Improving the approximation of the first and second order statistics of the response stochastic process to the random Legendre differential equation

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Abstract. In this paper, we deal with uncertainty quantification for the random Legendre differential equation, with input coefficient A and initial conditions X_0 and X_1. In a previous study [Calbo G. et al, Comput. Math. Appl., 61(9), 2782–2792 (2011)], a mean square convergent power series solution on $(-1/\mathrm{e}, 1/\mathrm{e})$ was constructed, under the assumptions of mean fourth integrability of X_0 and X_1, independence, and at most exponential growth of the absolute moments of A. In this paper, we relax these conditions to construct an L^p solution $(1 \leq p \leq \infty)$ to the random Legendre differential equation on the whole domain $(-1, 1)$, as in its deterministic counterpart. Our hypotheses assume no independence and less integrability of X_0 and X_1. Moreover, the growth condition on the moments of A is characterized by the boundedness of A, which simplifies the proofs significantly. We also provide approximations of the expectation and variance of the response process. The numerical experiments show the wide applicability of our findings. A comparison with Monte Carlo simulations and gPC expansions is performed.

1. Introduction

Random differential equations are differential equations in which randomness appears in the coefficients, forcing term, initial conditions and/or boundary conditions. The solution is a stochastic process that solves the differential equation in some probabilistic sense, usually in the sample path or L^p sense. For a theoretical approach to random differential equations, see [1, 2]. Uncertainty quantification consists in calculating the main statistics of the response process to the stochastic system [3]. The main methods used to deal with uncertainty quantification for random differential equations are Monte Carlo simulations [4], gPC based stochastic Galerkin technique [5, 6], finite difference schemes [7–15], Itô calculus [16] and L^p calculus [1, 17, 18]. In the concrete case of second-order random linear differential equations, the Fröbenius method has been successfully used to deal with particular equations: Airy [19], Hermite [20], Legendre [21], Bessel [22], etc. In [23–25], homotopy, Adomian decomposition and differential transformations techniques, respectively, have been extended to the random scenario to solve some particular second-order random linear differential equations.

In this paper, we will deal with the random Legendre differential equation:

$$\begin{cases} (1 - t^2)\ddot{X}(t) - 2t\dot{X}(t) + A(A + 1)X(t) = 0, \ |t| < 1, \\ X(0) = X_0, \\ \dot{X}(0) = X_1. \end{cases} \tag{1}$$

The coefficient A is a non-negative random variable and the initial conditions X_0 and X_1 are random variables. All of them are defined in a common complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the context of random differential equations, one usually works with random variables that belong to the so-called Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq \infty$. Recall that, given a random variable $Y : \Omega \to \mathbb{R}$, its norm in the space $L^p(\Omega)$ is defined as:

$$\|Y\|_{L^p(\Omega)} = (\mathbb{E}[|Y|^p])^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\|Y\|_{L^\infty(\Omega)} = \inf\{\sup\{|Y(\omega)| : \omega \in \Omega \backslash N\} : \mathbb{P}(N) = 0\}.$$

Here, $\mathbb{E}[\cdot]$ stands for the expectation operator.

Given a stochastic process $Y(t)$ with the property that $Y(t) \in L^p(\Omega)$ for each t, one can define the $L^p(\Omega)$ continuity, differentiability or analiticity of $Y(t)$, by taking limits in $L^p(\Omega)$. This $L^p(\Omega)$ random calculus is the setting in which one usually considers random differential equations such as (1). The particular case $p = 2$, which arises from working in the Hilbert space $L^2(\Omega)$ and with random variables with well-defined expectation $\mathbb{E}[\cdot]$ and variance $\mathbb{V}[\cdot]$, is the most extended in the literature, and it is usually referred to as mean square calculus. An exposition of these topics is presented, for instance, in [1, 17].

In [21], the authors constructed a mean square convergent power series solution $X(t)$ to (1) on $(-1/e, 1/e)$ under certain assumptions on the random

inputs A, X_0 and X_1. The goal of this article is to improve [21]: to weaken the hypotheses from [21], to simplify the proofs significantly and to obtain an $L^p(\Omega)$ random power series solution on the whole domain $(-1,1)$, as in the deterministic counterpart of (1). Numerical examples that could not be tackled via the hypotheses from [21] will be carried out in this paper, establishing a comparison with Monte Carlo simulations and a particular version of gPC expansions [5].

The structure of this paper is the following. In Section 2, we will review the techniques used in [21]. We will relax the assumptions from [21] and we will improve the conclusions of the results. In Section 3, we will show how to approximate the expectation and variance of the response process, under no independence assumption. In Section 4, we will perform a wide variety of examples and illustrate the potentiality of our findings by comparing the numerical results with Monte Carlo simulations and gPC expansions. Section 5 will draw conclusions.

