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A Higher Order Chebyshev-Halley-Type Family of Iterative Methods for Multiple Roots

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Abstract: The aim of this paper is to introduce new high order iterative methods for multiple roots of the nonlinear scalar equation; this is a demanding task in the area of computational mathematics and numerical analysis. Specifically, we present a new Chebyshev–Halley-type iteration function having at least sixth-order convergence and eighth-order convergence for a particular value in the case of multiple roots. With regard to computational cost, each member of our scheme needs four functional evaluations each step. Therefore, the maximum efficiency index of our scheme is 1.6818 for $\alpha = 2$, which corresponds to an optimal method in the sense of Kung and Traub’s conjecture. We obtain the theoretical convergence order by using Taylor developments. Finally, we consider some real-life situations for establishing some numerical experiments to corroborate the theoretical results.

Keywords: nonlinear equations; multiple roots; Chebyshev–Halley-type; optimal iterative methods; efficiency index

1. Introduction

One important field in the area of computational methods and numerical analysis is to find approximations to the solutions of nonlinear equations of the form:

$$f(x) = 0, \quad (1)$$

where $f : \mathbb{D} \subset \mathbb{C} \rightarrow \mathbb{C}$ is the analytic function in the enclosed region \mathbb{D} , enclosing the required solution. It is almost impossible to obtain the exact solution in an analytic way for such problems. Therefore, we concentrate on obtaining approximations of the solution up to any specific degree of accuracy by means of an iterative procedure, of course doing it also with the maximum efficiency. In [1], Kung and Traub conjectured that a method without memory that uses $n + 1$ functional evaluations per iteration can have at most convergence order $p = 2^n$. If this bound is reached, the method is said to be optimal.

For solving nonlinear Equation (1) by means of iterations, we have the well-known cubically-convergent family of Chebyshev–Halley methods [2], which is given by:

$$x_{n+1} = x_n - \left[1 + \frac{1}{2} \frac{L_f(x_n)}{1 - \alpha L_f(x_n)} \right] \frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}, \quad (2)$$

where $L_f(x_n) = \frac{f''(x_n)f(x_n)}{\{f'(x_n)\}^2}$. A great variety of iterative methods can be reported in particular cases. For example, the classical Chebyshev’s method [1,3], Halley’s method [1,3], and the super-Halley method [1,3] can be obtained if $\alpha = 0$, $\alpha = \frac{1}{2}$, and $\alpha = 1$, respectively. Despite the third-order convergence, the scheme (2) is considered less practical from a computational point of view because of the computation of the second-order derivative.

For this reason, several variants of Chebyshev–Halley’s methods free from the second-order derivative have been presented in [4–7]. It has been shown that these methods are comparable to the classical third-order methods of the Chebyshev–Halley-type in their performance and can also compete with Newton’s method. One family of these methods is given as follows:

$$\begin{aligned}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= x_n - \left(1 + \frac{f(y_n)}{f(x_n) - \alpha f(y_n)}\right) \frac{f(x_n)}{f'(x_n)}, \quad \alpha \in \mathbb{R}
 \end{aligned}
 \tag{3}$$

We can easily obtain some well-known third-order methods proposed by Potra and Pták [4] and Sharma [5] (the Newton-secant method (NSM)) for $\alpha = 0$ and $\alpha = 1$. In addition, we have Ostrowski’s method [8] having optimal fourth-order convergence, which is also a special case for $\alpha = 2$. This family is important and interesting not only because of not using a second- or higher order derivative. However, this scheme also converges at least cubically and has better results in comparison to the existing ones. Moreover, we have several higher order modifications of the Chebyshev–Halley methods available in the literature, and some of them can be seen in [9–12].

In this study, we focus on the case of the multiple roots of nonlinear equations. We have some fourth-order optimal and non-optimal modifications or improvements of Newton’s iteration function for multiple roots in the research articles [13–17]. Furthermore, we can find some higher order methods for this case, but some of them do not reach maximum efficiency [18–23]; so, this topic is of interest in the current literature.

We propose a new Chebyshev–Halley-type iteration function for multiple roots, which reaches a high order of convergence. Specifically, we get a family of iterative methods with a free parameter α , with sixth-order convergence. Therefore, the efficiency index is $6^{1/4}$, and for $\alpha = 2$, this index is $8^{1/4}$, which is the maximum value that one can get with four functional evaluations, reaching optimality in the sense of Kung and Traub’s conjecture. Additionally, an extensive analysis of the convergence order is presented in the main theorem.

