Robust Controller Design for Input-Delayed Systems using Predictive Feedback and an Uncertainty Estimator

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SUMMARY

This paper deals with the problem of stabilizing a class of input-delayed systems with (possibly) nonlinear uncertainties by using explicit delay compensation. It is well known that plain predictive schemes lack robustness with respect to uncertain model parameters. In this work, an uncertainty estimator is derived for input-delay systems and combined with a modified state predictor, which uses current available information of the estimated uncertainties. Furthermore, based on Lyapunov-Krasovskii functionals, a computable criterion to check robust stability of the closed-loop is developed and cast into a minimization problem constrained to a linear matrix inequality (LMI). Additionally, for a given input delay, an Iterative-LMI algorithm is proposed to design stabilizing tuning parameters. The main results are illustrated and validated using a numerical example with a second-order dynamic system.

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KEY WORDS: robust stabilization; input delay; state predictor; uncertainty estimator; LMI

1. INTRODUCTION

This paper deals with the problem of stabilizing a class of input-delayed systems with possibly nonlinear uncertainties. The stability analysis and control of time-delay systems is a problem of recurring interest because time delays are inherent to many engineering problems such as networked control systems, chemical processes, population dynamics or epidemic models [1]. Very often, time delays lead to poor performance of the controlled system, or even instability if they are not taken into account in the design process. The first solution to explicitly compensate delays was proposed in 1959 with the introduction of the Smith Predictor (SP) [2]. The original SP was formulated in frequency domain, applicable only to SISO open-loop stable plants, and exhibited poor robustness with respect to uncertain parameters [3, 4]. Over the last decades, many modifications to the SP have been proposed to overcome these limitations, see for example [5, 6, 7] and the references therein.

The same underlying idea of the SP was proposed in time domain for general MIMO stable/unstable systems, introducing the so-called state predictor [8], also known as the reduction-based approach [9]. The robustness of this technique to bounded parametric uncertainties was analyzed later in [10], and extended to unknown constant and time-varying delays in [11] and [12], respectively. Complementary results have been obtained in the discrete-time framework

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[13, 14, 15], and also for nonlinear systems [16, 17, 18, 19, 20], even when the plant is unstable. The aforementioned works were focused on proving the closed-loop stability. For linear systems, some researchers have been actively working on reducing the conservatism of the robust stability conditions [21, 22, 23], mainly formulated in terms of linear matrix inequalities (LMIs) [24]. In any case additional robust control structures were applied to improve robustness (which was a common practice in frequency-based methods). Works in this line are scattered. Regarding disturbance rejection, in [25] a low-pass filtered prediction is shown to improve disturbance rejection capabilities and, in [26], better disturbance rejection is achieved by introducing a modified state prediction. Regarding robustness improvement, which is problem this paper is concerned with, a sliding mode control with delay compensation is proposed in [27] to deal with matched uncertainties; in [28], adaptive schemes are introduced to estimate uncertain plant parameters and input delay; and a control strategy is proposed in [29] to deal with Euler-Lagrange-like nonlinear systems with time-varying input delay.

In this paper, the disturbance observer based control (DOBC) is adopted to improve the robustness of the state predictor. The basic idea behind this technique is to use a model of the system along with its input/output information to identify uncertainties and disturbances [30]. There are slightly different techniques that pursue the same goal, which have been recently summarized and explained in [31] (see also the references therein). Most of them make use of some sort of low-pass filter which is needed to obtain a realizable control law. In general, the bandwidth of this filter is desired to be as high as possible to estimate uncertainties in a wide range of frequencies. However, in [32], by means of the Bode's integral formula, it is shown that there is a severe limitation in the choice of the bandwidth if the plant includes time delay. Intuitively, the uncertainties can be estimated arbitrarily fast, but they cannot be counteracted in the same way. This inconvenient can be seen as analogous to the need of detuning PID controllers for open-loop unstable systems [33]. An attempt of combining the state predictor with a DOBC technique known as the ADRC (active disturbance rejection control) can be found in [34], but neither the analysis nor the design problems are tackled, presenting only numerical case studies. Also, a similar idea was explored in [35], where work in [36] is extended to the case of state delays.