2. Random Legendre differential equation

In [21], the authors constructed a mean square power series solution to the random Legendre differential equation (1) on the time interval $(-1/e, 1/e)$. The hypotheses assumed in [21] were that the absolute moments of A increased at most exponentially, that is, there exist two positive constants H and M such that

$$\mathbb{E}[|A|^n] \leq HM^n, \quad n \geq n_0; \tag{2}$$

that A is independent of the initial conditions X_0 and X_1; and that $X_0, X_1 \in L^4(\Omega)$. Hypothesis (2) has been of constant use in the extant literature to study significant linear random differential equations via the Fröbenius method: [19–21]. In [21], the explicit solution to (1) was obtained in the form of a random power series solution by means of the Fröbenius method:

$$X(t) = X_0 \tilde{X}_1(t) + X_1 \tilde{X}_2(t) \tag{3}$$

for $|t| < 1/e$, where

$$\tilde{X}_1(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} P_1(m) t^{2m}, \quad \tilde{X}_2(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} P_1(m) t^{2m+1}, \tag{4}$$

$$P_1(m) = \prod_{k=1}^{m} (A-2k+2)(A+2k-1), \quad P_2(m) = \prod_{k=1}^{m} (A-2k+1)(A+2k). \tag{5}$$

The series in (4) were proved to be mean fourth convergent for $|t| < 1/e$. Since $X_0, X_1 \in L^4(\Omega)$, it follows that (3) is a mean square solution to (1) on $(-1/e, 1/e)$.

To summarize, the main result obtained in [21, Th. 11] was stated as follows:

Theorem 2.1. *Suppose that $X_0, X_1 \in L^4(\Omega)$, that A satisfies the growth condition (2), and that A is independent of X_0 and X_1. Then the stochastic process defined by (3)–(5) is a mean square solution to the random initial value problem (1) on the time domain $(-1/e, 1/e)$.*

Our goal is to extend this theorem and to simplify its proof given in [21]. The growth condition (2) was established in order to demonstrate the mean fourth convergence of (4), by applying well-known inequalities: Hölder's inequality, c_s-inequality and arithmetic-geometric inequality. We will simplify the proof given in [21] by working with an equivalent but easier to manage form of (2), see Lemma 2.2. Moreover, the $L^\infty(\Omega)$ convergence of (4) (which implies mean fourth convergence) will be obtained on the whole interval $(-1, 1)$, see Theorem 2.4. This will provide the complete extension of the deterministic counterpart for the random Legendre differential equation.

Lemma 2.2. *The growth condition (2) is equivalent to the boundedness of A: $\|A\|_{L^\infty(\Omega)} < \infty$.*

Proof. If $\|A\|_{L^\infty(\Omega)} < \infty$, then $E[|A|^n] \leq \|A\|_{L^\infty(\Omega)}^n$, so that we can take $H = 1$ and $M = \|A\|_{L^\infty(\Omega)}$ and (2) is satisfied.

On the other hand, if (2) holds, then $\|A\|_{L^n(\Omega)} \leq H^{1/n}M$. By taking limits, $\|A\|_{L^\infty(\Omega)} = \lim_{n\to\infty} \|A\|_{L^n(\Omega)} \leq M < \infty$. □

Lemma 2.3. *Let $X(t) = \sum_{n=0}^{\infty} X_n t^n$ be a formal random power series on $(-1, 1)$. Let $1 \leq p \leq \infty$. Then the given series converges in $L^p(\Omega)$ for all $t \in (-1, 1)$, if and only if $\sum_{n=0}^{\infty} \|X_n\|_{L^p(\Omega)}|t|^n < \infty$ for all $t \in (-1, 1)$.*

Proof. If $\sum_{n=0}^{\infty} \|X_n\|_{L^p(\Omega)}|t|^n < \infty$ for all $t \in (-1, 1)$, then the series converges in $L^p(\Omega)$ for all $t \in (-1, 1)$, because in a Banach space, absolute convergence of a series implies convergence.

On the other hand, suppose that the series converges in $L^p(\Omega)$ for all $t \in (-1, 1)$. Fix $|t_0| < 1$. Let $|t_0| < |\rho| < 1$. Since $\sum_{n=0}^{\infty} \|X_n\|_{L^p(\Omega)}|\rho|^n < \infty$, then $\|X_n\|_{L^p(\Omega)}|\rho|^n \leq 1$, for $n \geq n_0$. Thus, $\|X_n\|_{L^p(\Omega)}|t_0|^n \leq (|t_0|/|\rho|)^n$, for $n \geq n_0$, with $\sum_{n=0}^{\infty}(|t_0|/|\rho|)^n < \infty$. By comparison, $\sum_{n=0}^{\infty} \|X_n\|_{L^p(\Omega)}|t_0|^n < \infty$. □

