Mean ergodicity of weighted composition operators on spaces of holomorphic functions

María J. Beltrán-Meneu[a], M. Carmen Gómez-Collado[b], Enrique Jordá[b], David Jornet[b,*]

[a] *Departament de Didàctica de la Matemàtica, Universitat de València, Avda. Tarongers, 4, E-46022 Valencia, Spain*
[b] *Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Camino de Vera, s/n, E-46022 Valencia, Spain*

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Dedicated to our friend Prof. Manuel Maestre on the occasion of his 60th birthday

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A B S T R A C T

Let φ be a self-map of the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ and let ψ be a holomorphic function on $\mathbb{D}$. We investigate the mean ergodicity and power boundedness of the weighted composition operator $C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$ with symbol φ and multiplier ψ on the space $H(\mathbb{D})$. We obtain necessary and sufficient conditions on the symbol φ and on the multiplier ψ which characterize when the weighted composition operator is power bounded and (uniformly) mean ergodic. One necessary condition is that the symbol φ has a fixed point in $\mathbb{D}$. If φ is not a rational rotation, the sufficient conditions are related to the modulus of the multiplier on the fixed point of φ. Some of our results are valid in an open connected set U of the complex plane.

1. Introduction

Let U be a connected open subset (= *domain*) of $\mathbb{C}$. We denote by $H(U)$ the space of analytic functions on U, which is a Fréchet Montel space endowed with the compact open topology τ_{co}.

Let φ and ψ be analytic functions on U such that $\varphi(U) \subseteq U$. These maps define on the space $H(U)$ the so-called *weighted composition operator* $C_{\varphi,\psi}$ by $C_{\varphi,\psi}(f) = \psi(f \circ \varphi)$, $f \in H(U)$. The function φ is called symbol and ψ is called multiplier. It combines the classical composition operator $C_\varphi : f \mapsto f \circ \varphi$ with the pointwise multiplication operator $M_\psi : f \mapsto \psi \cdot f$.

Given $\varphi : U \to U$ a continuous self-map on U, we say that φ has *stable orbits* on U if for every compact subset $K \subseteq U$ there is a compact subset $L \subseteq U$ such that $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$.

* Corresponding author.
E-mail addresses: maria.jose.beltran@uv.es (M.J. Beltrán-Meneu), mcgomez@mat.upv.es (M.C. Gómez-Collado), ejorda@mat.upv.es (E. Jordá), djornet@mat.upv.es (D. Jornet).

In 2011, Bonet and Domański characterized those composition operators C_φ defined on $H(U)$ which are power bounded and proved that this condition is equivalent to the composition operator being (uniformly) mean ergodic or the map φ having stable orbits [9, Proposition 1].

In a Montel DF or LF space we have that power bounded operators are (uniformly) mean ergodic [1, Proposition 2.8], and this implies that the operator is topologizable. In this paper we prove that $C_{\varphi,\psi}$ is topologizable if and only if the symbol φ has stable orbits. Thus, for composition operators, being power bounded, (uniformly) mean ergodic and topologizable are equivalent. We show that this is no longer true for weighted composition operators. One necessary condition that we get for $C_{\varphi,\psi}$ being mean ergodic is $|\psi(z_0)| \leq 1$ for each $z_0 \in U$ which is a fixed point of φ. The endomorphisms in U with stable orbits which are not automorphisms have an attracting fixed point $z_0 \in U$. In this case the condition $|\psi(z_0)| \leq 1$ is also sufficient for $C_{\varphi,\psi}$ being power bounded. In case $U = \mathbb{D}$, if φ is an automorphism with stable orbits then it is an elliptic automorphism. We prove that for non-periodic rotation symbols associated to a Diophantine irrational number and ψ a multiplier which is nonzero in $\mathbb{D}$, (uniformly) mean ergodic is equivalent to power bounded and equivalent to the condition $|\psi(z_0)| \leq 1$, z_0 being the (nonattractive) fixed point of φ. For rotations associated to rational numbers we provide examples showing that this is not true.

2. Notation and preliminaries

Let X be a locally convex Hausdorff space and $T : X \to X$ a continuous and linear operator on X. The iterates of T are denoted by $T^n := T \circ \overset{n)}{\cdots} \circ T$, $n \in \mathbb{N}$. For $x \in X$ we denote by $\mathrm{Orb}(T, x) := \{T^n x,\ n \in \mathbb{N}_0\}$ the *orbit* of x under T. If the sequence $(T^n)_{n\in\mathbb{N}}$ is equicontinuous in the space $L(X)$ of all continuous and linear operators from X to X, T is called *power bounded*.

Given $T \in L(X)$, we denote by

$$T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^m, \quad n \in \mathbb{N},$$

the *Cesàro means* of T. The operator T is *mean ergodic* precisely when $(T_{[n]})_{n=1}^{\infty}$ converges in the strong operator topology, i.e., for all $x \in X$, the limit $\lim_{n\to\infty} T_{[n]}(x)$ exists. If $(T_{[n]})_{n=1}^{\infty}$ converges uniformly on the bounded subsets of X, then T is called *uniformly mean ergodic*.

