



Article

# Rank Equalities Related to the Generalized Inverses $A \parallel (B_1, C_1)$ , $D \parallel (B_2, C_2)$ of Two Matrices $A$ and $D$

Wenjie Wang <sup>1</sup>, Sanzhang Xu <sup>1,\*</sup>  and Julio Benítez <sup>2,\*</sup> 

<sup>1</sup> Faculty of Mathematics and Physics, Huaiyin Institute of Technology, Huaian 223003, China; 15195389157@163.com

<sup>2</sup> Instituto de Matemática Multidisciplinar, Universitat Politècnica de València, 46022 Valencia, Spain

\* Correspondence: xusanzhang5222@126.com (S.X.); jbenitez@mat.upv.es (J.B.)

Received: 27 March 2019; Accepted: 10 April 2019; Published: 15 April 2019



**Abstract:** Let  $A$  be an  $n \times n$  complex matrix. The  $(B, C)$ -inverse  $A \parallel (B, C)$  of  $A$  was introduced by Drazin in 2012. For given matrices  $A$  and  $B$ , several rank equalities related to  $A \parallel (B_1, C_1)$  and  $B \parallel (B_2, C_2)$  of  $A$  and  $B$  are presented. As applications, several rank equalities related to the inverse along an element, the Moore-Penrose inverse, the Drazin inverse, the group inverse and the core inverse are obtained.

**Keywords:** Rank;  $(B, C)$ -inverse; inverse along an element

**MSC:** 15A09; 15A03

## 1. Introduction

The set of all  $m \times n$  matrices over the complex field  $\mathbb{C}$  will be denoted by  $\mathbb{C}^{m \times n}$ . Let  $A^*$ ,  $\mathcal{R}(A)$ ,  $\mathcal{N}(A)$  and  $\text{rank}(A)$  denote the conjugate transpose, column space, null space and rank of  $A \in \mathbb{C}^{m \times n}$ , respectively.

For  $A \in \mathbb{C}^{m \times n}$ , if  $X \in \mathbb{C}^{n \times m}$  satisfies  $AXA = A$ ,  $XAX = X$ ,  $(AX)^* = AX$  and  $(XA)^* = XA$ , then  $X$  is called a *Moore-Penrose inverse* of  $A$  [1,2]. This matrix  $X$  always exists, is unique and will be denoted by  $A^\dagger$ .

Let  $A \in \mathbb{C}^{n \times n}$ . It can be easily proved that exists a non-negative integer  $k$  for which  $\text{rank}(A^k) = \text{rank}(A^{k+1})$  holds. The *Drazin index* of  $A$  is the smallest non-negative  $k$  such that  $\text{rank}(A^k) = \text{rank}(A^{k+1})$ , and denoted by  $\text{ind}(A)$ . A matrix  $X \in \mathbb{C}^{n \times n}$  such that  $XA^{k+1} = A^k$ ,  $XAX = X$  and  $AX = XA$  hold, where  $k = \text{ind}(A)$ , is called a *Drazin inverse* of  $A$ . It can be proved, (see, e.g., [3] Chapter 4), that the Drazin inverse of any square matrix  $A$  exists, is unique, and will be denoted by  $A^D$ . If  $\text{ind}(A) \leq 1$ , then the Drazin inverse of  $A$  is called the *group inverse* and denoted by  $A^\#$ .

The core inverse of a complex matrix was introduced by Baksalary and Trenkler in [4]. Let  $A \in \mathbb{C}^{n \times n}$ , a matrix  $X \in \mathbb{C}^{n \times n}$  is called a *core inverse* of  $A$ , if it satisfies  $AX = P_{\mathcal{R}(A)}$  and  $\mathcal{R}(X) \subseteq \mathcal{R}(A)$ , where  $P_{\mathcal{R}(A)}$  denotes the orthogonal projector onto  $\mathcal{R}(A)$ . If such a matrix  $X$  exists, then it is unique and denoted by  $A^\oplus$ . In [4] it was proved that a square matrix  $A$  is core invertible if and only if  $\text{ind}(A) \leq 1$ .

In [5], Mary introduced a new type of generalized inverse, namely, the inverse along an element. Let  $A, D \in \mathbb{C}^{n \times n}$ . We say that  $A$  is *invertible along*  $D$  if there exists  $Y \in \mathbb{C}^{n \times n}$  such that

$$YAD = D = DAY, \quad \mathcal{R}(Y) \subseteq \mathcal{R}(D) \quad \text{and} \quad \mathcal{N}(D) \subseteq \mathcal{N}(Y). \quad (1)$$

If such  $Y$  exists, then it is unique and denoted by  $A^{\parallel D}$ . The inverse along an element extends some known generalized inverses, for example, the group inverse, the Drazin inverse and the Moore-Penrose inverse.

