Acoustic Metamaterial Models on the (2+1)D Schwarzschild Plane

Michael M. Tung[a,*], Ewa B. Weinmüller[b]

[a] *Instituto de Matemática Multidisciplinar*
Universitat Politècnica de València
Camino de Vera, s/n, 46022 Valencia, Spain
[b] *Institute for Analysis and Scientific Computing*
Vienna University of Technology
Wiedner Hauptstrasse 8–10/1152, 1040 Wien, Austria

Abstract

Recent developments in acoustic metamaterial engineering have led to the design and fabrication of devices with formidable properties, such as acoustic cloaking, superlenses and ultra-sound waves. Artificial materials of this type are generally absent in natural environments. In this work, we focus on feasible implementations of acoustic black holes on the 2D plane, that is, within (2+1)D spacetime. For an accurate description of planar black holes in transformation acoustics, we examine Schwarzschild-type models. After proposing an appropriate form for the Lorentzian metric of the underlying spacetime, we explore the geometric content and physical consequences of such models, which will turn out to have de Sitter and anti-de Sitter spacetime structure. For this purpose, we derive a general expression for its acoustic wave propagation. Next, a numerical simulation is carried out for prototype waves which probe these spacetime geometries. Finally, we discuss how to fine-tune the corresponding acoustic parameters for an implementation in the laboratory environment.

Keywords:
Acoustic analogue model of gravity, Acoustic black holes, Differential geometry, Variational principles of physics, Manifolds, Spacetime models
PACS: 43.20.Wd, 04.70.Bw
2000 MSC: 53Z05, 49S05, 49Q99, 83C99

*corresponding author
Email addresses: `mtung@imm.upv.es` (Michael M. Tung), `ewa.weinmueller@tuwien.ac.at` (Ewa B. Weinmüller)

May 22, 2018

1. Introduction

Metamaterials are artificially manufactured materials which by far surpass the properties of conventional materials found in nature. With their help researchers and engineers alike are presented with unique possibilities for the development of novel artificial devices with extraordinary characteristics. This does not merely entail a simple and gradual improvement of devices with already known features, but involves a paradigm shift, *e.g.* with optical metamaterials it has become possible to construct devices with negative refractive index [1]—a concept which was traditionally regarded as impossible, although already hypothesized in the late 1960's [2].

For almost two decades researchers now have focussed on optical metamaterials, whereas the manipulation of sound waves via acoustic metamaterials only recently has come under their spotlight [3, 4, 5, 6]. Acoustic metamaterials allow to model sophisticated acoustic phenomena via curved background spacetimes and make predictions for future laboratory experiments. Apart from the interesting technical applications, these models may also help to settle fundamental questions with far-reaching impact, as *e.g.* the existence and analysis of inverse Doppler effects [7] and Hawking radiation from acoustic black holes [8].

With regard to black holes, one of the first—and remarkably simple—solutions of Einstein's equations for the curved spacetime of the gravitational field with underlying static and spherical symmetry is the Schwarzschild solution [9]. In the beginning considered to be a mathematical curiosity and only of academic interest, it has now in the age of high-precision GPS navigation and black-hole astronomy become the center stage of many practical and important applications.

In order to implement 2D artificial black holes for acoustic waves, several experimental and theoretical pathways have been studied and proposed in the literature [10, 11, 12, 13]. In this context, acoustic black holes have been loosely defined as metadevices where sound is effectively trapped in a stream-like event horizon, however, without further precise assumption about the global spacetime structure. In the present work, we employ a different approach to model acoustic wave propagation on a curved spacetime with the characteristics of a genuine Schwarzschild spacetime structure. This approach is based on a variational principle in combination with the powerful framework of differential geometry [14, 15, 16, 17].

The Schwarzschild geometry is both a mathematically and physically intriguing non-euclidean geometry and as such a fascinating candidate for the implementation and study of an acoustic metamaterial. In general, for the $(n + 1)$D case, it represents a modification of flat Minkowski spacetime which imposes full spherical symmetry for the n-dimensional spatial part.

First, we will briefly review the field formulation of acoustics and its variational principle. Then, before beginning the discussion on the modelling of acoustic wave propagation on the Schwarzschild plane, we examine the feasibility of Schwarzschild-type geometries in (2+1) spacetime dimensions and give a classification of the possible solutions. Next, we outline how to derive within this framework the partial differential equation for the acoustic potential which simulates wave propagation on the Schwarzschild plane. A numerical simulation for prototype waves probing these spacetime and an analysis of their crucial geometric features follows. Finally, we will comment on the design and implementation of such a spacetime with acoustic metadevices. Employing the constitutive equations [15] will enable us to connect the Schwarzschild geometry with the acoustic parameters of the model.

2. Field formulation of acoustics and variational principle

This outline on the field formulation of acoustics and its variational principle closely follows Refs. [15, 17]. The importance of variational principles in classical and field mechanics, including optics and electrodynamics, lies in defining concisely and in a coordinate-independent manner the fundamental laws which they describe via a given scalar Lagrange function $\mathscr{L}$, *i.e.* they remain invariant with respect to arbitrary transformations of the coordinates. The equations of motion that fully determine the physical behaviour of the system correspond to the extremal solutions of the action integral $\mathscr{A}$ defined by $\mathscr{L}$. This Lagrangian approach allows to easily reveal the underlying symmetries and conservation laws of the theoretical model via Noether's theorem (see *e.g.* [18]). Furthermore, the laws describing linear physical phenomena will have their equivalent in equations of motion with self-adjoint differential operators acting on the related field variables [19, p. 301]. In principle, this yields separable partial differential equations which are Sturm-Liouville problems for one of the field variables with analytical or at least semi-analytical solutions.

