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Rank Equalities Related to a Class of Outer Generalized Inverse

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Abstract. In 2012, Drazin introduced a class of outer generalized inverse in a ring $\mathcal{R}$, the (b,c)-inverse of a for $a,b,c \in \mathcal{R}$ and denoted by $a^{\|(b,c)}$. In this paper, rank equalities of $A^kA^{\|(B,C)} - A^{\|(B,C)}A^k$ and $(A^*)^kA^{\|(B,C)} - A^{\|(B,C)}(A^*)^k$ are obtained. As applications, we investigate equivalent conditions for the equalities $(A^*)^kA^{\|(B,C)} = A^{\|(B,C)}(A^*)^k$ and $A^kA^{\|(B,C)} = A^{\|(B,C)}A^k$. As corollaries we obtain rank equalities related to the Moore-Penrose inverse, the core inverse, and the Drazin inverse. The paper finishes with some rank equalities involving different expressions containing $A^{\|(B,C)}$.

1. Introduction

There exist many generalized inverses of matrices in the literature, such as the group inverse, the Drazin inverse, the Moore-Penrose inverse, the core inverse, the inverse along an element and the outer inverse with prescribed range and null spaces. Many properties of such generalized inverses can be found in, for example, [1, 2, 4, 7, 12–17, 23]. The (B,C)-inverse of a matrix $A \in \mathbb{C}^{n\times m}$ (denoted by $A^{\|(B,C)}$ and it will be defined in next section) is a strong generalization of such inverses.

For $A \in \mathbb{C}^{n\times n}$ and $k \geq 1$, several authors have investigated the rank of $A^kX - XA^k$, $(A^*)^kX - X(A^*)^k$, $I \pm X$, $X \pm X^2$, $I - X^2$ and $X - X^3$, where X is some generalized inverse of A (see [12, 20]). In this paper, we extend these rank equalities to the (B,C)-inverse of A. The paper finishes studying the rank of $A^{\|(B,C)} - A^{\|(D,E)}$ (which permits obtain when several generalized inverses coincide) and generalizing the Schur complement to the (B,C)-inverse.

2. Preliminaries

The set of all $m \times n$ matrices over the complex field $\mathbb{C}$ will be denoted by $\mathbb{C}^{m\times n}$. Let A^*, $\mathcal{R}(A)$, $\mathcal{N}(A)$ and $\mathrm{rk}(A)$ denote the conjugate transpose, column space, null space and rank of $A \in \mathbb{C}^{m\times n}$, respectively. Moreover, I stands for the identity matrix of appropriate order.

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In [4, Definition 4.1], Benítez et al. extended the (B, C)-inverse from elements in a semigroup (see [8]) to rectangular matrices as follows: let $A \in \mathbb{C}^{m\times n}$ and $B, C \in \mathbb{C}^{n\times m}$, then the matrix A is said to be (B, C)-*invertible*, if there exists a matrix $Y \in \mathbb{C}^{n\times m}$ such that

$$YAB = B, \quad CAY = C, \quad \mathcal{R}(Y) \subseteq \mathcal{R}(B) \quad \text{and} \quad \mathcal{N}(C) \subseteq \mathcal{N}(Y). \tag{1}$$

If such a matrix Y exists, then it is unique and denoted by $A^{\|(B,C)}$. Many existence criteria and properties of the (B, C)-inverse can be found in, for example, [4, 6, 9–11, 18, 22, 24].

The (B, C)-inverse of A is a generalization of some well-known generalized inverses (see [8, p.1910]). The Moore-Penrose inverse of A, denoted by $A^\dagger$, is the (A^*, A^*)-inverse of A. The inverse along $D \in \mathbb{C}^{n\times m}$ of A, denoted by $A^{\|D}$, is the (D, D)-inverse of A. The group inverse of A, denoted by $A^\#$, is the (A, A)-inverse of A. The Drazin inverse of A, denoded by A^D, is the (A^k, A^k)-inverse of A, where $k = \mathrm{ind}(A)$, the index of A, is the smallest nonnegative integer for which $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$. By [19, Theorem 4.4], the (A, A^*)-inverse of A coincides with the core inverse of A, denoted by $A^{\circledast}$. Moreover, the following affirmations are equivalent: (a) the core inverse of A exists, (b) the group inverse exists, (c) $\mathrm{ind}(A) \leq 1$.

Observe that from (1) one gets

$$\mathcal{R}(A^{\|(B,C)}) = \mathcal{R}(B) \quad \text{and} \quad \mathcal{N}(A^{\|(B,C)}) = \mathcal{N}(C) \tag{2}$$

in case that $A^{\|(B,C)}$ exists.

The following lemmas will be used in the sequel.

Lemma 2.1. [4, Theorem 4.4] *If $A \in \mathbb{C}^{m\times n}$ and $B, C \in \mathbb{C}^{n\times m}$, then the following statements are equivalent:*

(1) *$A^{\|(B,C)}$ exists.*

(2) $\mathrm{rk}\,(B) = \mathrm{rk}\,(C) = \mathrm{rk}\,(CAB)$.

In this case, $A^{\|(B,C)} = B(CAB)^\dagger C$.

Lemma 2.2. [20, Theorem 2.2] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$, $C \in \mathbb{C}^{l\times n}$, $D \in \mathbb{C}^{l\times k}$, and*

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}.$$

If

$$\mathcal{R}(B_1) \subseteq \mathcal{R}(A_1), \quad \mathcal{R}(C_1^*) \subseteq \mathcal{R}(A_1^*), \quad \mathcal{R}(B_2) \subseteq \mathcal{R}(A_2), \quad \mathcal{R}(C_2^*) \subseteq \mathcal{R}(A_2^*),$$

then

$$\mathrm{rk}\left(D - C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2\right) = \mathrm{rk}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & D \end{bmatrix} - \mathrm{rk}(A_1) - \mathrm{rk}(A_2). \tag{3}$$

In addition, we shall also use the following several basic rank formulas in the sequel. Equalities (1) and (2) in the following lemma can be found in [20, Lemma 1.5].

Lemma 2.3. *If $A, B \in \mathbb{C}^{n\times n}$, then*

(1) $\mathrm{rk}(A - ABA) = \mathrm{rk}(A) + \mathrm{rk}(I - BA) - n$.