We recall that $\zeta \in \mathbb{C}$ is a multiple root of the equation $f(x) = 0$, if it is verified that:

$$f(\zeta) = 0, f'(\zeta) = 0, \dots, f^{(m-1)}(\zeta) = 0 \quad \text{and} \quad f^{(m)}(\zeta) \neq 0,$$

the positive integer ($m \geq 1$) being the multiplicity of the root.

We deal with iterative methods in which the multiplicity must be known in advance, because this value, m , is used in the iterative expression. However, we point out that these methods also work when one uses an estimation of the multiplicity, as was proposed in the classical study carried out in [24].

Finally, we consider some real-life situations that start from some given conditions to investigate and some standard academic test problems for numerical experiments. Our iteration functions here are found to be more comparable and effective than the existing methods for multiple roots in terms of residual errors and errors among two consecutive iterations, and also, we obtain a more stable computational order of convergence. That is, the proposed methods are competitive.

2. Construction of the Higher Order Scheme

In this section, we present the new Chebyshev–Halley-type methods for multiple roots of nonlinear equations, for the first time. In order to construct the new scheme, we consider the following scheme:

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= x_n - m \left(1 + \frac{\eta}{1 - \alpha\eta} \right) \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= z_n - H(\eta, \tau) \frac{f(x_n)}{f'(x_n)},
 \end{aligned}
 \tag{4}$$

where the function:

$$H(\eta, \tau) = \frac{\eta\tau (\beta - (\alpha - 2)^2\eta^2(\eta + 1) + \tau^3 + \tau^2)}{(\eta + 1)(\tau + 1)}$$

with:

$$\begin{aligned}
 \eta &= \left(\frac{f(y_n)}{f(x_n)} \right)^{\frac{1}{m}}, \\
 \tau &= \left(\frac{f(z_n)}{f(y_n)} \right)^{\frac{1}{m}}, \\
 \beta &= m ((\alpha(\alpha + 2) + 9)\eta^3 + \eta^2(\alpha(\alpha + 3) - 6\tau - 3) + \eta(\alpha + 8\tau + 1) + 2\tau + 1),
 \end{aligned}$$

where $\alpha \in \mathbb{R}$ is a free disposable variable. For $m = 1$, we can easily obtain the scheme (3) from the first two steps of the scheme (4).

In Theorem 1, we illustrate that the constructed scheme attains at least sixth-order convergence and for $\alpha = 2$, it goes to eighth-order without using any extra functional evaluation. It is interesting to observe that $H(\eta, \tau)$ plays a significant role in the construction of the presented scheme (for details, please see Theorem 1).

Theorem 1. *Let us consider $x = \zeta$ to be a multiple zero with multiplicity $m \geq 1$ of an analytic function $f : \mathbb{C} \rightarrow \mathbb{C}$ in the region containing the multiple zero ζ of $f(x)$. Then, the present scheme (4) attains at least sixth-order convergence for each α , but for a particular value of $\alpha = 2$, it reaches the optimal eighth-order convergence.*

Proof. We expand the functions $f(x_n)$ and $f'(x_n)$ about $x = \zeta$ with the help of a Taylor’s series expansion, which leads us to:

$$f(x_n) = \frac{f^{(m)}(\zeta)}{m!} e_n^m \left(1 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9) \right), \tag{5}$$

and:

$$\begin{aligned}
 f'(x_n) &= \frac{f^m(\zeta)}{m!} e_n^{m-1} \left(m + (m + 1)c_1 e_n + (m + 2)c_2 e_n^2 + (m + 3)c_3 e_n^3 + (m + 4)c_4 e_n^4 + (m + 5)c_5 e_n^5 \right. \\
 &\quad \left. + (m + 6)c_6 e_n^6 + (m + 7)c_7 e_n^7 + (m + 8)c_8 e_n^8 + O(e_n^9) \right),
 \end{aligned}
 \tag{6}$$

respectively, where $c_k = \frac{m!}{(m-1+k)!} \frac{f^{m-1+k}(\zeta)}{f^m(\zeta)}$, $k = 2, 3, 4, \dots, 8$ and $e_n = x_n - \zeta$ is the error in the n^{th} iteration.

Inserting the above expressions (5) and (6) into the first substep of scheme (4) yields:

$$y_n - \zeta = \frac{c_1}{m} e_n^2 + \sum_{i=0}^5 \phi_i e_n^{i+3} + O(e_n^9), \tag{7}$$

where $\phi_i = \phi_i(m, c_1, c_2, \dots, c_8)$ are given in terms of m, c_2, c_3, \dots, c_8 , for example $\phi_0 = \frac{1}{m^2}(2mc_2 - (m + 1)c_1^2)$ and $\phi_1 = \frac{1}{m^3}[3m^2c_3 + (m + 1)^2c_1^3 - m(3m + 4)c_1c_2]$, etc.