This work introduces a new approach to control input-delayed systems, in which the state predictor is modified and combined with an uncertainty estimator. The uncertain part of the model, possibly nonlinear, is neglected when designing the controller, and counteracted by using the information provided by the uncertainty estimator. Furthermore, the conventional state prediction is modified to include also this information. The closed-loop is analyzed using Lyapunov-Krasovskii functionals. A computable robust stability criterion is proved first in Theorem 1, which is cast into a minimization problem constrained to a linear matrix inequality (LMI). Additionally, for a given input delay, an Iterative-LMI (ILMI) algorithm is proposed to design stabilizing tuning parameters in Theorem 2. Once solved, it yields a feedback gain matrix, an observer bandwidth and a robustness index, which ensure stability for a prescribed upper bounded delay.

The rest of the paper is structured as follows. In Section 2, the problem formulation and the assumptions are presented. The building blocks of the proposed control structure are introduced in Section 3 where the closed-loop equations are also derived. The main results are shown in Section 4. Finally, the results are illustrated with a numerical example in Section 5.

Notation: The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$ while $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, and I is the identity matrix of appropriate dimensions. The notation $X > 0$ ($X \geq 0$) means that $X \in \mathbb{R}^{n\times n}$ is a real positive definite (semi-definite) matrix. The standard Euclidean vector norm as well as its induced matrix norm are represented by $\|\cdot\|$. The function $\phi(s) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is said to belong to $L_1[0, \infty)$ if the norm $\|\phi(t)\|_1 = \int_0^\infty |\phi(s)|\, \mathrm{d}s$ exists and is finite.

2. PROBLEM FORMULATION AND PRELIMINARIES

Consider the class of multivariable nonlinear input-delay systems represented by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[u(t-h) + w(x)] \\ u(s) &= u_0(s) \qquad s \in [-h, 0) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ are known matrices, h is a known constant input delay, $w : \mathcal{D} \to \mathbb{R}^m$ is an unknown (possibly nonlinear) function, and $\mathcal{D} \subset \mathbb{R}^n$ is an open connected set that contains the origin $x = 0$. The source of a system such as the one described by (1) could be a more general nonlinear system after an appropriate change of coordinates [37]. The system defined in (1) implies that the uncertain term $Bw\big(x(t)\big)$ is matched, that is, it belongs to the range space of the matrix B, which is a rather conventional assumption in robust control problems [31].

Assumption 1
The pair (A, B) is stabilizable

Assumption 2
The function $w(x)$: i.) vanishes at the origin; ii.) is differentiable on $\mathcal{D}$ with derivative locally bounded by $\nabla w_x^T \nabla w_x \leq \beta^2 H^T H$ for some $H \in \mathbb{R}^{m\times n}$, $\beta \geq 0$

The Assumption 1 is standard in analysis of both nonlinear and time-delay systems, as it implies the existence of a Lyapunov function for the nominal non-delayed closed-loop system $\dot{\mathcal{X}} = (A + BK)\mathcal{X}(t)$. This is necessary because the analysis techniques used throughout this paper to deal with delays and uncertainties require the stability of the nominal system. The Assumption 2 is slightly stronger than Lipschitz continuity (the latter allows also functions that are differentiable almost everywhere). The matrix H can be useful to describe how the uncertainty affects each of the states and thus reduce conservatism in the stability conditions.

3. PROPOSED CONTROL STRATEGY

The building blocks of the proposed control strategy are presented in this section. The uncertainty estimator for time-delay systems is derived first, whose output is denoted by $\hat{w}(t)$. The novel state prediction $\hat{x}(t+h)$ that includes information about the uncertainty estimation is introduced next. Finally, both elements are combined into a composite control law and the closed-loop equations are derived.

