



GENERALIZED CORE INVERSES OF MATRICES

SANZHANG XU, JIANLONG CHEN, JULIO BENÍTEZ, AND DINGGUO WANG

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Abstract. In this paper, we introduce two new generalized inverses of matrices, namely, the $\langle i, m \rangle$ -core inverse and the $\langle j, m \rangle$ -core inverse. The $\langle i, m \rangle$ -core inverse of a complex matrix extends the notions of the core inverse defined by Baksalary and Trenkler [1] and the core-EP inverse defined by Manjunatha Prasad and Mohana [10]. The $\langle j, m \rangle$ -core inverse of a complex matrix extends the notions of the core inverse and the DMP-inverse defined by Malik and Thome [9]. Moreover, the formulae and properties of these two new concepts are investigated by using matrix decompositions and matrix powers.

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1. INTRODUCTION

Let $\mathbb{C}^{m \times n}$ denote the set of all $m \times n$ complex matrices. Let A^* , $\mathcal{R}(A)$ and $\text{rk}(A)$ denote the conjugate transpose, column space, and rank of $A \in \mathbb{C}^{m \times n}$, respectively. For $A \in \mathbb{C}^{m \times n}$, if $X \in \mathbb{C}^{n \times m}$ satisfies $AXA = A$, $XAX = X$, $(AX)^* = AX$, and $(XA)^* = XA$, then X is called a *Moore-Penrose inverse* of A . This matrix X is unique and denoted by A^\dagger . A matrix $X \in \mathbb{C}^{n \times m}$ is called an *outer inverse* of A if it satisfies $XAX = X$; is called a $\{2, 3\}$ -inverse of A if it satisfies $XAX = X$ and $(AX)^* = AX$; is called a $\{1, 3\}$ -inverse of A if it satisfies $AXA = A$ and $(AX)^* = AX$; is called a $\{1, 2, 3\}$ -inverse of A if it satisfies $AXA = A$, $XAX = X$ and $(AX)^* = AX$.

The core inverse of a complex matrix was introduced by Baksalary and Trenkler [1]. Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a *core inverse* of A , if it satisfies $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, here P_A denotes the orthogonal projector onto $\mathcal{R}(A)$. If such a matrix exists, then it is unique and denoted by A^\oplus . For a square complex matrix A , one has that A is core invertible, A is group invertible, and $\text{rk}(A) = \text{rk}(A^2)$ are three equivalent conditions (see [2]). We denote $\mathbb{C}_n^{CM} = \{A \in \mathbb{C}^{n \times n} \mid \text{rk}(A) = \text{rk}(A^2)\}$.

The first author is the corresponding author.

Let $A \in \mathbb{C}^{n \times n}$. A matrix $X \in \mathbb{C}^{n \times n}$ such that $XA^{k+1} = A^k$, $XAX = X$ and $AX = XA$ is called the *Drazin inverse* of A and denoted by A^D . The Drazin inverse of a square matrix always exists and it is unique. Such integer k is called the Drazin index of A , denoted by $\text{ind}(A)$. If $\text{ind}(A) \leq 1$, then the Drazin inverse of A is called the *group inverse* and denoted by $A^\#$.

The DMP-inverse for a complex matrix was introduced by Malik and Thome [9]. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a *DMP-inverse* of A , if it satisfies $XAX = X$, $XA = A^D A$ and $A^k X = A^k A^\dagger$. It is unique and denoted by $A^{D,\dagger}$. Malik and Thome gave several characterizations of the DMP-inverse by using the decomposition of Hartwig and Spindelböck [8].

The notion of the core-EP inverse for a complex matrix was introduced by Manjunatha Prasad and Mohana [10]. A matrix $X \in \mathbb{C}^{n \times n}$ is a *core-EP inverse* of $A \in \mathbb{C}^{n \times n}$ if X is an outer inverse of A satisfying $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^k)$, where k is the index of A . The core-EP inverse is unique and denoted by A^\oplus .

In addition, $\mathbf{1}_n$ and $\mathbf{0}_n$ will denote the $n \times 1$ column vectors all of whose components are 1 and 0, respectively. $0_{m \times n}$ (abbr. 0) denotes the zero matrix of size $m \times n$. If \mathcal{S} is a subspace of \mathbb{C}^n , then $P_{\mathcal{S}}$ stands for the *orthogonal projector* onto the subspace \mathcal{S} . A matrix $A \in \mathbb{C}^{n \times n}$ is called an *EP matrix* if $\mathcal{R}(A) = \mathcal{R}(A^*)$, A is called *Hermitian* if $A^* = A$ and A is *unitary* if $AA^* = I_n$, where I_n denote the *identity matrix* of size n . Let \mathbb{N} denote the set of positive integers.

2. PRELIMINARIES

A related decomposition of the matrix decomposition of Hartwig and Spindelböck [8] was given in [2, Theorem 2.1] by Benítez, in [3] it can be found a simpler proof of this decomposition. Let us start this section with the concept of principal angles.

Definition 1 ([12]). Let \mathcal{S}_1 and \mathcal{S}_2 be two nontrivial subspaces of \mathbb{C}^n . We define the *principal angles* $\theta_1, \dots, \theta_r \in [0, \pi/2]$ between \mathcal{S}_1 and \mathcal{S}_2 by

$$\cos \theta_i = \sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}),$$

for $i = 1, \dots, r$, where $r = \min\{\dim \mathcal{S}_1, \dim \mathcal{S}_2\}$. The real numbers $\sigma_i(P_{\mathcal{S}_1} P_{\mathcal{S}_2}) \geq 0$ are the singular values of $P_{\mathcal{S}_1} P_{\mathcal{S}_2}$.

The following theorem can be found in [2, Theorem 2.1].

Theorem 1. Let $A \in \mathbb{C}^{n \times n}$, $r = \text{rk}(A)$, and let $\theta_1, \dots, \theta_p$ be the principal angles between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$ belonging to $]0, \pi/2[$. Denote by x and y the multiplicities of the angles 0 and $\pi/2$ as a canonical angle between $\mathcal{R}(A)$ and $\mathcal{R}(A^*)$, respectively. There exists a unitary matrix $U \in \mathbb{C}^{n \times n}$ such that

$$A = U \begin{bmatrix} MC & MS \\ 0 & 0 \end{bmatrix} U^*, \quad (2.1)$$

where $M \in \mathbb{C}^{r \times r}$ is nonsingular,

$$C = \text{diag}(\mathbf{0}_y, \cos \theta_1, \dots, \cos \theta_p, \mathbf{1}_x),$$

$$S = \begin{bmatrix} \text{diag}(\mathbf{1}_y, \sin \theta_1, \dots, \sin \theta_p) & 0_{p+y, n-(r+p+y)} \\ 0_{x, p+y} & 0_{x, n-(r+p+y)} \end{bmatrix},$$

and $r = y + p + x$. Furthermore, x and $y + n - r$ are the multiplicities of the singular values 1 and 0 in $P_{\mathcal{R}(A)} P_{\mathcal{R}(A^*)}$, respectively.