In ([6] Definition 4.1), Benítez et al. gave the following definition extending simultaneously the notion of the  $(B, C)$ -inverse from elements in rings [7] to rectangular matrices and the invertibility along an element. Let  $A \in \mathbb{C}^{m \times n}$  and  $B, C \in \mathbb{C}^{n \times m}$ , the matrix  $A$  is said to be  $(B, C)$ -invertible, if there exists a matrix  $Y \in \mathbb{C}^{n \times m}$  such that

$$YAB = B, \quad CAY = C, \quad \mathcal{R}(Y) \subseteq \mathcal{R}(B) \quad \text{and} \quad \mathcal{N}(C) \subseteq \mathcal{N}(Y). \tag{2}$$

If such a matrix  $Y$  exists, then it is unique and denoted by  $A^{\parallel(B,C)}$ . Many existence criteria and properties of the  $(B, C)$ -inverse can be found in, for example, [6,8–14]. From the definition of the  $(B, C)$ -inverse, it is evident that  $\mathcal{R}(Y) = \mathcal{R}(B)$  and  $\mathcal{N}(C) = \mathcal{N}(Y)$ . The  $(B, C)$ -inverse of  $A$  is a generalization of some well-known generalized inverses. By ([7] p. 1910), the Moore-Penrose inverse of  $A$  coincides with the  $(A^*, A^*)$ -inverse of  $A$ .  $A^{\parallel D}$  is the  $(D, D)$ -inverse of  $A$ ,  $A^D$  is the  $(A^k, A^k)$ -inverse of  $A$ , where  $k = \text{ind}(A)$  and  $A^\#$  is the  $(A, A)$ -inverse of  $A$ . By ([15] Theorem 4.4), we have that the  $(A, A^*)$ -inverse is the core inverse of  $A$ .

Let  $A \in \mathbb{C}^{m \times n}$  be a matrix of rank  $r$ , let  $T$  be a subspace of  $\mathbb{C}^n$  of dimension  $s \leq r$  and let  $S$  be a subspace of  $\mathbb{C}^m$  of dimension  $m - s$ . The matrix  $A$  has a  $\{2\}$ -inverse  $X$  such that  $\mathcal{R}(X) = T$  and  $\mathcal{N}(X) = S$  if and only if  $AT \oplus S = \mathbb{C}^n$  (see, e.g., [3] Section 2.6). In this case,  $X$  is unique and is denoted by  $A_{T,S}^{(2)}$ . Many properties of  $A_{T,S}^{(2)}$  can be found in, for example, [3,16–19].

The theory of the generalized inverses has many applications, as one can see in [3]. Another important application is the study of singular systems of differential equations (see, e.g., [20,21]).

The main purpose of the manuscript is twofold: to research the rank of the difference  $AA^{\parallel(B_1,C_1)} - DD^{\parallel(B_2,C_2)}$  and to apply this study to characterize when  $AA^{\parallel(B_1,C_1)} = DD^{\parallel(B_2,C_2)}$ . These results are contained in Sections 3 and 4. The paper finishes by particularizing the previous results to some standard generalized inverses.

## 2. Preliminaries

The following lemmas about the partitioned matrices  $[A, B]$  and  $\begin{bmatrix} A \\ B \end{bmatrix}$  will be useful in the sequel.

**Lemma 1.** Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{m \times k}$ .

(1) ([22] Theorem 5) For any  $A^- \in A\{1\}$  and  $B^- \in B\{1\}$ , we have

$$\text{rank}([A, B]) = \text{rank}(A) + \text{rank}([I_m - AA^-]B) = \text{rank}([I_m - BB^-]A) + \text{rank}(B);$$

(2)  $\text{rank}([A, B]) = \text{rank}(B)$  if and only if  $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ .

**Proof.** Since (1) was proved in ([22] Theorem 5), we only give the proof of (2). Observe that  $\text{rank}([A, B]) = \dim[\mathcal{R}(A) + \mathcal{R}(B)] = \dim \mathcal{R}(A) + \dim \mathcal{R}(B) - \dim[\mathcal{R}(A) \cap \mathcal{R}(B)]$ , which leads to  $\text{rank}([A, B]) = \text{rank}(B) \Leftrightarrow \dim \mathcal{R}(A) = \dim[\mathcal{R}(A) \cap \mathcal{R}(B)] \Leftrightarrow \mathcal{R}(A) \subseteq \mathcal{R}(B)$ .  $\square$