An operator $T \in L(X)$ is called *topologizable* if for every continuous seminorm $p \in cs(X)$ there is a continuous seminorm $q \in cs(X)$ such that for every $n \in \mathbb{N}$ there is $C_n \geq 0$ such that

$$p(T^n(x)) \leq C_n q(x) \text{ for all } x \in X.$$

This class of operators was defined and studied by Zelazko [29] (see also Bonet [8]). If T is an operator defined on a barrelled space such that there exists a sequence $(c_n)_n$ of positive numbers such that $(c_n T^n)_n$ is equicontinuous (pointwise bounded), then T is topologizable, and the constants C_n in the definition of topologizable do not depend on p. This is an immediate consequence of the Banach–Steinhaus theorem. Being topologizable is a weaker condition than being power bounded. As

$$\frac{1}{n} T^n = T_{[n]} - \frac{n-1}{n} T_{[n-1]},$$

if T is mean ergodic then $(\frac{1}{n} T^n)_n$ converges to 0 in the strong operator topology and therefore T is topologizable. In a Montel DF or LF space we have that power bounded operators are always uniformly mean ergodic (see [1, Proposition 2.8]).

There is a huge literature about the dynamical behavior of various linear continuous operators on Banach, Fréchet and more general locally convex spaces; see the survey paper by Grosse-Erdmann [17] and the recent books by Bayart and Matheron [5] and by Grosse-Erdmann and Peris [18]. For more details of mean ergodic operators on locally convex spaces, see [1,2], and the references therein.

Composition operators on various spaces of analytic functions have been the object for intense study in recent years, specially the problem of relating operator-theoretic properties of C_φ to function theoretic properties of the symbol φ. See the books of Cowen and MacCluer [13] and Shapiro [27] for discussions of composition operators on classical spaces of holomorphic functions. Several authors have studied dynamical properties on these operators. Bonet and Domański studied the mean ergodicity of composition operators acting on the space $H(U)$ of holomorphic functions defined on a domain U in a Stein manifold [9]. They characterized those composition operators which are power bounded and proved that this condition is equivalent to the operator being mean ergodic or uniformly mean ergodic. In [28], Wolf studied when the composition operator is power bounded or uniformly mean ergodic on the weighted Bergman spaces of infinite order $H_v^\infty(\mathbb{D})$. In [6] the authors obtain a characterization of the mean ergodicity and uniformly mean ergodicity of C_φ on the disc algebra $A(\mathbb{D})$ and on the Hardy space $H^\infty(\mathbb{D})$ looking at the asymptotic behavior of the iterates of the symbol. It is easy to see that on both Banach spaces the composition operator is always power bounded. Bonet and Ricker studied when multiplication operators are power bounded or uniformly mean ergodic on weighted Banach spaces of holomorphic functions in the unit disc [10].

Weighted composition operators appear in a natural way on different spaces of analytic functions also. For example, it is well known that isometries on most of the spaces of analytic functions are described as weighted composition operators (see the monographs of Fleming and Jamison [14,15]). In this paper we study the power boundedness and mean ergodicity of these operators on the Fréchet space $H(U)$. Other dynamical properties such as (frequently) hypercyclicity and supercyclicity have been studied in this context; see [7] and the references therein. The power boundedness and the compactness of weighted composition operators have been studied in [11] and [26] also. Some research on the spectra can be found in [3,19–21].

Our notation for topology and functional analysis is standard, see for example [25]. In what follows, given $z \in \mathbb{C}$ and $r > 0$ we denote by $B(z,r)$ and $\overline{B(z,r)}$ the open and closed balls centered at z with radius r.

3. Weighted composition operators on $H(U)$

For φ an analytic self-map of U and ψ a multiplier we have

$$(C_{\varphi,\psi})^n f(z) = \psi(z)\cdots\psi(\varphi^{n-1}(z))f(\varphi^n(z)) = \left(\prod_{m=0}^{n-1}(\psi\circ\varphi^m)(z)\right) f(\varphi^n(z))$$

for $n \in \mathbb{N}$, where for $n = 0$ we set $\varphi^0 := \mathrm{id}_U$, the identity function on U. Denote $\psi_{[n]}(z) := \prod_{m=0}^{n-1}\psi(\varphi^m(z))$.

In the next result, τ_{co} denotes the compact open topology in $H(U)$, as mentioned in the introduction.

Proposition 3.1. *Let U be a domain of $\mathbb{C}$.*

(i) *If $C_{\varphi,\psi} : H(U) \to H(U)$ is power bounded then $C_{\varphi,\psi}$ is uniformly mean ergodic and $(\prod_{m=0}^{n}(\psi\circ\varphi^m))_n$ is bounded on $H(U)$.*

(ii) *If $C_{\varphi,\psi} : H(U) \to H(U)$ is mean ergodic then $\lim_n \frac{1}{n}\prod_{m=0}^{n-1}(\psi\circ\varphi^m) = 0$ on τ_{co} and $C_{\varphi,\psi}$ is topologizable.*

Proof. (i) If $C_{\varphi,\psi}$ is power bounded then the set $\{C^n_{\varphi,\psi}(f) : n \in \mathbb{N}\}$ is bounded on $H(U)$ for every $f \in H(U)$. In particular, for $f \equiv 1$ we have that $\prod_{m=0}^{n-1}(\psi\circ\varphi^m)$ is bounded on $H(U)$. Moreover, since $H(U)$ is a Fréchet Montel space, the uniformly mean ergodicity is deduced from [1, Proposition 2.8].
(ii) If $C_{\varphi,\psi}$ is mean ergodic then $\lim_{n\to\infty}\frac{1}{n}C^n_{\varphi,\psi} = 0$ pointwise. As a consequence, $C_{\varphi,\psi}$ is topologizable and $\lim_{n\to\infty}\frac{1}{n}C^n_{\varphi,\psi}(1) = 0$, which yields the assertion. □

Proposition 3.2. *Let U be a domain of $\mathbb{C}$. The following assertions are equivalent:*

(i) *$C_{\varphi,\psi}$ is topologizable,*
(ii) *φ has stable orbits.*

Proof. (i) $\Rightarrow$ (ii): Consider a compact set $K \subseteq U$ such that K is equal to its U-holomorphic convex hull, i.e. $K = \hat{K} := \{z \in U\,;\, |f(z)| \leq \sup_{\zeta\in K} |f(\zeta)|,\ \forall\, f \in H(U)\}$ (see [24, Chapter VI]). By [24, Proposition 1.6 (v)], if $u \in U \setminus K$ then for any $M > 0$, $\epsilon > 0$ and $z \in U \setminus K$, there is an $f \in H(U)$ such that $\sup_{z\in K} |f(z)| < \epsilon$ and $|f(u)| > M$.