3.1. *Uncertainty estimation*

The uncertainty estimation is obtained by constructing a reduced-order observer from the system model (1). The observer derived next is an adaptation of the one proposed in [38]. Note that the unknown term can be obtained from (1) as $w\big(x(t)\big) = B^+[\dot{x}(t) - Ax(t) - Bu(t-h)]$, where $B^+ = (B^TB)^{-1}B^T$ is the pseudoinverse of the matrix B. However, the previous equation cannot be computed because the state derivative $\dot{x}(t)$ is not accessible. To circumvent this issue, a filtered estimation $\hat{w}(t) \in \mathbb{R}^m$ is proposed such that

$$\dot{\hat{w}}(t) \triangleq -\Omega\hat{w}(t) + \Omega B^+[\dot{x}(t) - Ax(t) - Bu(t-h)] \tag{2}$$

where $\Omega \in \mathbb{R}^{m\times m}$ is a positive definite diagonal matrix $\Omega \triangleq \operatorname{diag}\{\omega_1, \omega_2, \ldots, \omega_m\}$, with $\omega_j > 0$. In this way, the explicit state derivative $\dot{x}(t)$ can be removed from the observer by performing the change of variable $\hat{\xi}(t) \triangleq \hat{w}(t) - \Omega B^+x(t)$ in (2), which leads to the following reduced-order observer

$$\dot{\hat{\xi}}(t) = -\Omega\hat{\xi}(t) - (\Omega^2B^+ + \Omega B^+A)x(t) - \Omega u(t-h) \tag{3}$$

$$\hat{w}(t) = \hat{\xi}(t) + \Omega B^+x(t) \tag{4}$$

Remark 1
Notice that the observer (3)-(4) is computed using only information from the actual state $x(t)$, and the past control input $u(t-h)$. The Equation (2) can be seen as a low-pass filter with m channels, driven by the virtual input $w\big(x(t)\big) = B^{+}[\dot{x}(t) - Ax(t) - Bu(t-h)]$, and hence each parameter ω_j should be understood as the bandwidth of the j-th channel.

The error between the actual uncertainty and its estimation is defined as

$$e(t) \triangleq w\big(x(t)\big) - \hat{w}(t) \tag{5}$$

Differentiating (4), plugging (3) in, rearranging terms and using the system model (1), the following expression is obtained

$$\dot{\hat{w}}(t) = -\Omega\hat{w}(t) + \Omega w\big(x(t)\big) \tag{6}$$

The fact that $\hat{w}(t)$ is a low-passed filtered estimation of $w\big(x(t)\big)$, pointed out in Remark 1, is highlighted by (6). Now, it is straightforward to use (6) and (5) to obtain the dynamics of the estimation error as

$$\dot{e}(t) = -\Omega e(t) + \dot{w}\big(x(t)\big) \tag{7}$$

Remark 2
From (7), it can be seen that the estimation error decays, assuming $\dot{w}\big(x(t)\big)$ bounded, with an exponential rate $\alpha = \lambda_{max}(\Omega) = \max_j \omega_j$. Therefore, one would be interested in choosing the bandwidths ω_j as large as possible.

So far, the input delay does not play any role as it does not affect the uncertainty estimation. However, it is critical when attempting to cancel out its effect in the closed-loop. In contrast to the estimation error $e(t)$, the cancellation error is defined as

$$\sigma(t) \triangleq w\big(x(t)\big) - \hat{w}(t-h) \tag{8}$$

which will naturally arise later in the closed-loop equations.

3.2. Predictor-based feedback

For LTI time delay systems $\dot{\mathcal{X}}(t) = A\mathcal{X}(t) + B\mathcal{U}(t-h)$, the conventional predictive feedback $\mathcal{U}(t) = \mathcal{K}\hat{\mathcal{X}}(t+h)$, with $\hat{\mathcal{X}}(t+h) = e^{Ah}\mathcal{X}(t) + \int_{-h}^{0} e^{-As}B\mathcal{U}(t+s)\mathrm{d}s$ is widely known in the literature as the finite spectrum assignment (FSA), as it renders (in the nominal case) a closed-loop with a finite dimensional characteristic polynomial, for any arbitrarily large delay $h \geq 0$.