**Lemma 2.** Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{k \times n}$ .

(1) ([22] Theorem 5) For any  $A^- \in A\{1\}$  and  $B^- \in B\{1\}$ , we have

$$\text{rank}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) = \text{rank}(A) + \text{rank}(B[I_n - A^-A]) = \text{rank}(A[I_n - B^-B]) + \text{rank}(B);$$

$$(2) \quad \text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right) = \text{rank}(B) \text{ if and only if } \mathcal{N}(B) \subseteq \mathcal{N}(A).$$

**Proof.** Again, only the the proof of (2) will be given. Since  $\text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right) = \text{rank}([A^*, B^*])$ , by employing item (2) of Lemma 1 we get

$$\text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right) = \text{rank}(B) \Leftrightarrow \text{rank}([A^*, B^*]) = \text{rank}(B^*) \Leftrightarrow \mathcal{R}(A^*) \subseteq \mathcal{R}(B^*).$$

The proof finishes by recalling the equality  $\mathcal{R}(X^*) = \mathcal{N}(X)^\perp$  valid for any matrix  $X$ , where the superscript  $\perp$  denotes the orthogonal complement.  $\square$

**Lemma 3.** ([22], Theorem 5) Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{m \times k}$ .

(1) If there is a matrix  $Q \in \mathbb{C}^{n \times t}$  such that  $\mathcal{R}(A) \subseteq \mathcal{R}(AQ)$ , then

$$\text{rank}([AQ, B]) = \text{rank}([A, B]);$$

(2) If there is a matrix  $P \in \mathbb{C}^{k \times s}$  such that  $\mathcal{R}(B) \subseteq \mathcal{R}(BP)$ , then

$$\text{rank}([A, BP]) = \text{rank}([A, B]).$$

**Proof.** We only prove (1) since (2) is analogous. Observe that  $\mathcal{R}(A) \subseteq \mathcal{R}(AQ)$  and  $\mathcal{R}(A) = \mathcal{R}(AQ)$  are equivalent. Now, (1) is evident from the expression  $\text{rank}([X, Y]) = \dim \mathcal{R}(X) + \dim \mathcal{R}(Y) - \dim[\mathcal{R}(X) \cap \mathcal{R}(Y)]$  valid for any pair of matrices  $X$  and  $Y$  with the same number of rows.  $\square$

**Lemma 4.** ([22], Theorem 5) Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{k \times n}$ .

(1) If there is a matrix  $Q \in \mathbb{C}^{t \times m}$  such that  $\mathcal{N}(QA) \subseteq \mathcal{N}(A)$ , then

$$\text{rank} \left( \begin{bmatrix} QA \\ B \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right);$$

(2) If there is a matrix  $P \in \mathbb{C}^{s \times k}$  such that  $\mathcal{N}(PB) \subseteq \mathcal{N}(B)$ , then

$$\text{rank} \left( \begin{bmatrix} A \\ PB \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right).$$

**Proof.** The proof is similar than the proof of Lemma 2, item (2).  $\square$

From Lemma 3 and Lemma 4, we have the following two lemmas.

**Lemma 5.** Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{m \times k}$ . If there are matrices  $Q \in \mathbb{C}^{n \times t}$  and  $P \in \mathbb{C}^{k \times s}$  such that  $\mathcal{R}(A) \subseteq \mathcal{R}(AQ)$  and  $\mathcal{R}(B) \subseteq \mathcal{R}(BP)$ , then

$$\text{rank}([AQ, BP]) = \text{rank}([A, B]).$$

**Lemma 6.** Let  $A \in \mathbb{C}^{m \times n}$  and  $B \in \mathbb{C}^{k \times n}$ . If there are matrices  $Q \in \mathbb{C}^{t \times m}$  and  $P \in \mathbb{C}^{s \times k}$  such that  $\mathcal{N}(QA) \subseteq \mathcal{N}(A)$  and  $\mathcal{N}(PB) \subseteq \mathcal{N}(B)$ , then

$$\text{rank} \left( \begin{bmatrix} QA \\ PB \end{bmatrix} \right) = \text{rank} \left( \begin{bmatrix} A \\ B \end{bmatrix} \right).$$

**Lemma 7.** ([23] Theorem 2.1) Let  $P, Q \in \mathbb{C}^{n \times n}$  be any two idempotent matrices. The difference  $P - Q$  satisfies the following rank equality:

$$\text{rank}(P - Q) = \text{rank} \left( \begin{bmatrix} P \\ Q \end{bmatrix} \right) + \text{rank}([P, Q]) - \text{rank}(P) - \text{rank}(Q).$$

The following lemma gives the calculation and the characterization of the existence of the  $(B, C)$ -inverse of a matrix  $A$ .