We state and proof our main Theorem 2.4. It is a significant improvement of Theorem (2.1) stated and proved in [21]: for $p = 2$, we only require mean square integrability of X_0 and X_1, not mean fourth integrability; we do not need any independence assumption on A, X_0 and X_1; and we demonstrate mean square convergence of the series on the whole interval $(-1, 1)$, not just $(-1/e, 1/e)$. Moreover, our proof is much simpler, because the hypothesis of boundedness for A instead of the equivalent growth condition (2) allows simpler and more direct inequalities (we do not need Hölder's inequality, c_s-inequality, arithmetic-geometric inequality, etc.).

Theorem 2.4. *Suppose that $X_0, X_1 \in L^p(\Omega)$, for certain $1 \leq p \leq \infty$, and $\|A\|_{L^\infty(\Omega)} < \infty$. Then the stochastic process defined by (3)–(5) is the unique*

$L^p(\Omega)$ *solution to the random initial value problem (1) on the whole time domain* $(-1, 1)$.

Proof. From (3) and $X_0, X_1 \in L^p(\Omega)$, it suffices to see that the two series given in (4) converge in $L^\infty(\Omega)$ for $t \in (-1, 1)$. That is,

$$\sum_{m=0}^\infty \frac{1}{(2m)!}\|P_1(m)\|_{L^\infty(\Omega)}|t|^{2m} < \infty, \ \sum_{m=0}^\infty \frac{1}{(2m+1)!}\|P_1(m)\|_{L^\infty(\Omega)}|t|^{2m+1} < \infty,$$
(6)

for $t \in (-1, 1)$ (see Lemma 2.3). We will check (6) for the first series, as for the second one the reasoning is completely analogous.

Let $L = \|A\|_{L^\infty(\Omega)}$. We have

$$\|P_1(m)\|_{L^\infty(\Omega)} = \left\|\prod_{k=1}^m (A - 2k + 2)(A + 2k - 1)\right\|_{L^\infty(\Omega)} \leq \prod_{k=1}^m (L + 2k - 2)(L + 2k - 1)$$

$$\leq \prod_{k=1}^m (L + 2k - 1)^2 = \left(\frac{\prod_{k=1}^{2m-1}(L + k)}{\prod_{k=1}^{m-1}(L + 2k)}\right)^2 = \left(\frac{(L + 2m - 1)!}{L! \prod_{k=1}^{m-1}(L + 2k)}\right)^2$$

$$= \left(\frac{(L + 2m - 1)!}{L! 2^{m-1} \prod_{k=1}^{m-1}(L/2 + k)}\right)^2 = \left(\frac{(L + 2m - 1)!\Gamma(L/2 + 1)}{L! 2^{m-1}\Gamma(L/2 + m)}\right)^2,$$

where the property $\Gamma(x) = (x - 1)\Gamma(x - 1)$ of the Gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$ has been used. By the root test, if we check that

$$\lim_{m\to\infty} \left(\frac{(L + 2m - 1)!\Gamma(L/2 + 1)}{L! 2^{m-1}\Gamma(L/2 + m)(2m)!^{\frac{1}{2}}}\right)^{2/m} = 1,$$

then the first part of (6) will follow. By Stirling's formula, as $x \to \infty$, the asymptotic behavior of the Gamma function is $\Gamma(x) \sim \sqrt{2\pi x}(\frac{x-1}{e})^{x-1}$. As a consequence,

$$\lim_{m\to\infty} \left(\frac{(L + 2m - 1)!\Gamma(L/2 + 1)}{L! 2^{m-1}\Gamma(L/2 + m)(2m)!^{\frac{1}{2}}}\right)^{2/m}$$

$$= \lim_{m\to\infty} \left(\frac{\sqrt{2\pi(L + 2m - 1)}\left(\frac{L+2m-1}{e}\right)^{L+2m-1}\Gamma(L/2 + 1)}{L! 2^{m-1}\sqrt{2\pi(L/2 + m)}\left(\frac{L/2+m-1}{e}\right)^{L/2+m-1}\sqrt[4]{4\pi m}\left(\frac{2m}{e}\right)^m}\right)^{2/m}$$

$$= \lim_{m\to\infty} \frac{\left(\frac{L+2m-1}{e}\right)^4}{4\left(\frac{L/2+m-1}{e}\right)^2\left(\frac{2m}{e}\right)^2} = 1.$$

As a conclusion, the stochastic process defined by (3)–(5) is an $L^p(\Omega)$ solution to (1) on $(-1, 1)$.