In this decomposition, one has $C^2 + SS^* = I_r$. Recall that A^\dagger always exists. We have that $A^\#$ exists if and only if C is nonsingular in view of [2, Theorem 3.7]. The following equalities hold

$$A^\dagger = U \begin{bmatrix} CM^{-1} & 0 \\ S^*M^{-1} & 0 \end{bmatrix} U^*, \quad A^\# = U \begin{bmatrix} C^{-1}M^{-1} & C^{-1}M^{-1}C^{-1}S \\ 0 & 0 \end{bmatrix} U^*.$$

By [3, Theorem 2], we have that

$$A^D = U \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} U^*. \tag{2.2}$$

We also have

$$AA^\dagger = U \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{2.3}$$

$$A^\oplus = A^\#AA^\dagger = U \begin{bmatrix} C^{-1}M^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{2.4}$$

Lemma 1 ([13, Theorem 3.1]). *Let $A \in \mathbb{C}^{n \times n}$. Then A is core invertible if and only if there exists $X \in \mathbb{C}^{n \times n}$ such that $(AX)^* = AX$, $XA^2 = A$ and $AX^2 = X$. In this situation, we have $A^\oplus = X$.*

Lemma 2. *Let $A \in \mathbb{C}^{n \times n}$. If there exists $X \in \mathbb{C}^{n \times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then for $m \in \mathbb{N}$ we have*

- (1) $A^k = X^m A^{k+m}$;
- (2) $X^k = A^m X^{k+m}$;
- (3) $A^k X^k = A^{k+m} X^{k+m}$;
- (4) $X^k A^k = X^{k+m} A^{k+m}$;
- (5) $A^k = A^m X^m A^k$;
- (6) $X^k = X^m A^m X^k$.

Proof. (1). For $m = 1$, it is clear by the hypotheses. If the formula is true for $m \in \mathbb{N}$, then $X^{m+1} A^{k+m+1} = XX^m A^{k+m} A = XA^k A = XA^{k+1} = A^k$.

(3). It is easy to check that $A^k X^k = A^{k+1} X^{k+1}$ by $AX^{k+1} = X^k$. It is not difficult to check the equality $A^k X^k = A^{k+m} X^{k+m}$ by induction.

(5). From (1) we have $A^k = X^k A^{2k}$. Thus by $AX^{k+1} = X^k$, we have $A^k = X^k A^{2k} = AX^{k+1} A^{2k} = AX^k XA^{2k} = A(AX^{k+1})XA^{2k} = A^2 X^{k+2} A^{2k} = A^2 X^2 X^k A^{2k} = \dots = A^m X^m X^k A^{2k} = A^m X^m A^k$.

The proofs of (2), (4), and (6) are similar to the proofs of (1), (3), (5), respectively. \square

Lemma 3. *Let $A \in \mathbb{C}^{n \times n}$. If there exists $X \in \mathbb{C}^{n \times n}$ such that $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$ for some $k \in \mathbb{N}$, then $A^D = X^{k+1}A^k$.*

Proof. Since A is Drazin invertible. We will check that $A^D = X^{k+1}A^k$. Have in mind, $AX^{k+1} = X^k$ and $XA^{k+1} = A^k$, thus

$$A(X^{k+1}A^k) = X^kA^k = X^k(XA^{k+1}) = X^{k+1}A^kA. \quad (2.5)$$

That is, $X^{k+1}A^k$ and A commute. Then by (1) and (4) in Lemma 2, we have that

$$\begin{aligned} (X^{k+1}A^k)A(X^{k+1}A^k) &= X^{k+1}A^{k+1}X^{k+1}A^k = X^kA^k(X^{k+1}A^k) \\ &= X^kX^{k+1}A^kA^k = X^{k+1}X^kA^{2k} = X^{k+1}A^k. \end{aligned} \quad (2.6)$$

From (1) in Lemma 2, we have that

$$(X^{k+1}A^k)A^{k+1} = X(X^kA^{2k})A = XA^{k+1} = A^k. \quad (2.7)$$

Thus we have $A^D = X^{k+1}A^k$ by the definition of the Drazin inverse and in view of (2.5), (2.6), and (2.7). \square

Remark 1. From the proofs of Lemma 2 and Lemma 3, it is obvious that Lemma 2 and Lemma 3 are valid for rings. Moreover, we can get that for an element $a \in R$, a is Drazin invertible if and only if there exist $x \in R$ and $k \in \mathbb{N}$ such that $ax^{k+1} = x^k$ and $xa^{k+1} = a^k$, where R is a ring.

The following lemma is similar to [9, Theorem 2.5].

Lemma 4. *Let $A \in \mathbb{C}^{n \times n}$ be the form (2.1). Then*

$$A^{D,\dagger} = U \begin{bmatrix} (MC)^D & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (2.8)$$

Proof. By (2.2), (2.3) and the definition of DMP-inverse we have

$$\begin{aligned} A^{D,\dagger} &= A^DAA^\dagger \\ &= U \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} U^* = U \begin{bmatrix} (MC)^D & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned} \quad \square$$

Lemma 5 ([11, Corollary 3.3]). *Let $A \in \mathbb{C}^{n \times n}$ be a matrix of index k . Then $AA^\oplus = A^k(A^k)^\dagger$.*

3. $\langle i, m \rangle$ -CORE INVERSE

Let us start this section by introducing the definition of the $\langle i, m \rangle$ -core inverse.

Definition 2. Let $A \in \mathbb{C}^{n \times n}$ and $m, i \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called an $\langle i, m \rangle$ -core inverse of A , if it satisfies

$$X = A^D AX \quad \text{and} \quad A^m X = A^i (A^i)^\dagger. \tag{3.1}$$

It will be proved that if X exists, then it is unique and denoted by $A_{i,m}^\oplus$.

Theorem 2. Let $A \in \mathbb{C}^{n \times n}$. If exists $X \in \mathbb{C}^{n \times n}$ such that (3.1) holds, then X is unique.

Proof. Assume that X satisfies the system in (3.1), that is $X = A^D AX$ and $A^m X = A^i (A^i)^\dagger$. Thus $X = A^D AX = (A^D)^m A^m X = (A^D)^m A^i (A^i)^\dagger$. Therefore, X is unique by the uniqueness of A^D and $A^i (A^i)^\dagger$. \square

Theorem 3. The system in (3.1) is consistent if and only if $i \geq \text{ind}(A)$. In this case, the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.

Proof. Assume that $i \geq \text{ind}(A)$. Let $X = (A^D)^m A^i (A^i)^\dagger$. We have

$$\begin{aligned} A^D AX &= A^D A (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^D A A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = X \\ A^m X &= A^m (A^D)^m A^i (A^i)^\dagger = A^D A A^i (A^i)^\dagger = A^i (A^i)^\dagger. \end{aligned}$$

Thus, the system in (3.1) is consistent and the solution of (3.1) is $X = (A^D)^m A^i (A^i)^\dagger$.