**Lemma 8** ([6] Theorem 4.4). Let  $A \in \mathbb{C}^{m \times n}$  and  $B, C \in \mathbb{C}^{n \times m}$ . The following statements are equivalent:

- (1) the  $(B, C)$ -inverse of  $A$  exists;
- (2)  $\text{rank}(B) = \text{rank}(C) = \text{rank}(CAB)$ .

In this case,  $A^{\parallel(B,C)} = B(CAB)^{\dagger}C$ .

**Lemma 9** ([6] Corollary 7.2). Let  $A \in \mathbb{C}^{n \times m}$  and  $G \in \mathbb{C}^{m \times n}$ . The following statements are equivalent:

- (1) the matrix  $A$  is invertible along  $G$ ;
- (2) the outer inverse  $A_{\mathcal{R}(G), \mathcal{N}(G)}^{(2)}$  exists.

### 3. Main Theorem

In this section, several rank equalities related to the generalized inverse  $A^{\parallel(B_1, C_1)}$  and  $D^{\parallel(B_2, C_2)}$  of  $A$  and  $D$  are derived, where  $A, D \in \mathbb{C}^{n \times m}$  and  $B_1, B_2, C_1, C_2 \in \mathbb{C}^{m \times n}$ .

**Theorem 1.** Let  $A, D \in \mathbb{C}^{n \times m}$ ,  $B_1, B_2, C_1, C_2 \in \mathbb{C}^{m \times n}$ . If  $A$  is  $(B_1, C_1)$  invertible and  $D$  is  $(B_2, C_2)$  invertible with  $A^{\parallel(B_1, C_1)}$  is the  $(B_1, C_1)$ -inverse of  $A$  and  $D^{\parallel(B_2, C_2)}$  is the  $(B_2, C_2)$ -inverse of  $D$ , then

$$\begin{aligned} & \text{rank}(AA^{\parallel(B_1, C_1)} - DD^{\parallel(B_2, C_2)}) \\ &= \text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) + \text{rank}([AB_1, DB_2]) - \text{rank}(B_1) - \text{rank}(B_2) \\ &= \text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) + \text{rank}([AB_1, DB_2]) - \text{rank}(C_1) - \text{rank}(C_2). \end{aligned}$$

**Proof.** Since  $AA^{\parallel(B_1, C_1)}$  and  $DD^{\parallel(B_2, C_2)}$  are outer inverse of  $A$  and  $D$  we have have that  $AA^{\parallel(B_1, C_1)}$  and  $DD^{\parallel(B_2, C_2)}$  are idempotents. By Lemma 7, we have

$$\begin{aligned} \text{rank}(AA^{\parallel(B_1, C_1)} - DD^{\parallel(B_2, C_2)}) &= \text{rank} \left( \begin{bmatrix} AA^{\parallel(B_1, C_1)} \\ DD^{\parallel(B_2, C_2)} \end{bmatrix} \right) + \\ & \text{rank}([AA^{\parallel(B_1, C_1)}, DD^{\parallel(B_2, C_2)}]) - \text{rank}(AA^{\parallel(B_1, C_1)}) - \text{rank}(DD^{\parallel(B_2, C_2)}). \end{aligned} \quad (3)$$

Since  $A^{\parallel(B_1, C_1)}AB_1 = B_1$  and  $\mathcal{R}(A^{\parallel(B_1, C_1)}) = \mathcal{R}(B_1)$ , thus

$$\text{rank}(B_1) \leq \text{rank}(A^{\parallel(B_1, C_1)}A) \leq \text{rank}(A^{\parallel(B_1, C_1)}) = \text{rank}(B_1). \quad (4)$$

Similarly, the expressions  $C_1AA^{\parallel(B_1, C_1)} = C_1$  and  $\mathcal{N}(A^{\parallel(B_1, C_1)}) = \mathcal{N}(C_1)$  imply

$$\text{rank}(C_1) \leq \text{rank}(AA^{\parallel(B_1, C_1)}) \leq \text{rank}(A^{\parallel(B_1, C_1)}) = \text{rank}(C_1). \quad (5)$$

From (4) and (5), we have

$$\text{rank}(AA^{\parallel(B_1, C_1)}) = \text{rank}(A^{\parallel(B_1, C_1)}A) = \text{rank}(B_1) = \text{rank}(C_1). \quad (6)$$

In an analogous manner, for the  $(B_2, C_2)$ -invertible matrix  $D$ , we have

$$\text{rank}(DD^{\|(B_2, C_2)}) = \text{rank}(D^{\|(B_2, C_2)}D) = \text{rank}(B_2) = \text{rank}(C_2). \quad (7)$$