If the operator $C_{\varphi,\psi}$ is topologizable, considering $(K_n)_n$ a fundamental sequence of compact sets of U, with each K_n being equal to its U-holomorphic convex hull (this can be done since U is an open and connected subset of $\mathbb{C}$), we get that for every $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exists $D_n > 0$ such that

$$\sup_{z\in K_i} |C^n_{\varphi,\psi}(f)(z)| \leq D_n \sup_{z\in K_j} |f(z)| \text{ for all } f \in H(U). \tag{3.1}$$

Suppose that there exists n such that $\varphi^n(K_i) \nsubseteq K_j$. Since the zero set of an holomorphic function is discrete, we can get $z_0 \in U$ such that $z_0 \in K_i$ and $\varphi^n(z_0) \notin K_j$ with $\psi_{[n]}(z_0) \neq 0$. If we get a function $f \in H(U)$ such that $|f(z)| \leq 1$ for all $z \in K_j$ and $|f(\varphi^n(z_0))| > D_n/|\psi_{[n]}(z_0)|$ then we get a contradiction with (3.1).

(ii) $\Rightarrow$ (i): Let K be a compact set in $\mathbb{C}$. Consider L a compact satisfying $\varphi^n(K) \subseteq L$ for every $n \in \mathbb{N}$. Let $C_n = \sup_{z\in K} \left|\prod_{m=0}^{n-1} (\psi \circ \varphi^m)(z)\right|$. Then

$$\sup_{z\in K} |(C_{\varphi,\psi})^n f(z)| = \sup_{z\in K} \left\{ \left(\prod_{m=0}^{n-1} (\psi \circ \varphi^m)(z) \right) f(\varphi^n(z)) \right\} \leq$$
$$\leq C_n \sup_{z\in K} |f(\varphi^n(z))| \leq C_n \sup_{z\in L} |f(z)|. \qquad \square$$

Theorem 3.3. *Let U be a domain of $\mathbb{C}$. The following assertions are equivalent:*

(i) *$C_{\varphi,\psi}$ is power bounded,*
(ii) *φ has stable orbits and $(\prod_{m=0}^{n} (\psi \circ \varphi^m))_n$ is a bounded sequence on $H(U)$.*

Proof. (i) $\Rightarrow$ (ii) is a consequence of Propositions 3.1 and 3.2.
(ii) $\Rightarrow$ (i): Let K be a compact subset of U and consider

$$C := \sup_{n\in\mathbb{N}} \sup_{z\in K} \left| \prod_{m=0}^{n-1} (\psi \circ \varphi^m)(z) \right| < \infty.$$

Since φ has stable orbits, given K we take a compact subset $L \subseteq U$ such that $\varphi^n(K) \subseteq L$ for $n \in \mathbb{N}$. Then

$$\sup_{z\in K} \left|(C^n_{\varphi,\psi}) f(z)\right| \leq C \sup_{z\in K} |(f \circ \varphi^n)(z)| \leq C \sup_{z\in L} |f(z)|, \quad \forall\, n \in \mathbb{N}.$$

Hence $((C^n_{\varphi,\psi})(f))_n$ is bounded on $H(U)$ for every $f \in H(U)$. $\square$

Remark 3.4. A similar proof to that of Theorem 3.3 shows that $(C^n_{\varphi,\psi})$ is convergent to 0 in $L(H(U))$ if and only if φ has stable orbits and $(\psi_{[n]})_n$ converges to 0 in $H(U)$.

Corollary 3.5. *Let U be a domain of $\mathbb{C}$. Let φ and ψ be analytic functions on U such that $\varphi(U) \subseteq U$. If $C_{\varphi,\psi}$ is power bounded then the composition operator C_φ is power bounded.*

Proof. If $C_{\varphi,\psi}$ is power bounded, by Theorem 3.3, φ has stable orbits and then we can conclude by an application of [9, Proposition 1]. $\square$

Corollary 3.6. *Let U be a domain of $\mathbb{C}$. Let φ and ψ be analytic functions on U such that $\varphi(U) \subseteq U$. If φ has stable orbits and $\psi(U) \subseteq \overline{\mathbb{D}}$, then $C_{\varphi,\psi}$ is power bounded.*

Condition $\psi(U) \subseteq \overline{\mathbb{D}}$ in Corollary 3.6 becomes also necessary for multiplication operators associated to the multiplier ψ, i.e., for the operator $C_{\varphi,\psi}$ when the symbol is the identity $\varphi(z) = z$, $z \in U$. Actually, in this case the condition on the multiplier in Theorem 3.3 becomes $(\psi^n)_n$ bounded in $H(U)$, and this is trivially equivalent to the condition $\psi(U) \subseteq \overline{\mathbb{D}}$. In this paper we prove with several examples that, in general, this condition is not necessary when the symbol differs from the identity. In this context the necessary condition is generalized considering the set of fixed points of φ instead of the whole domain U:

