES score	38	37
YES average time	0.04 sec.	0.79 sec.

Table 15.4: Summary of benchmarks for innermost termination of rewriting (based on cycles)

uses Fernández's criterion and if it fails then it tries to solve it in the usual way, that is, without any replacement map (tool MU-TERM Fernández cycle-based). The second one tries to *force* MU-TERM to solve the cycle with Fernández's criterion; No other option is allowed (tool MU-TERM Fernández cycle-based only). In both cases the results obtained are similar. With the previous implementation of MU-TERM (MU-TERM iDPs) we solve 39 examples and with these two configurations we obtain better results than with Corollary 78; however, they do not improve the performance of MU-TERM iDPs. The results are summarized in Tables 15.3 and 15.4.

Therefore, it seems that using Corollary 78 to prove innermost termination of rewriting is not as good idea (at least with the considered set of examples) since we loose some examples due to a too restrictive new replacement map and the average time is the same. In the case of applying it to cycles, we obtain better results but no essential improvement since we also loose some examples.
Full details for the benchmarks summarized in Table 15.3 can be found here:

`http://www.dsic.upv.es/~balarcon/MasterThesis/Fer/benchmarks/FerInnermost.htm`

In the following URL:

`http://www.dsic.upv.es/~balarcon/MasterThesis/Fer/Innermost/benchmarks.html`

more information can be found regarding the benchmarks summarized in Table 15.4. All this shows that we do not obtain any real improvement over the basic technique of dependency pairs for proving innermost termination at least for the set of considered examples.

15.3 Proving innermost termination of *CSR*: relaxing monotonicity requirements

For our experiments about proving termination of innermost *CSR* by means of a new replacement map which imposes less monotonicity requirements we have used the set of examples mentioned in Section 15.1 . Since Corollary 83 only applies to conservative systems, we restrict the attention to the 27 conservative examples. We solve all of them with an average time of 0.025 seconds (MU-TERM Fernández). Further details can be found here:

`http://www.dsic.upv.es/~balarcon/MasterThesis/Fer/benchmarks/FerICSR.html`

On the other hand, we have also implemented the use of Theorem 86 to deal with nonconservative systems. We have compared the same configurations explained above: the first one (MU-TERM Fernández cycle-based) tries to solve each μ-conservative cycle (with associated μ-conservative usable rules) by using CS-usable arguments as the new replacement map. If it fails then uses the normal configuration of MU-TERM (MU-TERM ICS-DPs). The second one only applies CS-usable arguments on cycles when searching for a compatible μ-reduction pair (MU-TERM Fernández cycle-based only). All these versions of MU-TERM succeed over the same 70 examples, the same number of examples that we had already solved using the innermost version of the context-sensitive dependency pairs. The time average rates has no exhibit substantial differences. Further details can be found here:

`http://www.dsic.upv.es/~balarcon/MasterThesis/Fer/iCSR/benchmarks.html`

15.4 Transforming CS-dependency pairs

We have also implemented both μ-narrowing and innermost μ-narrowing in MU-TERM. Due to the possibility of performing an unbounded number of narrowing steps, the μ-narrowing transformation could be infinite (this also happens in the standard approach). In order to implement the transformation, we have chosen to use *one-step μ-narrowing* only if the (innermost) context-sensitive dependency graph obtained has less cycles and arcs than the original one. One of the best advantages of using μ-narrowing lies in the possibility of dismissing some CS-DPs if the right-hand sides do not unify with any left-hand side of another (possible renamed) CS-DP and they have no μ-narrowings. In fact, MU-TERM 4.4 which run in the 2007 termination competition already implements these features. MU-TERM was the winner of the context-sensitive category after solving 68 examples out from 90. Two of the tested examples require the μ-narrowing transformation to obtain a proof (as also happens in Example 68).

16
Conclusions

The results of this thesis are revised and extended versions of the results published in [AL07a, AL07b, AL08, AEFG+08].

Theoretical contributions. We have investigated the structure of infinite innermost context-sensitive rewrite sequences starting from minimal innermost non-μ-terminating terms (Theorem 23). This knowledge has been used to provide an appropriate definition of innermost context-sensitive dependency pair (Definition 24), and the related notion of innermost chain (Definition 27). We have proved that it can be used to characterize innermost μ-termination (Theorems 35 and 39). We have provided a suitable adaptation of Giesl et al.'s *dependency pair framework* to *CSR* by defining appropriate notions of *CS-termination problem* (Definition 43) and *CS-processor* (Definition 46). In this setting we have described a number of sound and (most of them) complete CS-processors which can be used in any practical implementation of the ICSDP-framework. In particular, we have introduced the notion of (estimated) *innermost context-sensitive (dependency) graph* (Definitions 49 and 54) and the associated CS-processor (Theorem 50). We have also shown how to automatically prove innermost μ-termination by means of the ICS-dependency graph (Theorem 50). We have formulated the notion of basic usable rules showing how to use them in proofs of innermost termination of *CSR* (Definition 60, Theorem 64) Narrowing context-sensitive dependency pairs has also been investigated. It can also be helpful to simplify or restructure the dependency graph and eventually simplify the proof of (innermost) termination (Theorems 71 and 74). We have also shown how to relax monotonicity requirements for proving innermost termination of context-sensitive rewriting. We have adapted Fernández's approach [Fer05] to be used for proving innermost termination of context-sensitive rewriting (Theorems 82 and 86).

Applications and practical impact. We have implemented these ideas as part of the termination tool MU-TERM [AGIL07, Luc04]. The implementation and practical use of the developed techniques yield a novel and powerful framework which improves the current state-of-the-art of methods for proving termination of *CSR*.

Actually, ICS-DPs were an essential ingredient for MU-TERM in winning the context-sensitive subcategory of the 2007 competition of termination tools.

Up to our contributions, no direct method has been proposed to prove termination of innermost *CSR*. So this is the first proposal of a direct method for proving termination of innermost *CSR*. We have extended Arts and Giesl's approach to prove innermost termination of TRSs to *CSR* (thus also extending [AGL06, AGL07]). Our benchmarks show that the use of ICS-DPs dramatically improves the performance of existing (transformational) methods for proving termination of innermost *CSR*.

Future work. As remarked in the introduction, we aim at applying all previous developments to deal with termination of Maude programs. Since its computational mechanism can be thought of as kind of "context-sensitive call by value", we believe that our research is a essential contribution to the development of tools for proving termination of Maude programs. However, a lot of further work is necessary before being able to achieve this main goal.

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