Let acoustics be described by the acoustic potential $\phi : M \to \mathbb{R}$, where M is a smooth spacetime endowed with a Lorentzian metric $\boldsymbol{g}$ having negative signature, *i.e.* $g = \det \boldsymbol{g} < 0$. Then, we postulate that the following action integral is stationary with respect to variations of the potential [15]:

$$\frac{\delta}{\delta\phi}\mathscr{A}[\phi] = \frac{\delta}{\delta\phi}\int\limits_{\Omega} d\mathrm{vol}_g\,\mathscr{L}(x,\phi,\nabla\phi) = 0 \quad \text{so that} \quad \frac{\delta}{\delta\phi}\mathscr{A}[\phi] = 0. \tag{1}$$

The integration domain $\Omega \subseteq M$ is an open, connected subset of spacetime with smooth boundary $\partial\Omega$, and the corresponding invariant volume element is denoted by $d\mathrm{vol}_g = \sqrt{-g}\,dx^0 \wedge \ldots \wedge dx^3$, where $x \in M$. Here, in general, the Lagrangian is a function $\mathscr{L} : TM \to \mathbb{R}$, where TM is the tangent bundle of coordinate space M. The form of $\mathscr{L}$ is severely constrained by fundamental symmetry requirements: energy-momentum conservation, locality, and free-wave propagation. Its simplest possible choice is [15]

$$\mathscr{L}(\nabla\phi) = \tfrac{1}{2}\,\boldsymbol{g}(\nabla\phi,\nabla\phi) = \tfrac{1}{2}\,g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}. \tag{2}$$

Note that if $\boldsymbol{v}$ denotes the local fluid velocity, p the acoustic pressure, ϱ_0 the density, and $c > 0$ the time-independent wave speed of the acoustic metamaterial, the gradient or covariant derivative appearing in Eq. (2) will be

$$\phi_{,\mu} = \begin{pmatrix} p/(c\varrho_0) \\ -\boldsymbol{v} \end{pmatrix}. \tag{3}$$

Here, Greek tensor indices refer to the full range of spacetime components, whereas Latin indices will only indicate spatial components. We also adopt the standard notation using a comma and semicolon before the indices to denote partial and covariant derivatives, respectively. In this notation, for a scalar ϕ, the (covariant) components of the derivative $\nabla\phi$ are identically $\phi_{;\mu}$ or $\phi_{,\mu}$.

Expression Eq. (3) encapsulates elementary relations of acoustics [20] and is valid within a fixed laboratory frame. Note that in classical acoustics the four-vector $\phi_{,\mu}$ obviously cannot be fully relativistic, meaning that one cannot transform from one arbitrary inertial laboratory frame to another. However, the 4-vector will transform with a subgroup of the full Lorentz group, according to $\mathrm{SO}^{+}(1,3)/\mathrm{H}^3$, which excludes all boosts (corresponding to rotations in real hyperbolic 3-space H^3) from the restricted Lorentz group.

After substituting Eq. (2) into Eq. (1), we obtain the Euler-Lagrange equation for the acoustic potential. This equation directly gives the wave equation of the acoustic system for the spacetime $(M,\boldsymbol{g})$ under investigation. In the laboratory (*physical space*), the acoustic engineer who wishes to implement spacetime $(M,\boldsymbol{g})$ has to calibrate the mass-density tensor ϱ and bulk modulus κ relating them to their magnitude in the corresponding space with known acoustic wave propagation (*virtual space*). For convenience, we will denote all quantities in virtual by an overbar. In explicit form, both of these spaces are linked by the *constitutive relations* [15]. The underlying symmetry for the $(n+1)$D Schwarzschild geometry obviously implies SO(n) symmetry for the n spatial coordinates. This group symmetry obviously reflects itself in the constitutive relations, and for $n = 2$ we have

$$\kappa = \frac{\sqrt{\gamma}}{\sqrt{\bar{\gamma}}}\,\bar{\kappa}, \qquad \rho_0\rho^{ij} = \frac{\sqrt{\bar{\gamma}}}{\sqrt{\gamma}}\,\bar{g}^{ij}, \tag{4}$$

where we have employed the usual shorthand notation $\gamma = \det(g^{ij})$ for the determinant of the spatial metric components.

3. Schwarzschild-type spacetime geometries

In $(n+1)$D spacetime geometry, the static and spherically symmetric metric of Schwarzschild-type takes the following form

$$\boldsymbol{g} = -h(r)\,c^2 dt \otimes dt + h^{-1}(r) dr \otimes dr + r^2 d\Omega_{n-1} \otimes d\Omega_{n-1}, \tag{5}$$

where t and r are the local time and radial coordinates, respectively. It is a generalized form of the Friedmann-Lemaître-Robertson-Walker (FLRW) metric with positive curvature including the radial fudge factor $h(r)$. For $h \equiv 1$, one recovers the FLRW form in hyperspherical coordinates. Here, as usual, the solid angle Ω_n comprises all n angles φ_i of the hypersphere S^n, and it is defined by

$$d\Omega_n^2 = d\varphi_1^2 + \sin^2\varphi_1\, d\varphi_2^2 + \cdots + \left(\prod_{i=1}^{n-1} \sin^2\varphi_i\right) d\varphi_n^2, \tag{6}$$

where $\varphi_1 \in [0, 2\pi[$ and $\varphi_i \in [0, \pi[$ for $i = 2, 3, \ldots, n$.

The radial function implies the inverse correlation between relativistic time dilation and space contraction, and thus it will satisfy $0 < h(r) \leq 1$ for all $r > 0$. Evidently, the specific solution for $h(r)$ will depend on dimension n and the physical conditions imposed.

For $n = 3$, that is for usual 4-dimensional spacetime, the vacuum solution of Einstein's field equations (*viz.* Eq. (16) with $\Lambda = 0$) yields the conventional Schwarzschild metric [9], which is the only possible solution with asymptotic flatness[1] and the correct Newtonian limit. It is

$$h(r) = 1 - \frac{r_s}{r}, \quad r_s = \frac{2GM}{c^2}, \tag{7}$$

where G is the gravitational constant, and M the mass of the object. The Schwarzschild radius is r_s, and flat spacetime is recovered for $r_s = 0$.