(2) $\mathrm{rk}(I - A^2) = \mathrm{rk}(I + A) + \mathrm{rk}(I - A) - n$.

(3) $\mathrm{rk}(A^*A^2 - A^*A) = \mathrm{rk}(A) + \mathrm{rk}(I - A) - n$.

(4) $\mathrm{rk}(A^*A^2 + A^*A) = \mathrm{rk}(A) + \mathrm{rk}(I + A) - n$.

Proof. (3) Since

$$\begin{bmatrix} I & 0 \\ -A^*A & I \end{bmatrix}\begin{bmatrix} I & A \\ A^*A & A^*A \end{bmatrix}\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & A^*A - A^*A^2 \end{bmatrix}$$

and

$$\begin{bmatrix} I & -(A^\dagger)^* \\ 0 & I \end{bmatrix}\begin{bmatrix} I & A \\ A^*A & A^*A \end{bmatrix}\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} = \begin{bmatrix} I - A & 0 \\ 0 & A^*A \end{bmatrix},$$

we have $\mathrm{rk}(A^*A^2 - A^*A) + n = \mathrm{rk}(A^*A) + \mathrm{rk}(I - A)$. In addition, it is known (and simple to prove) that $\mathrm{rk}(A) = \mathrm{rk}(A^*A)$ holds. Therefore, the proof of (3) is finished. To prove (4) it is enough to change A by $-A$ in (3). □

3. Main results

Let $A \in \mathbb{C}^{n\times n}$ be of rank r and let T, S be subspaces of $\mathbb{C}^n$ with $\dim T \leq r$ and $\dim T + \dim S = n$. In [2, Theorem 14, Chapter 2] it was characterized when exists a matrix $X \in \mathbb{C}^{n\times n}$ such that X is an outer inverse of A, $\mathcal{R}(X) = T$, and $\mathcal{N}(X) = S$. These conditions force to the uniqueness of such X (when X exists) and X is denoted $A^{(2)}_{T,S}$. Assume that $A^{(2)}_{T,S}$ exists. In [12, Theorem 2.3], Liu proved the following rank equality

$$\mathrm{rk}\left(A^k A^{(2)}_{T,S} - A^{(2)}_{T,S}A^k\right) = \mathrm{rk}\begin{bmatrix} AG \\ GA^k \end{bmatrix} + \mathrm{rk}\left[A^kG \mid GA\right] - 2\,\mathrm{rk}(AG).$$

where $G \in \mathbb{C}^{n\times n}$ satisfies $\mathcal{R}(G) = T$, $\mathcal{N}(G) = S$ and $k \geq 1$.

In [4, Theorem 7.1], Benítez et al. showed that $A^{(2)}_{\mathcal{R}(D),\mathcal{N}(E)} = A^{\|(D,E)}$ when $A \in \mathbb{C}^{n\times m}$, $D, E \in \mathbb{C}^{m\times n}$ and $A^{\|(D,E)}$ exists. According this, we can consider when

$$\mathrm{rk}\left(AA^{\|(B,C)} - A^{\|(B,C)}A\right) = \mathrm{rk}\begin{bmatrix} AB \\ CA \end{bmatrix} + \mathrm{rk}\,[AB \mid CA] - \mathrm{rk}(AB) - \mathrm{rk}(CA) \tag{4}$$

holds for $B, C \in \mathbb{C}^{n\times n}$. The following example shows that (4) does not hold in general.

Example 3.1. Consider the following matrices

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

By using [4, Theorem 4.4], it is easy to prove that $A^{\|(B,C)}$ exists and

$$A^{\|(B,C)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Moreover, one has that $\mathrm{rk}\begin{bmatrix} AB \\ CA \end{bmatrix} = 3$, $\mathrm{rk}\,[AB \mid CA] = 3$, $\mathrm{rk}(AB) = 2$, $\mathrm{rk}(CA) = 2$, and $\mathrm{rk}\left(AA^{\|(B,C)} - A^{\|(B,C)}A\right) = 1$.

Next two lemmas will be useful in the sequel.

Lemma 3.2. [21, Theorem 2.19] *Let $A \in \mathbb{C}^{m\times n}$, $P^2 = P \in \mathbb{C}^{m\times m}$ and $Q^2 = Q \in \mathbb{C}^{n\times n}$. The difference $PA - AQ$ satisfies the following rank equality*

$$\mathrm{rk}(PA - AQ) = \mathrm{rk}\begin{bmatrix} Q \\ PA \end{bmatrix} + \mathrm{rk}\,[AQ \mid P] - \mathrm{rk}(P) - \mathrm{rk}(Q).$$

Lemma 3.3. *If $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{k\times n}$, then*

(1) $\operatorname{rk}[A \mid B] = \operatorname{rk}(B)$ *if and only if* $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

(2) $\operatorname{rk}\begin{bmatrix} A \\ C \end{bmatrix} = \operatorname{rk}(C)$ *if and only if* $\mathcal{N}(C) \subseteq \mathcal{N}(A)$.

Proof. (1) follows from the equality $\operatorname{rk}[A \mid B] = \operatorname{rk}(A) + \operatorname{rk}(B) - \dim(\mathcal{R}(A) \cap \mathcal{R}(B))$.
(2) follows from (1) and $\mathcal{N}(X)^{\perp} = \mathcal{R}(X^*)$ valid for any matrix X. □

Rank equality of the commutator $A^k A^{\|(B,C)} - A^{\|(B,C)} A^k$ will be considered in the following theorem.

Theorem 3.4. *Let $A, B, C \in \mathbb{C}^{n\times n}$. If $A^{\|(B,C)}$ exists, then for any $k \geq 1$, we have*

(1) $\operatorname{rk}\left(A^k A^{\|(B,C)} - A^{\|(B,C)} A^k\right) = \operatorname{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix} + \operatorname{rk}\left[A^k B \mid B\right] - 2\operatorname{rk}(B)$.

(2) *The following statements are equivalent:*

(a) $A^k A^{\|(B,C)} = A^{\|(B,C)} A^k$.

(b) $\operatorname{rk}\left[A^k B \mid B\right] = \operatorname{rk}(B)$ *and* $\operatorname{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix} = \operatorname{rk}(C)$.

(c) $\mathcal{R}(A^k B) \subseteq \mathcal{R}(B)$ *and* $\mathcal{N}(C) \subseteq \mathcal{N}(CA^k)$.