By Lemma 8, we have  $A^{\|(B_1, C_1)} = B_1(C_1AB_1)^{\dagger}C_1$  and  $D^{\|(B_2, C_2)} = B_2(C_2DB_2)^{\dagger}C_2$ . Thus

$$\text{rank}\left(\begin{bmatrix} AA^{\|(B_1, C_1)} \\ DD^{\|(B_2, C_2)} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} AB_1(C_1AB_1)^{\dagger}C_1 \\ DB_2(C_2DB_2)^{\dagger}C_2 \end{bmatrix}\right) \quad (8)$$

$$\text{rank}([AA^{\|(B_1, C_1)}, DD^{\|(B_2, C_2)}]) = \text{rank}([AB_1(C_1AB_1)^{\dagger}C_1, DB_2(C_2DB_2)^{\dagger}C_2]). \quad (9)$$

By ([6] Theorem 3.4), we have  $\mathcal{R}(C_1) = \mathcal{R}(C_1AB_1)$  and  $\mathcal{N}(B_1) = \mathcal{N}(C_1AB_1)$ . Then we have

$$C_1 = C_1AB_1(C_1AB_1)^{\dagger}C_1 \quad \text{and} \quad B_1 = B_1(C_1AB_1)^{\dagger}C_1AB_1. \quad (10)$$

In a analogous manner, for the  $(B_2, C_2)$ -invertible matrix  $D$ , we have

$$C_2 = C_2DB_2(C_2DB_2)^{\dagger}C_2 \quad \text{and} \quad B_2 = B_2(C_2DB_2)^{\dagger}C_2DB_2. \quad (11)$$

Since the conditions in (10) and (11) imply that  $\mathcal{N}(AB_1(C_1AB_1)^{\dagger}C_1) \subseteq \mathcal{N}(C_1)$  and  $\mathcal{N}(DB_2(C_2DB_2)^{\dagger}C_2) \subseteq \mathcal{N}(C_2)$ , respectively. Thus, Lemma 6 and the condition in (8) imply

$$\text{rank}\left(\begin{bmatrix} AA^{\|(B_1, C_1)} \\ DD^{\|(B_2, C_2)} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\right). \quad (12)$$

By (10) and (11), we have

$$AB_1 = AB_1(C_1AB_1)^{\dagger}C_1AB_1 \quad \text{and} \quad DB_2 = DB_2(C_2DB_2)^{\dagger}C_2DB_2. \quad (13)$$

The expressions in (13) imply that  $\mathcal{R}(AB_1) \subseteq \mathcal{R}(AB_1(C_1AB_1)^{\dagger}C_1)$  and  $\mathcal{R}(DB_2) \subseteq \mathcal{R}(DB_2(C_2DB_2)^{\dagger}C_2)$ . Thus, Lemma 5 and the expression in (9) imply

$$\text{rank}([AA^{\|(B_1, C_1)}, DD^{\|(B_2, C_2)}]) = \text{rank}([AB_1, DB_2]). \quad (14)$$

The proof is completed by using (3), (6), (7), (13) and (14).  $\square$

**Example 1.** Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

By using Lemma 8 we get that  $A$  is  $(B_1, C_1)$ -invertible,  $D$  is  $(B_2, C_2)$ -invertible, and

$$A^{\|(B_1, C_1)} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad D^{\|(B_2, C_2)} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 1 \end{bmatrix}.$$

These latter computations can be easily performed by a numerical software, e.g., Octave. Now,

$$\text{rank}\left(AA^{\|(B_1, C_1)} - DD^{\|(B_2, C_2)}\right) = \text{rank}\left(\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\right) = 1.$$

$$\text{Evidently, } \text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) = 1, \text{rank}(B_1) = \text{rank}(B_2) = 1, \text{ and}$$

$$\text{rank}([AB_1, DB_2]) = \text{rank} \left( \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix} \right) = 2,$$

which exemplifies Theorem 1.

#### 4. Applications

In this section, as an application of the preceding section, we will characterize when  $AA^{\|(B_1, C_1)} = DD^{\|(B_2, C_2)}$  holds. Also, several rank equalities for the Moore-Penrose inverse, the Drazin inverse, the group inverse, the core inverse and the inverse along an element will be presented in view of the rank equalities for the  $(B, C)$ -inverse presented in Theorem 1.