To demonstrate the uniqueness, we use [1, Th. 5.1.2], [2, Th. 5]. Rewrite (1) as $\dot{Z}(t) = B(t)Z(t)$, where

$$Z(t) = \begin{pmatrix} X(t) \\ \dot{X}(t) \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 & 1 \\ \frac{A(A+1)}{1-t^2} & \frac{-2t}{1-t^2} \end{pmatrix}.$$

We say that the random vector $Z = (Z_1, Z_2)$ belongs to $L_2^p(\Omega)$ if $\|Z\|_{L_2^p(\Omega)} :=$ $\max\{\|Z_1\|_{L^p(\Omega)}, \|Z_2\|_{L^p(\Omega)}\} < \infty$. Consider the random matrix norm $|||B||| :=$

$\max_i \sum_j \|b_{ij}\|_{\mathrm{L}^\infty(\Omega)}$. If $Z, Z' \in \mathrm{L}_2^p(\Omega)$, then $\|B(t)Z - B(t)Z'\|_{\mathrm{L}_2^p(\Omega)} \leq \||B(t)\|| \cdot \|Z - Z'\|_{\mathrm{L}_2^p(\Omega)}$, where

$$\int_{-a}^a \||B(t)\|| \, \mathrm{d}t = \int_{-a}^a \frac{\|A\|_{\mathrm{L}^\infty(\Omega)}(\|A\|_{\mathrm{L}^\infty(\Omega)} + 1) + 2|t|}{1 - t^2} \, \mathrm{d}t < \infty$$

for each $a \in (0, 1)$. Then the assumptions of [1, Th. 5.1.2], [2, Th. 5] hold. $\square$

The hypothesis $\|A\|_{\mathrm{L}^\infty(\Omega)} < \infty$ is satisfied by some standard probability distributions: Uniform, Beta, Binomial, etc. If one wants A to follow an unbounded distribution, the truncation method permits bounding the support of A (see [26]). For example, the truncated Normal or Gamma distributions can be given to A. See Example 4.3 for a test of this methodology.

Remark 2.5. If $\|A\|_{\mathrm{L}^\infty(\Omega)} = \infty$, then (6) does not hold for any $t \in (-1, 1)\backslash\{0\}$. Indeed,

$$\sum_{m=0}^\infty \frac{1}{(2m)!}\|P_1(m)\|_{\mathrm{L}^\infty(\Omega)}|t|^{2m} \geq \frac{1}{2}\|P_1(1)\|_{\mathrm{L}^\infty(\Omega)}t^2 = \frac{1}{2}\|A(A+1)\|_{\mathrm{L}^\infty(\Omega)}t^2 = \infty.$$

By Lemma 2.3, the two series given in (4) do not converge in $\mathrm{L}^\infty(\Omega)$, for any $t \in (-1, 1)\backslash\{0\}$.

Remark 2.6. If $X_0, X_1, A \in \mathrm{L}^\infty(\Omega)$, then the response process $X(t)$ defined by (3)–(5) is the unique $\mathrm{L}^\infty(\Omega)$ solution to (1) on $(-1, 1)$. In particular, $X(t)$ is the unique solution in the sample path sense [1, Appendix A].

3. Approximation of the expectation and variance of the response process

Let $X_0, X_1 \in \mathrm{L}^2(\Omega)$ and A be a bounded random variable, not necessarily independent. By Theorem 2.4, the stochastic process $X(t)$ defined by (3)–(5) is an $\mathrm{L}^2(\Omega)$ solution to the random initial value problem (1) on the whole time domain $(-1, 1)$. If we consider $X^M(t) = X_0 \tilde{X}_1^M(t) + X_1 \tilde{X}_2^M(t)$, where

$$\tilde{X}_1^M(t) = \sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(-1)^m}{(2m)!} P_1(m) t^{2m}, \quad \tilde{X}_2^M(t) = \sum_{m=0}^{\lfloor \frac{M-1}{2} \rfloor} \frac{(-1)^m}{(2m+1)!} P_2(m) t^{2m+1},$$

we know that $X^M(t) \to X(t)$ in $\mathrm{L}^2(\Omega)$ as $M \to \infty$, for each $t \in (-1, 1)$. This mean square convergence allows us to approximate the expectation and variance of $X(t)$ by using

$$\mathbb{E}[X(t)] = \lim_{M \to \infty} \mathbb{E}[X^M(t)], \quad \mathbb{V}[X(t)] = \lim_{M \to \infty} \mathbb{V}[X^M(t)], \tag{7}$$

see [1, Th. 4.2.1, Th. 4.3.1].