Using the Taylor series expansion and the expression (7), we have:

$$f(y_n) = f^{(m)}(\xi)e_n^{2m} \left[\frac{\left(\frac{c_1}{m}\right)^m}{m!} + \frac{(2mc_2 - (m + 1)c_1^2)\left(\frac{c_1}{m}\right)^m e_n}{m!c_1} + \left(\frac{c_1}{m}\right)^{1+m} \frac{1}{2m!c_1^3} \{ (3 + 3m + 3m^2 + m^3)c_1^4 - 2m(2 + 3m + 2m^2)c_1^2c_2 + 4(m - 1)m^2c_2^2 + 6m^2c_1c_3 \} e_n^2 + \sum_{i=0}^5 \bar{\phi}_i e_n^{i+3} + O(e_n^9) \right]. \tag{8}$$

We obtain the following expression by using (5) and (8):

$$\eta = \frac{c_1 e_n}{m} + \frac{2mc_2 - (m + 2)c_1^2}{m^2} e_n^2 + \theta_0 e_n^3 + \theta_1 e_n^4 + \theta_2 e_n^5 + O(e_n^6), \tag{9}$$

where $\theta_0 = \frac{(2m^2+7m+7)c_1^3+6m^2c_3-2m(3m+7)c_1c_2}{2m^3}$, $\theta_1 = -\frac{1}{6m^4} [12m^2(2m + 5)c_1c_3 + 12m^2((m + 3)c_2^2 - 2mc_4) - 6m(4m^2 + 16m + 17)c_1^2c_2 + (6m^3 + 29m^2 + 51m + 34)c_1^4]$ and $\theta_2 = \frac{1}{24m^5} [12m^2(10m^2 + 43m + 49)c_1^2c_3 - 24m^3((5m + 17)c_2c_3 - 5mc_5) + 12m^2((10m^2 + 47m + 53)c_2^2 - 2m(5m + 13)c_4)c_1 - 4m(30m^3 + 163m^2 + 306m + 209)c_1^3c_2 + (24m^4 + 146m^3 + 355m^2 + 418m + 209)c_1^5]$.

With the help of Expressions (5)–(9), we obtain:

$$z_n - \xi = -\frac{(\alpha - 2)c_1^2}{m^2} e_n^3 + \sum_{i=0}^4 \psi_i e_n^{i+4} + O(e_n^9), \tag{10}$$

where $\psi_i = \psi_i(\alpha, m, c_1, c_2, \dots, c_8)$ are given in terms of $\alpha, m, c_2, c_3, \dots, c_8$ with the first two coefficients explicitly written as $\psi_0 = -\frac{1}{2m^3} [2\alpha^2 - 10\alpha + (7 - 4\alpha)m + 11]c_1^3 + 2m(4\alpha - 7)c_1c_2]$ and $\psi_1 = \frac{1}{6m^4} [(-6\alpha^3 + 42\alpha^2 - 96\alpha + (29 - 18\alpha)m^2 + 6(3\alpha^2 - 14\alpha + 14)m + 67)c_1^4 + 12m^2(5 - 3\alpha)c_1c_3 + 12m^2(3 - 2\alpha)c_2^2 + 12m(-3\alpha^2 + 14\alpha + (5\alpha - 8)m - 14)c_1^2c_2]$.

By using the Taylor series expansion and (10), we have:

$$f(z_n) = f^{(m)}(\xi)e_n^{3m} \left[\frac{\left(-\frac{(\alpha-2)c_1^2}{m^2}\right)^m}{m!} + \sum_{i=1}^5 \bar{\psi}_i e_n^i + O(e_n^6) \right]. \tag{11}$$

From Expressions (8) and (11), we further have:

$$\tau = -\frac{(\alpha - 2)c_1}{m} e_n + \frac{((-2\alpha^2 + 8\alpha + (2\alpha - 3)m - 7)c_1^2 + 2m(3 - 2\alpha)c_2)}{2m^2} e_n^2 + \gamma_1 e_n^3 + \gamma_2 e_n^4 + O(e_n^5), \tag{12}$$