In 1983, O'Neill [21, pp. 152–153] introduced the toy model with the geometry of Eq. (5) for $n = 1$. In this case all solutions are automatically vacuum solutions. With infinite choices for $h(r)$, the null geodesics (paths of light rays) in the tr-plane are here given by

$$-h\left(\frac{dt}{d\lambda}\right)^2 + h^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0, \tag{8}$$

where λ is the usual affine parameter on light-like curves. This describes radial light rays whose light cones get tipped over when approaching the Schwarzschild radius [21]. Although this model has all the essential building blocks of the simplest relativistic model of spacetime around a significantly massive object, it is lacking otherwise interesting features and is still rather limited.

In the following, we will propose and examine the considerably richer case for $n = 2$—a spacetime which we term the (2 + 1)D *Schwarzschild plane*. In particular, we first will discuss its admissible solutions and then proceed in the next section with its wave simulation and geometrical properties.

For this purpose, we rewrite the Schwarzschild-type metric Eq. (5) for $n = 2$ in terms of the nonholonomic basis 1-forms θ^μ, which yields the following expression for the metric:

$$\boldsymbol{g} = -(\sqrt{h(r)}\, c dt) \otimes \underbrace{(\sqrt{h(r)}\, c dt)}_{\theta^0} + \frac{dr}{\sqrt{h(r)}} \otimes \underbrace{\frac{dr}{\sqrt{h(r)}}}_{\theta^1} + (r\, d\varphi) \otimes \underbrace{(r\, d\varphi)}_{\theta^2}. \tag{9}$$

By construction, in the nonholonomic frame $(\theta^0, \theta^1, \theta^2)$ local flatness and orthonormality hold, such that $\hat{\boldsymbol{g}} = -\theta^0 \otimes \theta^0 + \theta^1 \otimes \theta^1 + \theta^2 \otimes \theta^2 = \eta_{\mu\nu}\, \theta^\mu \otimes \theta^\nu$, where $\eta_{\mu\nu}$

[1] A metric for a spacetime manifold is *asymptotically flat* if at sufficiently large distances from some region (here the origin and with increasing distances $r \to \infty$) its geometry becomes indistinguishable from flat (Minkowski) spacetime.

locally agrees with the components of the Minkowski metric. Within this formalism, Cartan's structure equations will enable us to compute the corresponding curvature forms ω and Ω in a straightforward manner. As we shall see, these explicit results are necessary for the application of additional constraints on the spacetime structure at hand, which will eventually give shape to the acoustic model.

Cartan's first structure equation expresses that the pseudo-Riemannian geometry, in this case the (2+1)D Schwarzschild plane, is torsion-free so that the exterior covariant derivative of the dual base vanishes, *i.e.* $D\theta^\mu = d\theta^\mu + \omega^\mu{}_\nu \wedge \theta^\nu = 0$. Then it is not difficult to use Eq. (9) in order to find the curvature 1-forms within this frame:

$$\omega^\mu{}_\nu = \begin{pmatrix} 0 & \frac{h'}{2\sqrt{h}}\,\theta^0 & 0 \\ \frac{h'}{2\sqrt{h}}\,\theta^0 & 0 & -\frac{\sqrt{h}}{r}\,\theta^2 \\ 0 & \frac{\sqrt{h}}{r}\,\theta^2 & 0 \end{pmatrix}. \tag{10}$$

Similarly, by applying once more the exterior covariant derivative to Eq. (10), Cartan's second structure equation produces the curvature 2-forms

$$\Omega^\mu{}_\nu = D\omega^\mu{}_\nu = -\frac{1}{2r}\begin{pmatrix} 0 & rh''\,\theta^0\wedge\theta^1 & h'\,\theta^0\wedge\theta^2 \\ rh''\,\theta^0\wedge\theta^1 & 0 & h'\,\theta^1\wedge\theta^2 \\ h'\,\theta^0\wedge\theta^2 & -h'\,\theta^1\wedge\theta^2 & 0 \end{pmatrix}, \tag{11}$$

and one can observe that there are only three non-vanishing and independent results: $\Omega^0{}_1 = (-h''/2)\,\theta^0\wedge\theta^1$, $\Omega^0{}_2 = (-h'/2r)\,\theta^0\wedge\theta^2$, and $\Omega^1{}_2 = (-h'/2r)\,\theta^1\wedge\theta^2$. In the following step, from the definition $\Omega^\mu{}_\nu = \frac{1}{2}\hat{R}^\mu{}_{\nu\rho\sigma}\,\theta^\rho\wedge\theta^\sigma$, all components of the Riemann tensor in the Cartan frame can easily be identified. Note that the circumflex notation in $\hat{R}^\mu{}_{\nu\rho\sigma}$, and later on for other tensors, indicates that we are working in the Cartan frame with nonholonomic basis 1-forms θ^μ, $\mu = 0, 1, 2$. Carrying out the usual contraction readily yields the Ricci tensor

$$\hat{\boldsymbol{R}} = \frac{1}{2}\left(h'' + \frac{h'}{r}\right)\theta^0\otimes\theta^0 - \frac{1}{2}\left(h'' + \frac{h'}{r}\right)\theta^1\otimes\theta^1 - \frac{h'}{r}\theta^2\otimes\theta^2 \tag{12}$$

and the corresponding curvature scalar is

$$R = \hat{R} = \eta^{\mu\nu}\hat{R}_{\mu\nu} = -h'' - \frac{2h'}{r}. \tag{13}$$

Employing Eqs. (12) and (13) gives the following Einstein tensor

$$\hat{\boldsymbol{G}} = \hat{\boldsymbol{R}} - \tfrac{1}{2}R\,\hat{\boldsymbol{g}} = -\frac{h'}{2r}\theta^0\otimes\theta^0 + \frac{h'}{2r}\theta^1\otimes\theta^1 + \frac{1}{2}h''\theta^2\otimes\theta^2. \tag{14}$$

Due to the vacuum condition, $\boldsymbol{G}$ is zero and Eq. (16) has to vanish in all frames. Hence, there exists only one trivial solution exactly then when $h(r) = C_1$ is a constant, and the corresponding metric is Minkowski flat (so that $\boldsymbol{R} = \boldsymbol{0}$ and $R = 0$). After selecting the appropriate time scale $h \equiv 1$, the metric agrees with the FLRW metric. Consequently, classical black holes are forbidden to exist on the conventional 2D plane, and it would not make much sense either to try implementing its acoustic analogue.