Proposition 3.7. *Let U be a domain of $\mathbb{C}$. Let φ and ψ be analytic functions on U such that $\varphi(U) \subseteq U$ and z_0 is a fixed point of φ. If $|\psi(z_0)| > 1$, then the weighted composition operator $C_{\varphi,\psi}$ is not mean ergodic (and thus, it is not power bounded).*

Proof. Since $|\psi(z_0)| > 1$, we get $\frac{1}{n}\prod_{m=0}^{n-1} |\psi \circ \varphi^m(z_0)| = \frac{1}{n}\prod_{m=0}^{n-1} |\psi(z_0)|$ diverges to infinity. Therefore, by Proposition 3.1, $C_{\varphi,\psi}$ is not mean ergodic. $\square$

Remark 3.8. Let U be a connected domain of holomorphy in $\mathbb{C}^d$ (or even in a Stein manifold). All the above results remain valid for weighted composition operators defined on $H(U)$ endowed with the compact open topology, since $H(U)$ is a Fréchet Montel space and U admits a fundamental sequence $(K_n)_n$ of compact subsets of U such that each K_n coincides with its U-holomorphic convex hull (i.e., $K_n = \{\omega \in U : |f(\omega)| \leq \sup_{z \in K_n} |f(z)|, \forall f \in H(U)\}$). In fact, this is the setting of Bonet and Domański [9]. Our Proposition 3.2 combined with [9, Proposition 1] means that, for composition operators (case $\psi \equiv 1$), being power bounded, being mean ergodic and being topologizable are equivalent.

Propositions 3.1 and 3.2 and Theorem 3.3 show that if $C_{\varphi,\psi}$ is mean ergodic or power bounded then the stability of the orbits of φ is a necessary condition. For composition operators, Bonet and Domański [9] proved that it is also a sufficient condition for the power boundedness. In what follows we assume φ has stable orbits and we find conditions on ψ to obtain the power boundedness of the operator $C_{\varphi,\psi}$. If U is a domain in $\mathbb{C}$, $U \neq \mathbb{C}$, by a theorem due to Abate [23, Theorem 5.5.4] (see also [9, Theorem 1] for a complement of this result for Stein manifolds), if φ has stable orbits we have two possibilities:

- (a) there is an attractive fixed point $z_0 \in U$ of φ such that $(\varphi^n)_n$ converges to the constant function $\alpha(z) := z_0$ in $(H(U), \tau_{co})$, or
- (b) there exists a subsequence $(\varphi^{n_k})_k$ which converges to id_U in $(H(U), \tau_{co})$. In this case φ is an automorphism and φ^{-1} has stable orbits.

3.1. Non-automorphic symbols with stable orbits

In this section we study case (a), that is, the case in which φ has an attracting fixed point $z_0 \in U$. We prove that in this context the mean ergodicity of $C_{\varphi,\psi}$ depends on the modulus of ψ at the fixed point z_0.

Lemma 3.9. *Let U be a domain in $\mathbb{C}$, $U \neq \mathbb{C}$ and let $\varphi : U \to U$ be holomorphic with a fixed point $z_0 \in U$ and φ different from an automorphism of U. Then for every compact set $K \subseteq U$, there are $0 < \rho < 1$ and $L > 0$ such that*

$$|\varphi^n(z) - z_0| < L\rho^n, \quad z \in K,\ n \in \mathbb{N}.$$

In particular, $(\varphi^n)_n$ converges to the constant function $\alpha(z) := z_0$ uniformly on compact subsets of U.

Proof. Take $h : \mathbb{D} \to U$ a Riemann map of the domain U such that $h(0) = z_0$. Let K be a compact set in U. Since φ has stable orbits, there is a compact set Q such that $\varphi^n(K) \subseteq Q$ for all $n \in \mathbb{N}$. Then there is a constant $L > 0$ such that

$$|h(z) - h(w)| \leq L|z - w|, \quad z, w \in \widetilde{Q} := h^{-1}(Q \cup K) \cup \{0\}.$$

Consider the holomorphic map $\psi : \mathbb{D} \to \mathbb{D}$ given by $\psi = h^{-1} \circ \varphi \circ h$. Clearly $\psi(0) = 0$ and ψ is not an automorphism of the unit disc. By a standard argument, there is $0 < \rho < 1$ such that

$$|\psi^n(w)| \leq \rho^n, \quad w \in \widetilde{Q},\ n \in \mathbb{N}.$$

Hence for all $z \in K$, $\psi^n(h^{-1}(z)) = h^{-1}(\varphi^n(z)) \subseteq \widetilde{Q}$ and $|\psi^n(h^{-1}(z))| \leq \rho^n$, for all $n \in \mathbb{N}$. Consequently

$$|\varphi^n(z) - z_0| = |h \circ \psi^n(h^{-1}(z)) - h(0)| \leq L|\psi^n(h^{-1}(z)) - 0| \leq L\rho^n, \quad z \in K,\ n \in \mathbb{N}. \qquad \square$$

In the following theorem we prove that for a symbol φ with attractive interior fixed point, the behavior of the multiplier ψ in the attractor completely characterizes the mean ergodicity of $C_{\varphi,\psi}$.