In this paper, a new state prediction is introduced that includes additional information about the estimated uncertainty $\hat{w}(t)$. Such prediction is defined by

$$\hat{x}(t+h) \triangleq e^{Ah}x(t) + \int_{-h}^{0} e^{-As}B[u(t+s) + \hat{w}(t+s)]\mathrm{d}s \tag{9}$$

Notice that the exact prediction of the state from t to $t+h$ can be obtained from (1) as

$$x(t+h) = e^{Ah}x(t) + \int_{-h}^{0} e^{-As}B\left[u(t+s) + w\big(x(t+s+h)\big)\right]\mathrm{d}s \tag{10}$$

Of course, (10) cannot be computed because $w\big(x(t)\big)$ is unknown. Using (9)-(10) and the definition (8), the error between the exact and the proposed prediction is given by

$$x(t+h) - \hat{x}(t+h) = \int_{-h}^{0} e^{-As}B\sigma(t+s+h)\mathrm{d}s \tag{11}$$

3.3. Control law

The state prediction and the uncertainty estimation are brought together into a composite control law defined by

$$u(t) = K\hat{x}(t+h) - \hat{w}(t), \tag{12}$$

with $K \in \mathbb{R}^{m\times n}$. The predicted feedback is used to stabilize the nominal part of the system and compensate for the delay, while the estimation $\hat{w}(t)$ is fed back to the system in order to cancel out the effect of the uncertainties. Notice that this is a rather conventional approach when attenuating matched uncertainties [31].

3.4. Closed-loop equations

Using the control law (12) and (11) into the system (1) yields the following closed-loop dynamics

$$\dot{x}(t) \;\; = \;\; (A+BK)x(t) + B\sigma(t) + B\int_{-h}^{0} G(s)\sigma(t+s)\,\mathrm{d}s \tag{13}$$

where $G(s) = -Ke^{-As}B \in \mathbb{R}^{m\times m}$ has been defined. As mentioned above, the cancellation error $\sigma(t)$, arises in the closed-loop equation due to the input delay. The dynamics of the cancellation error is obtained by differentiating (8), using (6) evaluated at $t-h$, and adding and subtracting $\Omega w\big(x(t)\big)$, which yields

$$\dot{\sigma}(t) \;\; = \;\; -\Omega\sigma(t) + \Omega\left[w\big(x(t)\big) - w\big(x(t-h)\big)\right] + \dot{w}(x,t) \tag{14}$$

Now, for analysis purposes, it is convenient to get rid of the of the term $[w\big(x(t)\big) - w\big(x(t-h)\big)]$ in (14), so that the entire system is driven only by $\dot{w}(x,t)$. To do so, the estimation error dynamics (7) is integrated and used to rewrite

$$w\big(x(t)\big) - w\big(x(t-h)\big) = e(t) - e(t-h) + \Omega\int_{-h}^{0} e(t+s)\,\mathrm{d}s \tag{15}$$

Let us denote by $g_{ij}(s)$ the (i,j)-entry of $G(s)$, which can be expressed as a sum of scalar functions as $G(s) = \sum_{i=1}^{m}\sum_{j=1}^{m}\delta_{ij}g_{ij}(s)$, with $\delta_{ij} \in \mathbb{R}^{m\times m}$ a matrix such that its (i,j)-entry is equal one while the rest are zero. This formulation is chosen so that the remaining kernel in the distributed delay is scalar. Defining the augmented state vector $\eta(t) \triangleq [x^T(t),\ \sigma^T(t),\ e^T(t)]^T$ and gathering (7), (13)-(15) leads to

$$\begin{aligned}\dot{\eta}(t) \;\; = \;\; & \begin{bmatrix} A+BK & B & 0 \\ 0 & -\Omega & \Omega \\ 0 & 0 & -\Omega \end{bmatrix}\eta(t) + \sum_{i=1}^{m}\sum_{j=1}^{m}\begin{bmatrix} 0 & B\delta_{ij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\int_{-h}^{0} g_{ij}(s)\eta(t+s)\,\mathrm{d}s \\ & + \int_{-h}^{0}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega^2 \\ 0 & 0 & 0 \end{bmatrix}\eta(t+s)\,\mathrm{d}s + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\Omega \\ 0 & 0 & 0 \end{bmatrix}\eta(t-h) + \begin{bmatrix} 0 \\ I_m \\ I_m \end{bmatrix}\dot{w}(x) \end{aligned} \tag{16}$$

The system (16) is the result of controlling (1) with (3)-(4), (9) and (12). In what follows, a practically computable criterion to check the robust stability of (16) is derived, in the form of a minimization problem subject to LMI constraints (Theorem 1). Additionally, for a given input delay, an Iterative-LMI (ILMI) algorithm is proposed to design stabilizing tuning parameters K and Ω (Theorem 2).