The expectation of $X^M(t)$ is given by

$$\mathbb{E}[X^M(t)] = \sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} \frac{(-1)^m}{(2m)!}\mathbb{E}[X_0 P_1(m)]t^{2m} + \sum_{m=0}^{\lfloor \frac{M-1}{2} \rfloor} \frac{(-1)^m}{(2m+1)!}\mathbb{E}[X_1 P_2(m)]t^{2m+1},$$

where

$$\mathbb{E}[X_0 P_1(m)] = \int_{(0,\infty)\times\mathbb{R}} x_0 \left(\prod_{j=1}^{m}(a - 2j + 2)(a + 2j - 1) \right) \mathbb{P}_{(A,X_0)}(\mathrm{d}a, \mathrm{d}x_0),$$

$$\mathbb{E}[X_1 P_2(m)] = \int_{(0,\infty)\times\mathbb{R}} x_1 \left(\prod_{j=1}^{m}(a - 2j + 1)(a + 2j) \right) \mathbb{P}_{(A,X_1)}(\mathrm{d}a, \mathrm{d}x_1).$$

Here, $\mathbb{P}_Z$ represents the probability law of the random vector Z, which comprises the different cases of absolute continuity, discrete support, etc.

On the other hand, the variance of $X_M(t)$ is given by

$$\mathbb{V}[X_M(t)] = \mathbb{E}[X_M(t)^2] - (\mathbb{E}[X_M(t)])^2,$$

so that we need to compute $\mathbb{E}[X_M(t)^2]$. Let

$$X_{2m} = X_0 \frac{(-1)^m}{(2m)!} P_1(m), \quad X_{2m+1} = X_1 \frac{(-1)^m}{(2m+1)!} P_2(m).$$

We have

$$\mathbb{E}[X_M(t)^2] = \mathbb{E}\left[\left(\sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} X_{2m} t^{2m} \right)^2 \right] + \mathbb{E}\left[\left(\sum_{m=0}^{\lfloor \frac{M-1}{2} \rfloor} X_{2m+1} t^{2m+1} \right)^2 \right]$$

$$+ 2 \sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} \sum_{n=0}^{\lfloor \frac{M-1}{2} \rfloor} \mathbb{E}[X_{2m} X_{2n+1}] t^{2(m+n)+1},$$

where

$$\mathbb{E}\left[\left(\sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} X_{2m} t^{2m} \right)^2 \right] = \sum_{m=0}^{\lfloor \frac{M}{2} \rfloor} \sum_{n=0}^{\lfloor \frac{M}{2} \rfloor} \mathbb{E}[X_{2m} X_{2n}] t^{2(m+n)},$$

$$\mathbb{E}\left[\left(\sum_{m=0}^{\lfloor \frac{M-1}{2} \rfloor} X_{2m+1} t^{2m+1} \right)^2 \right] = \sum_{m=0}^{\lfloor \frac{M-1}{2} \rfloor} \sum_{n=0}^{\lfloor \frac{M-1}{2} \rfloor} \mathbb{E}[X_{2m+1} X_{2n+1}] t^{2(m+n)+2}.$$

The expectations involved in these expressions can be computed as follows:

$$\mathbb{E}[X_{2m} X_{2n}] = \frac{(-1)^{m+n}}{(2m)!(2n)!} \mathbb{E}[X_0^2 P_1(m) P_1(n)]$$

$$= \frac{(-1)^{m+n}}{(2m)!(2n)!} \int_{(0,\infty)\times\mathbb{R}} x_0^2 \left(\prod_{j=1}^{m}(a - 2j + 2)(a + 2j - 1) \right)$$

$$\cdot \left(\prod_{j=1}^{n}(a - 2j + 2)(a + 2j - 1) \right) \mathbb{P}_{(A,X_0)}(\mathrm{d}a, \mathrm{d}x_0),$$

$$\mathbb{E}[X_{2m+1}X_{2n+1}] = \frac{(-1)^{m+n}}{(2m+1)!(2n+1)!}\mathbb{E}[X_1^2 P_2(m)P_2(n)]$$

$$= \frac{(-1)^{m+n}}{(2m+1)!(2n+1)!}\int_{(0,\infty)\times\mathbb{R}} x_1^2 \left(\prod_{j=1}^m (a-2j+1)(a+2j)\right)$$

$$\cdot \left(\prod_{j=1}^n (a-2j+1)(a+2j)\right)\mathbb{P}_{(A,X_1)}(\mathrm{d}a,\mathrm{d}x_1),$$

$$\mathbb{E}[X_{2m}X_{2n+1}] = \frac{(-1)^{m+n}}{(2m)!(2n+1)!}\mathbb{E}[X_0 X_1 P_1(m)P_2(n)]$$

$$= \frac{(-1)^{m+n}}{(2m)!(2n)!}\int_{(0,\infty)\times\mathbb{R}\times\mathbb{R}} x_0 x_1 \left(\prod_{j=1}^m (a-2j+2)(a+2j-1)\right)$$

$$\cdot \left(\prod_{j=1}^n (a-2j+1)(a+2j)\right)\mathbb{P}_{(A,X_0,X_1)}(\mathrm{d}a,\mathrm{d}x_0,\mathrm{d}x_1).$$

4. Numerical experiments

In this section we perform several numerical experiments. Since in [21] the authors carried out numerical examples when A, X_0 and X_1 are independent random variables, we will show three more examples in which A, X_0 and X_1 are not independent. To assess the reliability of the approximations obtained for the expectation and variance by using (7), we will compare them with Monte Carlo simulations and a generalized Polynomial Chaos (gPC) approach.