Proof. (1) Let $X = A^{\|(B,C)}$. Since X is an outer inverse of A, we have that both AX and XA are idempotents. Note that $A^k X - XA^k$ can be written as $-(XAA^{k-1} - A^{k-1}AX)$. Thus, by Lemma 3.2, we have

$$\operatorname{rk}(A^k X - XA^k) = \operatorname{rk}\begin{bmatrix} AX \\ XA^k \end{bmatrix} + \operatorname{rk}\left[A^k X \mid XA\right] - \operatorname{rk}(AX) - \operatorname{rk}(XA).$$

Since $XAB = B$ and $\mathcal{R}(X) = \mathcal{R}(B)$,

$$\operatorname{rk}(B) = \operatorname{rk}(XAB) \leq \operatorname{rk}(AB) = \operatorname{rk}(AXAB) \leq \operatorname{rk}(AX) \leq \operatorname{rk}(X) = \operatorname{rk}(B). \tag{5}$$

Similarly, the conditions $CAX = C$ and $\mathcal{N}(X) = \mathcal{N}(C)$ imply

$$\operatorname{rk}(C) = \operatorname{rk}(CAX) \leq \operatorname{rk}(CA) = \operatorname{rk}(CAXA) \leq \operatorname{rk}(XA) \leq \operatorname{rk}(X) = \operatorname{rk}(C). \tag{6}$$

From (5) and (6), we have

$$\operatorname{rk}(AB) = \operatorname{rk}(AX) = \operatorname{rk}(CA) = \operatorname{rk}(XA) = \operatorname{rk}(B) = \operatorname{rk}(C).$$

By Lemma 2.1, we have $X = B(CAB)^{\dagger}C$. Now, $\mathcal{R}(A^k B) = \mathcal{R}(A^k XAB) \subseteq \mathcal{R}(A^k X) = \mathcal{R}(A^k B(CAB)^{\dagger}C) \subseteq \mathcal{R}(A^k B)$, therefore, $\mathcal{R}(A^k B) = \mathcal{R}(A^k X)$. Also we have $\mathcal{R}(XA) = \mathcal{R}(B(CAB)^{\dagger}CA) \subseteq \mathcal{R}(B) = \mathcal{R}(XAB) \subseteq \mathcal{R}(XA)$, so $\mathcal{R}(XA) = \mathcal{R}(B)$. Therefore,

$$\operatorname{rk}\left[A^k X \mid XA\right] = \dim\left(\mathcal{R}(A^k X) + \mathcal{R}(XA)\right) = \dim\left(\mathcal{R}(A^k B) + \mathcal{R}(B)\right) = \operatorname{rk}\left[A^k B \mid B\right].$$

In addition, $\mathcal{N}(AX) \subseteq \mathcal{N}(CAX) = \mathcal{N}(C) \subseteq \mathcal{N}(AB(CAB)^{\dagger}C) = \mathcal{N}(AX)$ and $\mathcal{N}(CA^k) \subseteq \mathcal{N}(B(CAB)^{\dagger}CA^k) = \mathcal{N}(XA^k) \subseteq \mathcal{N}(CAXA^k) = \mathcal{N}(CA^k)$ imply

$$\begin{aligned} \operatorname{rk}\begin{bmatrix} AX \\ XA^k \end{bmatrix} &= \operatorname{rk}\left[(AX)^* \mid (XA^k)^*\right] = \dim\left(\mathcal{R}[(AX)^*] + \mathcal{R}[(XA^k)^*]\right) \\ &= \dim\left(\mathcal{N}(AX)^{\perp} + \mathcal{N}(XA^k)^{\perp}\right) = \dim\left(\mathcal{N}(C)^{\perp} + \mathcal{N}(CA^k)^{\perp}\right) = \operatorname{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix}. \end{aligned}$$

The proof of (1) is finished.

(2) (a)⇔(b). By (1) and using $\mathrm{rk}(B) = \mathrm{rk}(C)$ one has

$$\mathrm{rk}(A^k A^{\|(B,C)} - A^{\|(B,C)} A^k) = \left(\mathrm{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix} - \mathrm{rk}(C)\right) + \left(\mathrm{rk}\begin{bmatrix} A^k B \mid B \end{bmatrix} - \mathrm{rk}(B)\right).$$

The proof of (a)⇔(b) follows by this last equality and having in mind that $\mathrm{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix} - \mathrm{rk}(C)$ and $\mathrm{rk}\begin{bmatrix} A^k B \mid B \end{bmatrix} - \mathrm{rk}(B)$ are always nonnegative.

(b)⇔(c) Since $\mathrm{rk}\begin{bmatrix} C \\ CA^k \end{bmatrix} = \mathrm{rk}\begin{bmatrix} CA^k \\ C \end{bmatrix}$, it is obvious by Lemma 3.3. □

As a trivial corollary it is obtained an equivalent condition for $AA^{\|(B,C)} = A^{\|(B,C)}A$. When $B = C$ and in rings, the same equivalence was given in [3, Theorem 7.3].

Since the Moore-Penrose inverse of A coincides with the (A^*, A^*)-inverse of A ([15, Theorem 11]), by letting $B = C = A^*$ in Theorem 3.4, the following corollary is obtained.

Corollary 3.5. [20, Theorem 2.8] *Let $A \in \mathbb{C}^{n\times n}$. For any $k \geq 1$, we have*

(1) $\mathrm{rk}\left(A^k A^\dagger - A^\dagger A^k\right) = \mathrm{rk}\begin{bmatrix} A^* \\ A^k \end{bmatrix} + \mathrm{rk}[A^k \mid A^*] - 2\,\mathrm{rk}(A)$.

(2) *The following statements are equivalent:*

   (a) $A^k A^\dagger = A^\dagger A^k$.

   (b) $\mathrm{rk}(A) = \mathrm{rk}\left[(A^k)^* A \mid A\right] = \mathrm{rk}\left[A^k A^* \mid A^*\right]$.

   (c) $\mathcal{R}(A^k A^*) \subseteq \mathcal{R}(A^*)$ *and* $\mathcal{N}(A^*) \subseteq \mathcal{N}(A^* A^k)$.