A necessary requirement for Schwarzschild-type spacetimes is that the Ricci curvature scalar R must be constant. Upon integrating Eq. (13), we obtain

$$h(r) = C_1 + \frac{C_2}{r} - \frac{R}{6} r^2, \tag{15}$$

and we note again that $C_2 = 0$ and $R = 0$ (Ricci flatness) is the only possible solution. However, the metric of Eq. (9) with condition Eq. (15) represents the Kottler metric [22, 23], which is well-known to be the only spherically symmetric solution of Einstein's vacuum field equations with a cosmological constant Λ:

$$\hat{\boldsymbol{G}} + \Lambda \hat{\boldsymbol{g}} = \boldsymbol{0}. \tag{16}$$

In three spacetime dimensions one has $\Lambda = R/6$. For $\Lambda \neq 0$, we still maintain the previous choice $C_1 = 1$ due to asymptotic flatness. Since $\hat{G}_{00} = -\hat{G}_{11}$ in Eq. (16), we additionally have to require for consistency that $C_2 = 0$. Altogether this gives for Eq. (15):

$$h(r) = 1 - \Lambda r^2, \quad \Lambda \neq 0. \tag{17}$$

Choosing the natural length scale $\ell > 0$, we may identify

$$\Lambda = \pm \frac{1}{\ell^2}, \tag{18}$$

so that $R = \pm 6/\ell^2$, indicating that both cases are not Ricci-flat. For the positive sign this gives *de Sitter spacetime* (dS_{2+1}) with positive scalar curvature. Similarly, for a negative sign it gives *anti-de Sitter spacetime* (AdS_{2+1}) with negative scalar curvature (see *e.g.* [23, p. 66]). This leaves us with two possible spacetime candidates in (2+1)D for implementing and simulating the acoustic analogues of black holes.

4. Acoustic wave simulation of 2D black holes and properties

A description of wave propagation for acoustic black holes in dS_{2+1} and AdS_{2+1} spacetime is obtained by applying the variational principle, Eq. (1), with the

Lagrangian function, Eq. (2), where the underlying metric $\boldsymbol{g}$ explicitly contains $h(r) = 1 \pm r^2/\ell^2$ [*viz.* Eqs. (17)–(18)], so that

$$\boldsymbol{g} = -\left(1 \pm \frac{r^2}{\ell^2}\right) c^2 dt \otimes dt + \left(1 \pm \frac{r^2}{\ell^2}\right)^{-1} dr \otimes dr + r^2 d\varphi \otimes d\varphi. \tag{19}$$

Note that the AdS_{2+1} metric, choosing the positive sign in Eq. (19), was obtained by Bañados *et al.* from the (2+1)D Einstein-Hilbert action containing a cosmological constant and negative mass [24]. Our approach, however, is based on feasible solutions of the Einstein field equations, Eq. (16), for a metric of (2+1)D-Schwarzschild type, *viz.* Eq. (9).

The wave equation corresponding to Eq. (19)) is the Euler-Lagrange equation and involves the Laplace-Beltrami operator for manifold M (see Refs. [15, 16]),

$$\Delta_M \phi = \frac{1}{\sqrt{-g}} \Big(\sqrt{-g}\, g^{\mu\nu} \phi_{,\mu} \Big)_{,\nu} \phi = 0. \tag{20}$$

Here, M either has the dS_{2+1} or the AdS_{2+1} spacetime structure, where the metric components $g_{\mu\nu}$ correspond to Eq. (19) with appropriate signs, and it is $g = \det \boldsymbol{g} = -r^2$ for both cases. Eq. (20) is a self-adjoint partial differential equation for coordinates $x^0 = ct$, $x^1 = r$, and $x^2 = \varphi$. To solve it, the standard procedure is to use the separation of variables method and assume

$$\phi(t, r, \varphi) = \phi_0(t)\, \phi_1(r)\, \phi_2(\varphi). \tag{21}$$

The time dependence of the potential will display a simple harmonic behaviour, and we let $\phi_0(n, t) = A_n \cos(nct) + B_n \sin(nct)$ with $n \in \mathbb{N}_0$. Moreover, to probe the prominent features of these spacetimes, it will be reasonable to focus for this study on concentric wave propagation, *i.e.* $\phi_2(\varphi) \equiv 1$.

Therefore, in this model only the non-trivial radial dependence $\phi_1(r)$ remains to be examined and completely determines the nontrivial behaviour of the wave propagation. A detailed but straightforward calculation yields the following differential equations for the AdS_{2+1} potential $\phi_1^-(r)$, and the dS_{2+1} potential $\phi_1^+(r)$:

$$\mathrm{AdS}_{2+1}: \quad r\left(1 + \frac{r^2}{\ell^2}\right)^2 \phi_1^{-\prime\prime} + \left(1 + \frac{r^2}{\ell^2}\right)\left(1 + 3\frac{r^2}{\ell^2}\right)\phi_1^{-\prime} + n^2 r \phi_1^- = 0, \tag{22}$$

$$\mathrm{dS}_{2+1}: \quad r\left(1 - \frac{r^2}{\ell^2}\right)^2 \phi_1^{+\prime\prime} + \left(1 - \frac{r^2}{\ell^2}\right)\left(1 - 3\frac{r^2}{\ell^2}\right)\phi_1^{+\prime} + n^2 r \phi_1^+ = 0. \tag{23}$$