Monte Carlo simulations generate samples of $X(t)$ by computing realizations of A, X_0 and X_1 and solving the corresponding deterministic problem (1). Although it is an effective and easy to implement approach to quantify the uncertainty, the slowness to get accurately the digits in the computations makes this technique computationally expensive, [4], [6, pp. 53–54].

Our gPC approach is based on the computational algorithm presented in [5], which works when the random input parameters are non-independent and jointly absolutely continuous. Due to the spectral convergence of the Galerkin projections in $\mathrm{L}^2(\Omega)$ [6, 27–29], for small orders of bases m (see [5]) the approximations for the expectation and variance are very accurate, especially for small t. However, increasing the order m of the bases may entail numerical errors, see [28, 29] and Example 4.3.

Example 4.1. We consider the random differential equation (1) with

$$(A, X_0, X_1) \sim \mathrm{Dirichlet}(5, 1, 2, 3).$$

Since X_0, X_1 and A are bounded random variables, Theorem 2.4 implies that the stochastic process $X(t)$ defined by (3)–(5) is the unique $\mathrm{L}^\infty(\Omega)$ solution to (1) on $(-1, 1)$. In Table 1, we show $\mathbb{E}[X^M(t)]$ for different orders M, which approximates $\mathbb{E}[X(t)]$ by (7). We observe that the approximations achieved are more accurate for small M when t is near 0, because the random power

series is centered at 0 and the process $X(t)$ is known at 0. For $t \leq 0.8$, stabilization of the results has been achieved for $M = 80$. For $t = 0.9$, a larger M would be needed. We notice that Monte Carlo simulations with $500,000$ realizations give an approximate result up to three significant figures. To obtain more exact approximations, more simulations and computational cost are needed. In general, the approximations via Monte Carlo simulations are worse than via our Fröbenius method. Concerning gPC approximations, the results obtained are as accurate as via the Fröbenius method, due to its spectral convergence. Table 2 provides analogous results for the variance, where $\mathbb{V}[X^M(t)]$ approximates $\mathbb{V}[X(t)]$ by (7). For $t \leq 0.7$ stabilization of the approximations has been reached for $M = 80$. In general, a larger M is required to achieve nearly exact approximations for the variance. The results obtained agree with Monte Carlo simulations and gPC expansions.

t	$\mathbb{E}[X^{10}(t)]$	$\mathbb{E}[X^{20}(t)]$	$\mathbb{E}[X^{40}(t)]$	$\mathbb{E}[X^{80}(t)]$	MC $500,000$	gPC $m = 3$
0	0.0909091	0.0909091	0.0909091	0.0909091	0.0906787	0.0909091
0.1	0.108855	0.108855	0.108855	0.108855	0.108648	0.108855
0.2	0.126491	0.126491	0.126491	0.126491	0.126308	0.126491
0.3	0.144059	0.144059	0.144059	0.144059	0.143903	0.144059
0.4	0.161835	0.161835	0.161835	0.161835	0.161709	0.161835
0.5	0.180166	0.180172	0.180172	0.180172	0.180080	0.180172
0.6	0.199548	0.199591	0.199592	0.199592	0.199540	0.199592
0.7	0.220733	0.220998	0.221002	0.221002	0.221000	0.221002
0.8	0.24491	0.246266	0.246352	0.246352	0.246416	0.246352
0.9	0.273962	0.280100	0.281585	0.281693	0.281863	0.281694

TABLE 1. Approximation of the expectation of the solution stochastic process. Example 4.1.