**Theorem 2.** Let  $A, D \in \mathbb{C}^{n \times m}$ ,  $B_1, B_2, C_1, C_2 \in \mathbb{C}^{m \times n}$ . If  $A$  is  $(B_1, C_1)$  invertible and  $D$  is  $(B_2, C_2)$  invertible, then the following statements are equivalent:

- (1)  $AA^{\|(B_1, C_1)} = DD^{\|(B_2, C_2)}$ ;
- (2)  $\mathcal{R}(AB_1) \subseteq \mathcal{R}(DB_2)$  and  $\mathcal{N}(C_1) \subseteq \mathcal{N}(C_2)$ ;
- (3)  $\mathcal{R}(DB_2) \subseteq \mathcal{R}(AB_1)$  and  $\mathcal{N}(C_2) \subseteq \mathcal{N}(C_1)$ ;
- (4)  $\mathcal{N}(A^{\|(B_1, C_1)}) \subseteq \mathcal{N}(DD^{\|(B_2, C_2)})$  and  $\mathcal{R}(AA^{\|(B_1, C_1)}) \subseteq \mathcal{R}(DB_2)$ ;
- (5)  $\mathcal{N}(D^{\|(B_2, C_2)}) \subseteq \mathcal{N}(AA^{\|(B_2, C_2)})$  and  $\mathcal{R}(DD^{\|(B_2, C_2)}) \subseteq \mathcal{R}(AB_1)$ .

**Proof.** By  $B_1 = A^{\|(B_1, C_1)}AB_1$ , we have  $\text{rank}(AB_1) \leq \text{rank}(B_1) = \text{rank}(A^{\|(B_1, C_1)}AB_1) \leq \text{rank}(AB_1)$ , that is  $\text{rank}(AB_1) = \text{rank}(B_1)$ . Similarly, we have  $\text{rank}(DB_2) = \text{rank}(B_2)$ . By Lemma 2, we have

$$\text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) = \text{rank}(C_1) + \text{rank}(C_2[I_n - C_1^-C_1]). \quad (15)$$

By Lemma 1, we have

$$\text{rank}([AB_1, DB_2]) = \text{rank}(DB_2) + \text{rank}([I_n - DB_2(DB_2)^-]AB_1), \quad (16)$$

where  $(DB_2)^-$  is any inner inverse of  $DB_2$ .

Assume that (1) holds. By Theorem 1 we have

$$\text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) + \text{rank}([AB_1, DB_2]) - \text{rank}(B_1) - \text{rank}(B_2) = 0. \quad (17)$$

By employing  $\text{rank}(DB_2) = \text{rank}(B_2)$  and Lemma 8, the equality in (17) can be written as

$$\text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) + \text{rank}([AB_1, DB_2]) - \text{rank}(C_1) - \text{rank}(DB_2) = 0. \quad (18)$$

Having in mind (18), (15), and (16), we have that  $\text{rank}(C_2[I_n - C_1^-C_1]) = 0$  and  $\text{rank}([I_n - DB_2(DB_2)^-]AB_1) = 0$ . That is  $\mathcal{R}(AB_1) \subseteq \mathcal{R}(DB_2)$  and  $\mathcal{N}(C_1) \subseteq \mathcal{N}(C_2)$ , we have just obtained (2).

By using  $\text{rank}(AB_1) = \text{rank}(B_1)$ , Lemma 8 and (17) we get

$$\text{rank} \left( \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right) + \text{rank}([AB_1, DB_2]) - \text{rank}(AB_1) - \text{rank}(C_2) = 0.$$

The proof of (1)  $\Rightarrow$  (3) finishes as the proof of (1)  $\Rightarrow$  (2).

(2) ⇒ (1). The hypotheses are clearly equivalent to  $AB_1 = DB_2(DB_2)^-AB_1$  and  $C_2 = C_2C_1^-C_1$ , which in view of (15), (16), and Theorem 1 lead to (1).

(3) ⇒ (1). The proof is similar than the proof of (2) ⇒ (1).

(1) ⇔ (4). From the proof of Theorem 1 and  $\text{rank}(DB_2) = \text{rank}(B_2)$ , we have

$$\begin{aligned} \text{rank}(AA^{||_{(B_1,C_1)}} - DD^{||_{(B_2,C_2)}}) &= \text{rank}\left(\begin{bmatrix} AA^{||_{(B_1,C_1)}} \\ DD^{||_{(B_2,C_2)}} \end{bmatrix}\right) + \\ \text{rank}([AA^{||_{(B_1,C_1)}}, DB_2]) &- \text{rank}(AA^{||_{(B_1,C_1)}}) - \text{rank}(DB_2). \end{aligned} \tag{19}$$

By  $A^{||_{(B_1,C_1)}}AA^{||_{(B_1,C_1)}} = A^{||_{(B_1,C_1)}}$ , we have  $\mathcal{N}(A^{||_{(B_1,C_1)}}A) \subseteq \mathcal{N}(A^{||_{(B_1,C_1)}})$ . Thus, by Lemma 2, the expression (19) can be written as

$$\begin{aligned} \text{rank}(AA^{||_{(B_1,C_1)}} - DD^{||_{(B_2,C_2)}}) &= \text{rank}\left(\begin{bmatrix} A^{||_{(B_1,C_1)}} \\ DD^{||_{(B_2,C_2)}} \end{bmatrix}\right) + \\ \text{rank}([AA^{||_{(B_1,C_1)}}, DB_2]) &- \text{rank}(AA^{||_{(B_1,C_1)}}) - \text{rank}(DB_2). \end{aligned} \tag{20}$$