where $\gamma_1 = \frac{1}{3m^3} [(-3\alpha^3 + 18\alpha^2 - 30\alpha + (4 - 3\alpha)m^2 + 3(2\alpha^2 - 7\alpha + 5)m + 11)c_1^3 + 3m^2(4 - 3\alpha)c_3 + 3m(-4\alpha^2 + 14\alpha + 3\alpha m - 4m - 10)c_1c_2]$ and $\gamma_2 = \frac{1}{24m^4} [24m^2(-6\alpha^2 + 20\alpha + (4\alpha - 5)m - 14)c_1c_3 + 12m^2((-8\alpha^2 + 24\alpha + 4\alpha m - 5m - 13)c_2^2 + 2m(5 - 4\alpha)c_4) - 12m(12\alpha^3 - 66\alpha^2 + 100\alpha + 2(4\alpha - 5)m^2 + (-20\alpha^2 + 64\alpha - 41)m - 33)c_1^2c_2 + (-24\alpha^4 + 192\alpha^3 - 492\alpha^2 + 392\alpha + 6(4\alpha - 5)m^3 + (-72\alpha^2 + 232\alpha - 151)m^2 + 6(12\alpha^3 - 66\alpha^2 + 100\alpha - 33)m + 19)c_1^4]$.

By using Expressions (9) and (12), we obtain:

$$H(\eta, \tau) = -\frac{(\alpha - 2)c_1^2}{m^2} e_n^2 + \lambda_1 e_n^3 + \lambda_2 e_n^4 + O(e_n^5) \tag{13}$$

where $\lambda_1 = \frac{c_1}{2m^3} \left[c_1^2 (-2\alpha^2 + 8\alpha + (4\alpha - 7)m - 7) + 2(7 - 4\alpha)c_2m \right]$ and $\lambda_2 = \frac{1}{6m^3} \left[c_1^4 (-6\alpha^3 + 36\alpha^2 - 66\alpha + (29 - 18\alpha)m^2 + 3(6\alpha^2 - 22\alpha + 17)m + 34) + 12(5 - 3\alpha)c_3c_1m^2 + 12(3 - 2\alpha)c_2^2m^2 + 6c_2c_1^2m (-6\alpha^2 + 22\alpha + 2(5\alpha - 8)m - 17) \right]$.

Now, we use the expressions (5)–(13) in the last substep of Scheme (4), and we get:

$$e_{n+1} = \sum_{i=1}^3 L_i e_n^{i+5} + O(e_n^9), \tag{14}$$

where $L_1 = \frac{(\alpha-2)^2c_1^3}{m^6} \left[c_1^2 (\alpha^2 - \alpha + m^2 - (\alpha^2 + 4\alpha - 17)m - 3) - 2c_2(m - 1)m \right]$, $L_2 = (\alpha - 2)c_1^2 \left[-12c_2c_1^2m \{ 10\alpha^3 - 24\alpha^2 - 39\alpha + (16\alpha - 27)m^2 - (10\alpha^3 + 27\alpha^2 - 262\alpha + 301)m + 91 \} + 12c_3c_1m^2(-4\alpha + (4\alpha - 7)m + 8) + 12c_2^2m^2(-12\alpha + 4(3\alpha - 5)m + 21) + c_1^4 \{ -24\alpha^4 + 168\alpha^3 - 156\alpha^2 - 662\alpha + (52\alpha - 88)m^3 - (60\alpha^3 + 162\alpha^2 - 1616\alpha + 1885)m^2 + 2(18\alpha^4 - 12\alpha^3 - 711\alpha^2 + 2539\alpha - 2089)m + 979 \} \right]$ and $L_3 = \frac{c_1}{24m^8} \left[-24c_2c_3c_1m^3((42\alpha^2 - 146\alpha + 125)m - 6(7\alpha^2 - 26\alpha + 24)) - 24c_2^3m^3(-24\alpha^2 + 84\alpha + (24\alpha^2 - 80\alpha + 66)m - 73) + 12c_3c_1^3m^2 \{ 2(15\alpha^4 - 63\alpha^3 - 5\alpha^2 + 290\alpha - 296) + (54\alpha^2 - 190\alpha + 165)m^2 + (-30\alpha^4 - 28\alpha^3 + 968\alpha^2 - 2432\alpha + 1697)m \} + 12c_1^2m^2 \{ c_2^2(80\alpha^4 - 304\alpha^3 - 226\alpha^2 + 1920\alpha + 2(81\alpha^2 - 277\alpha + 234)m^2 + (-80\alpha^4 - 112\alpha^3 + 2712\alpha^2 - 6410\alpha + 4209)m - 1787) - 4(\alpha - 2)c_4m(-3\alpha + (3\alpha - 5)m + 6) \} - 2c_2c_1^4m \{ -3(96\alpha^5 - 804\alpha^4 + 1504\alpha^3 + 2676\alpha^2 - 10612\alpha + 8283) + 4(177\alpha^2 - 611\alpha + 521)m^3 - 3(220\alpha^4 + 280\alpha^3 - 7556\alpha^2 + 18400\alpha - 12463)m^2 + 4(108\alpha^5 - 234\alpha^4 - 4302\alpha^3 + 22902\alpha^2 - 38593\alpha + 20488)m \} + c_1^6 \{ 48\alpha^6 - 480\alpha^5 + 996\alpha^4 + 5472\alpha^3 - 29810\alpha^2 + 50792\alpha + (276\alpha^2 - 956\alpha + 818)m^4 + (-360\alpha^4 - 448\alpha^3 + 12434\alpha^2 - 30518\alpha + 20837)m^3 + (432\alpha^5 - 1236\alpha^4 - 16044\alpha^3 + 92306\alpha^2 - 161292\alpha + 88497)m^2 + (-168\alpha^6 + 888\alpha^5 + 5352\alpha^4 - 55580\alpha^3 + 173290\alpha^2 - 224554\alpha + 97939)m - 29771 \} \right]$.