If the system in (3.1) is consistent, then exists X_0 such that $X_0 = A^D AX_0$ and $A^m X_0 = A^i (A^i)^\dagger$. Then $X_0 = A^D AX_0 = (A^D)^m A^m X_0 = (A^D)^m A^i (A^i)^\dagger$ and $A^i (A^i)^\dagger = A^m X_0 = A^m (A^D)^m A^i (A^i)^\dagger = A A^D A^i (A^i)^\dagger$. Hence $A^i = A^i (A^i)^\dagger A^i = A A^D A^i (A^i)^\dagger A^i = A A^D A^i$, that is $i \geq \text{ind}(A)$. \square

Example 1. We will give an example that shows if $i < \text{ind}(A)$, then the system in (3.1) is not consistent. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. It is easy to get $\text{ind}(A) = 2$ and $A^D = 0$. Let $i = 1$ and suppose that X is the solution of system in (3.1), then $X = A^D AX = 0$, which gives $AA^\dagger = A^m X = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Remark 2. If $i \geq \text{ind}(A)$, then $A_{i,m+1}^\oplus = A^D A_{i,m}^\oplus$.

Remark 3. The $\langle i, m \rangle$ -core inverse is a generalization of the core inverse and the core-EP inverse. More precisely, we have the following statements:

- (1) If $m = i = \text{ind}(A) = 1$, then the $\langle 1, 1 \rangle$ -core inverse coincides with the core inverse;
- (2) If $m = 1$ and $i = \text{ind}(A)$, then the $\langle i, 1 \rangle$ -core inverse coincides with the core-EP inverse.

For the convenience of the readers, in the following, we give some notes of (1) and (2) in Remark 3.

- (1) If $m = i = \text{ind}(A) = 1$, then A is group invertible and $A^D = A^\#$ and (3.1) is equivalent to $X = A^\#AX$ and $AX = AA^\dagger$. Thus $X = A^\#AX = A^\#AA^\dagger$, $(AX)^* = (AA^\dagger)^* = AA^\dagger = AX$, $AX^2 = AA^\#AA^\dagger A^\#AA^\dagger = AA^\#AA^\dagger AA^\#A^\dagger = AA^\#A^\dagger = X$ and $XA^2 = A^\#AA^\dagger A^2 = A^\#A^2 = A$. Hence, $(1, 1)$ -core inverse coincides with the core inverse by Lemma 1. Note that if A is group invertible, then we have that X is the core inverse of A if and only if $X = A^\#AX$ and $AX = AA^\dagger$.
- (2) If $m = 1$ and $i = \text{ind}(A)$, then by Theorem 3.3, $A_{i,1}^\oplus$ exists and $A_{i,1}^\oplus = A^D A^i (A^i)^\dagger$. Let us denote $X = A_{i,1}^\oplus = A^D A^i (A^i)^\dagger$. Observe that $AX = A^i (A^i)^\dagger$ is Hermitian. Now,

$$XAX = A^D A^i (A^i)^\dagger A^i (A^i)^\dagger = A^D A^i (A^i)^\dagger = X,$$

that is X is an outer inverse of A . From

$$A^i = A^D A^{i+1} = A^D A^i (A^i)^\dagger A^i A = XA^{i+1}$$

we get $\mathcal{R}(A^i) \subseteq \mathcal{R}(X)$. Also,

$$\begin{aligned} AX^2 &= (AX)X = A^i (A^i)^\dagger A^D A^i (A^i)^\dagger \\ &= A^i (A^i)^\dagger A^i A^D (A^i)^\dagger = A^D A^i (A^i)^\dagger = X, \end{aligned}$$

which implies $X = (AX)^*X \in \mathcal{R}(X^*)$, therefore, $\mathcal{R}(X) \subseteq \mathcal{R}(X^*)$. Finally, $X^* = [A^D A^i (A^i)^\dagger]^* = A^i (A^i)^\dagger (A^D)^*$ implies $\mathcal{R}(X^*) \subseteq \mathcal{R}(A^i)$. Hence $\mathcal{R}(X) = \mathcal{R}(X^*) = \mathcal{R}(A^i)$. Therefore, the $(i, 1)$ -core inverse coincides with the core-EP inverse by the definition of the core-EP inverse.

From the above statement, we have the following theorem.

Theorem 4. *Let $A \in \mathbb{C}^{n \times n}$ with $i = \text{ind}(A)$. Then X is the core-EP inverse of A if and only if $X = A^D AX$ and $AX = A^i (A^i)^\dagger$.*

Corollary 1. *Let $A \in \mathbb{C}^{n \times n}$ with $1 = \text{ind}(A)$. Then X is the core inverse of A if and only if $X = A^\#AX$ and $AX = AA^\dagger$.*

For any $A \in \mathbb{C}^{n \times n}$, either $A^l = 0$ for some $l \in \mathbb{N}$, or $A^l \neq 0$ for all positive integers. Moreover, if $\text{ind}(A) = k$, then $G_k B_k$ is nonsingular (see [5–7]), where $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$, $l = 1, \dots, k-1$. When $A^k \neq 0$, then it can be written as

$$A^k = \prod_{l=1}^k B_l \prod_{l=1}^k G_{k+1-l}. \quad (3.2)$$

We have the following results, (see [5, Theorem 4] or [4, Theorem 7.8.2]):

$$\text{ind}(A) = \begin{cases} k & \text{when } G_k B_k \text{ is nonsingular,} \\ k+1 & \text{when } G_k B_k = 0. \end{cases}$$

and

$$A^D = \begin{cases} \prod_{l=1}^k B_l (G_k B_k)^{-k-1} \prod_{l=1}^k G_{k+1-l} & \text{when } G_k B_k \text{ is nonsingular,} \\ 0 & \text{when } G_k B_k = 0. \end{cases} \quad (3.3)$$

In the sequel, we always assume that $A^k \neq 0$.

It is well-known that if $A = EF$ is a full rank factorization of A , where $r = \text{rk}(A)$, $E \in \mathbb{C}^{n \times r}$ and $F \in \mathbb{C}^{r \times n}$, then (see [4, Theorem 1.3.2])

$$A^\dagger = F^* (FF^*)^{-1} (E^* E)^{-1} E^*. \quad (3.4)$$

Remark 4. The notations and results in above paragraph will be used many times in the sequel.

We will investigate the $\langle i, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using Remark 4.

Theorem 5. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. If $i \geq k$, then $A_{i,m}^\oplus = A_{k,m}^\oplus$.*

Proof. Since $\text{ind}(A) = k$, we have $\mathcal{R}(A^k) = \mathcal{R}(A^i)$ for any $i \geq k$, and therefore, $A^k (A^k)^\dagger = A^i (A^i)^\dagger$. Now, the conclusion follows from Theorem 3. \square

Remark 5. The proof of Theorem 5 also can be proved as follows. Since the proof in this remark will be used several times in the sequel, we write this proof here.