4. MAIN RESULTS

In this section, sufficient stability conditions for the closed-loop stability based on Lyapunov-Krasovskii functionals are derived. In order to reduce conservatism, the system (16) is transformed into a descriptor form [39]. The terms $\eta(t-h)$ are replaced by $\eta(t) - \int_{-h}^{0} \mu(t+s)\,\mathrm{d}s$ using the Newton-Leibnitz formula, which leads to

$$\begin{aligned} \dot{\eta}(t) &= \mu(t) &\quad (17)\\ 0 &= -\mu(t) + A_1\eta(t) + A_2\int_{-h}^{0}\eta(t+s)\,\mathrm{d}s + A_3\int_{-h}^{0}\mu(t+s)\,\mathrm{d}s \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m} A_{4_{ij}}\int_{-h}^{0} g_{ij}(s)\eta(t+s)\,\mathrm{d}s + \Gamma\dot{w}(x) &\quad (18)\end{aligned}$$

with

$$A_1 = \begin{bmatrix} A+BK & B & 0 \\ 0 & -\Omega & 0 \\ 0 & 0 & -\Omega \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega^2 \\ 0 & 0 & 0 \end{bmatrix} \qquad A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \Omega \\ 0 & 0 & 0 \end{bmatrix} \quad (19)$$

$$A_{4_{ij}} = \begin{bmatrix} 0 & B\delta_{ij} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Gamma = \begin{bmatrix} 0 \\ I_m \\ I_m \end{bmatrix} \quad (20)$$

Theorem 1
Consider some prescribed tuning for the controller parameters K and Ω. Under the Assumptions 1 and 2, the system (1) controlled with (3)-(4), (9) and (12) is robustly asymptotically stable for any $0 \le \beta \le \beta^* \triangleq (\gamma^*)^{1/2}$ and any delay $0 \le h \le \bar{h}$, if there exist symmetric positive definite matrices $P_1, R, S, W_{ij} \in \mathbb{R}^{(n+2m)\times(n+2m)}$ and real matrices $N_2, N_3 \in \mathbb{R}^{(n+2m)\times(n+2m)}$ such that the following problem is feasible

$$\begin{aligned} \gamma^* = \ & \underset{\gamma>0}{\text{argmin}} \quad \gamma \\ & \text{subject to} \quad \Psi^0 < 0 \end{aligned} \quad (21)$$

with

$$\Psi^0 = \begin{bmatrix} \Psi^0_{11} & \Psi^0_{12} & \bar{h}N_2^TA_2 & \bar{h}N_2^TA_3 & \bar{g}_{11}N_2^TA_{4_{11}} & \dots & \bar{g}_{mm}N_2^TA_{4_{mm}} & N_2^T\Gamma & 0 \\ (*) & \Psi^0_{22} & \bar{h}N_3^TA_2 & \bar{h}N_3^TA_3 & \bar{g}_{11}N_3^TA_{4_{11}} & \dots & \bar{g}_{mm}N_3^TA_{4_{mm}} & N_3^T\Gamma & (H\Phi)^T \\ (*) & (*) & -\bar{h}R & 0 & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & -\bar{h}S & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & -\bar{g}_{11}W_{11} & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\bar{g}_{mm}W_{mm} & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & -I_m & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & -\gamma I_m \end{bmatrix} \quad (22)$$

$$\begin{aligned} \Psi^0_{11} &= N_2^TA_1 + A_1^TN_2 + \bar{h}R + \sum_{i=1}^{m}\sum_{j=1}^{m}\bar{g}_{ij}W_{ij} & \quad \bar{g}_{ij} &= \int_{-\bar{h}}^{0}|g_{ij}(s)|\,\mathrm{d}s \\ \Psi^0_{12} &= P_1 - N_2^T + A_1^TN_3 & \Phi &= [I_n, 0_{n\times m}, 0_{n\times m}] \\ \Psi^0_{22} &= -N_3 - N_3^T + \bar{h}S \end{aligned}$$