Theorem 3.10. *Let U be a domain in $\mathbb{C}$, $U \neq \mathbb{C}$ and let $\varphi : U \to U$ be a symbol with a fixed point $z_0 \in U$ and such that φ is not an automorphism. Let $\psi : U \to \mathbb{C}$ be a multiplier.*

(i) *If $|\psi(z_0)| \leq 1$, then $C_{\varphi,\psi}$ is power bounded,*
(ii) *If $|\psi(z_0)| > 1$, then $C_{\varphi,\psi}$ is not mean ergodic.*

Proof. We only need to prove (i), since (ii) follows from Proposition 3.7. First, suppose that $|\psi(z_0)| < 1$ and consider $0 < \delta < 1$ such that $|\psi(z_0)| < \delta < 1$. Since $(\varphi^n)_n$ converges to z_0 in $(H(U), \tau_{co})$, $(\psi \circ \varphi^n)_n$ converges to $\psi(z_0)$ in $(H(U), \tau_{co})$. Given a compact set $K \subseteq U$, there is $n_0 \in \mathbb{N}$ such that

$$|(\psi \circ \varphi^n)(z)| < \delta < 1 \quad \forall\, n \geq n_0,\ z \in K.$$

Then, for all $n > n_0$,

$$\sup_{z \in K} \prod_{m=0}^{n-1} |\psi(\varphi^m(z))| < \sup_{z \in K} \prod_{m=0}^{n_0} |\psi(\varphi^m(z))|$$

and so, $\{\prod_{m=0}^{n-1} |\psi \circ \varphi^m| : \ n \in \mathbb{N}\}$ is bounded on $H(U)$. By Theorem 3.3, $C_{\varphi,\psi}$ is power bounded. In this case, it is not hard to prove that $(C^n_{\varphi,\psi})_n$ is even convergent to 0 in $L_b(H(U))$.

Now, consider the case $|\psi(z_0)| = 1$. By Theorem 3.3, it suffices to show that $\{\prod_{m=0}^{n-1} |\psi(\varphi^m)| : \ n \in \mathbb{N}\}$ is bounded on $H(U)$. We can assume $\psi(z_0) = 1$ because $\{\prod_{m=0}^{n-1} |\psi(\varphi^m)| : \ n \in \mathbb{N}\}$ is bounded if and only if $\{\prod_{m=0}^{n-1} |\alpha\psi(\varphi^m)| : \ n \in \mathbb{N}\}$ is bounded on $H(U)$ for all $|\alpha| = 1$. By Lemma 3.9 there is $0 < \rho < 1$ and $M_1 \in \mathbb{N}$ such that

$$|\varphi^m(z) - z_0| \leq M_1\rho^m \quad \forall z \in K,\ \forall m \in \mathbb{N}.$$

Consider $h(z) := \frac{\psi(z)-1}{z-z_0} \in H(U)$, $z \in U \setminus \{z_0\}$, and $h(z_0) = \psi'(z_0)$. Given a compact set $K \subseteq U$, by hypothesis there is a compact set $L \subseteq U$ such that $\varphi^n(K) \subseteq L$ for all $n \in \mathbb{N}$. Put $M_2 := \sup_{z\in L} |h(z)| < \infty$ and $M = M_1M_2$. Hence, for all $z \in K$, $m \in \mathbb{N}$,

$$|\psi(\varphi^m(z)) - 1| = |\varphi^m(z) - z_0||h(\varphi^m(z))| \leq M\rho^m.$$

From this it follows that

$$\sum_{m=0}^{\infty} |\psi(\varphi^m(z)) - 1| < \infty$$

and applying [4, Lemma 6.1.2 and 6.1.4] we deduce that

$$\prod_{m=0}^{\infty} \psi(\varphi^m(z)) = \prod_{m=0}^{\infty} (1 + (\psi(\varphi^m(z)) - 1)),$$

converges absolutely on τ_{co}. □

3.2. Automorphic symbols with stable orbits

If φ has stable orbits but does not have an attractive fixed point in U, by [23, Theorem 5.5.4] we obtain that φ is an automorphism and φ^{-1} has stable orbits.

In this case, the following result is satisfied:

Proposition 3.11. *Let U be a domain in $\mathbb{C}$, $U \neq \mathbb{C}$ and let $\psi : U \to \mathbb{C}$ be an analytic function such that $\psi^{-1}(\overline{\mathbb{D}})$ is a compact set. If $\varphi : U \to U$ is an analytic function with stable orbits and has no attractive fixed point, then $C_{\varphi,\psi}$ cannot be mean ergodic.*

Proof. By [9, Proposition 1(e)], there is a fundamental family of connected compact sets $(K_n)_n$ in U such that $\varphi(K_n) \subseteq K_n$ for every $n \in \mathbb{N}$. So, there is a compact set K such that $\psi^{-1}(\overline{\mathbb{D}}) \subseteq \mathring{K}$ and $\varphi(K) \subseteq K$. This implies that for every $z \notin K$ there exists $\delta > 0$ such that $|\psi(\varphi^n(z))| > 1 + \delta$ for every $n \in \mathbb{N}$. Otherwise, there exists $z \notin K$ such that, for every $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that

$$|\psi(\varphi^{n_j}(z))| \leq 1 + \frac{1}{j}. \tag{3.2}$$

Since φ has stable orbits, there is a compact $L \subseteq U$ such that $(\varphi^{n_j}(z))_j \subseteq L$ for every $j \in \mathbb{N}$. Therefore, there exists $\omega \in L$ and a subsequence $(n_{j_k})_k$ such that $\varphi^{n_{j_k}}(z)$ converges to ω. Thus, $\psi(\varphi^{n_{j_k}}(z))$ converges to $\psi(\omega)$. By (3.2), $\omega \in \psi^{-1}(\overline{\mathbb{D}}) \subseteq \mathring{K}$. Hence, there is $k_0 \in \mathbb{N}$ such that $\varphi^{n_{j_{k_0}}}(z) \in \mathring{K}$, and so, $\varphi^n(z) \in K$ for every $n \geq n_{j_{k_0}}$, since $\varphi(K) \subseteq K$. But this contradicts the existence of a sequence $(\varphi^{m_k})_k$ which converges to id_U in $(H(U), \tau_{co})$. Therefore,