Proof. If A is nilpotent, then $A^D = 0$, hence by Theorem 3, one has $A_{i,m}^\oplus = A_{k,m}^\oplus = 0$. Therefore, we can assume that $A^k \neq 0$. By equality (3.2), we have

$$A^k = \prod_{l=1}^k B_l \prod_{l=1}^k G_{k+1-l}. \quad (3.5)$$

where $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$, $l = 1, \dots, k-1$. Let $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$. Now, we will show that

$$A^i = \prod_{l=1}^k B_l (G_k B_k)^{i-k} \prod_{l=1}^k G_{k+1-l} = M L^{i-k} N.$$

In fact,

$$\begin{aligned}
A^i &= \prod_{l=1}^i B_l \prod_{l=1}^i G_{k+1-l} \\
&= B_1 \cdots B_i G_i \cdots G_1 \\
&= B_1 \cdots B_{i-1} (B_i G_i) G_{i-1} \cdots G_1 \\
&= B_1 \cdots B_{i-1} (G_{i-1} B_{i-1}) G_{i-1} \cdots G_1 \\
&= B_1 \cdots B_{i-2} (G_{i-2} B_{i-2})^2 G_{i-2} \cdots G_1 \\
&= \cdots \\
&= B_1 \cdots B_k (G_k B_k)^{i-k} G_k \cdots G_1 = ML^{i-k} N.
\end{aligned} \tag{3.6}$$

If we let $M_1 = ML^{i-k}$, then $A^i = ML^{i-k} N = M_1 N$ is a full rank factorization of A^i (see [7, p.183]). Thus

$$(A^i)^\dagger = N^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^*. \tag{3.7}$$

Note that $NM = \prod_{l=1}^k G_{k+1-l} \prod_{l=1}^k B_l = L^k$. By Theorem 3, (3.3) and (3.7) we have

$$\begin{aligned}
A_{i,1}^\oplus &= A^D A^i (A^i)^\dagger \\
&= ML^{-k-1} NML^{i-k} N (A^i)^\dagger \\
&= ML^{-k-1} NML^{i-k} NN^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} NN^* (NN^*)^{-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} (M_1^* M_1)^{-1} M_1^* \\
&= ML^{i-k-1} [(L^{i-k})^* M^* ML^{i-k}]^{-1} (L^{i-k})^* M^* \\
&= ML^{i-k-1} L^{k-i} (M^* M)^{-1} [(L^{i-k})^*]^{-1} (L^{i-k})^* M^* \\
&= ML^{-1} (M^* M)^{-1} M^*.
\end{aligned} \tag{3.8}$$

The last expression does not depend on i , then $A_{i,1}^\oplus = A_{k,1}^\oplus$. Thus, by Remark 2, we have $A_{i,m}^\oplus = A^D A_{i,m-1}^\oplus = A^D (A^D A_{i,m-2}^\oplus) = (A^D)^2 A_{i,m-2}^\oplus = \cdots = (A^D)^{m-1} A_{i,1}^\oplus = (A^D)^{m-1} A_{k,1}^\oplus = A_{k,m}^\oplus$. \square

Remark 6. By Theorem 5, it is enough to investigate the $i = \text{ind}(A) = k$ case, when we discuss the $\langle i, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$. That is, the Theorem 5 is a key theorem.

Theorem 6. Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$ and $k, m \in \mathbb{N}$. If $A = B_1 G_1$ is a full rank factorization of A and $G_l B_l = B_{l+1} G_{l+1}$ is a full rank factorization of $G_l B_l$,

$l = 1, \dots, k-1$, then $A_{k,m}^\oplus = ML^{-m}M^\dagger$, where $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$.

Proof. By the proof of Remark 5, we have $A_{k,1}^\oplus = ML^{-1}(M^*M)^{-1}M^*$ and $NM = L^k$. Now, we will prove $(A^D)^s A_{k,1}^\oplus = ML^{-s-1}(M^*M)^{-1}M^*$ for any $s \in \mathbb{N}$. By (3.3) we have $A^D = \prod_{l=1}^k B_l (G_k B_k)^{-k-1} \prod_{l=1}^k G_{k+1-l} = ML^{-k-1}N$. When $s = 1$, we have

$$\begin{aligned} A^D A_{k,1}^\oplus &= ML^{-k-1}NML^{-1}(M^*M)^{-1}M^* = ML^{-k-1}(NM)L^{-1}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^k L^{-1}(M^*M)^{-1}M^* = ML^{-2}(M^*M)^{-1}M^*. \end{aligned}$$

Assume that $(A^D)^{s-1} A_{k,1}^\oplus = ML^{-s}(M^*M)^{-1}M^*$. Then

$$\begin{aligned} (A^D)^s A_{k,1}^\oplus &= A^D (A^D)^{s-1} A_{k,1}^\oplus = A^D ML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}NML^{-s}(M^*M)^{-1}M^* \\ &= ML^{-k-1}L^k L^{-s}(M^*M)^{-1}M^* \\ &= ML^{-s-1}(M^*M)^{-1}M^*. \end{aligned}$$

Thus by Remark 2, we have

$$A_{k,m}^\oplus = (A^D)^{m-1} A_{k,1}^\oplus = ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^\dagger.$$

□

In the following theorem, we will give a canonical form for the $\langle k, m \rangle$ -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using the matrix decomposition in Theorem 1. We will also use the following simple fact: Let $X \in \mathbb{C}^{n \times m}$ and $\mathbf{b} \in \mathbb{C}^n$. If $\mathbf{y} \in \mathbb{C}^m$ satisfies $X^*X\mathbf{y} = X^*\mathbf{b}$, then $XX^\dagger\mathbf{b} = X\mathbf{y}$.

Theorem 7. Let $A \in \mathbb{C}^{n \times n}$ have the form (2.1) with $\text{ind}(A) = k$ and $m \in \mathbb{N}$. Then

$$A_{k,m}^\oplus = U \begin{bmatrix} (MC)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (3.9)$$

Proof. Let r be the rank of A . By Theorem 3 we have

$$A_{k,m}^\oplus = (A^D)^m A^k (A^k)^\dagger. \quad (3.10)$$

Since A has the form given in Theorem 1 we have

$$A^k = U \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix} U^*. \quad (3.11)$$

Let $\mathbf{b} \in \mathbb{C}^n$ be arbitrary and let us decompose $\mathbf{b} = U \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$, where $\mathbf{b}_1 \in \mathbb{C}^r$. Let $\mathbf{x}_0 \in \mathbb{C}^n$ satisfy $(A^k)^* A^k \mathbf{x}_0 = (A^k)^* \mathbf{b}$ [this \mathbf{x}_0 always exists because the normal

equations always have a solution]. We can decompose \mathbf{x}_0 by writing $\mathbf{x}_0 = U \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$, where $\mathbf{x}_1 \in \mathbb{C}^r$. Let us denote $N = (MC)^{k-1}M$. Using (3.11),

$$U \begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix} \begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = U \begin{bmatrix} CN^* & 0 \\ S^*N^* & 0 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}.$$

Therefore,

$$CN^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = CN^*\mathbf{b}_1 \quad \text{and} \quad S^*N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = S^*N^*\mathbf{b}_1.$$

Premultiplying the first equality by C and the second equality by S and after, adding them, we get $N^*N(C\mathbf{x}_1 + S\mathbf{x}_2) = N^*\mathbf{b}_1$, and hence, $N(C\mathbf{x}_1 + S\mathbf{x}_2) = NN^\dagger\mathbf{b}_1$. Now,

$$\begin{aligned} A^k(A^k)^\dagger\mathbf{b} &= A^k\mathbf{x}_0 = U \begin{bmatrix} NC & NS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \\ &= U \begin{bmatrix} NC\mathbf{x}_1 + NS\mathbf{x}_2 \\ \mathbf{0} \end{bmatrix} = U \begin{bmatrix} NN^\dagger\mathbf{b}_1 \\ \mathbf{0} \end{bmatrix} = U \begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*\mathbf{b}. \end{aligned}$$

Since \mathbf{b} is arbitrary,

$$A^k(A^k)^\dagger = U \begin{bmatrix} NN^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Now we will prove $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$. Recall that we have $N = (MC)^{k-1}M$, and so, $\mathcal{R}(N) \subseteq \mathcal{R}((MC)^{k-1})$. Since M is nonsingular, $\text{rk}(N) = \text{rk}((MC)^{k-1})$, and thus, $\mathcal{R}(N) = \mathcal{R}((MC)^{k-1})$. Since $(MC)^{k-1}[(MC)^{k-1}]^\dagger$ and NN^\dagger are the orthogonal projectors onto $\mathcal{R}((MC)^{k-1})$ and $\mathcal{R}(N)$, respectively, we get $NN^\dagger = (MC)^{k-1}[(MC)^{k-1}]^\dagger$.