Proof. (1) Since $\mathcal{R}(A^k) = \mathcal{R}(A^{k-1} A A^\dagger A) = \mathcal{R}(A^k A^* (A^\dagger)^*) \subseteq \mathcal{R}(A^k A^*) \subseteq \mathcal{R}(A^k)$ we get

$$\mathrm{rk}\left[A^k A^* \mid A^*\right] = \dim\left(\mathcal{R}(A^k A^*) + \mathcal{R}(A^*)\right) = \dim\left(\mathcal{R}(A^k) + \mathcal{R}(A^*)\right) = \mathrm{rk}\left[A^k \mid A^*\right].$$

By using this last equality for $B = A^*$, we have

$$\mathrm{rk}\begin{bmatrix} A^* \\ A^* A^k \end{bmatrix} = \mathrm{rk}[A \mid (A^*)^k A] = \mathrm{rk}[B^* \mid B^k B^*] = \mathrm{rk}[B^* \mid B^k] = \mathrm{rk}\begin{bmatrix} B \\ (B^*)^k \end{bmatrix} = \mathrm{rk}\begin{bmatrix} A^* \\ A^k \end{bmatrix},$$

and the proof of (1) is finished.

(2) Let $B = C = A^*$ in item (2) of Theorem 3.4. □

In particular, by letting $k = 1$ in Corollary 3.5 we obtain the following corollary.

Corollary 3.6. *Let $A \in \mathbb{C}^{n\times n}$. Then*

(1) [20, Theorem 2.1] $\mathrm{rk}(AA^\dagger - A^\dagger A) = 2\,\mathrm{rk}[A \mid A^*] - 2\,\mathrm{rk}(A)$.

(2) *The following statements are equivalent:*

   (a) $AA^\dagger = A^\dagger A$, *that is, A is an EP matrix.*

   (b) $\mathrm{rk}[A \mid A^*] = \mathrm{rk}(A)$.

   (c) $\mathcal{R}(A^*) \subseteq \mathcal{R}(A)$.

   (d) $\mathcal{R}(A) \subseteq \mathcal{R}(A^*)$.

   (e) $\mathcal{R}(A) = \mathcal{R}(A^*)$.

Definition 3.7. [5, Definition 1.2] *Let $A \in \mathbb{C}^{n\times n}$. We call that A is* co-EP *if $AA^\dagger - A^\dagger A$ is nonsingular.*

By Corollary 3.6 and Definition 3.7, we have the following corollary.

Corollary 3.8. *Let $A \in \mathbb{C}^{n\times n}$. Then A is co-EP if and only if $n = 2\,\mathrm{rk}[A \mid A^*] - 2\,\mathrm{rk}(A)$.*

According to [19, Theorem 4.4] (i), an equivalent condition for $A \in \mathbb{C}^{n\times n}$ to be core invertible is that A is (A, A^*)-invertible, and in this case, $A^{\#\!\!\!\!\bigcirc} = A^{\|(A,A^*)}$. Also, it is known that a matrix $A \in \mathbb{C}^{n\times n}$ is core invertible if and only if the index of A is less or equal than 1 (see [1]).

Corollary 3.9. *Let $A \in \mathbb{C}^{n\times n}$ be core invertible. For any $k \geq 1$, we have*

(1) $\mathrm{rk}(A^kA^{\#\!\!\!\!\bigcirc} - A^{\#\!\!\!\!\bigcirc}A^k) = \mathrm{rk}\begin{bmatrix} A^* \\ A^k \end{bmatrix} + \mathrm{rk}[A^{k+1} \mid A] - 2\,\mathrm{rk}(A)$.

(2) *The following statements are equivalent:*

  (a) $A^kA^{\#\!\!\!\!\bigcirc} = A^{\#\!\!\!\!\bigcirc}A^k$.

  (b) $\mathrm{rk}\begin{bmatrix} A^* \\ A^k \end{bmatrix} = \mathrm{rk}(A^*)$.

  (c) $\mathcal{N}(A^*) \subseteq \mathcal{N}(A^k)$.

Proof. It follows from Lemma 3.3, and Theorem 3.4. □

Lemma 3.10. *Let $A, B_1, B_2, C_1, C_2 \in \mathbb{C}^{n\times n}$. If $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$ and $\mathcal{R}(C_1^*) \subseteq \mathcal{R}(C_2^*)$, then*

$$\mathrm{rk}\begin{bmatrix} A & 0 & C_1 \\ 0 & 0 & C_2 \\ B_1 & B_2 & 0 \end{bmatrix} = \mathrm{rk}(A) + \mathrm{rk}(B_2) + \mathrm{rk}(C_2).$$

Proof. Since $\mathcal{R}(B_1) \subseteq \mathcal{R}(B_2)$, there exists $X \in \mathbb{C}^{n\times n}$ such that $B_1 = B_2X$. Since $\mathcal{R}(C_1^*) \subseteq \mathcal{R}(C_2^*)$, there exists $Y \in \mathbb{C}^{n\times n}$ such that $C_1 = YC_2$. The equality

$$\begin{bmatrix} I & Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} A & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & B_2 \end{bmatrix}\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ X & 0 & I \end{bmatrix} = \begin{bmatrix} A & C_1 & 0 \\ 0 & C_2 & 0 \\ B_1 & 0 & B_2 \end{bmatrix}$$

finishes the proof of this lemma. □

In [20], Tian gave rank equalities of the commutator $A^*A^\dagger - A^\dagger A^*$, in [12], Liu investigated rank equalities of the commutator $A^*A^{(2)}_{T,S} - A^{(2)}_{T,S}A^*$. We will extend these equalities to the (B, C)-inverse of A, i.e., we will study the rank of $(A^*)^kA^{\|(B,C)} - A^{\|(B,C)}(A^*)^k$ in the following theorem, where $k \geq 1$.

Theorem 3.11. *Let $A, B, C \in \mathbb{C}^{n\times n}$. If $A^{\|(B,C)}$ exists, then for any $k \geq 1$, we have*

(1) $\mathrm{rk}\left[(A^*)^kA^{\|(B,C)} - A^{\|(B,C)}(A^*)^k\right] = \mathrm{rk}\begin{bmatrix} C\left[(A^*)^kA - A(A^*)^k\right]B & 0 & C(A^*)^k \\ 0 & 0 & C \\ (A^*)^kB & B & 0 \end{bmatrix} - 2\,\mathrm{rk}(B)$.