It is noteworthy that we reached at least sixth-order convergence for all α . In addition, we can easily obtain $L_1 = L_2 = 0$ by using $\alpha = 2$.

Now, by adopting $\alpha = 2$ in Expression (14), we obtain:

$$e_{n+1} = \frac{A_0(12c_3c_1m^3 - 12c_2c_1^2m(3m^2 + 30m - 1) + 12c_2^2m^2(2m - 1) + c_1^4(10m^3 + 183m^2 + 650m - 3))}{24m^8} e_n^8 + O(e_n^9), \tag{15}$$

where $A_0 = (c_1^3(m + 1) - 2c_1c_2m)$. The above Expression (15) demonstrates that our proposed Scheme (4) reaches eighth-order convergence for $\alpha = 2$ by using only four functional evaluations per full iteration. Hence, it is an optimal scheme for a particular value of $\alpha = 2$ according to the Kung–Traub conjecture, completing the proof. \square

3. Numerical Experiments

In this section, we illustrate the efficiency and convergence behavior of our iteration functions for particular values $\alpha = 0$, $\alpha = 1$, $\alpha = 1.9$, and $\alpha = 2$ in Expression (4), called *OM1*, *OM2*, *OM3*, and *OM4*, respectively. In this regards, we choose five real problems having multiple and simple zeros. The details are outlined in the examples (1)–(3).

For better comparison of our iterative methods, we consider several existing methods of order six and the optimal order eight. Firstly, we compare our methods with the two-point family of sixth-order methods proposed by Geum et al. in [18], and out of them, we pick Case 4c, which is mentioned as follows:

$$y_n = x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m > 1, \tag{16}$$

$$x_{n+1} = y_n - \left[\frac{m + a_1u_n}{1 + b_1u_n + b_2u_n^2} \times \frac{1}{1 + c_1s_n} \right] \frac{f(y_n)}{f'(y_n)},$$

where:

$$\begin{aligned}
 a_1 &= \frac{2m(4m^4 - 16m^3 + 31m^2 - 30m + 13)}{(m-1)(4m^2 - 8m + 7)}, & b_1 &= \frac{4(2m^2 - 4m + 3)}{(m-1)(4m^2 - 8m + 7)}, \\
 b_2 &= -\frac{4m^2 - 8m + 3}{4m^2 - 8m + 7}, & c_1 &= 2(m-1), \\
 u_n &= \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}, & s_n &= \left(\frac{f'(y_n)}{f'(x_n)}\right)^{\frac{1}{m-1}},
 \end{aligned}$$

called GM1.

In addition, we also compare them with one more non-optimal family of sixth-order iteration functions given by the same authors of [19], and out of them, we choose Case 5YD, which is given by:

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \quad m \geq 1, \\
 w_n &= x_n - m \left[\frac{(u_n - 2)(2u_n - 1)}{(u_n - 1)(5u_n - 2)} \right] \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= x_n - m \left[\frac{(u_n - 2)(2u_n - 1)}{(5u_n - 2)(u_n + v_n - 1)} \right] \frac{f(x_n)}{f'(x_n)},
 \end{aligned} \tag{17}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$ and $v_n = \left(\frac{f(w_n)}{f(x_n)}\right)^{\frac{1}{m}}$, and this method is denoted as GM2.