Apparently, in the limit $r \to \infty$ the asymptotic behaviour of the AdS_{2+1} and dS_{2+1} solution is similar, and in both cases one has to examine

$$r^4 \phi_1'' + 3r^3 \phi_1' + n^2 \ell^4 \phi_1 = 0. \tag{24}$$

Using in Eq. (24) the substitutions $r = 1/\xi$ and $\phi_1(r) = \phi_1(1/\xi) = \xi f(\xi)$, we arrive at

$$\xi^2 f'' + \xi f' + \left(n^2 \ell^4 \xi^2 - 1\right) f = 0, \tag{25}$$

which is a differential equation reducible to Bessel's equation [25, pp. 108]. Thus, for $r \to \infty$ the general asymptotic solution is

$$\phi_1(\ell, n, r) \sim C_1 \frac{J_1\left(n\ell^2/r\right)}{r} + C_2 \frac{Y_1\left(n\ell^2/r\right)}{r}, \tag{26}$$

with integration constants C_1, C_2, and the Bessel functions J_1, Y_1 of first order of the first and second kinds, respectively. Applying the identities of Ref. [25, pp. 131–132], we have in the $r \to \infty$ regime

$$\begin{aligned} \frac{J_1\left(n\ell^2/r\right)}{r} &\sim \frac{n\ell^2}{2r^2} \to 0, \\ \frac{Y_1\left(n\ell^2/r\right)}{r} &\sim -\frac{2}{\pi n \ell^2}. \end{aligned} \tag{27}$$

Obviously, integration constant C_2 can be chosen to normalize $Y_1\left(n\ell^2/r\right)/r \to 1$.

For obtaining numerical results and approximations of Eqs. (22) and (23), the full concentric wave solution can be expressed in the general form

$$\phi^{\pm}(t, r) = \sum_{n=0}^{\infty} \left[A_n \cos(nct) + B_n \sin(nct)\right] \phi_1^{\pm}(\ell, n, r). \tag{28}$$

In this expansion, the $n = 1$ mode is of particular interest for probing AdS_{2+1} and dS_{2+1} spacetime. Besides, we can set the scale parameter $\ell = 1$ without loss of generality. Figures 1 and 2 provide illustrations for the radial behaviour of these prototype waves in AdS_{2+1} and dS_{2+1} spacetime, respectively. The numerical estimates were carried out with Mathematica and the `sbvp` solver Matlab [26, 27].

The code `sbvp` was originally designed to solve singular BVPs in ODEs. Since the code is implemented for BVPs in the general form [26]

$$\boldsymbol{y}'(t) = f(t, \boldsymbol{y}(t)) \text{ for } t \in]a, b[, \quad \boldsymbol{g}(\boldsymbol{y}(a), \boldsymbol{y}(b)) = \boldsymbol{0},$$

it can also be applied to solve problems without singular points. The development of a reliable code taking into account the specific difficulties caused by the singularity is strongly motivated by applications from physics, chemistry, and mechanics (buckling of spherical shells), as well as research activities in related areas.

We chose collocation as the basic method for the numerical solution of the singular ODE, because of its advantageous convergence properties for this class of problems—the global error of a collocation solution does not suffer from an order reduction, provided that the data of the analytical problem and its solution are sufficiently smooth. Such order reductions are typically observed for many other direct discretization methods or the well-known acceleration techniques [26]. Collocation also proved to be a reliable and robust solver for regular ODEs. To enhance the code efficiency grid adaptation strategy based on the equidistribution of the *global error estimate* has been implemented. We estimate the global error using the defect correction principle. The reason for choosing to control the global error, instead of monitoring the local error is the non-smoothness of the latter near the singular point and the order reductions it suffers from. For an extensive discussion of this phenomenon and numerical experiments see [28]. Moreover, global error is the quantity users wish to be controlled, so it makes sense to adapt the grids in such a way that this error satisfies the prescribed tolerance.

Figure 1 clearly demonstrates the existence of an effective potential well—an outstanding feature of AdS_{2+1} geometries already known from cosmological and string-theory models [29, 30]. On the other hand, as Figure 2 shows, for dS_{2+1} spacetime a typical event horizon emerges as r approaches the fixed scale $\ell > 0$, and therefore represents an acoustic black hole.[2] Typical frequency shifts will occur as an acoustic wave enters or departs the event horizon. Because the wave appears to slow down when approaching the horizon, a redshift is observed. In other words, waves near the outer region of the event horizon travelling away, and observed from at a greater distance, suffer redshift due to its motion up the (acoustic) potential. Likewise a blueshift occurs for the opposite direction of motion. To illustrate this effect, Figure 3 uses the numerical estimates of the acoustic potential in Figure 2 and shows the changes of the pressure-wave frequency for

[2]Recall that a *black hole* is a spacetime region exhibiting an *event horizon*. This horizon is a boundary in spacetime separating regions which are causally disconnected (communication via signals is impossible at least in one direction). The geometry of genuine black holes possesses spatial rotational symmetry and has to be a solution of the Einstein field equations, *viz.* Eq. (16). Thus, an *acoustic black hole* is an object endowed with such an event horizon for sound waves. It is important to observe that black holes may or may not contain singularities in their interior.