(2) *If $\mathcal{R}\left[(A^*)^kB\right] \subseteq \mathcal{R}(B)$ and $\mathcal{R}(A^kC^*) \subseteq \mathcal{R}(C^*)$, then*

$$\mathrm{rk}\left[(A^*)^kA^{\|(B,C)} - A^{\|(B,C)}(A^*)^k\right] = \mathrm{rk}\left[C[(A^*)^kA - A(A^*)^k]B\right].$$

Proof. (1) By Lemma 2.1, we have $A^{\|(B,C)} = B(CAB)^\dagger C$. Hence

$$\begin{aligned}(A^*)^kA^{\|(B,C)} - A^{\|(B,C)}(A^*)^k &= (A^*)^kB(CAB)^\dagger C - B(CAB)^\dagger C(A^*)^k \\ &= -C_1A_1^\dagger B_1 - C_2A_2^\dagger B_2,\end{aligned}$$

where $C_1 = -(A^*)^k B$, $A_1 = A_2 = CAB$, $B_1 = C$, $C_2 = B$, and $B_2 = C(A^*)^k$.

We claim that

$$\mathcal{R}(B_1) \subseteq \mathcal{R}(A_1), \quad \mathcal{R}(C_1^*) \subseteq \mathcal{R}(A_1^*), \quad \mathcal{R}(B_2) \subseteq \mathcal{R}(A_2), \quad \mathcal{R}(C_2^*) \subseteq \mathcal{R}(A_2^*).$$

Two useful equalities to prove this claim are $B(CAB)^\dagger CAB = B$ and $CAB(CAB)^\dagger C = C$. In fact, applying $A^{\|(B,C)} = B(CAB)^\dagger C$ and the definition of the (B,C)-inverse we obtain, $B(CAB)^\dagger CAB = A^{\|(B,C)}AB = B$ and $CAB(CAB)^\dagger C = CAA^{\|(B,C)} = C$. Now, we have

$$\mathcal{R}(B_1) = \mathcal{R}(C) = \mathcal{R}\left(CAB(CAB)^\dagger C\right) = \mathcal{R}(A_1A_1^\dagger C) \subseteq \mathcal{R}(A_1).$$

From

$$\mathcal{N}(A_1) \subseteq \mathcal{N}(C_1A_1^\dagger A_1) = \mathcal{N}\left((A^*)^k B(CAB)^\dagger CAB\right) = \mathcal{N}\left((A^*)^k B\right) = \mathcal{N}(C_1),$$

we get $\mathcal{R}(C_1^*) \subseteq \mathcal{R}(A_1^*)$. Also we have

$$\mathcal{R}(B_2) = \mathcal{R}\left(C(A^*)^k\right) = \mathcal{R}\left(CAB(CAB)^\dagger C(A^*)^k\right) = \mathcal{R}\left(A_2A_2^\dagger B_2\right) \subseteq \mathcal{R}(A_2).$$

From

$$\mathcal{N}(A_2) \subseteq \mathcal{N}\left(C_2A_2^\dagger A_2\right) = \mathcal{N}\left(B(CAB)^\dagger CAB\right) = \mathcal{N}(B) = \mathcal{N}(C_2),$$

we deduce $\mathcal{R}(C_2^*) \subseteq \mathcal{R}(A_2^*)$.

Applying equality (3) of Lemma 2.2, we have

$$\begin{aligned}
\operatorname{rk}\left[(A^*)^k A^{\|(B,C)} - A^{\|(B,C)}(A^*)^k\right] &= \operatorname{rk}\left[C_1A_1^\dagger B_1 + C_2A_2^\dagger B_2\right] \\
&= \operatorname{rk}\begin{bmatrix} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ C_1 & C_2 & 0 \end{bmatrix} - \operatorname{rk}(A_1) - \operatorname{rk}(A_2) \\
&= \operatorname{rk}\begin{bmatrix} CAB & 0 & C \\ 0 & CAB & C(A^*)^k \\ -(A^*)^k B & B & 0 \end{bmatrix} - 2\operatorname{rk}(CAB).
\end{aligned}$$

Now we apply block Gaussian elimination.

$$\begin{aligned}
\operatorname{rk}\begin{bmatrix} CAB & 0 & C \\ 0 & CAB & C(A^*)^k \\ -(A^*)^k B & B & 0 \end{bmatrix} &\overset{(a)}{=} \operatorname{rk}\begin{bmatrix} 0 & 0 & C \\ -C(A^*)^k AB & CAB & C(A^*)^k \\ -(A^*)^k B & B & 0 \end{bmatrix} \\
&= \operatorname{rk}\begin{bmatrix} 0 & 0 & C \\ C(A^*)^k AB & CAB & C(A^*)^k \\ (A^*)^k B & B & 0 \end{bmatrix} \\
&\overset{(b)}{=} \operatorname{rk}\begin{bmatrix} 0 & 0 & C \\ C\left[(A^*)^k A - A(A^*)^k\right]B & 0 & C(A^*)^k \\ (A^*)^k B & B & 0 \end{bmatrix} \\
&= \operatorname{rk}\begin{bmatrix} C\left[(A^*)^k A - A(A^*)^k\right]B & 0 & C(A^*)^k \\ 0 & 0 & C \\ (A^*)^k B & B & 0 \end{bmatrix},
\end{aligned}$$

where the non evident steps are

(a) 1st column is replaced by 1st column − 3rd column $*$ AB.

(b) 2^{nd} row is replaced by 2^{nd} row $- CA * 3^{\text{rd}}$ row.

By Lemma 2.1, we have $\text{rk}(B) = \text{rk}(CAB)$. Thus, (1) is obtained.

(2) Assume that $\mathcal{R}\left[(A^*)^k B\right] \subseteq \mathcal{R}(B)$, $\mathcal{R}\left[A^k C^*\right] \subseteq \mathcal{R}(C^*)$. By Lemma 3.10, we have

$$\text{rk}\begin{bmatrix} C\left[(A^*)^k A - A(A^*)^k\right]B & 0 & C(A^*)^k \\ 0 & 0 & C \\ (A^*)^k B & B & 0 \end{bmatrix} = \text{rk}\left[C[(A^*)^k A - A(A^*)^k]B\right] + \text{rk}(B) + \text{rk}(C). \quad (7)$$

The proof of (2) follows by (1), Lemma 2.1, and (7). $\square$

Since $A^{\dagger} = A^{\|(A^*,A^*)}$, Theorem 3.11 implies next corollary.