Proof
See Appendix A. $\square$

Theorem 2
Under the Assumptions 1-2, the system (1) can be robustly asymptotically stabilized with (3)-(4), (9) and (12), for any $0 \leq \beta \leq (\gamma^*)^{-1/2}$ and any delay $0 \leq h \leq \bar{h}$, if there exist symmetric positive definite matrices $Y_1, Y_R, Y_S, Y_{W_{ij}} \in \mathbb{R}^{(n+2m)\times(n+2m)}$, positive scalars $\gamma, \kappa_x^{(0)}, \ldots, \kappa_x^{(m)}, \kappa_y^{(0)}, \ldots, \kappa_y^{(m)} \in \mathbb{R}_{>0}$, adjustable weights λ_j, a tuning parameter $\epsilon \in \mathbb{R}$, and block-diagonal matrices $Y_2 = \operatorname{diag}\left\{Y_2^{(0)}, Y_2^{(1)}, \ldots, Y_2^{(m)}, Y_2^{(1)}, \ldots, Y_2^{(m)}\right\}$, $X = \operatorname{diag}\left\{X^{(0)}, X^{(1)}, \ldots, X^{(m)}, X^{(1)}, \ldots, X^{(m)}\right\}$, with a real matrix $X^{(0)} \in \mathbb{R}^{n\times m}$, a positive definite matrix $Y_2^{(0)} \in \mathbb{R}^{n\times n}$, positive scalars $Y_2^{(1)}, \ldots, Y_2^{(m)} \in \mathbb{R}_{>0}$, and scalars $X^{(1)}, \ldots, X^{(m)} \in \mathbb{R}$, such that the following problem is feasible

$$
\begin{aligned}
\gamma^* = \underset{\gamma>0}{\operatorname{argmin}} \quad & \lambda_\gamma \gamma + \sum_{j=0}^{m} \lambda_j (\kappa_y^{(j)} + \kappa_x^{(j)}) \\
\text{subject to} \quad & \begin{cases} \Psi < 0 \\ \begin{bmatrix} -\kappa_x^{(j)} I & (X^{(j)})^T \\ (*) & -I \end{bmatrix} < 0, \quad \begin{bmatrix} Y_2^{(j)} & I \\ (*) & \kappa_y^{(j)} \end{bmatrix} > 0, \quad j = 1, \ldots, m \end{cases}
\end{aligned} \tag{23}
$$

with

$$
\Psi = \begin{bmatrix}
\Psi_{11} & \Psi_{12} & \bar{h}\tilde{K}^T B_{20} X & \bar{h} B_{30} X & \bar{g}_{11} A_{4_{11}} Y_2 & \ldots & \bar{g}_{mm} A_{4_{mm}} Y_2 & \Gamma & 0 \\
(*) & \Psi_{22} & \epsilon \bar{h}\tilde{K}^T B_{20} X & \epsilon \bar{h} B_{30} X & \epsilon \bar{g}_{11} A_{4_{11}} Y_2 & \ldots & \epsilon \bar{g}_{mm} A_{4_{mm}} Y_2 & \epsilon\Gamma & Y_2^T (H\Phi)^T \\
(*) & (*) & -\bar{h} Y_R & 0 & 0 & \ldots & 0 & 0 & 0 \\
(*) & (*) & (*) & -\bar{h} Y_S & 0 & \ldots & 0 & 0 & 0 \\
(*) & (*) & (*) & (*) & -\bar{g}_{11} Y_{W_{11}} & \ldots & 0 & 0 & 0 \\
(*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & -\bar{g}_{mm} Y_{W_{mm}} & 0 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & -I_m & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & -\gamma I_m
\end{bmatrix} < 0 \tag{24}
$$