By (2.2) we have

$$A^D = U \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} U^*. \quad (3.12)$$

Thus, we have

$$(A^D)^m = U \begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1}MS \\ 0 & 0 \end{bmatrix} U^*.$$

Since $\text{ind}(A) = k$, we have $A^D A^{k+1} = A^k$. By using the above representations of A^D and A^k given in (3.11) and (3.12), respectively,

$$\begin{aligned} \begin{bmatrix} (MC)^D & [(MC)^D]^2MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{k+1} & (MC)^kMS \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} (MC)^k & (MC)^{k-1}MS \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore,

$$(MC)^D(MC)^kM[C \mid S] = (MC)^{k-1}M[C \mid S]. \quad (3.13)$$

Have in mind that we have $C^2 + SS^* = I_r$. Thus, postmultiplying (3.13) by $\begin{bmatrix} C \\ S^* \end{bmatrix}$ gives us $(MC)^D(MC)^k M = (MC)^{k-1} M$ and from the nonsingularity of M we obtain $(MC)^D(MC)^k = (MC)^{k-1}$, and so, $\text{ind}(MC) \leq k - 1$. Therefore we have

$$\begin{aligned} A_{k,m}^\oplus &= (A^D)^m A^k (A^k)^\dagger \\ &= U \begin{bmatrix} [(MC)^D]^m & [(MC)^D]^{m+1} MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{k-1} ((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} [(MC)^D]^m (MC)^{k-1} ((MC)^{k-1})^\dagger & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

Remark 7. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the $\langle k, m \rangle$ -core inverse of A is

$$A_{k,m}^\oplus = U \begin{bmatrix} (\Sigma K)_{k-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which is similar to the expression of $A_{k,m}^\oplus$ in Theorem 7. Since the proof of this result can be proved as the proof of Theorem 7, we omit this proof.

Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n \times n}$ is nonsingular and $J \in \mathbb{C}^{n \times n}$ is a block diagonal matrix composed of Jordan blocks. In the following theorem, we will compute the $\langle k, m \rangle$ -core inverse by using the Jordan Canonical form of A .

Theorem 8. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$, then $A_{k,m}^\oplus = P_1 D^{-m} P_1^\dagger$, where $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$ with $D \in \mathbb{C}^{r \times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n \times r}$.*

Proof. The Jordan Canonical form of A is $P^{-1}AP = J$, where $P \in \mathbb{C}^{n \times n}$ is nonsingular and $J \in \mathbb{C}^{n \times n}$ is a block diagonal matrix. Rearrange the elements of J such that $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$, where D is nonsingular and N is nilpotent. It is well-known that $A^D = P \begin{bmatrix} D^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$ and $A^k = P \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} P^{-1}$. If we let $P = [P_1 \mid P_2]$ and $P^{-1} = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$, then

$$(A^D)^m A^k = [P_1 \mid P_2] \begin{bmatrix} (D^{-1})^m & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D^k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = P_1 D^{k-m} Q_1.$$

Observe that $A^k = (P_1 D^k) Q_1$ is a full rank factorization of A^k . Hence by (3.4) we have

$$\begin{aligned} (A^k)^\dagger &= (P_1 D^k Q_1)^\dagger \\ &= Q_1^* (Q_1 Q_1^*)^{-1} [(P_1 D^k)^* P_1 D^k]^{-1} (P_1 D^k)^* \\ &= Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} (P_1^* P_1)^{-1} [(D^k)^*]^{-1} (D^k)^* P_1^* \\ &= Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} (P_1^* P_1)^{-1} P_1^* \\ &= Q_1^\dagger D^{-k} P_1^\dagger. \end{aligned}$$

By Theorem 3, we have $A_{k,m}^\oplus = (A^D)^m A^k (A^k)^\dagger$. Thus we have

$$\begin{aligned} A_{k,m}^\oplus &= (A^D)^m A^k (A^k)^\dagger = P_1 D^{k-m} Q_1 Q_1^\dagger D^{-k} P_1^\dagger \\ &= P_1 D^{k-m} Q_1 Q_1^* (Q_1 Q_1^*)^{-1} D^{-k} P_1^\dagger = P_1 D^{-m} D^k D^{-k} P_1^\dagger = P_1 D^{-m} P_1^\dagger. \end{aligned}$$

□

Proposition 1. *Let $A \in \mathbb{C}^{n \times n}$. If $i \geq \text{ind}(A)$, then $A^m A_{i,m}^\oplus$ is the projector onto $\mathcal{R}(A^i)$ along $\mathcal{R}(A^i)^\perp$.*

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the $\langle i, m \rangle$ -core inverse.

Proposition 2. *Let $A \in \mathbb{C}^{n \times n}$, $m, i \in \mathbb{N}$. If $i \geq \text{ind}(A)$, then*

- (1) $A_{i,m}^\oplus$ is a $\{2, 3\}$ -inverse of A^m ;
- (2) $A_{i,m}^\oplus = (A^D)^m P_{A^i}$;
- (3) $(A_{i,m}^\oplus)^n = (A^D)^{m(n-1)} A_{i,m}^\oplus = (A^D)^{mn} P_{A^i}$;
- (4) $A^i A_{i,m}^\oplus = A_{i,m}^\oplus A^i$ if and only if $\mathcal{R}(A^i)^\perp \subseteq \mathcal{N}(A^i)$;
- (5) $A_{i,m}^\oplus = A$ implies that A is EP.

Proof. (1). By Theorem 3 we have $A_{i,m}^\oplus = (A^D)^m A^i (A^i)^\dagger$, thus

$$\begin{aligned} A_{i,m}^\oplus A^m A_{i,m}^\oplus &= (A^D)^m A^i (A^i)^\dagger A^m (A^D)^m A^i (A^i)^\dagger \\ &= (A^D)^m A^i (A^i)^\dagger A^i A^m (A^D)^m (A^i)^\dagger \\ &= (A^D)^m A^i A^m (A^D)^m (A^i)^\dagger = (A^D)^m A^m (A^D)^m A^i (A^i)^\dagger \\ &= A^D A (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger = A_{i,m}^\oplus. \end{aligned}$$

Thus $A_{i,m}^\oplus$ is a $\{2, 3\}$ -inverse of A^m in view of $A^m A_{i,m}^\oplus = A^i (A^i)^\dagger$.
(2) is trivial.