Corollary 3.12. *If $A \in \mathbb{C}^{n\times n}$ and $k \geq 1$, then*

$$\text{rk}\left[(A^*)^k A^{\dagger} - A^{\dagger}(A^*)^k\right] = \text{rk}\left[(A^*)^{k+1}AA^* - A^*A(A^*)^{k+1}\right].$$

Having in mind that if $A \in \mathbb{C}^{n\times n}$ is core invertible, then $A^{\#} = A^{\|(A,A^*)}$, Theorem 3.11 implies next result.

Corollary 3.13. *Let $A \in \mathbb{C}^{n\times n}$ be core invertible and $k \geq 1$. We have*

(1) $\text{rk}\left[(A^*)^k A^{\#} - A^{\#}(A^*)^k\right] = \text{rk}\begin{bmatrix} A^*\left((A^*)^k A - A(A^*)^k\right)A & 0 & (A^*)^{k+1} \\ 0 & 0 & A^* \\ (A^*)^k A & A & 0 \end{bmatrix} - 2\,\text{rk}(A).$

(2) *If $\mathcal{R}\left[(A^*)^k A\right] \subseteq \mathcal{R}(A)$, then*

$$\text{rk}\left[(A^*)^k A^{\#} - A^{\#}(A^*)^k\right] = \text{rk}\left[A^*((A^*)^k A - A(A^*)^k)A\right].$$

In [12, Section 3], Liu presented some rank equalities of matrix expressions involving powers of the generalized inverse $A^{(2)}_{T,S}$ of a matrix. We will extend this study to the (B,C)-inverse of a matrix.

Theorem 3.14. *Let $A, B, C \in \mathbb{C}^{n\times n}$. If $A^{\|(B,C)}$ exists, then*

(1) $\text{rk}(I \pm A^{\|(B,C)}) = \text{rk}(CAB \pm CB) - \text{rk}(B) + n.$

(2) $\text{rk}\left(A^{\|(B,C)} \pm (A^{\|(B,C)})^2\right) = \text{rk}(CAB \pm CB).$

In particular, $A^{\|(B,C)} = \left(A^{\|(B,C)}\right)^2$ if and only if $CAB = CB$.

Proof. (1) By Lemma 2.1, we obtain $\text{rk}(I - A^{\|(B,C)}) = \text{rk}(I - B(CAB)^{\dagger}C)$. We claim that $\mathcal{R}(C) \subseteq \mathcal{R}(CAB)$ and $\mathcal{R}(B^*) \subseteq \mathcal{R}((CAB)^*)$. Indeed, since $\mathcal{R}(CAB) \subseteq \mathcal{R}(C)$ and Lemma 2.1 (ii) we get $\mathcal{R}(CAB) = \mathcal{R}(C)$, and also $\mathcal{N}(B) \subseteq \mathcal{N}(CAB)$ and Lemma 2.1 (ii) imply $\mathcal{N}(B) = \mathcal{N}(CAB)$, which leads to $\mathcal{R}(B^*) = \mathcal{R}((CAB)^*)$. Thus, applying equation (3) in Lemma 2.2, we have

$$\text{rk}\left(I - B(CAB)^{\dagger}C\right) = \text{rk}\begin{bmatrix} CAB & 0 & C \\ 0 & 0 & 0 \\ B & 0 & I \end{bmatrix} - \text{rk}(CAB) = \text{rk}\begin{bmatrix} CAB & C \\ B & I \end{bmatrix} - \text{rk}(CAB).$$

The evident equality

$$\begin{bmatrix} I & -C \\ 0 & I \end{bmatrix}\begin{bmatrix} CAB & C \\ B & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} = \begin{bmatrix} CAB - CB & 0 \\ 0 & I \end{bmatrix}$$

proves

$$\mathrm{rk}\begin{bmatrix} CAB & C \\ B & I \end{bmatrix} = \mathrm{rk}(CAB - CB) + n,$$

which together with Lemma 2.1 (2) finishes the proof of $\mathrm{rk}(I - A^{\|(B,C)}) = \mathrm{rk}(CAB - CB) - \mathrm{rk}(B) + n$.

Lemma 2.1 and the existence of $A^{\|(B,C)}$ lead to the existence of $(-A)^{\|(B,C)}$ and $(-A)^{\|(B,C)} = -A^{\|(B,C)}$. Apply what has been proved for $-A$ instead of A to get $\mathrm{rk}(I + A^{\|(B,C)}) = \mathrm{rk}(CAB + CB) - \mathrm{rk}(B) + n$.

(2) Replacing A by $A^{\|(B,C)}$ and B by I in the equality (1) of Lemma 2.3 leads to $\mathrm{rk}\left(A^{\|(B,C)} - (A^{\|(B,C)})^2\right) = \mathrm{rk}\left(A^{\|(B,C)}\right) + \mathrm{rk}\left(I - A^{\|(B,C)}\right) - n$. Observe that (2) implies $\mathrm{rk}(A^{\|(B,C)}) = \mathrm{rk}(B)$. Use (1) to get $\mathrm{rk}\left(A^{\|(B,C)} - (A^{\|(B,C)})^2\right) = \mathrm{rk}(CAB - CB)$. Replace A by $-A$ in this last equality to obtain $\mathrm{rk}\left(A^{\|(B,C)} + (A^{\|(B,C)})^2\right) = \mathrm{rk}(CAB + CB)$. $\square$

Setting $B = C = A^*$ for a given $A \in \mathbb{C}^{n\times n}$ in Theorem 3.14 leads to

$$\mathrm{rk}(I \pm A^\dagger) = \mathrm{rk}(AA^*A \pm A^2) - \mathrm{rk}(A) + n, \qquad \mathrm{rk}(A^\dagger \pm (A^\dagger)^2) = \mathrm{rk}(AA^*A \pm A^2).$$

Notice that these equalities where given in [20, Theorem 6.15] and in [20, Theorem 6.16].