$$\left| \frac{1}{n} \prod_{m=1}^{n} (\psi \circ \varphi^k)(z) \right| > \frac{(1+\delta)^n}{n},$$

and $C_{\varphi,\psi}$ cannot be mean ergodic by Proposition 3.1. □

Corollary 3.12. *Let U be a bounded domain in $\mathbb{C}$. Let $\varphi : U \to U$ be a symbol with stable orbits and no attractive fixed point and $\psi : U \to \mathbb{C}$ a multiplier. If $C_{\varphi,\psi}$ is mean ergodic, then $\psi^{-1}(\overline{\mathbb{D}}) \cap \delta(U) \neq \emptyset$, where $\delta(U)$ denotes the boundary of U.*

In what follows we focus on automorphisms of the unit disc, since they are completely characterized (see [19, page 837]). The ones with stable orbits are called *elliptic* automorphisms, and have a fixed point inside $\mathbb{D}$. By [19, Lemma 3.0.5], if z_0 is its fixed point, $|\varphi'(z_0)| = 1$ and

$$\left(\frac{z_0 - z}{1 - \bar{z}_0 z}\right)^{-1} \circ \varphi \circ \frac{z_0 - z}{1 - \bar{z}_0 z} = \varphi'(z_0)z.$$

Lemma 3.13. *Let E, F be two locally convex spaces and $S : E \to E$, $T : F \to F$ two conjugate operators, that is, there exists a isomorphism $\chi : E \to F$ such that $S = \chi^{-1} \circ T \circ \chi$. Then, T is power bounded ((uniformly) mean ergodic) if and only if S is so.*

In the following until the end of the paper, we present our results for $U = \mathbb{D}$. Proceeding as in [19, Lemma 3.0.6], we get:

Lemma 3.14. *Let $C_{\varphi,\psi}$ be a weighted composition operator on $H(\mathbb{D})$, where φ is an elliptic automorphism of the unit disc with fixed point $z_0 \in \mathbb{D}$. Then $C_{\varphi,\psi}$ is conjugated to a weighted composition operator with composition map $\widetilde{\varphi}(z) = \varphi'(z_0)z$.*

As a consequence of Lemmas 3.13 and 3.14 we get that, if we want to study the power boundedness and (uniformly) mean ergodicity of a weighted composition operator $C_{\varphi,\psi}$ such that φ is an elliptic automorphism on the disc, we can consider, without loss of generality, that φ is a rotation of the unit disc centered at zero.

As in Proposition 3.10 we observe that, under some conditions on ψ and φ, the dynamical behavior of the associated weighted composition operator depends on the modulus of ψ at 0, the fixed point of φ. To do this, we need to introduce the following definition [22, 2.8.1].

Definition 3.15. A number α is called Diophantine of type (c, s) if for any nonzero $p, q \in \mathbb{Z}$ we have $|q\alpha - p| > cq^{-s}$. α is called *Diophantine* if there exist $c > 0$, $s > 1$ such that α is Diophantine of type (c, s).

It is well known that the union of all Diophantine irrational numbers has measure 1 in $(0, 1)$ [22, exercise 7.1.5, p. 290]. The following statement is also well known (see [22, p. 95]).

Lemma 3.16. *If $\alpha \in (0, 1)$ is Diophantine and $\lambda = e^{2\pi\alpha i}$ then there exists $C > 0$ and $s \in \mathbb{N}$ such that for all $n \in \mathbb{N}$,*

$$|\lambda^n - 1| \geq \frac{C}{n^s}.$$

We consider now multipliers which are nonzero in the disc. Inspired in Gottschalk and Hedlund's coboundary theorem [16] and Gunatillake's recent work [19], where the spectra of $C_{\varphi,\psi}$ is studied, we obtain a complete characterization of the mean ergodicity of weighted composition operators in terms of the value of the multiplier at 0 when the symbol is a rotation of the disc associated to a Diophantine irrational number.

Theorem 3.17. *Consider the automorphism of the disc $\varphi(z) = \lambda z$, $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$, with $|\lambda| = 1$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, and let $\psi \in H(\mathbb{D})$ nonvanishing. The weighted composition operator $C_{\varphi,\psi}$ satisfies:*

(i) $C_{\varphi,\psi}$ *is not mean ergodic if* $|\psi(0)| > 1$,
(ii) $C_{\varphi,\psi}$ *is convergent to* 0 *in* $L_b(H(\mathbb{D}))$ *(and then it is power bounded and uniformly mean ergodic) if* $|\psi(0)| < 1$,
(iii) *If* $\lambda = e^{2\pi\alpha i}$ *with* $\alpha \in (0,1)$ *being a Diophantine irrational number, then* $C_{\varphi,\psi}$ *is power bounded, and thus uniformly mean ergodic if* $|\psi(0)| = 1$.

Proof. (i) follows by Proposition 3.7.

(ii) If $|\psi(0)| < 1$, applying [19, Lemma 3.2.1] to each function $\psi((1-1/n)z)$, $z \in \overline{\mathbb{D}}$, we get that $\left(\prod_{m=0}^{N-1} |\psi \circ \varphi^m|\right)^{1/N}$ converges uniformly on $D(0,1-1/n)$ to $|\psi(0)| < 1$ on the unit disc as N tends to infinity, for each $n \in \mathbb{N}$. Then, $\prod_{m=0}^{N-1} |\psi \circ \varphi^m|$ converges to 0 uniformly on the compact sets of $\mathbb{D}$. By Proposition 3.3, $C_{\varphi,\psi}$ is power bounded with $C_{\varphi,\psi}^n \longrightarrow 0$, and thus uniformly mean ergodic.