$$
\begin{aligned}
\Psi_{11} &= A_{10} Y_2 + B_{10} X + X^T B_{10}^T + Y_2^T A_{10}^T + \bar{h} Y_R + \sum_{i=1}^{m}\sum_{j=1}^{m} \bar{g}_{ij} Y_{W_{ij}} \qquad \bar{g}_{ij} = \int_{-\bar{h}}^{0} |g_{ij}(s)|\, \mathrm{d}s \\
\Psi_{12} &= Y_1 - Y_2 + \epsilon (X^T B_{10}^T + Y_2^T A_{10}^T) \qquad \Phi = [I_n, 0_{n\times m}, 0_{n\times m}] \\
\Psi_{22} &= -\epsilon (Y_2^T - Y_2) + \bar{h} Y_S
\end{aligned}
$$

$$
A_{10} = \begin{bmatrix} A & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_{10} = \begin{bmatrix} B & 0 & 0 \\ 0 & -I_m & 0 \\ 0 & 0 & -I_m \end{bmatrix} \quad B_{20} = B_{30} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & I_m \\ 0 & 0 & 0 \end{bmatrix} \tag{25}
$$

and $\tilde{K} = \operatorname{diag}\{K, \Omega, \Omega\}$. Furthermore, the stabilizing feedback control gain is given by $K = X^{(0)}(Y_2^{(0)})^{-1}$ and each of the observer bandwidths is given by $\omega_j = X^{(j)}(Y_2^{(j)})^{-1}$, satisfying $\|K\| < \kappa_y^{(0)}\sqrt{\kappa_x^{(0)}}$ and $|\omega_j| < \kappa_y^{(j)}\sqrt{\kappa_x^{(j)}}$, respectively.

Proof
See Appendix B. □

The matrix $\tilde{K}$, which is an output of the minimization problem (23), appears in (24), and thus the problem has to be solved by iterating. Notice that (24) becomes a standard LMI if $\tilde{K}$ is fixed. The following Iterative-LMI algorithm, similar to [40], is used to solve this problem. **Step 1**: Choose $\tilde{K}_0 = 0$ initially and let $k = 1$. **Step 2**: Solve the optimization problem (23) with $\tilde{K} = \tilde{K}_{k-1}$ to obtain X_k, Y_{2_k} and compute $\tilde{K}_k = X_k Y_{2_k}^{-1}$. **Step 3**: If $||\tilde{K}_k - \tilde{K}_{k-1}|| < \delta$ for some sufficiently small $\delta > 0$, or the maximum number of iterations is reached, then stop. Otherwise, let $k \to k+1$ and go to Step 2.

Remark 3
The weights in the minimization problem (23) can be adjusted to reach a trade-off between robustness, feedback gain norm, and observer bandwidth (in practice, if measurements are noisy, having a low observer bandwidth is convenient). Furthermore, there is another user-supplied parameter, ϵ, which arises when manipulating the matrix inequality. The optimal value of this parameter can be easily found using by numerical optimization because the cost exhibits a convex behavior with respect to ϵ, as explained in Remark 5 of [21]. This fact is demonstrated below using a numerical example and illustrated in Fig. 1.c.

5. NUMERICAL EXAMPLE

In order to illustrate the proposed strategy, an uncertain second-order system with delayed input $\ddot{y}(t) = u(t-h) + w(y, \dot{y})$ is chosen, which can be expressed in the form of (1) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and $x = [x_1, x_2]^T$ and $x_1 = y$, $x_2 = \dot{y}$.

5.1. Linear uncertainty

Let us consider a long time delay $h = 1$ s, and a linear uncertainty $w(x) = \beta x_1$. Since $w(x)$ is linear, its gradient does not depend on x and it is simply given by $\nabla w = [\beta, 0]$. Assumption 2 is then satisfied globally with $H = [1, 0]$. In this case, a controller designed by means of Theorem 2 will ensure global stability if $\beta < \beta^*$.

Solving the minimization problem (23) in Theorem 2 with weights† $\lambda_\gamma = 0.5, \lambda_0 = \lambda_1 = 0.25$ yields an optimal $\gamma^* = 189.6$ and matrices

$$Y_2 = \begin{bmatrix} 373.59 & -51.08 & 0 & 0 \\ -51.08 & 47.98 & 0 & 0 \\ 0 & 0 & 6.14 & 0 \\ 0 & 0 & 0 & 6.14 \end{bmatrix} \qquad X = \begin{bmatrix} -17.64 & -65.43 & 0 & 0 \\ 0 & 0 & 3.96 & 0 \\ 0 & 0 & 0 & 3.96 \end{bmatrix}$$