t	$\mathbb{V}[X^{10}(t)]$	$\mathbb{V}[X^{20}(t)]$	$\mathbb{V}[X^{40}(t)]$	$\mathbb{V}[X^{80}(t)]$	MC $500,000$	gPC $m = 3$
0	0.00688705	0.00688705	0.00688705	0.00688705	0.00685105	0.00688705
0.1	0.00670461	0.00670461	0.00670461	0.00670461	0.00666882	0.00670461
0.2	0.00672130	0.00672130	0.00672130	0.00672130	0.00668621	0.00672130
0.3	0.00697044	0.00697045	0.00697045	0.00697045	0.00693658	0.00697045
0.4	0.00751088	0.00751091	0.00751091	0.00751091	0.00747887	0.00751091
0.5	0.00844437	0.00844482	0.00844482	0.00844482	0.00841536	0.00844482
0.6	0.00995308	0.00995823	0.00995825	0.00995825	0.00993237	0.00995825
0.7	0.0123829	0.0124269	0.0124276	0.0124276	0.0124068	0.0124276
0.8	0.0164346	0.0167508	0.0167712	0.0167714	0.0167582	0.0167714
0.9	0.0236175	0.0256974	0.0262304	0.0262699	0.0262712	0.0262702

TABLE 2. Approximation of the variance of the solution stochastic process. Example 4.1.

Example 4.2. We set a joint discrete distribution to (A, X_0, X_1):

$$(A, X_0, X_1) \sim \mathrm{Multinomial}(10; 0.2, 0.3, 0.5).$$

Since X_0, X_1 and A are bounded random variables, Theorem 2.4 entails that the response process $X(t)$ defined by (3)–(5) is the unique $\mathrm{L}^\infty(\Omega)$ solution to (1) on $(-1, 1)$. Expression (7) allows approximating $\mathbb{E}[X(t)]$ and $\mathbb{V}[X(t)]$ via $\mathbb{E}[X^M(t)]$ and $\mathbb{V}[X^M(t)]$, respectively. Analogous comments to the previous example apply here, and the results are presented in Table 3 and Table 4. We point out that, since (A, X_0, X_1) is discrete, the computational method from [5] to apply gPC expansions does not work in this case. The results obtained via our Fröbenius method are accurate.

t	$\mathbb{E}[X^{10}(t)]$	$\mathbb{E}[X^{20}(t)]$	$\mathbb{E}[X^{40}(t)]$	$\mathbb{E}[X^{80}(t)]$	MC $500,000$
0	3	3	3	3	3.00207
0.1	3.39965	3.39965	3.39965	3.39965	3.40154
0.2	3.59067	3.59067	3.59067	3.59067	3.59226
0.3	3.57194	3.57194	3.57194	3.57194	3.57322
0.4	3.35661	3.35661	3.35661	3.35661	3.35768
0.5	2.97154	2.97154	2.97154	2.97154	2.97259
0.6	2.45625	2.45623	2.45623	2.45623	2.45738
0.7	1.86122	1.86112	1.86111	1.86111	1.86226
0.8	1.24584	1.24523	1.24515	1.24515	1.24558
0.9	0.675881	0.672543	0.670722	0.670550	0.668041

TABLE 3. Approximation of the expectation of the solution stochastic process. Example 4.2.

t	$\mathbb{V}[X^{10}(t)]$	$\mathbb{V}[X^{20}(t)]$	$\mathbb{V}[X^{40}(t)]$	$\mathbb{V}[X^{80}(t)]$	MC $500,000$
0	2.1	2.1	2.1	2.1	2.10094
0.1	1.81331	1.81331	1.81331	1.81331	1.81300
0.2	1.78089	1.78089	1.78089	1.78089	1.77973
0.3	2.43304	2.43304	2.43304	2.43304	2.43226
0.4	4.15996	4.15996	4.15996	4.15996	4.16027
0.5	7.12080	7.12077	7.12077	7.12077	7.12084
0.6	11.1844	11.1838	11.1838	11.1838	11.1812
0.7	16.1150	16.1090	16.1090	16.1090	16.1046
0.8	22.1109	22.0920	22.0932	22.0932	22.0965
0.9	31.1044	31.4557	31.6189	31.6324	31.6569

TABLE 4. Approximation of the variance of the solution stochastic process. Example 4.2.

Example 4.3. We set a truncated Multinormal distribution for the random input parameters:

$$(A, X_0, X_1) \sim \text{Multinormal}\left(\begin{pmatrix} 10 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & 0.01 & -0.02 \\ 0.01 & 4 & 2 \\ -0.02 & 2 & 4 \end{pmatrix}\right)\Big|_{[6,14] \times \mathbb{R} \times \mathbb{R}}.$$

Since $X_0, X_1 \in \mathrm{L}^p(\Omega)$ for all $1 \le p < \infty$ and A is bounded in $[6, 14]$, Theorem 2.4 shows that the stochastic process $X(t)$ defined by (3)–(5) is the

unique $L^p(\Omega)$ solution to (1) on $(-1, 1)$, for each $1 \leq p < \infty$. Analogously to the previous two examples, Table 5 and Table 6 show the results. Observe that stabilization of the results for $t \leq 0.7$ is achieved for $M = 80$. Notice also that, for $M \leq 20$ and $t \geq 0.4$, the approximation of the expectation and variance is not good. The results obtained from the Fröbenius method for $M \geq 80$ agree with the statistics calculated via Monte Carlo simulations. On the other hand, the approximations performed by gPC expansions are not good. This is due to the accumulation of numerical errors, which invalidates the corresponding results. See [28, 29] for an analysis of computational errors when working with gPC expansions. Thus, the Fröbenius method proves to be the best uncertainty quantification technique for this example.

