A Caristi fixed point theorem for complete quasi-metric spaces by using mw-distances

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Abstract

In this paper we give a quasi-metric version of Caristi's fixed point theorem by using mw-distances. Our theorem generalizes a recent result obtained by Karapinar and Romaguera in [7].

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1 Introduction and preliminaries

In 1976, Caristi [3] stated the following result which is one of the most important generalizations of the Banach contraction principle.

Theorem A. (Caristi fixed point theorem) *Let T be a self mapping of a complete metric space (X, d). If there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \tag{1}$$

for all $x \in X$, then T has a fixed point.

It is well known that this theorem is equivalent to Ekeland variational principle ([5]) which is nowadays an important tool in nonlinear analysis. Due to its application, Caristi's fixed point theorem has been investigated, extended, generalized and improved in several directions. Very recently, in [7] Karapinar and Romaguera proved, among other interesting results, the following quasi-metric generalization of Theorem A.

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Theorem B. ((1)$\to$ (2) in Theorem 2 of [7]) *Let T be a self mapping of a right K-sequentially complete quasi-metric space (X,d). If there exists a proper bounded below and nearly lower semicontinuous for τ_d, $\varphi : X \to \mathbb{R}\cup\{\infty\}$ such that $d(Tx,x)+\varphi(Tx) \leq \varphi(x)$, for all $x \in X$, then there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $d(Tz,z) = 0$.*

On the other hand, in [6] Kada et al. introduced the notion of w-distance on a metric space (X,d) as follows.

A function $q : X \times X \to \mathbb{R}^+$ is a w-distance on (X,d) if it satisfies the following conditions:

(W1) $q(x,y) \leq q(x,z) + q(z,y)$, for all $x,y,z \in X$;

(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous for τ_d for all $x \in X$;

(W3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(x,y) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

Clearly the metric d is a w-distance on (X,d).

In Theorem 2 of [6], the authors obtained the following generalization of Theorem A by using w-distances.

Theorem C. *Let T be a self mapping of a complete metric space (X,d) and let q a $w-$distance on (X,d). If there exists a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that*

$$q(x,Tx) \leq \varphi(x) - \varphi(Tx),$$

for all $x \in X$, then T has a fixed point.

Later on, Park in [10] extended the notion of w-distance to quasi-metric spaces and this concept has been used in some directions in order to obtain fixed point results on complete quasi-metric spaces ([2], [8], [9]).

Since a quasi-metric d is not in general a w-distance on the quasi-metric space (X,d), in [1] we introduced the notion of mw-distance which generalizes the concept of quasi-metric and we obtained fixed point theorems for generalized contractions with respect to mw-distances on complete quasi-metric spaces.

Definition 1. *(Definition 3 of [1]) An mw-distance on a quasi-metric space (X,d) is a function $q : X \times X \to \mathbb{R}^+$ satisfying the following conditions:*
(W1) $q(x,y) \leq q(x,z) + q(z,y)$ for all $x,y,z \in X$;
(W2) $q(x,\cdot) : X \to \mathbb{R}^+$ is lower semicontinuous on $(X,\tau_{d^{-1}})$ for all $x \in X$;
(mW3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y,x) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

Note that the concepts of w-distance and mw-distance are independent (see examples of [1]) both in quasi-metric spaces and metric spaces.

In this paper we prove a quasi-metric version of Caristi's fixed point theorem by using mw-distances which generalizes Theorem B. We also obtain a generalization of Theorem A similar to Theorem C but using mw-distances instead of w-distances.

We start by recalling several notions and properties of the theory of quasi-metric spaces. Our basic reference is [4].

A quasi-metric on a set X is a function $d : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$; (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.

Each quasi-metric d on a set X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If d is a quasi-metric on X then τ_d is a T_1 topology if and only if $d(x, y) = 0$ implies $x = y$.

Given a quasi-metric d on X, the function d^{-1} defined by $d^{-1}(x, y) = d(y, x)$ for all $x, y \in X$, is also a quasi-metric on X, called conjugate quasi-metric, and the function d^s defined by $d^s(x, y) = \max\{d(x, y), d(y, x)\}$ for all $x, y \in X$, is a metric on X.

There exist several different notions of Cauchy sequence and quasi-metric completeness in the literature (see e.g. [4]). Here we will consider the following ones.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a quasi-metric (X, d) is said to be left (right) K-Cauchy if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n \leq m$ $(n_0 \leq m \leq n)$.

A quasi-metric space (X, d) is d^{-1}-complete if every left K-Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in (X, d) converges with respect to the topology $\tau_{d^{-1}}$, i.e., there exists $z \in X$ such that $d(x_n, z) \to 0$.

Note that our notion of d^{-1}-completeness of (X, d) coincides with the usual notion of right K-sequential completeness of (X, d^{-1}).