Moreover, we compare our methods with the optimal eighth-order iterative methods proposed by Zafar et al. [21]. We choose the following two schemes out of them:

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
 w_n &= y_n - mu_n(6u_n^3 - u_n^2 + 2u_n + 1) \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= w_n - mu_n v_n (1 + 2u_n)(1 + v_n) \left(\frac{2w_n + 1}{A_2 P_0} \right) \frac{f(x_n)}{f'(x_n)}
 \end{aligned} \tag{18}$$

and:

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
 w_n &= y_n - mu_n \left(\frac{1 - 5u_n^2 + 8u_n^3}{1 - 2u_n} \right) \frac{f(x_n)}{f'(x_n)}, \\
 x_{n+1} &= w_n - mu_n v_n (1 + 2u_n)(1 + v_n) \left(\frac{3w_n + 1}{A_2 P_0 (1 + w_n)} \right) \frac{f(x_n)}{f'(x_n)},
 \end{aligned} \tag{19}$$

where $u_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $v_n = \left(\frac{f(w_n)}{f(y_n)}\right)^{\frac{1}{m}}$, $w_n = \left(\frac{f(w_n)}{f(x_n)}\right)^{\frac{1}{m}}$, and these iterative methods are denoted in our tables as ZM1 and ZM2, respectively.

Finally, we demonstrate their comparison with another optimal eighth-order iteration function given by Behl et al. [22]. However, we consider the following the best schemes (which was claimed by them):

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - m \frac{f(x_n)}{f'(x_n)} h_n (1 + 2h_n), \\
 x_{n+1} &= z_n + m \frac{f(x_n)}{f'(x_n)} \frac{t_n h_n}{1 - t_n} \left[-1 - 2h_n - h_n^2 + 4h_n^3 - 2k_n \right]
 \end{aligned} \tag{20}$$

and:

$$\begin{aligned}
 y_n &= x_n - m \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - m \frac{f(x_n)}{f'(x_n)} h_n (1 + 2h_n), \\
 x_{n+1} &= z_n - m \frac{f(x_n)}{f'(x_n)} \frac{t_n h_n}{1 - t_n} \left[\frac{1 + 9h_n^2 + 2k_n + h_n(6 + 8k_n)}{1 + 4h_n} \right],
 \end{aligned}
 \tag{21}$$

with $h_n = \left(\frac{f(y_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $k_n = \left(\frac{f(z_n)}{f(x_n)}\right)^{\frac{1}{m}}$, $t_n = \left(\frac{f(z_n)}{f(y_n)}\right)^{\frac{1}{m}}$, which are denoted *BM1* and *BM2*, respectively.

In order to compare these schemes, we perform a numerical experience, and in Tables 1 and 2, we display the difference between two consecutive iterations $|x_{n+1} - x_n|$, the residual error in the corresponding function $|f(x_n)|$, and the computational order of convergence (ρ) (we used the formula given by Cordero and Torregrosa [25]):

$$\rho \approx \frac{\ln(|x_{k+1} - x_k| / |x_k - x_{k-1}|)}{\ln(|x_k - x_{k-1}| / |x_{k-1} - x_{k-2}|)}
 \tag{22}$$

We make our calculations with several significant digits (a minimum of 3000 significant digits) to minimize the round-off error. Moreover, the computational order of convergence is provided up to five significant digits. Finally, we display the initial guess and approximated zeros up to 25 significant digits in the corresponding example where an exact solution is not available.

All computations have been performed using the programming package Mathematica 11 with multiple precision arithmetic. Further, the meaning of $a(\pm b)$ is shorthand for $a \times 10^{(\pm b)}$ in the numerical results.

Example 1. *Population growth problem:*

The law of population growth is defined as follows:

$$\frac{dN(t)}{dt} = \gamma N(t) + \eta,$$

where $N(t)$ = the population at time t , η = the fixed/constant immigration rate, and γ = the fixed/constant birth rate of the population. We can easily obtain the following solution of the above differential equation:

$$N(t) = N_0 e^{\gamma t} + \frac{\eta}{\gamma} (e^{\gamma t} - 1),$$

where N_0 is the initial population.

For a particular case study, the problem is given as follows: Suppose a certain population contains 1,000,000 individuals initially, that 300,000 individuals immigrate into the community in the first year, and that 1,365,000 individuals are present at the end of one year. Find the birth rate (γ) of this population.

To determine the birth rate, we must solve the equation:

$$f_1(x) = 1365 - 1000e^x - \frac{300}{x}(e^x - 1).
 \tag{23}$$

wherein $x = \gamma$ and our desired zero of the above function f_1 is 0.05504622451335177827483421. The reason for considering the simple zero problem is to confirm that our methods also work for simple zeros. We choose the starting point as $x_0 = 0.5$.

Example 2. The van der Waals equation of state:

$$\left(P + \frac{a_1 n^2}{V^2}\right) (V - na_2) = nRT,$$

explains the behavior of a real gas by introducing in the ideal gas equations two parameters, a_1 and a_2 , specific for each gas. The determination of the volume V of the gas in terms of the remaining parameters requires the solution of a nonlinear equation in V ,

$$PV^3 - (na_2P + nRT)V^2 + a_1n^2V - a_1a_2n^2 = 0.$$

Given the constants a_1 and a_2 of a particular gas, one can find values for n, P , and T , such that this equation has three simple roots. By using the particular values, we obtain the following nonlinear function:

$$f_2(x) = x^3 - 5.22x^2 + 9.0825x - 5.2675. \tag{24}$$

which has three zeros; out of them, one is the multiple zero $\alpha = 1.75$ of multiplicity two, and the other is the simple zero $\alpha = 1.72$. Our desired root is $\alpha = 1.75$, and we chose $x_0 = 1.8$ as the initial guess.