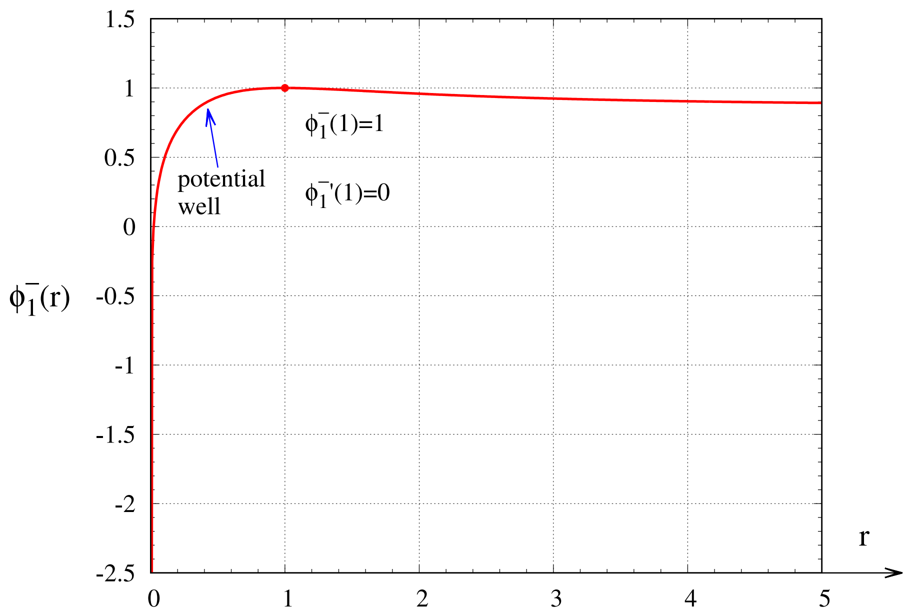

Figure 1: Simulation of acoustic AdS_{2+1} spacetime with prototype waves ($n = 1$). Numerical estimates for the nontrivial radial dependence $\phi_1^-(r)$ obeying Eq. (22), with scale $\ell = 1$ and boundary conditions $\phi_1^-(1) = 1$, $\phi_1^{-\prime}(1) = 0$. A potential well is observed while any event horizons are absent.

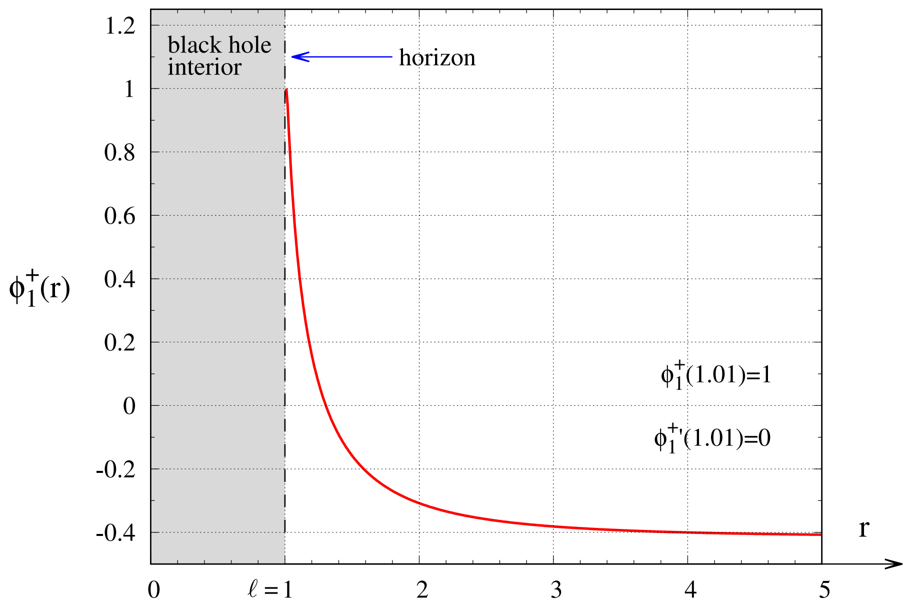

Figure 2: Simulation of acoustic dS_{2+1} black hole with prototype waves ($n = 1$). Numerical estimates for the nontrivial radial dependence $\phi_1^+(r)$ obeying Eq. (23), with scale $\ell = 1$ and boundary conditions $\phi_1^+(1.01) = 1$, $\phi_1^{+\prime}(1.01) = 0$. A typical event horizons emerges as r approaches $\ell = 1$.

the prototype wave ($n = 1$).

Apart from horizons, other remarkable properties of the simulated spacetime geometries can be identified by employing standard techniques of the differential-geometric toolbox (see *e.g.* Refs. [21, 31]).

To determine whether the AdS_{2+1} and dS_{2+1} spacetimes are conformally flat, we have to confirm that the Cotton tensor $\boldsymbol{C}$ vanishes. In local coordinates, and for any dimension $n \geq 3$, the components of the Cotton tensor generally can be expressed in terms of the components of the Schouten tensor, $S_{\mu\nu}$, by

$$C_{\lambda\mu\nu} = (n-2)\left[S_{\lambda\mu;\nu} - S_{\lambda\nu;\mu}\right]. \tag{29}$$

Note also that the Weyl-Schouten theorem states that a manifold with dimension $n \geq 3$ is conformally flat if and only if the Schouten tensor is Codazzi, *i.e.* $S_{\lambda\mu;\nu} - S_{\lambda\nu;\mu} = 0$, see *e.g.* [32, p. 63] and [33]. So Eq. (29) shows that this statement is identical with $\boldsymbol{C} = \boldsymbol{0}$.

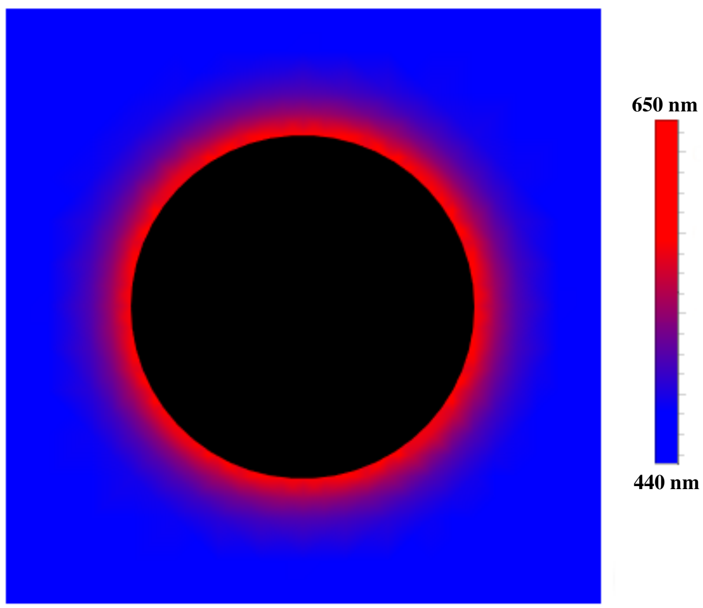

Figure 3: Graphical representation of the acoustic frequency redshift for the dS_{2+1} black hole with the acoustic potential of Figure 2. The color profile of the gradient corresponds to a wavelength range from 440 nm (in the asymptotically flat domain) to 650 nm (close to the event horizon).