2 The results

The following lemma is necessary to prove our main result (Theorem 1 below).

Lemma 1. *Let (X, d) be a quasi-metric space, q an $mw-$distance on (X, d) and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. If for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $q(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n < m$, then $(x_{2n})_{n\in\mathbb{N}}$ and $(x_{2n-1})_{n\in\mathbb{N}}$ are left K-Cauchy sequences in (X, d).*

Proof. Let $\varepsilon > 0$. By $(mW3)$, there exists $\delta > 0$ such that if $q(y,x) \leq \delta$ and $q(x,z) \leq \delta$ then $d(y,z) \leq \varepsilon$.

By hypothesis, there exists n_0 such that $q(x_n, x_m) \leq \delta$ whenever $n_0 \leq n < m$. Then, $q(x_{2n}, x_{2n+1}) \leq \delta$ and $q(x_{2n+1}, x_{2m}) \leq \delta$ whenever $n_0 \leq n < m$. Consequently, $d(x_{2n}, x_{2m}) \leq \varepsilon$ whenever $n_0 \leq n \leq m$.

In a similar way it is proved that $(x_{2n-1})_{n\in\mathbb{N}}$ is a left K-Cauchy sequence. □

Recall that if X is a (nonemptyset) set, a function $f : X \to \mathbb{R} \cup \{\infty\}$ is said to be proper if there exists $x \in X$ such that $f(x) < \infty$.

In [7], the authors introduced the notion of nearly lower semicontinuity which is a generalization of the concept of lower semicontinuity. A proper function $f : X \to \mathbb{R} \cup \{\infty\}$ is nearly semicontinuous on the quasi-metric space (X, d) if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence of distinct points of X that τ_d converges to some $x \in X$ then $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Theorem 1. *Let T be a self mapping of a d^{-1}-complete quasi-metric space (X, d) and let q be an $mw-$distance on (X, d). If there exists a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ such that $q(x, Tx) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$, then there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $q(z, Tz) = 0$.*

Proof. For each $x \in X$ let

$$S(x) = \{y \in X : q(x, y) + \varphi(y) \leq \varphi(x)\}.$$

Since $Tx \in S(x)$, we have that $S(x) \neq \emptyset$ for all $x \in X$. Let

$$i(x) = inf\{\varphi(y) : y \in S(x)\}.$$

Let $x_1 \in X$ such that $\varphi(x_1) < \infty$. There exists $x_2 \in S(x_1)$ such that $\varphi(x_2) \leq i(x_1) + 1$. Following this process we obtain a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that

$$x_{n+1} \in S(x_n),$$

$$\varphi(x_{n+1}) < \infty,$$

and

$$\varphi(x_{n+1}) \leq i(x_n) + \frac{1}{n}$$

for all $n \in \mathbb{N}$. Since $q(x_n, x_{n+1}) + \varphi(x_{n+1}) \leq \varphi(x_n)$, the sequence $(\varphi(x_n))_{n\in\mathbb{N}}$ is non-increasing. So, $\lim_{n\to\infty} \varphi(x_n)$ exists. Put $l = \lim_{n\to\infty} \varphi(x_n)$.

Now we prove that $(x_{2n})_{n\in\mathbb{N}}$ is a left K-Cauchy sequence in (X, d).

If $m > n$, then

$$q(x_n, x_m) \leq \sum_{i=n}^{m-1} q(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} (\varphi(x_i) - \varphi(x_{i+1})) = \varphi(x_n) - \varphi(x_m)$$

Since $(\varphi(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that if $n_0 \leq n \leq m$ then $\varphi(x_n) - \varphi(x_m) < \varepsilon$. Therefore $q(x_n, x_m) \leq \varepsilon$ whenever $n_0 \leq n < m$. From Lemma 1, $(x_{2n})_{n\in\mathbb{N}}$ is a left K-Cauchy sequence.

Without loss of generality, we distinguish the following two cases.

Case 1. The sequence $(x_{2n})_{n\in\mathbb{N}}$ is eventually constant. Then there exists $n_0 \in \mathbb{N}$ such that $x_{2n} = x_{2n_0}$ for all $n \geq n_0$. Since

$$\varphi(x_{2n+2}) - \frac{1}{2n} \leq \varphi(x_{2n+1}) - \frac{1}{2n} \leq i(x_{2n}) \leq \varphi(x_{2n+1}) \leq \varphi(x_{2n}),$$

then

$$\varphi(x_{2n_0}) - \frac{1}{2n} \leq i(x_{2n_0}) \leq \varphi(x_{2n_0}),$$

for all $n \geq n_0$. Taking limits, we obtain that $i(x_{2n_0}) = \varphi(x_{2n_0})$. Since $Tx_{2n_0} \in S(x_{2n_0})$, then $i(x_{2n_0}) \leq \varphi(Tx_{2n_0}) \leq \varphi(x_{2n_0})$, so $\varphi(Tx_{2n_0}) = \varphi(x_{2n_0})$ and $q(x_{2n_0}, Tx_{2n_0}) = 0$.