(3). By

$$\begin{aligned} (A_{i,m}^\oplus)^n &= (A^D)^m A^i (A^i)^\dagger (A^D)^m A^i (A^i)^\dagger (A_{i,m}^\oplus)^{n-2} \\ &= (A^D)^m (A^D)^m A^i (A^i)^\dagger (A_{i,m}^\oplus)^{n-2} \\ &= (A^D)^m A_{i,m}^\oplus (A_{i,m}^\oplus)^{n-2} = (A^D)^m (A_{i,m}^\oplus)^{n-1}, \end{aligned}$$

it is easy to check (3).

(4). By $\mathcal{R}[(I_n - A^i (A^i)^\dagger)] = \mathcal{N}[(A^i)^\dagger]$, we have

$$\begin{aligned} A^i A_{i,m}^\oplus &= A_{i,m}^\oplus A^i \Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i (A^i)^\dagger A^i \\ &\Leftrightarrow A^i (A^D)^m A^i (A^i)^\dagger = (A^D)^m A^i \\ &\Leftrightarrow A^i (A^D)^m (I_n - A^i (A^i)^\dagger) = 0 \\ &\Leftrightarrow \mathcal{R}[I_n - A^i (A^i)^\dagger] \subseteq \mathcal{N}[A^i (A^D)^m] \\ &\Leftrightarrow \mathcal{N}[(A^i)^\dagger] \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{N}[(A^i)^*] \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[(A^D)^m] \\ &\Leftrightarrow \mathcal{R}(A^i)^\perp \subseteq \mathcal{N}[A^i]. \end{aligned}$$

(5). Let A be written in the form (2.1). We have $A_{i,m}^\oplus = U \begin{bmatrix} (MC)_{i-1,m}^\oplus & 0 \\ 0 & 0 \end{bmatrix} U^*$ by Theorem 7. Thus, $A_{i,m}^\oplus = A$ implies $MS = 0$. From the nonsingularity of M , we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. \square

4. (j, m) -CORE INVERSE

Let us start this section by introducing the definition of the (j, m) -core inverse.

Definition 3. Let $A \in \mathbb{C}^{n \times n}$ and $m, j \in \mathbb{N}$. A matrix $X \in \mathbb{C}^{n \times n}$ is called a (j, m) -core inverse of A , if it satisfies

$$X = A^D AX \quad \text{and} \quad A^m X = A^m (A^j)^\dagger. \tag{4.1}$$

Theorem 9. Let $A \in \mathbb{C}^{n \times n}$. If the system in (4.1) is consistent, then the solution is unique.

Proof. Assume that X satisfies (4.1), that is $X = A^D AX$ and $A^m X = A^m (A^j)^\dagger$. Then $X = A^D AX = (A^D)^m A^m X = (A^D)^m A^m (A^j)^\dagger = A^D A (A^j)^\dagger$. Thus X is unique. \square

By Theorem 9 if X exists, then it is unique and denoted by $A_{j,m}^\ominus$.

Theorem 10. Let $A \in \mathbb{C}^{n \times n}$ and $m, j \in \mathbb{N}$.

- (1) If $m \geq \text{ind}(A)$, then the system in (4.1) is consistent and the solution is $X = A^D A (A^j)^\dagger$.

(2) If the system in (4.1) is consistent, then $\text{ind}(A) \leq \max\{j, m\}$.

Proof. (1). Let $X = A^D A(A^j)^\dagger$. We have $A^D AX = A^D AA^D A(A^j)^\dagger = A^D A(A^j)^\dagger = X$ and $A^m X = A^m A^D A(A^j)^\dagger = A^D AA^m(A^j)^\dagger = A^m(A^j)^\dagger$.

(2). If the system in (4.1) is consistent, then exists $X_0 \in \mathbb{C}^{n \times n}$ such that $X_0 = A^D AX_0 = (A^D)^m A^m X_0 = (A^D)^m A^m(A^j)^\dagger = A^D A(A^j)^\dagger$ and $A^m(A^j)^\dagger = A^m X_0 = A^m A^D A(A^j)^\dagger = A^m(A^D)^j A^j(A^j)^\dagger$. Thus

$$A^m(A^j)^\dagger A^j = A^m(A^D)^j A^j(A^j)^\dagger A^j = A^m(A^D)^j A^j = A^m A^D A.$$

If $m \geq j$, then $A^m A^D A = A^m(A^j)^\dagger A^j = A^{m-j} A^j(A^j)^\dagger A^j = A^{m-j} A^j = A^m$. That is, $\text{ind}(A) \leq m$. If $j > m$, then $A^j = A^j(A^j)^\dagger A^j = A^{j-m} A^m(A^j)^\dagger A^j = A^{j-m} A^m A^D A = A^j A^D A$. That is, $\text{ind}(A) \leq j$. Thus, $\text{ind}(A) \leq \max\{j, m\}$. \square

Example 2. We will give an example that shows if $m < \text{ind}(A)$, then the system in (4.1) is not consistent. Let A be the same matrix in Example 1. It is easy to get $\text{ind}(A) = 2$ and $A^D = 0$. Let $m = j = 1$ and suppose that X is the solution of system in (4.1), then $X = A^D AX = 0$, which gives $AA^\dagger = AX = 0$, thus $A = AA^\dagger A = 0$, this is a contradiction.

Example 3. The converse of Theorem 10 (1) is not true. Let $m = 1$ and $j = 3$. If we let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

then $\text{ind}(A) = 3$ and $A^3 = 0$. Hence $X = 0$ is a solution of (4.1), but $m < \text{ind}(A)$.

Example 4. If $\text{ind}(A) \leq \max\{j, m\}$, then the system in (4.1) may be not consistent. If we let

$$A = \begin{bmatrix} 2 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$A^3 = A^2 = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$A^D = A^2$ and $\text{ind}(A) = 2$. Let $m = 1$ and $j = 2$, then $\text{ind}(A) \leq \max\{j, m\}$. It is easy to check that

$$(A^2)^\dagger = \frac{1}{15} \begin{bmatrix} 2 & -1 & 0 \\ 2 & -1 & 0 \\ 2 & -1 & 0 \end{bmatrix}.$$

If the system in (4.1) has a solution X_0 , then $X_0 = A^D A X_0 = A^D A(A^2)^\dagger$ and $A(A^2)^\dagger = A X_0 = A A^D A(A^2)^\dagger = A^4(A^2)^\dagger = A^2(A^2)^\dagger$ would hold. But

$$A(A^2)^\dagger = \frac{1}{15} \begin{bmatrix} 10 & -5 & 0 \\ -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq \frac{1}{15} \begin{bmatrix} 12 & -6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^2(A^2)^\dagger.$$

Thus, the system in (4.1) is not consistent.

Remark 8. If $m \geq \text{ind}(A) = k$, it is not difficult to see that $A_{j,m}^\ominus = A_{j,m+1}^\ominus$. That is to say, the (j, m) -core inverse of A coincides with the $(j, m + 1)$ -core inverse of A . Thus, in the sequel, we only discuss the $m = \text{ind}(A)$ case.