To evaluate the Schouten tensor defined by

$$\boldsymbol{S} = \frac{1}{n-2}\left[\boldsymbol{R} - \frac{R}{2(n-1)}\boldsymbol{g}\right]$$

in the $(\theta^0, \theta^1, \theta^2)$ coframe with $n = 3$, we use the results from Eq. (12) and substitute our particular metric, Eq. (9). After some computation this yields

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{R}} - \tfrac{1}{4}R\,\hat{\boldsymbol{g}} = \frac{1}{4}\left[h''\theta^0\otimes\theta^0 - h''\theta^1\otimes\theta^1 + \left(h'' - \frac{2h'}{r}\right)\theta^2\otimes\theta^2\right] = \frac{1}{2\ell^2}\hat{\boldsymbol{g}}, \quad (30)$$

where in the last step $h(r) = 1 \pm r^2/\ell^2$ was necessary for simplification. If the Schouten tensor is a constant multiple of the metric, then it is called *trivial*. Since the metric is covariantly constant, Eq. (29) immediately implies that $\boldsymbol{C} = \boldsymbol{0}$. So all spacetimes with trivial Schouten tensors are necessarily conformally flat, which includes both spacetimes of Eq. (19).

For a closer examination of spacetime singularities, we consider the Kretschmann scalar K, which is is the quadratic invariant formed by the Riemann tensor. If the spacetime geometry contains any curvature singularities, the Kretschmann invariant is at least for one spacetime point infinite. After carrying out a Ricci decomposition [31] and assuming 3D conformally flat geometries, we obtain in the coframe:

$$K = 4\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} - R^2. \quad (31)$$

Then, with the help of Eqs. (12)–(13) and $h(r) = 1 \pm r^2/\ell^2$, it is straightforward to show that

$$K = h''^2 + \frac{2h'^2}{r^2} = \frac{12}{\ell^4} > 0 \quad (32)$$

holds for both geometries, AdS_{2+1} and dS_{2+1} spacetimes. Therefore both spacetime geometries are singularity free, as opposed to the conventional 4D Schwarzschild case with a gravitational singularity at the centre of the black hole. For the AdS_{2+1} case, the absence of both, event horizon and singularity, was already mentioned in Ref. [24].

Table 1 provides an overview of the crucial geometric and physical properties of the AdS_{2+1} and dS_{2+1} spacetime geometries in comparison with the conventional black hole described by the (3+1)D Schwarzschild geometry.

After the in-depth discussion of the most significant geometric and physical features of black-hole acoustics on the plane, we now have to come to the laboratory setting which enables the engineer to contrive metadevices with such select

black-hole type	singularity $r \to 0$	asymptotic limit $r \to \infty$	event horizon	Ricci flatness	conformal flatness
conventional	yes	flat	yes	yes	no
AdS_{2+1}	no	simple	no	no	yes
dS_{2+1}	no	simple	yes	no	yes

Table 1: Summary of the essential geometric properties for admissible Schwarzschild-type (2+1)D spacetimes with metric Eq. (9) and $h(r) = 1 \pm r^2/\ell^2$, compared to the conventional (3+1)D Schwarzschild geometry. With *simple* asymptotic behaviour we refer to the elementary formulas Eqs. (26) and (27).

properties. The analogous acoustic space will be implemented by a suitable choice of the physical parameters ϱ and κ. For this purpose, we only require knowledge of the components of metric $\boldsymbol{g}$, which are immediately read off from Eq. (9) with Eqs. (17)–(18), and also the constitutive equations given by Eqs. (4). Then, we obtain the following simple prescription in polar coordinate basis for the acoustic analogue of Schwarzschild-de Sitter black holes and Schwarzschild-AdS spacetime:

$$\kappa = \frac{\bar{\kappa}}{\sqrt{1 \pm r^2/\ell^2}}, \qquad \rho_0 \rho^{ij} = \sqrt{1 \pm r^2/\ell^2} \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}. \tag{33}$$

Again, from these physical quantities it becomes clear that only the expressions with a negative sign (dS_{2+1}) will give rise to a spacetime geometry with an event horizon.

5. Conclusions

Our closer examination of (2+1)D Schwarzschild black holes naturally led us to spacetimes with a non-vanishing cosmological constant, since conventional Schwarzschild solutions on the 2D plane are forbidden. The dS_{2+1} or Schwarzschild-deSitter black hole is not only physically but also mathematically very intriguing, because there is no comparable solution with a noncompact event horizon in asymptotic Minkowski space. On the other hand, AdS_{2+1} solutions have no event horizons. However, we have seen that their radial potential takes the form of a potential well.

In order to derive the corresponding wave equation for the acoustic potential, we have employed a covariant variational principle, and in principle we have arrived at a unified description of acoustic wave phenomena on the (2+1)D Schwarzschild plane.

For the numerical simulation we have probed these spacetime geometries with concentric waves which possess a harmonic time dependence so that all of the non-trivial propagation behaviour occurs in the radial direction. Standard techniques yield useful approximate solutions for this r-dependence in terms of the Bessel functions. Additionally, our numerical predictions of the model are complemented by a comprehensive analysis of the geometric features of these spacetimes in order to exhibit their similarities and differences compared to the conventional black hole within 4-dimensional spacetime.

In summary, we conclude that the acoustic analogues of both, dS_{2+1} and AdS_{2+1} spacetime, are particularly interesting and might serve as natural candidates for a laboratory implementation with the aim to explore important phenomena such as acoustic Hawking radiation.

It is our hope that the variational spacetime approach to transformation acoustics and the corresponding constitutive relations supply a powerful tool for the study and design of acoustic metadevices and may help to open up new research pathways in this field.

Acknowledgements

M. M. T. acknowledges financial support by the Spanish *Ministerio de Economía y Competitividad*, the European Regional Development Fund under grant TIN2014-59294-P, and the *Generalitat Valenciana* (BEST2017). He also wishes to thank for the cordial reception and hospitality at the *Institute for Analysis and Scientific Computing* of the Vienna University of Technology where part of the present work was established.