By Lemma 1 and Lemma 2, we have

$$\text{rank}\left(\begin{bmatrix} A^{||_{(B_1,C_1)}} \\ DD^{||_{(B_2,C_2)}} \end{bmatrix}\right) = \text{rank}(A^{||_{(B_1,C_1)}}) + \text{rank}(DD^{||_{(B_2,C_2)}}[I_n - (A^{||_{(B_1,C_1)}})^-A^{||_{(B_1,C_1)}}])$$

and

$$\text{rank}([AA^{||_{(B_1,C_1)}}, DB_2]) = \text{rank}(DB_2) + \text{rank}([I_n - DB_2(DB_2)^-]AA^{||_{(B_1,C_1)}}).$$

By (6) and  $\mathcal{R}(B_1) = \mathcal{R}(A^{||_{(B_1,C_1)}})$ , we have

$$\text{rank}(AA^{||_{(B_1,C_1)}}) = \text{rank}(A^{||_{(B_1,C_1)}}). \tag{21}$$

Thus by (20) and (21), we have that  $AA^{||_{(B_1,C_1)}} = DD^{||_{(B_2,C_2)}}$  if and only if both

$$\text{rank}(DD^{||_{(B_2,C_2)}}[I_n - (A^{||_{(B_1,C_1)}})^-A^{||_{(B_1,C_1)}}]) = 0 \tag{22}$$

and

$$\text{rank}([I_n - DB_2(DB_2)^-]AA^{||_{(B_1,C_1)}}) = 0 \tag{23}$$

hold. It is easy to see that  $\text{rank}(DD^{||_{(B_2,C_2)}}[I_n - (A^{||_{(B_1,C_1)}})^-A^{||_{(B_1,C_1)}}]) = 0$  is equivalent to  $\mathcal{N}(A^{||_{(B_1,C_1)}}) \subseteq \mathcal{N}(DD^{||_{(B_2,C_2)}})$  and  $\text{rank}([I_n - DB_2(DB_2)^-]AA^{||_{(B_1,C_1)}}) = 0$  is equivalent to  $\mathcal{R}(AA^{||_{(B_1,C_1)}}) \subseteq \mathcal{R}(DB_2)$ .

The proof of (1) ⇔ (5) is similar to the proof of (1) ⇔ (4). □

Since the inverse of a matrix  $A$  along  $D$  coincides with the  $(D, D)$ -inverse of  $A$ , Theorem 1 and Theorem 2 lead to the following corollaries.

**Corollary 1.** Let  $A, B, D_1, D_2 \in \mathbb{C}^{n \times n}$ . If  $A$  is invertible along  $D_1$  and  $B$  is invertible along  $D_2$ , then

$$\begin{aligned} &\text{rank}(AA^{||_{D_1}} - BB^{||_{D_2}}) \\ &= \text{rank}\left(\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right) + \text{rank}([AD_1, BD_2]) - \text{rank}(D_1) - \text{rank}(D_2) \\ &= \text{rank}\left(\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}\right) + \text{rank}([AD_1, BD_2]) - \text{rank}(D_1) - \text{rank}(D_2). \end{aligned}$$

**Corollary 2.** Let  $A, B, D_1, D_2 \in \mathbb{C}^{n \times n}$ . If  $A$  is invertible along  $D_1$  and  $B$  is invertible along  $D_2$ , then the following statements are equivalent:

- (1)  $AA^{\parallel D_1} = BB^{\parallel D_2}$ ;
- (2)  $\mathcal{R}(AD_1) \subseteq \mathcal{R}(BD_2)$  and  $\mathcal{N}(D_1) \subseteq \mathcal{N}(D_2)$ ;
- (3)  $\mathcal{R}(BD_2) \subseteq \mathcal{R}(AD_1)$  and  $\mathcal{N}(D_2) \subseteq \mathcal{N}(D_1)$ ;
- (4)  $\mathcal{N}(A^{\parallel D_1}) \subseteq \mathcal{N}(B^{\parallel D_2})$  and  $\mathcal{R}(AA^{\parallel D_1}) \subseteq \mathcal{R}(BD_2)$ ;
- (5)  $\mathcal{N}(B^{\parallel D_2}) \subseteq \mathcal{N}(AA^{\parallel D_1})$  and  $\mathcal{R}(BB^{\parallel D_2}) \subseteq \mathcal{R}(AD_1)$ .