Case 2. $x_{2n} \neq x_{2m}$ for all $n, m \in \mathbb{N}$ with $n \neq m$. Since (X, d) is d^{-1}-complete there exists $z \in X$ such that (x_{2n}) converges to z in $(X, \tau_{d^{-1}})$.

Next we show that $z \in S(x_{2n})$ for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. Since $q(x_{2n}, \cdot)$ is a lower semicontinuous function on $(X, \tau_{d^{-1}})$ and φ is a nearly lower semicontinuous function on $(X, \tau_{d^{-1}})$, there exists $m_0 > n$ such that if $m \geq m_0$ then

$$q(x_{2n}, z) - q(x_{2n}, x_{2m}) < \varepsilon$$

and

$$\varphi(z) - \varphi(x_{2m}) < \varepsilon.$$

Then

$$q(x_{2n}, z) < q(x_{2n}, x_{2m}) + \varepsilon \leq \varphi(x_{2n}) - \varphi(x_{2m}) + \varepsilon < \varphi(x_{2n}) - \varphi(z) + 2\varepsilon.$$

Therefore

$$q(x_{2n}, z) + \varphi(z) \leq \varphi(x_{2n}),$$

i.e., $z \in S(x_{2n})$ for all $n \in \mathbb{N}$.

Since $0 \leq q(x_{2n}, z) \leq \varphi(x_{2n}) - \varphi(z)$, we have that $\varphi(z) \leq \varphi(x_{2n})$, for all $n \in \mathbb{N}$. So $\varphi(z) \leq l$.

Since $\varphi(z) \geq i(x_{2n})$, for all $n \in \mathbb{N}$, and $l = \lim_{n\to\infty} i(x_n)$ because $\varphi(x_{n+1}) \leq i(x_n) + \frac{1}{n} \leq \varphi(x_{n+1}) + \frac{1}{n}$, we obtain that $\varphi(z) \geq l$. Hence $l = \varphi(z)$.

On the other hand,

$$q(x_{2n}, Tz) \leq q(x_{2n}, z) + q(z, Tz) \leq \varphi(x_{2n}) - \varphi(z) + \varphi(z) - \varphi(Tz) = \varphi(x_{2n}) - \varphi(Tz).$$

Therefore, $Tz \in S(x_{2n})$ for all $n \in \mathbb{N}$.

By using a similar argument to the one given above we obtain that $l = \varphi(Tz)$. Hence $\varphi(z) = \varphi(Tz)$ and, consequently, $q(z, Tz) = 0$.

□

Since every quasi-metric d on X is an $mw-$distance on (X, d), we obtain the following corollary.

Corollary 1. *Let T be a self mapping of a d^{-1}-complete quasi-metric space (X, d). If there exists a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ such that $d(x, Tx) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$, then there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $d(z, Tz) = 0$.*

This corollary is equivalent to Theorem B because a quasi-metric space (X, d) is right K-sequentially complete if and only if (X, d^{-1}) is $d-$complete. Theorem B can be obtained directly from Theorem 1 taking $q = d^{-1}$.

On the other hand, since the class of the nearly lower semicontinuos functions on a metric space (X, d) coincides with the class of the lower semicontinuous functions on (X, d), we obtain a generalization of Caristi's fixed point theorem in the same direction as Theorem B.

Corollary 2. *Let T be a self mapping of a complete metric space (X, d) and let q be an $mw-$distance on (X, d). If there exists a proper bounded below and lower semicontinuous function $\varphi : X \to \mathbb{R} \cup \{\infty\}$ such that $q(x, Tx) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$, then T has a fixed point.*

Proof. By Theorem 1, there exists $z \in X$ such that $\varphi(Tz) = \varphi(z)$ and $q(z, Tz) = 0$. Now we are going to prove that $z = Tz$.

Given $\varepsilon > 0$ there exists $\delta > 0$ such that if $q(y, x) \leq \delta$ and $q(x, z) \leq \delta$ then $d(y, z) \leq \varepsilon$. Since $q(x_n, z) \leq \varphi(x_n) - \varphi(z)$, for all $n \in \mathbb{N}$ and $l = \varphi(z)$, there exists $n_0 \in \mathbb{N}$ such that $q(x_n, z) \leq \delta$ for all $n \geq n_0$. Since $q(z, Tz) = 0 < \delta$, we have that $d(x_n, Tz) < \varepsilon$ for all $n \geq n_0$. Therefore $(x_n)_{n\in\mathbb{N}}$ converges to Tz. Consequently $Tz = z$.