Theorem 11. *Let $A, X \in \mathbb{C}^{n \times n}$, $k, j \in \mathbb{N}$. If $\text{ind}(A) = k$ and X is the (j, k) -core inverse of A , then we have $X^j A^j X^j = (A^D)^{j(j-1)} X^j$ and $X A^j = A^D A$.*

Proof. By the definition of the (j, k) -core inverse, we have $X = A^D A X$ and $A^k X = A^k (A^j)^\dagger$. By $X = A^D A (A^j)^\dagger$, it is easy to check that $X^{n+1} = (A^D)^j X^n$ for arbitrary $n \in \mathbb{N}$, which gives that $X^j = (A^D)^{j(j-1)} X$.

$$\begin{aligned} X A^j &= A^D A (A^j)^\dagger A^j = (A^D)^j A^j (A^j)^\dagger A^j = (A^D)^j A^j = A^D A; \\ X^j A^j X^j &= (A^D)^{j(j-1)} X A^j X^j = (A^D)^{j(j-1)} A^D A X^j = (A^D)^{j(j-1)} X^j. \end{aligned}$$

□

Corollary 2. *Let $A, X \in \mathbb{C}^{n \times n}$ and $\text{ind}(A) = k$. If X is the $(1, k)$ -core inverse of A , then we have $X A X = X$ and $X A = A^D A$.*

The (j, m) -core inverse is a generalization of the core inverse and the DMP-inverse in view of Theorem 11.

Remark 9. When $j = m = 1 = \text{ind}(A)$, the equations in (4.1) are equivalent to $X A X = X$, $X A = A^\# A$, and $A X = A A^\dagger$. Thus $A X = A A^\dagger$ implies that $(A X)^* = A X$; $X A = A^\# A$ gives that $X A^2 = A$ and $A X A = A$; and $X = X A X = A^\# A X = A A^\# X$, which means that $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, then $X = A Y$ for some $Y \in \mathbb{C}^{n \times n}$, thus $X = A Y = A X A Y = A X^2$. Therefore, we have $A^\oplus = X$ by Lemma 1. In a word, the $(1, 1)$ -core inverse coincides with the usual core inverse.

Remark 10. If we let $j = 1$ and $m = \text{ind}(A)$, then the equations in (4.1) are equivalent to $X A X = X$, $X A = A^D A$, and $A^k X = A^k A^\dagger$ by Theorem 11. Thus $(1, k)$ -core inverse coincides with the DMP-inverse.

From Remark 10, Theorem 11 and the definition of the (j, k) -core inverse, we have the following theorem, which says that the conditions $X A X = X$, and $X A = A^D A$ in the definition of the DMP-inverse can be replaced by $X = A^D A X$.

Theorem 12. *Let $A \in \mathbb{C}^{n \times n}$ with $k = \text{ind}(A)$. Then $X \in \mathbb{C}^{n \times n}$ is the DMP-inverse of A if and only if $X = A^D A X$ and $A^k X = A^k A^\dagger$.*

In the following theorem, we will give a canonical form for the (j, k) -core inverse of a matrix $A \in \mathbb{C}^{n \times n}$ by using the matrix decomposition in Theorem 1.

Theorem 13. *Let $A \in \mathbb{C}^{n \times n}$ have the form (2.1) with $\text{ind}(A) = k$ and $j \in \mathbb{N}$. Then*

$$A_{j,k}^{\ominus} = U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*.$$

Proof. By Theorem 10 and the idempotency of $A^D A$ we have

$$A_{j,k}^{\ominus} = A^D A (A^j)^{\dagger} = (A^D)^j A^j (A^j)^{\dagger}. \quad (4.2)$$

From the proof of Theorem 7, we have

$$A^j (A^j)^{\dagger} = U \begin{bmatrix} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^*. \quad (4.3)$$

By (2.2) we have

$$(A^D)^j = U \begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1} MS \\ 0 & 0 \end{bmatrix} U^*. \quad (4.4)$$

By the proof of Theorem 7, we have $\text{ind}(MC) \leq k - 1 < k$. From (4.2), (4.3) and (4.4), we have

$$\begin{aligned} A_{j,k}^{\ominus} &= (A^D)^j A^j (A^j)^{\dagger} \\ &= U \begin{bmatrix} [(MC)^D]^j & [(MC)^D]^{j+1} MS \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} [(MC)^D]^j (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D [(MC)^D]^{j-1} (MC)^{j-1} ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D (MC)^D MC ((MC)^{j-1})^{\dagger} & 0 \\ 0 & 0 \end{bmatrix} U^* \\ &= U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*. \end{aligned}$$

□

Remark 11. If we use the decomposition of Hartwig and Spindelböck in [8, Corollary 6], then an expression of the (j, k) -core inverse of A is

$$A_{j,k}^{\ominus} = U \begin{bmatrix} (\Sigma K)^D (\Sigma K)_{j-1,k}^{\ominus} & 0 \\ 0 & 0 \end{bmatrix} U^*,$$

which is similar to the expression of $A_{j,k}^{\ominus}$ in Theorem 13. Since the proof of this result can be proved like the proof of Theorem 13, we omit this proof.

Theorem 14. Let $A \in \mathbb{C}^{n \times n}$ and $\text{ind}(A) = k$. If $(A^k X^k)^* = A^k X^k$, $A X^{k+1} = X^k$ and $X A^{k+1} = A^k$, then A is (k, k) -core invertible and $A_{k,k}^\ominus = X^k$.

Proof. By Lemma 2 and Lemma 3, we have $A^k X^k A^k = A^k$, $X^k A^k X^k = X^k$, $A^k = X^k A^{2k}$, and $A^D = X^{k+1} A^k$. Equalities $(A^k X^k)^* = A^k X^k$ and $A^k X^k A^k = A^k$ imply that X^k is a $\{1, 3\}$ -inverse of A^k . From $A^D = X^{k+1} A^k$, we can obtain $(A^D)^k = X^{k-1} A^D$ by induction. Thus

$$\begin{aligned} A_{k,k}^\ominus &= A^D A(A^k)^\dagger = (A^D)^k A^k (A^k)^\dagger = (A^D)^k A^k (A^k)^{(1,3)} \\ &= (A^D)^k A^k X^k = (X^{k+1} A^k)^k A^k X^k = X^{k-1} X^{k+1} A^k A^k X^k \\ &= X^{2k} A^{2k} X^k = X^k (X^k A^{2k}) X^k = X^k A^k X^k = X^k. \end{aligned}$$

□

Proposition 3. Let $A \in \mathbb{C}^{n \times n}$ be a matrix with $j \geq \text{ind}(A) = k$. If A is (j, k) -core invertible, then $A^j A_{j,k}^\ominus$ is the projector onto $\mathcal{R}(A^j)$ along $\mathcal{R}(A^j)^\perp$.

Proof. It is trivial. □

In the following proposition, we will investigate some properties of the (j, k) -core inverse.

Proposition 4. Let $A \in \mathbb{C}^{n \times n}$ with $j \geq \text{ind}(A) = k$. If A is (j, k) -core invertible, then

- (1) $A_{j,k}^\ominus$ is a $\{1, 2, 3\}$ -inverse of A^j ;
- (2) $A_{j,k}^\ominus = (A^D)^j P_{A^j}$;
- (3) $(A_{j,k}^\ominus)^n = \begin{cases} [(A^D)^j (A^j)^\dagger]^{n/2} & \text{if } n \text{ is even,} \\ A^j [(A^D)^j (A^j)^\dagger]^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$
- (4) $A_{j,k}^\ominus A^D = (A^D)^{j+1}$;
- (5) $A^j A_{j,k}^\ominus = A_{j,k}^\ominus A^j$ if and only if $\mathcal{R}(A^j)^\perp \subseteq \mathcal{N}(A^j)$;
- (6) $A_{j,k}^\ominus = A$ implies that A is EP.