Example 3. Eigenvalue problem:

For this, we choose the following 8×8 matrix:

$$A = \begin{bmatrix} -12 & -12 & 36 & -12 & 0 & 0 & 12 & 8 \\ 148 & 129 & -397 & 147 & -12 & 6 & -109 & -74 \\ 72 & 62 & -186 & 66 & -8 & 4 & -54 & -36 \\ -32 & -24 & 88 & -36 & 0 & 0 & 24 & 16 \\ 20 & 13 & -45 & 19 & 8 & 6 & -13 & -10 \\ 120 & 98 & -330 & 134 & -8 & 24 & -90 & -60 \\ -132 & -109 & 333 & -115 & 12 & -6 & 105 & 66 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}.$$

The corresponding characteristic polynomial of this matrix is as follows:

$$f_3(x) = (x - 4)^3(x + 4)(x - 8)(x - 20)(x - 12)(x + 12).$$

The above function has one multiple zero at $\alpha = 4$ of multiplicity three. In addition, we consider $x_0 = 2.7$ as the starting point.

Example 4. Let us consider the following polynomial equation:

$$f_4(z) = \left((x - 1)^3 - 1\right)^{50}. \tag{25}$$

The desired zero of the above function f_4 is $\alpha = 2$ with multiplicity of order 50, and we choose initial guess $x_0 = 2.1$ for this problem.

Table 1. Comparison on the basis of the difference between two consecutive iterations $|x_{n+1} - x_n|$ for the functions f_1 - f_4 .

<i>f</i>	<i>n</i>	OM1	OM2	OM3	OM4	GM1	GM2	ZM1	ZM2	BM1	BM2
<i>f</i> ₁	1	2.3 (−3)	8.4 (−4)	9.3 (−5)	3.5 (−5)	*	3.6 (−5)	1.6 (−4)	2.3 (−4)	7.6 (−5)	3.7 (−5)
	2	2.0 (−16)	9.0 (−20)	8.8 (−28)	2.0 (−37)	*	1.4 (−29)	4.2 (−31)	8.9 (−30)	2.6 (−34)	5.0 (−37)
	3	9.7 (−95)	1.3 (−115)	6.4 (−166)	2.5 (−295)	*	5.4 (−173)	1.0 (−243)	5.5 (−233)	5.4 (−270)	5.7 (−292)
	ρ	5.9997	6.0000	6.0001	8.0000	*	6.0000	8.0000	8.0000	8.0000	8.0000
<i>f</i> ₂	1	1.3 (−3)	8.2 (−4)	4.0 (−3)	3.5 (−4)	9.5 (−4)	3.9 (−4)	3.9 (−4)	4.1 (−3)	2.7 (−4)	2.6 (−4)
	2	2.5 (−10)	4.2 (−12)	6.4 (−16)	8.7 (−18)	2.7 (−11)	1.0 (−14)	5.2 (−17)	9.8 (−17)	1.1 (−18)	1.4 (−19)
	3	2.0 (−50)	8.7 (−62)	6.5 (−87)	1.5 (−126)	2.0 (−56)	3.9 (−78)	5.9 (−120)	1.2 (−117)	6.3 (−134)	1.0 (−141)
	ρ	5.9757	5.9928	6.0214	7.9963	5.9836	5.9975	7.9945	7.9941	7.9971	8.0026
<i>f</i> ₃	1	9.1 (−5)	3.6 (−5)	8.0 (−6)	6.0 (−6)	8.5 (−5)	4.8 (−5)	4.9 (−6)	5.2 (−6)	2.0 (−6)	1.8 (−6)
	2	1.8 (−28)	1.4 (−31)	9.8 (−38)	2.0 (−47)	1.0 (−28)	5.0 (−31)	6.0 (−48)	1.0 (−47)	1.5 (−51)	2.8 (−52)
	3	1.2 (−170)	4.4 (−190)	3.3 (−229)	2.5 (−379)	3.1 (−172)	5.8 (−187)	2.7 (−383)	2.3 (−381)	1.4 (−412)	1.3 (−418)
	ρ	6.0000	6.0000	6.0000	8.0000	6.0000	6.0000	8.0000	8.0000	8.0000	8.0000
<i>f</i> ₄	1	2.4 (−5)	7.1 (−6)	4.2 (−7)	1.4 (−7)	1.8 (−5)	2.0 (−7)	4.8 (−7)	6.5 (−7)	1.9 (−7)	6.3 (−8)
	2	1.5 (−26)	1.7 (−30)	3.9 (−40)	6.7 (−54)	1.1 (−26)	1.8 (−41)	5.7 (−49)	8.4 (−48)	8.0 (−53)	4.2 (−57)
	3	7.5 (−154)	3.2 (−178)	2.6 (−438)	1.7 (−424)	6.6 (−154)	1.0 (−245)	2.2 (−384)	6.6 (−375)	9.6 (−416)	5.9 (−169)
	ρ	6.0000	6.0000	6.0000	8.0000	6.0000	6.0000	8.0000	8.0000	8.0000	2.2745