□

Remark 1. As mentioned above Caristi's fixed point theorem for metric spaces is a generalization of the Banach contraction principle. This is because if T is a contractive self mapping of a metric space (X, d), then $\varphi(x) = \frac{1}{1-r} d(x, Tx)$, where r is the contractivity constant, is a lower semicontinuous function on X and $d(x, Tx) + \varphi(Tx) \leq \varphi(x)$. This is not the case in the quasi-metric framework. In fact, the Banach contraction principle is not fulfilled if the complete metric space is replaced by a d^{-1}-complete quasi-metric space. For instance, if $X = \{1/n : n \in \mathbb{N}\}$ and d is the quasi-metric on X given by $d(x, x) = 0$ and $d(x, y) = x$ if $x \neq y$ then (X, d) is d^{-1}-complete and the self mapping of X given by $Tx = x/2$ is contractive but it has not fixed point. Note that T is not a Caristi type mapping because if that was the case, by Corollary 1, there exists $z \in X$ such that $d(z, Tz) = 0$ and then T has a fixed point since (X, τ_d) is a T_1 topological space.

Remark 2. As was expected, in Theorem 1 the condition $q(x,Tx) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$, can not be replaced by the condition $q(Tx,x) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$. Indeed, if $X = \{1/n : n \in \mathbb{N}\}$, d is the quasi-metric on X given by $d(x,y) = y - x$ if $x \leq y$ and $d(x,y) = 1$ if $x > y$, $q = d$ and φ is a function on X given by $\varphi(x) = x$, the self mapping of X given by $Tx = x/2$, satisfies that

$$q(Tx,x) + \varphi(Tx) = \frac{x}{2} + \frac{x}{2} = \varphi(x)$$

and nevertheless $Tz \neq z$ for every $z \in X$.

Finally, we give a characterization of d^{-1}-completeness in terms of the quasi-metric version of Caristi's fixed point theorem given in Theorem 1. For this purpose, we give the following definition.

Definition 2. *Let T a self mapping of the quasi-metric space (X,d). We say that T is (q,φ)-Caristi if q is an $mw-$distance on (X,d) and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$ such that $q(x,Tx) + \varphi(Tx) \leq \varphi(x)$, for all $x \in X$.*

The following example shows that if q is an $mw-$distance on the quasi-metric space (X,d) and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ is a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, then there exist (q,φ)-Caristi self mappings of X which are not (d,φ)-Caristi.

Example 1. *Let $X = \mathbb{N}$ and let d be the quasi-metric on X given by $d(x,x) = 0$ and $d(x,y) = x$ for all $x,y \in X$. Clearly, τ_d is the discrete topology on X and $\tau_{d^{-1}} = \tau_d$. Let q be the mw-distance on (X,d) given by $q(1,1) = 0$ and $q(x,y) = 1/2$ otherwise. Define $T : X \to X$ as $T1 = 1$ and $Tx = x - 1$ for all $x > 1$. If we consider the function $\varphi : X \to \mathbb{R}$ given by $\varphi(x) = x$, then φ is nearly lower semicontinuous for $\tau_{d^{-1}}$, $q(1,T1) = 0 = \varphi(1) - \varphi(T1)$ and if $x > 1$, then*

$$q(x,Tx) = 1/2 < 1 = \varphi(x) - \varphi(Tx).$$

Therefore T is (q,φ)-Caristi. Nevertheless T is not (d,φ)-Caristi because $d(x,Tx) > \varphi(x) - \varphi(Tx)$, for all $x > 1$.

Theorem 2. *Let (X,d) be a quasi-metric space. Then (X,d) is d^{-1}-complete if and only for every (q,φ)-Caristi self mapping T of X exists $z \in X$ such that $\varphi(z) = \varphi(Tz)$ and $q(z,Tz) = 0$.*

Proof. From Theorem 1 we have the direct. For the converse, we suppose that X is not d^{-1}-complete. Then (X,d^{-1}) is not right K-sequentially complete. By (2)$\to$(1) of Theorem 2 of [7], there exist a self mapping T of X and a proper bounded below and nearly lower semicontinuous function for $\tau_{d^{-1}}$, $\varphi : X \to \mathbb{R} \cup \{\infty\}$ such that $d^{-1}(Tx,x) + \varphi(Tx) \leq \varphi(x)$, for all

$x \in X$ and $\varphi(Tz) \neq \varphi(z)$ for all $z \in X$. Therefore T is a $(d, \varphi)-$Caristi self mapping of X such that for all $z \in X$, $\varphi(Tz) \neq \varphi(z)$ and this is a contradiction.

□

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