Let  $A \in \mathbb{C}^{n \times n}$ . By ([7] p. 1910), we have that the Moore-Penrose inverse of  $A$  coincides with the  $(A^*, A^*)$ -inverse of  $A$ , the Drazin inverse of  $A$  coincides with the  $(A^k, A^k)$ -inverse of  $A$  for some integer  $k$  and  $A$  is group invertible if and only if  $A$  is  $(A, A)$ -invertible. By ([15] Theorem 4.4), we have that the  $(A, A^*)$ -inverse coincides with the core inverse of  $A$ . We have that the  $(A^*, A)$ -inverse coincides with the dual core inverse of  $A$ . Thus, by Theorem 1 and Theorem 2, more results of the inverse along an element, the Moore-Penrose inverse, Drazin inverse, core inverse and dual core inverse can be obtained. We give some characterizations of these results as follows, and leaving the remaining parts to the reader to research. Also, some rank characterizations of the EP elements can be got by the following Corollary 3.

**Corollary 3.** Let  $A, B \in \mathbb{C}^{n \times n}$ . Then

- (1) Let  $A^\dagger$  and  $B^\dagger$  be the Moore-Penrose inverse of  $A$  and  $B$ , respectively. We have

$$\text{rank}(AA^\dagger - BB^\dagger) = \text{rank}\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) + \text{rank}([A, B]) - \text{rank}(A) - \text{rank}(B);$$

- (2) Let  $\text{ind}(A) = k, \text{ind}(B) = l$  and  $A^D, B^D$  be the Drazin inverse of  $A$  and  $B$ , respectively. We have

$$\text{rank}(AA^D - BB^D) = \text{rank}\left(\begin{bmatrix} A^k \\ B^l \end{bmatrix}\right) + \text{rank}([A^{k+1}, B^{l+1}]) - \text{rank}(A^k) - \text{rank}(B^l);$$

- (3) Let  $\text{ind}(A) = \text{ind}(B) = 1$  and  $A^\#, B^\#$  be the group inverse of  $A$  and  $B$ , respectively. We have

$$\text{rank}(AA^\# - BB^\#) = \text{rank}\left(\begin{bmatrix} A \\ B \end{bmatrix}\right) + \text{rank}([A^2, B^2]) - \text{rank}(A) - \text{rank}(B);$$

- (4) Let  $\text{ind}(A) = \text{ind}(B) = 1$  and  $A^\circ, B^\circ$  be the core inverse of  $A$  and  $B$ , respectively. We have

$$\text{rank}(AA^\circ - BB^\circ) = \text{rank}\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) + \text{rank}([A^2, B^2]) - \text{rank}(A) - \text{rank}(B).$$

**Proof.** (1). Since  $A^\dagger$  coincides with the  $(A^*, A^*)$ -inverse of  $A$  and  $B^\dagger$  coincides with the  $(B^*, B^*)$ -inverse of  $B$ , by Theorem 1, we have

$$\text{rank}(AA^\dagger - BB^\dagger) = \text{rank}\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) + \text{rank}([AA^*, BB^*]) - \text{rank}(A^*) - \text{rank}(B^*).$$

That is

$$\text{rank}(AA^\dagger - BB^\dagger) = \text{rank}\left(\begin{bmatrix} A^* \\ B^* \end{bmatrix}\right) + \text{rank}([A, B]) - \text{rank}(A) - \text{rank}(B),$$

by the following obvious facts  $\text{rank}([AA^*, BB^*]) = \text{rank}([A, B])$ ,  $\text{rank}(A^*) = \text{rank}(A)$ ,  $\text{rank}(B^*) = \text{rank}(B)$ .



(2), (3), (4) are obvious.  $\square$

Some particular cases can be obtained from the previous corollary by setting the matrix  $B$  to some concrete generalized inverses. For example, when  $B = A^\dagger$  we have  $\text{rank}(AA^\dagger - A^\dagger A) = 2[\text{rank}(A, A^\dagger) - \text{rank}(A)]$ , which can be used to get a characterization of the EP matrices ( $AA^\dagger = A^\dagger A$ ) or the co-EP matrices ( $AA^\dagger - A^\dagger A$  is nonsingular, see [24]).

**Author Contributions:** Writing-original draft preparation, W.W.; writing-review and editing, W.W. and S.X.; methodology, S.X. and J.B.; supervision, S.X. and J.B.

**Funding:** This research received no external funding.

**Acknowledgments:** The second author is grateful to China Scholarship Council for giving him a scholarship for his further study in Universitat Politècnica de València, Spain.

**Conflicts of Interest:** The authors declare no conflict of interest.

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