* means that the corresponding method does not work.

Table 2. Comparison on the basis of residual errors $|f(x_n)|$ for the functions f_1 – f_4 .

<i>f</i>	<i>n</i>	OM1	OM2	OM3	OM4	GM1	GM2	ZM1	ZM2	BM1	BM2
<i>f</i> ₁	1	2.7	1.0	1.1 (−1)	4.2 (−2)	*	4.4 (−2)	1.9 (−1)	2.7 (−1)	9.2 (−2)	4.4 (−2)
	2	2.4 (−13)	1.1 (−16)	1.1 (−24)	2.4 (−34)	*	1.7 (−26)	5.1 (−28)	1.1 (−26)	3.2 (−31)	6.0 (−34)
	3	1.2 (−91)	1.6 (−112)	7.8 (−163)	3.0 (−292)	*	5.4 (−173)	1.2 (−240)	6.7 (−230)	6.5 (−267)	7.0 (−289)
<i>f</i> ₂	1	5.0 (−8)	2.1 (−8)	4.8 (−9)	3.6 (−9)	2.8 (−8)	4.6 (−9)	4.6 (−9)	5.1 (−9)	2.3 (−9)	2.0 (−9)
	2	1.8 (−21)	5.3 (−25)	1.2 (−32)	2.3 (−36)	2.2 (−23)	3.2 (−30)	8.0 (−35)	2.9 (−34)	3.4 (−38)	5.9 (−40)
	3	1.2 (−101)	2.2 (−124)	1.3 (−174)	6.9 (−254)	1.2 (−113)	4.6 (−157)	1.1 (−240)	4.3 (−236)	1.2 (−268)	3.1 (−284)
<i>f</i> ₃	1	4.9 (−8)	3.1 (−9)	3.1 (−11)	1.4 (−11)	4.1 (−8)	7.4 (−9)	7.8 (−12)	9.1 (−12)	5.2 (−13)	3.6 (−13)
	2	3.9 (−79)	1.8 (−88)	6.1 (−107)	4.9 (−136)	7.1 (−80)	8.0 (−87)	1.4 (−137)	6.9 (−137)	2.1 (−148)	1.5 (−150)
	3	1.0 (−505)	5.6 (−564)	2.4 (−681)	1.1 (−1131)	1.9 (−510)	1.2 (−554)	1.3 (−1143)	7.5 (−1138)	1.9 (−1231)	1.3 (−1249)
<i>f</i> ₄	1	1.2 (−207)	2.7 (−234)	1.1 (−295)	3.3 (−319)	3.5 (−214)	1.0 (−311)	6.6 (−293)	2.3 (−286)	1.8 (−313)	6.2 (−337)
	2	1.9 (−1268)	2.6 (−1465)	3.8 (−1947)	1.6 (−2635)	1.9 (−1274)	9.8 (−2014)	3.4 (−2389)	9.4 (−2331)	9.8 (−2582)	1.1 (−2795)
	3	4.2 (−7633)	2.3 (−8851)	7.5 (−11856)	6.1 (−21166)	6.0 (−7636)	7.3 (−12226)	1.6 (−19159)	7.1 (−18686)	8.8 (−20728)	3.4 (−8388)

4. Conclusions

We presented an eighth-order modification of the Chebyshev–Halley-type iteration scheme having optimal convergence to obtain the multiple solutions of the scalar equation. The proposed scheme is optimal in the sense of the classical Kung–Traub conjecture. Thus, the efficiency index of the present methods is $E = \sqrt[4]{8} \approx 1.682$, which is better than the classical Newton’s method $E = \sqrt[3]{2} \approx 1.414$. Finally, the numerical experience corroborates the theoretical results about the convergence order, and moreover, it can be concluded that our proposed methods are highly efficient and competitive.

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