(iii) Assume now that λ is a Diophantine irrational number and that $|\psi(0)| = 1$. Since $(\prod_{i=1}^n (\psi \circ \varphi^i))_n$ is bounded in $H(\mathbb{D})$ if and only if $(\prod_{i=1}^n a(\psi \circ \varphi^i))_n$ is bounded in $H(\mathbb{D})$ for each $|a| = 1$, then we can assume also $\psi(0) = 1$. Since $\psi \neq 0$ in $\mathbb{D}$, there exists $h \in H(\mathbb{D})$ with $h(0) = 0$ such that $\psi(z) = e^{h(z)}$ for each $z \in \mathbb{D}$ [12, Corollary 6.17]. That is, $h(z) = \sum_{n=1}^\infty a_n z^n$, $z \in \mathbb{D}$, with radius of convergence $R = \frac{1}{\limsup_n |a_n|^{\frac{1}{n}}} \geq 1$. Now, observe that $(\prod_{i=1}^n (\psi \circ \varphi^i))_n$ is bounded in $H(\mathbb{D})$ if $(\sum_{i=1}^n (h \circ \varphi^i))_n$ is bounded in $H(\mathbb{D})$. This is clearly satisfied if

$$\sup_{n\in\mathbb{N}} \sup_{z \in D(0,1-1/j)} \left| \sum_{i=1}^n h(\lambda^i z) \right| < \infty \tag{3.3}$$

holds for each $j \in \mathbb{N}$. Define $b_n = a_n/(1-\lambda^n)$ if $a_n \neq 0$ and $b_n = 0$ elsewhere. Consider the series $g(z) = \sum_{n=1}^\infty b_n z^n$. Take s and C satisfying the conditions of Lemma 3.16 for λ. The radius of convergence $\widehat{R}$ of g satisfies

$$\widehat{R} = \frac{1}{\limsup_n |b_n|^{\frac{1}{n}}} \geq \frac{1}{\limsup_n |a_n|^{1/n} (\frac{n^s}{C})^{1/n}} = \frac{1}{\limsup_n |a_n|^{\frac{1}{n}}} = R \geq 1.$$

Hence g is holomorphic in $\mathbb{D}$ and $h(z) = g(z) - g(\lambda z)$ for each $z \in \mathbb{D}$. This gives us the conclusion, since the sums in (3.3) are now telescopic. □

Remark 3.18. The proof of Theorem 3.17 shows that $\sum_{i=1}^n h(\lambda^i z)$ has a telescopic form when λ is associated to a Diophantine irrational number. From this it follows that $(\prod_{i=1}^n \psi(\lambda^i \cdot))_n$ is a sequence which is bounded away from 0, i.e., there is no sequence $(n_k)_k$ of natural numbers such that $(\prod_{i=1}^{n_k} \psi(\lambda^i \cdot))_k$ is convergent to 0 in $H(\mathbb{D})$. Indeed, we get $|\prod_{i=1}^n \psi(\lambda^i z)| = e^{h(\lambda z) - h(\lambda^{n+1} z)}$ for each $n \in \mathbb{N}$ and $z \in \mathbb{D}$, and so, $\min_{z\in K} |(\prod_{i=1}^n \psi(\lambda^i z))| \geq e^{-2\max\{|h(z)|:\ z \in K\}} > 0$, for each $n \in \mathbb{N}$ and each compact set K in $\mathbb{D}$.

In the following proposition we see that if ψ is not zero free in $\mathbb{D}$ and φ is any irrational rotation, then the power boundedness of $C_{\varphi,\psi}$ is equivalent to the convergence to 0 of $(C_{\varphi,\psi}^n)_n$ in $L(H(\mathbb{D}))$.

Proposition 3.19. *Let us consider the automorphism of the disc* $\varphi(z) = \lambda z$, $z \in \mathbb{D}$, $\lambda \in \mathbb{C}$ *with* $|\lambda| = 1$ *and* $\lambda^n \neq 1$ *for every* $n \in \mathbb{N}$, *and let* ψ *be analytic on the unit disc with* $\psi(z_0) = 0$ *for some* $z_0 \in \mathbb{D}$. *The following are equivalent:*

(i) *The operator* $C_{\varphi,\psi}$ *is power bounded, and then uniformly mean ergodic in* $H(\mathbb{D})$.
(ii) $(\prod_{i=1}^n \psi(\lambda^i \cdot))_n$ *converges to* 0 *in* $H(\mathbb{D})$.
(iii) $(C_{\varphi,\psi}^n)_n$ *converges to* 0 *in* $L(H(\mathbb{D}))$.

Proof. Since (ii) and (iii) are equivalent by Remark 3.4 and (ii) implies (i) trivially, we only need to prove (i)⇒(ii). Assume $C_{\varphi,\psi}$ is power bounded, that is, the sequence $(\prod_{i=1}^n \psi(\lambda^i\cdot))_n$ is bounded, and then relatively compact in $H(\mathbb{D})$ (see Theorem 3.3).

If $\psi(0)=0$ then there exists $0<r<1$ such that $|\psi(z)|<1/2$ for each $z\in D(0,r)$. This implies that $(\prod_{i=1}^n \psi(\lambda^i\cdot))_n$ converges uniformly to 0 in $D(0,r)$. From this it follows that any subsequence $(\prod_{i=1}^{n_k} \psi(\lambda^i\cdot))_{n_k}$ which is convergent in $H(\mathbb{D})$ has limit 0. Since $(\prod_{i=1}^n \psi(\lambda^i\cdot))_n$ is relatively compact in $H(\mathbb{D})$, we conclude that $(\prod_{i=1}^n \psi(\lambda^i\cdot))_n$ converges to 0 in $H(\mathbb{D})$.