If the index of $A \in \mathbb{C}^{n\times n}$ is k, then it is evident that $\mathrm{rk}(A^{2k+1} \pm A^{2k}) = \mathrm{rk}(A^{k+1} \pm A^k)$. Setting $B = C = A^k$ in Theorem 3.14 we get

$$\mathrm{rk}(I \pm A^D) = \mathrm{rk}(A^{2k+1} \pm A^{2k}) - \mathrm{rk}(A^k) + n = \mathrm{rk}(A^{k+1} \pm A^k) - \mathrm{rk}(A^k) + n$$

and

$$\mathrm{rk}\left(A^D \pm (A^D)^2\right) = \mathrm{rk}(A^{2k+1} \pm A^{2k}) = \mathrm{rk}(A^{k+1} \pm A^k).$$

These equalities were given in [20, Theorem 13.1] and [20, Theorem 13.6].

Corollary 3.15. *If $A \in \mathbb{C}^{n\times n}$ is core invertible, then*

(1) $\mathrm{rk}(I \pm A^{\#}) = \mathrm{rk}(I \pm A)$.

(2) $\mathrm{rk}(A^{\#} \pm (A^{\#})^2) = \mathrm{rk}(A^*A^2 \pm A^*A)$.

In particular, $A^{\#} = (A^{\#})^2$ if and only if $A^*A^2 = A^*A$.

Proof. Firstly, recall $A^{\#} = A^{\|(A,A^*)}$.

(1) By using Theorem 3.14 and the equality (3) in Lemma 2.3, we get $\mathrm{rk}(I - A^{\#}) = \mathrm{rk}(A^*A^2 - A^*A) - \mathrm{rk}(A) + n = \mathrm{rk}(I - A)$. Replacing A by $-A$ in what has been proved we get $\mathrm{rk}(I + A^{\#}) = \mathrm{rk}(I + A)$.

(2) Set $B = A$ and $C = A^*$ in item (2) of Theorem 3.14. $\square$

Theorem 3.16. *Let $A \in \mathbb{C}^{n\times n}$. If exists $A^{\|(B,C)}$, then*

(1) $\mathrm{rk}\left(I - (A^{\|(B,C)})^2\right) = \mathrm{rk}(CAB + CB) + \mathrm{rk}(CAB - CB) - 2\,\mathrm{rk}(B) + n$.

(2) $\mathrm{rk}\left(A^{\|(B,C)} - (A^{\|(B,C)})^3\right) = \mathrm{rk}(CAB + CB) + \mathrm{rk}(CAB - CB) - \mathrm{rk}(B)$.

In particular, $A^{\|(B,C)} = \left(A^{\|(B,C)}\right)^3$ if and only if $\mathrm{rk}(CAB + CB) + \mathrm{rk}(CAB - CB) = \mathrm{rk}(B)$.

Proof. (1) Apply the equality (2) of Lemma 2.3 and item (1) of Theorem 3.14.

(2) By item (1) of Lemma 2.3 we have $\mathrm{rk}(A - A^3) = \mathrm{rk}(A) + \mathrm{rk}(I - A^2) - n$, and now, by item (2) of Lemma 2.3, $\mathrm{rk}(A - A^3) = \mathrm{rk}(A) + \mathrm{rk}(I + A) + \mathrm{rk}(I - A) - 2n$. Replace A by $A^{\|(B,C)}$ in this last equality, use item (1) of Theorem 3.14, and item (2) of Lemma 2.1 to get

$$\begin{aligned} &\mathrm{rk}(A^{\|(B,C)} - (A^{\|(B,C)})^3) \\ &\quad = \mathrm{rk}(B) + [\mathrm{rk}(CAB + CB) - \mathrm{rk}(B) + n] + [\mathrm{rk}(CAB - CB) - \mathrm{rk}(B) + n] - 2n \\ &\quad = \mathrm{rk}(CAB + CB) + \mathrm{rk}(CAB - CB) - \mathrm{rk}(B). \end{aligned}$$

The proof is finished. $\square$

Setting $B = C = A^*$ in Theorem 3.16 leads to

$$\operatorname{rk}\left(I - (A^\dagger)^2\right) = \operatorname{rk}(AA^*A + A^2) + \operatorname{rk}(AA^*A - A^2) - 2\operatorname{rk}(A) + n$$

and

$$\operatorname{rk}\left(A^\dagger - (A^\dagger)^3\right) = \operatorname{rk}(AA^*A + A^2) + \operatorname{rk}(AA^*A - A^2) - \operatorname{rk}(A).$$

These equalities were given in [20, Theorem 6.15] and in [20, Theorem 6.17].

Let k be the index of $A \in \mathbb{C}^{n\times n}$. By setting $B = C = A^k$ in Theorem 3.16 and using $\operatorname{rk}(A^{2k+1} \pm A^{2k}) = \operatorname{rk}(A^{k+1} \pm A^k)$ we obtain

$$\operatorname{rk}\left(I - (A^D)^2\right) = \operatorname{rk}(A^{k+1} + A^k) + \operatorname{rk}(A^{k+1} - A^k) - 2\operatorname{rk}(A^k) + n$$

and

$$\operatorname{rk}\left(A^D - (A^D)^3\right) = \operatorname{rk}(A^{k+1} + A^k) + \operatorname{rk}(A^{k+1} - A^k) - \operatorname{rk}(A^k).$$

These rank equalities were given in [20, Theorem 13.1] and [20, Theorem 13.7].

Corollary 3.17. *If $A \in \mathbb{C}^{n\times n}$ is core invertible, then*

(1) $\operatorname{rk}\left(I - (A^{\text{#}})^2\right) = \operatorname{rk}(I - A^2)$.

(2) $\operatorname{rk}(A^{\text{#}} - (A^{\text{#}})^3) = \operatorname{rk}(A - A^3)$.

In particular, $A^{\text{#}} = (A^{\text{#}})^3$ if and only if $A = A^3$.