Proof. (1). By Theorem 10 we have $A_{j,k}^\ominus = A^D A(A^j)^\dagger = (A^D)^j A^j (A^j)^\dagger$, thus

$$\begin{aligned} A^j A_{j,k}^\ominus A^j &= A^j (A^D)^j A^j (A^j)^\dagger A^j = A^j (A^D)^j A^j = A^j A^D A = A^j; \\ A_{j,k}^\ominus A^j A_{j,k}^\ominus &= (A^D)^j A^j (A^j)^\dagger A^j A_{j,k}^\ominus = A^D A A_{j,k}^\ominus \\ &= A^D A A^D A(A^j)^\dagger = A^D A(A^j)^\dagger = A_{j,k}^\ominus; \\ A^j A_{j,k}^\ominus &= A^j (A^D)^j A^j (A^j)^\dagger = A^j (A^j)^\dagger. \end{aligned}$$

(2) is trivial.

(3). By $(A_{j,k}^\ominus)^2 = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j (A^j)^\dagger = (A^D)^j (A^j)^\dagger$ and induction it is easy to check (3).

(4). $A_{j,k}^\ominus A^D = (A^D)^j A^j (A^j)^\dagger A^D = (A^D)^j A^j (A^j)^\dagger (A^D)^j A^j A^D = (A^D)^{j+1}$.

(5). By $\mathcal{R}[I_n - A^j (A^j)^\dagger] = \mathcal{N}[(A^j)^\dagger]$ and $\mathcal{N}(A^D A) = \mathcal{N}(A^D)$, we have

$$\begin{aligned} A^j A_{j,k}^\ominus &= A_{j,k}^\ominus A^j \Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j (A^j)^\dagger A^j \\ &\Leftrightarrow A^j (A^D)^j A^j (A^j)^\dagger = (A^D)^j A^j \\ &\Leftrightarrow A^j (A^D)^j [I_n - A^j (A^j)^\dagger] = 0 \\ &\Leftrightarrow \mathcal{R}[I_n - A^j (A^j)^\dagger] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^\dagger] \subseteq \mathcal{N}(A^D A) \\ &\Leftrightarrow \mathcal{N}[(A^j)^\dagger] \subseteq \mathcal{N}(A^D) \\ &\Leftrightarrow \mathcal{R}(A^j)^\perp \subseteq \mathcal{N}(A^j). \end{aligned}$$

(6). Let A be written in the form (2.1). We have

$$A_{j,k}^\ominus = U \begin{bmatrix} (MC)^D (MC)_{j-1,k}^\ominus & 0 \\ 0 & 0 \end{bmatrix} U^*$$

by Theorem 13. Thus, $A_{j,k}^\ominus = A$ implies $MS = 0$. From the nonsingularity of M , we have $S = 0$, which is equivalent to say that A is EP in view of [2, Theorem 3.7]. \square

In the following proposition, we shall give the the relationship between the (j, k) -core inverse and DMP-inverse and core-EP inverse.

Proposition 5. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. Then*

$$A_{k,k}^\ominus = A^{D,\dagger} (A^D)^{k-1} A A^\oplus.$$

Proof. We have that $A^k (A^k)^\dagger = A A^\oplus$ by Lemma 5 and $A^{D,\dagger} = A^D A A^\dagger$. Thus

$$\begin{aligned} A_{k,k}^\ominus &= A^D A (A^k)^\dagger = (A^D)^k A^k (A^k)^\dagger = A^D A^k (A^D)^{k-1} (A^k)^\dagger \\ &= A^D A A^\dagger A^k (A^D)^{k-1} (A^k)^\dagger = A^{D,\dagger} (A^D)^{k-1} A^k (A^k)^\dagger \\ &= A^{D,\dagger} (A^D)^{k-1} A A^\oplus. \end{aligned}$$

\square

In the following theorem, we will give a relationship between the (i, m) -core inverse and (j, m) -core inverse.

Theorem 15. *Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k$. Then $A_{k,m}^\oplus = A_{m,k}^\ominus$ for any $m \geq k$.*

Proof. By Theorem 10, we have $A_{m,k}^\ominus = A^D A(A^m)^\dagger = (A^D)^k A^k (A^m)^\dagger$. By the proof of Remark 5, we have $A^k = MN$ and $NM = L^k$, where $M = \prod_{l=1}^k B_l$, $N = \prod_{l=1}^k G_{k+1-l}$ and $L = G_k B_k$. It is easy to see that $(A^D)^s = ML^{-k-s}N$ for any $s \in \mathbb{N}$ by $NM = L^k$. Thus $(A^D)^k = ML^{-2k}N$ and

$$(A^D)^k A^k = ML^{-2k}NMN = ML^{-2k}L^kN = ML^{-k}N.$$

By the proof of Remark 5, we have $A^m = ML^{m-k}N = M_1N$ is a full rank factorization of A^m , where $M_1 = ML^{m-k}$ and

$$(A^m)^\dagger = N^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^*.$$

By Theorem 6, we have $A_{k,m}^\oplus = ML^{-m}M^\dagger$. In the following steps, we will show that $A_{m,k}^\ominus = ML^{-m}M^\dagger$. From $A_{m,k}^\ominus = (A^D)^k A^k (A^m)^\dagger$, we have

$$\begin{aligned} A_{k,m}^\ominus &= (A^D)^k A^k (A^m)^\dagger = ML^{-k}NN^*(NN^*)^{-1}(M_1^*M_1)^{-1}(M_1)^* \\ &= ML^{-k}(M_1^*M_1)^{-1}(M_1)^* = ML^{-k}[(L^{m-k})^*M^*ML^{m-k}]^{-1}(L^{m-k})^*M^* \\ &= ML^{-k}L^{k-m}(M^*M)^{-1}[(L^{m-k})^*]^{-1}(L^{m-k})^*M^* \\ &= ML^{-m}(M^*M)^{-1}M^* = ML^{-m}M^\dagger. \end{aligned}$$

□

Theorem 16. Let $A \in \mathbb{C}^{n \times n}$ with $i \geq \text{ind}(A) = k$, then $A_{i,k}^\ominus = P_1 D^{-i} P_1^\dagger$, where $A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1}$ with $D \in \mathbb{C}^{r \times r}$ is nonsingular, N is nilpotent and $P = [P_1 \mid P_2]$ with $P_1 \in \mathbb{C}^{n \times r}$.

Proof. It is easy to see that by Theorem 8 and Theorem 15. □

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Authors’ addresses

Sanzhang Xu

Faculty of Mathematics and Physics, Huaiyin Institute of Technology, 223003 Huaian, China

E-mail address: xusanzhang5222@126.com

Jianlong Chen

School of Mathematics, Southeast University, 210096 Nanjing, China

E-mail address: E-mail: jlchen@seu.edu.cn

Julio Benítez

Universitat Politècnica de València, Instituto de Matemática Multidisciplinar, 46022 Valencia, Spain

E-mail address: jbenitez@mat.upv.es

Dingguo Wang

School of Mathematical Sciences, Qufu Normal University, 273165 Qufu, China

E-mail address: dingguo95@126.com