Let us assume now that $z_0\neq 0$. Now, for each $k\in\mathbb{N}$ and $n\geq k$, $\lambda^{-k}z_0$ is a zero of $\prod_{j=1}^n \psi(\lambda^n\cdot)$. Hence, by Kronecker theorem, $\{\lambda^k z_0:\ k\in\mathbb{N}\}$ is a dense set in the circle $|z|=|z_0|$ in which every limit point of $(\prod_{i=1}^n \psi(\lambda^i\cdot))_n$ vanishes. From this it follows that the unique limit point is 0. $\square$

Remark 3.20.

(i) The proof of (iii) in Theorem 3.17 yields that $C_{\varphi,\psi}$ is power bounded when $\psi(z)=e^{P(z)}$, $P(z)$ being an homogeneous polynomial and $\varphi(z)=\lambda z$, with λ not a root of unity. Further, the result is true if $\psi(z)=e^{h(z)}$ with $h(z)=g(z)-g(\lambda z)$ for some g continuous in $\mathbb{D}$, i.e. h is coboundary in the sense of Gottschalk and Hedlund [16] for the composition operator with symbol φ, being the symbol a rotation $\varphi(z)=\lambda z$, in each closed disc $r\overline{\mathbb{D}}$ for $0<r<1$.

(ii) The hypothesis ψ different from zero in the unit disc cannot be dropped in Proposition 3.17(ii). Observe that if we take $\psi(z)=2z+\frac{1}{2}$, $z\in\mathbb{D}$, we get $|\psi(0)|<1$ but $C_{\varphi,\psi}$ cannot be power bounded. Indeed, $\psi^{-1}(\overline{\mathbb{D}})=\overline{B(\frac{-1}{4},\frac{1}{2})}$ is compact in $\mathbb{D}$, and by Proposition 3.11, $C_{\varphi,\psi}$ cannot be mean ergodic. For this function, $\psi(-1/4)=0$.

Remark 3.21. Condition $\psi^{-1}(\overline{\mathbb{D}})\cap\partial\mathbb{D}\neq\emptyset$ in Corollary 3.12 is not a sufficient condition in order to be mean ergodic. For instance, if we consider $\varphi(z)=\lambda z$, $z\in\mathbb{D}$, $\lambda\in\mathbb{C}$ with $|\lambda|=1$ and $\lambda^n\neq 1$ for every $n\in\mathbb{N}$, and $\psi:\mathbb{D}\to\mathbb{C}$, $\psi(z)=z+3/2$, we get that $-1\in\psi^{-1}(\overline{\mathbb{D}})$ but $C_{\varphi,\psi}$ is not mean ergodic by Theorem 3.17.

In the next example we show that condition $|\psi(0)|\leq 1$ in Proposition 3.17 does not necessarily imply power boundedness of $C_{\varphi,\psi}$ in the case φ is a rotation of the unit disc associated to a rational number. We use the next result:

Proposition 3.22. *Let $\varphi(z)=\lambda z$, $z\in\mathbb{D}$, $\lambda\in\mathbb{C}$ with $\lambda^k=1$ for some $k\in\mathbb{N}$ and let ψ be analytic on the unit disc. Then:*

(i) *$C_{\varphi,\psi}$ is not mean ergodic if there exists $z_0\in\mathbb{D}$ such that $\prod_{m=0}^{k-1}|\psi\circ\varphi^m(z_0)|>1$.*

(ii) *Otherwise, $C_{\varphi,\psi}$ is power bounded. In this case, if $\prod_{m=0}^{k-1}\psi\circ\varphi^m(z)$ is not constant, then $(C_{\varphi,\psi}^n)_n$ is convergent to 0 in $L(H(\mathbb{D}))$.*

Proof. Given $n\in\mathbb{N}$ we can find $p,q\in\mathbb{N}$, $0\leq q<k$ such that $n=kp+q$. Then, the assertions follow from the identity

$$\prod_{m=0}^{n}(\psi\circ\varphi^m)=\left(\prod_{m=0}^{k-1}(\psi\circ\varphi^m)\right)^p\prod_{m=0}^{q}(\psi\circ\varphi^m)$$

and Proposition 3.1, Theorem 3.3 and Remark 3.4. $\square$

Example 3.23. Let $\varphi(z)=-z$, $z\in\mathbb{D}$, and $\psi(z)=\frac{z}{2}+a$ with $\sqrt{3}/2<a<1$. The function $\psi(z)$ is analytic and different from zero in the closed unit disc, it satisfies $|\psi(0)|<1$, but the weighted composition operator $C_{\varphi,\psi}$ is not mean ergodic by Proposition 3.22, since $\psi(i)\psi(-i)>1$.

Example 3.24. Let $\psi(z) = 1 + z$.

(a) $C_{\varphi,\psi}$ is power bounded if $\varphi(z) = \lambda z$ with $\lambda = e^{2\pi\alpha i}$, with $\alpha \in (0, 1)$ being an irrational Diophantine number,

(b) $C_{\varphi,\psi}$ is not mean ergodic for $\varphi(z) = -z$.

In this concluding example (a) is a direct consequence of Theorem 3.17 and (b) follows from Proposition 3.22, since $h(z) := \psi(z)\psi(-z) = 1 - z^2$ and $h(ai) = 1 + a^2$ for each $0 < a < 1$.

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