Proof. Apply Theorem 3.16 and the equalities (2), (3), and (4) of Lemma 2.3 to get

$$\begin{aligned}\operatorname{rk}\left(I - (A^{\text{#}})^2\right) &= \operatorname{rk}(A^*A^2 + A^*A) + \operatorname{rk}(A^*A^2 - A^*A) - 2\operatorname{rk}(A) + n\\ &= \operatorname{rk}(I + A) + \operatorname{rk}(I - A) - n = \operatorname{rk}(I - A^2).\end{aligned}$$

and

$$\begin{aligned}\operatorname{rk}\left(A^{\text{#}} - (A^{\text{#}})^3\right) &= \operatorname{rk}(A^*A^2 + A^*A) + \operatorname{rk}(A^*A^2 - A^*A) - \operatorname{rk}(A)\\ &= \operatorname{rk}(A) + \operatorname{rk}(I + A) - n + \operatorname{rk}(A) + \operatorname{rk}(I - A) - n - \operatorname{rk}(A)\\ &= \operatorname{rk}(I + A) + \operatorname{rk}(I - A) + \operatorname{rk}(A) - 2n\\ &= \operatorname{rk}(I - A^2) + \operatorname{rk}(A) - n.\end{aligned}$$

Finally, item (1) of Lemma 2.3 implies $\operatorname{rk}(A - A^3) = \operatorname{rk}(I - A^2) + \operatorname{rk}(A) - n$. The proof of the corollary is finished. □

Next result concerns to the rank of $A^{\|(B,C)} - A^{\|(D,E)}$ obtaining as corollary several characterizations for the equality of several generalized inverses of A.

Theorem 3.18. *If $A \in \mathbb{C}^{n\times m}$ and $B, C, D, E \in \mathbb{C}^{m\times n}$ are such that $A^{\|(B,C)}$ and $A^{\|(D,E)}$ exist, then*

(1) $\operatorname{rk}\left(A^{\|(B,C)} - A^{\|(D,E)}\right) = \operatorname{rk}\begin{bmatrix} C \\ E \end{bmatrix} + \operatorname{rk}[B \mid D] - \operatorname{rk}(B) - \operatorname{rk}(D)$.

(2) *The following statements are equivalent:*

   (a) $A^{\|(B,C)} = A^{\|(D,E)}$.

   (b) $\mathcal{R}(D) \subseteq \mathcal{R}(B)$ *and* $\mathcal{N}(C) \subseteq \mathcal{N}(E)$.

   (c) $\mathcal{R}(D) = \mathcal{R}(B)$ *and* $\mathcal{N}(C) = \mathcal{N}(E)$.

(d) $\mathcal{R}(B) \subseteq \mathcal{R}(D)$ *and* $\mathcal{N}(E) \subseteq \mathcal{N}(C)$.

Proof. (1) Recall that $A^{\|(B,C)}$ and $A^{\|(D,E)}$ are both outer inverses of A ([4, Theorem 6.1]). Hence applying [20, Theorem 5.1] we get

$$\mathrm{rk}\left(A^{\|(B,C)} - A^{\|(D,E)}\right) = \mathrm{rk}\begin{bmatrix} A^{\|(B,C)} \\ A^{\|(D,E)} \end{bmatrix} + \mathrm{rk}\left[A^{\|(B,C)} \mid A^{\|(D,E)}\right] - \mathrm{rk}\left(A^{\|(B,C)}\right) - \mathrm{rk}\left(A^{\|(D,E)}\right).$$

Observe that equality (2) implies that

$$\mathrm{rk}\left[A^{\|(B,C)} \mid A^{\|(D,E)}\right] = \dim\left(\mathcal{R}\left(A^{\|(B,C)}\right) + \mathcal{R}\left(A^{\|(D,E)}\right)\right) = \dim(\mathcal{R}(B) + \mathcal{R}(D)) = \mathrm{rk}[B \mid D].$$

We use again (2) to get $\mathcal{R}\left[(A^{\|(B,C)})^*\right] = \mathcal{N}(A^{\|(B,C)})^{\perp} = \mathcal{N}(C)^{\perp} = \mathcal{R}(C^*)$. Similarly, $\mathcal{R}\left[(A^{\|(D,E)})^*\right] = \mathcal{R}(E^*)$, and therefore, $\mathcal{R}[(A^{\|(B,C)})^* \mid (A^{\|(D,E)})^*] = \mathcal{R}[C^* \mid E^*]$. Using item (2) of Lemma 2.1 finishes the proof of (1).

(2) (a) $\Leftrightarrow$ (b). Observe that we can write $\mathrm{rk}(A^{\|(B,C)} - A^{\|(D,E)}) = \alpha + \beta$, where $\alpha = \mathrm{rk}\begin{bmatrix} C \\ E \end{bmatrix} - \mathrm{rk}(C)$ and $\beta = \mathrm{rk}([B \mid D]) - \mathrm{rk}(D)$ are nonnegative. Trivially, $A^{\|(B,C)} = A^{\|(D,E)}$ if and only if $\alpha = \beta = 0$, which by Lemma 3.3, is equivalent to (b).

Interchanging $B \leftrightarrow D$ and $C \leftrightarrow E$ in (a) $\Leftrightarrow$ (b) we obtain (a) $\Leftrightarrow$ (d). Now, (a) $\Leftrightarrow$ (c) is trivial. $\square$

Next result concerns with a generalized Schur complement and it generalizes [20, Theorem 2.1].

Theorem 3.19. *Let* $X \in \mathbb{C}^{n\times m}$, $Y \in \mathbb{C}^{n\times k}$, $Z \in \mathbb{C}^{l\times m}$, *and* $T \in \mathbb{C}^{l\times k}$, $B, C \in \mathbb{C}^{m\times n}$ *be given. If* $X^{\|(B,C)}$ *exists, then*

$$\mathrm{rk}(T - ZX^{\|(B,C)}Y) = \mathrm{rk}\begin{bmatrix} CXB & CY \\ ZB & T \end{bmatrix} - \mathrm{rk}(B).$$

Proof. Observe that by item (2) of Lemma 2.1 one has $\mathcal{R}(CY) \subset \mathcal{R}(C) = \mathcal{R}(CXB)$ and $\mathcal{R}((ZB)^*) = \mathcal{R}(B^*Z^*) \subseteq \mathcal{R}(B^*) = \mathcal{N}(B)^{\perp} = \mathcal{N}(CXB)^{\perp} = \mathcal{R}((CXB)^*)$. By Lemma 2.1 and Lemma 2.2,

$$\mathrm{rk}\begin{bmatrix} CXB & CY \\ ZB & T \end{bmatrix} = \mathrm{rk}(CXB) + \mathrm{rk}(T - ZB(CXB)^{\dagger}CY) = \mathrm{rk}(B) + \mathrm{rk}\left(T - ZX^{\|(B,C)}Y\right).$$

This finishes the proof. $\square$

Observe that this last corollary permits give the generalized Schur complements concerning the Drazin inverse, group inverse, and core inverse with no effort. We leave the details to the interested reader.

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