References

[1] R. A. Shelby, D. R. Smith, S. Schultz, Experimental verification of a negative index of refraction, Science 292 (2001) 77–97.

[2] V. G. Veselago, The electrodynamics of substances with simultaneously negative values of ϵ and μ, Sov. Phys. Uspekhi 10 (4) (1968) 509–514.

[3] S. A. Cummer, Transformation acoustics, in: V. R. Craster, S. Guenneau (Eds.), Acoustic Metamaterials: Negative Refraction, Imaging, Lensing and Cloaking, Springer Netherlands, Dordrecht, 2013, pp. 197–218.

[4] S. Cummer, A sound future for acoustic metamaterials, J. Acoust. Soc. Am. 141 (5) (2017) 3451.

[5] M. Haberman, M. Guild, Acoustic metamaterials, Physics Today 69 (6) (2016) 42–48.

[6] G. Ma, P. Sheng, Acoustic metamaterials: From local resonances to broad horizons, Sci. Adv. 2 (2016) e1501595, 16 pp.

[7] S. L. Zhai, X. P. Zhao, S. Liu, F. L. Shen, L. L. Li, C. R. Luo, Inverse doppler effects in broadband acoustic metamaterials, Sci. Rep. 6 (2016) 32388.

[8] B. Horstmann, B. Reznik, S. Fagnocchi, J. I. Cirac, Hawking radiation from an acoustic black hole on an ion ring, Phys. Rev. Lett. 104 (2010) 250403.

[9] K. Schwarzschild, On the gravitational field of a mass point according to Einstein's theory, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) (1916) 189–196.

[10] R. Q. Li, X. F. Zhu, B. Liang, Y. Li, X. Y. Zou, J. C. Cheng, A broadband acoustic omnidirectional absorber comprising positive-index materials, Appl. Phys. Lett. 99 (19) (2011) 193507.

[11] A. Climente, D. Torrent, J. Sánchez-Dehesa, Noise reduction by perfect absorbers based on acoustic metamaterials, in: Volume 8: Mechanics of Solids, Structures and Fluids; Vibration, Acoustics and Wave Propagation, American Society of Mechanical Engineers (ASME), Denver, CO, 2011, pp. 947–952.

[12] L. Zigoneanu, Design and experimental applications of acoustic metamaterials, Ph.D. thesis, Department of Electrical and Computer Engineering, Duke University, Durham, NC (2013).

[13] S. Hossenfelder, A relativistic acoustic metric for planar black holes, Phys. Lett. B 752 (2016) 13–17.

[14] M. M. Tung, J. Peinado, A covariant spacetime approach to transformation acoustics, in: M. Fontes, M. Günther, N. Marheineke (Eds.), Progress in Industrial Mathematics at ECMI 2012, Vol. 19 of Mathematics in Industry, Springer, Berlin, 2014, pp. 335–340.

[15] M. M. Tung, A fundamental Lagrangian approach to transformation acoustics and spherical spacetime cloaking, Europhys. Lett. 98 (2012) 34002–34006.

[16] M. M. Tung, E. B. Weinmüller, Gravitational frequency shifts in transformation acoustics, Europhys. Lett. 101 (2013) 54006–54011.

[17] M. M. Tung, Modelling acoustics on the Poincaré half-plane, J. Comput. Appl. Math., in press (2017). doi:10.1016/j.cam.2017.10.037.

[18] P. J. Olver, Applications of Lie Groups to Differential Equations, no. 107 in Graduate Texts in Mathematics, Springer-Verlag, New York, NY, 1998.

[19] C. Lanczos, The Variational Principles of Mechanics, Dover Publications, New York, NY, 1970.

[20] F. P. Mechel, Formulas of Acoustics, Springer-Verlag, Berlin, 2002.

[21] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, no. 103 in Pure and Applied Mathematics, Academic Press, San Diego, CA, 1983.

[22] F. Kottler, Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie, Ann. Phys. 4 (56) (1918) 401–462.

[23] V. Perlick, Gravitational lensing from a spacetime perspective, Living Rev. Rel. 7 (9) (2014) 401–462.

[24] M. Bañados, C. Teitelboim, J. Zanelli, Black hole in three-dimensional spacetime, Phys. Rev. Lett. 69 (13) (1992) 1849–1851.

[25] W. W. Bell, Special Functions for Scientists and Engineers, Dover Publications, Mineola, NY, 2004.

[26] W. Auzinger, G. Kneisl, O. Koch, E. Weinmüller, A collocation code for singular boundary value problems in ordinary differential equations, Numer. Algorithms 33 (2003) 27–39.

[27] G. Kitzhofer, O. Koch, G. Pulverer, C. Simon, E. B. Weinmüller, The new MATLAB code `bvpsuite` for the solution of singular implicit BVPs, JNA-IAM J. Numer. Anal. Ind. Appl. Math. 5 (2010) 113–134.

[28] W. Auzinger, P. Kofler, E. Weinmüller, Steuerungsmaßnahmen bei der numerischen Lösung singulärer Anfangswertaufgaben, Tech. Rep. 124/98, Vienna Univ. of Technology (1998).

[29] O. J. C. Dias, Pair creation of anti-de Sitter black holes on a cosmic string background, Phys. Rev. D 70 (2004) 024007.

[30] V. Balasubramanian, P. Hořava, D. Minic, Deconstructing de Sitter, J. High Energ. Phys. 2001 (5) (2001) 043.

[31] C. Cherubini, D. Bini, S. Capozziello, R. Ruffini, Second order scalar invariants of the Riemann tensor: Applications to black hole spacetimes, Int. J. Mod. Phys. D 11 (2002) 827–841.

[32] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević, R. Vázquez-Lorenzo, The Geometry of Walker Manifolds, no. 5 in Synthesis Lectures on Mathematics and Statistics, Morgan & Claypool, Williston, VT, 2009.

[33] G. Catino, C. Mantegazza, L. Mazzieri, Locally conformally flat ancient Ricci flows, Anal. PDE 8 (2) (2015) 365–371.