t	$\mathbb{E}[X^{10}(t)]$	$\mathbb{E}[X^{20}(t)]$	$\mathbb{E}[X^{40}(t)]$	$\mathbb{E}[X^{80}(t)]$	MC 500,000	gPC $m = 3$	gPC $m = 4$
0	-2.01642	-2.01642	-2.01642	-2.01642	-2.00100	-2.01642	-2.01642
0.1	-0.905676	-0.905676	-0.905676	-0.905676	-0.905209	-0.917349	-0.91765
0.2	1.10885	1.10884	1.10884	1.10884	1.11031	1.10512	1.10054
0.3	1.94955	1.94909	1.94909	1.94909	1.94966	1.94172	1.93194
0.4	0.656784	0.643176	0.643176	0.643176	0.641893	0.616062	1.26544
0.5	-1.20831	-1.39804	-1.39804	-1.39804	-1.39941	-1.34976	16.2519
0.6	0.111123	-1.57901	-1.57903	-1.57903	-1.57838	-0.565463	286.226
0.7	10.8410	0.602665	0.602084	0.602084	0.594087	8.20453	1524.28
0.8	51.7915	1.58890	1.57617	1.57615	1.57588	50.6284	-211410
0.9	203.700	-0.987211	-1.20468	-1.20776	-1.20091	291.704	$-2.51516 \cdot 10^7$

TABLE 5. Approximation of the expectation of the solution stochastic process. Example 4.3.

t	$\mathbb{V}[X^{10}(t)]$	$\mathbb{V}[X^{20}(t)]$	$\mathbb{V}[X^{40}(t)]$	$\mathbb{V}[X^{80}(t)]$	MC 500,000	gPC $m = 3$	gPC $m = 4$
0	3.96931	3.96931	3.96931	3.96931	4.00268	3.96931	3.96931
0.1	1.23016	1.23016	1.23016	1.23016	1.22715	1.17839	1.16808
0.2	1.16167	1.16166	1.16166	1.16166	1.14804	0.816119	-7.83909
0.3	3.86797	3.87079	3.87079	3.87079	3.86348	-1.71661	-494.045
0.4	1.72091	1.76984	1.76984	1.76984	1.76343	-54.3357	156807
0.5	2.59759	2.75802	2.75802	2.75802	2.71796	-187.680	$1.98436 \cdot 10^7$
0.6	53.8179	3.79667	3.79665	3.79665	3.79030	7387.73	$-9.73065 \cdot 10^9$
0.7	1774.74	3.88379	3.87941	3.87941	3.88103	244717	$-1.46846 \cdot 10^{12}$
0.8	40373.8	5.25517	5.27273	5.27282	5.17336	$2.49059 \cdot 10^6$	$8.46091 \cdot 10^{15}$
0.9	658630	4.79558	7.67724	7.76726	7.73295	$-7.31059 \cdot 10^8$	$-9.69951 \cdot 10^{19}$

TABLE 6. Approximation of the variance of the solution stochastic process. Example 4.3.

5. Conclusions

In this article we have studied the random Legendre differential equation with input coefficient A and initial conditions X_0 and X_1. In [Calbo G. et

al, Comput. Math. Appl., 61(9), 2782–2792 (2011)], a mean square convergent random power series solution $X(t)$ on $(-1/e, 1/e)$ was constructed via the Fröbenius method. The authors proved that, under the assumption that the absolute moments of A grow at most exponentially, under mean fourth integrability of X_0 and X_1, and under independence of A and the initial conditions, the random power series becomes a mean square solution to the random Legendre differential equation on $(-1/e, 1/e)$. We have extended this result by assuming less integrability of X_0 and X_1 and no independence between the random inputs. Moreover, the growth condition on the absolute moments of A has been characterized in terms of the boundedness of A. This has permitted a simpler proof of our result, as no probabilistic inequalities (Hölder, c_s, etc.) have been required. Moreover, our random power series solution converges on the whole $(-1, 1)$, as it occurs with its deterministic counterpart. We have provided expressions for the approximate expectation and variance of $X(t)$, by truncating the random power series. In the numerical examples, we have illustrated the improvements developed by working with non-independent random inputs. Our approach has improved the approximations done by Monte Carlo simulations and gPC expansions.

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Conflict of Interest Statement

The authors declare that there is no conflict of interests regarding the publication of this article.

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