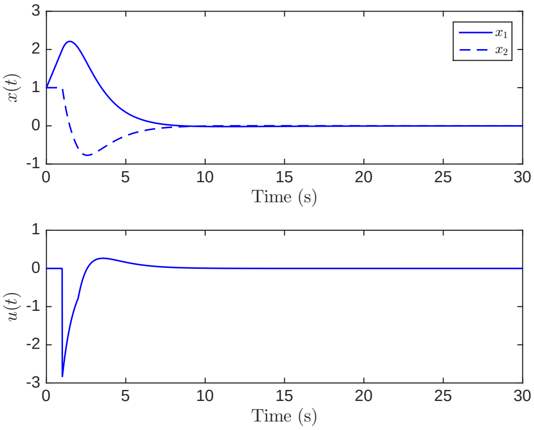
$$\tilde{K} = X(Y_2)^{-1} = \begin{bmatrix} -0.27 & -1.65 & 0 & 0 \\ 0 & 0 & 0.65 & 0 \\ 0 & 0 & 0 & 0.65 \end{bmatrix}$$

Therefore, the controller parameters are $K = [-0.27, -1.65]$ and $\omega_1 = 0.65$ rad/s, and the maximum allowable uncertainty is $\beta^* = (\gamma^*)^{-1/2} = 0.073$. Simulation results are presented in Fig. 1, showing both nominal and robust responses in Figs. 1.a and 1.b, respectively. The convex behavior of the cost with respect to the parameter ϵ is also depicted in Fig. 1.c. In this case, an optimal value of $\epsilon = 1.2$ is found. As an example, for that value of ϵ, the history of the convergence criterion $\left\|\tilde{K}_k - \tilde{K}_{k-1}\right\|$ can be seen in Fig. 1.d, which is stopped at $k = 20$.

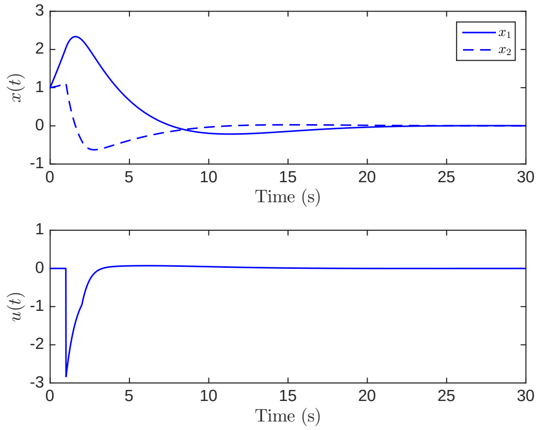
The decomposition process used in Theorem 2 introduces some conservatism in the resulting robustness index. Using Theorem 1, an almost four times larger value $\beta^* = 0.27$ is obtained. It can be seen in Fig. 2 that this value is quite close to the actual stability limit.

Among the existing references, very few of them follow the idea of combining predictive feedback with an additional robustifying structure. The approach presented in [26] is suitable for a numerical comparison, as it is a slight modification that improves the conventional predictive feedback [8]. A simulation in the same scenario as above shows that the proposed controller can keep the system

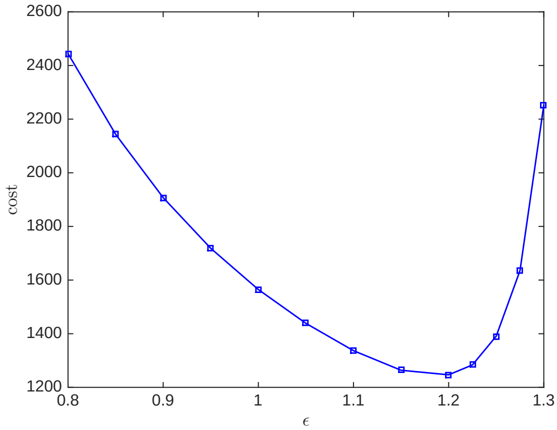
†Since (23) is a multi-objective minimization problem, it is reasonable to choose the weights such that $\lambda_\gamma + \sum_j \lambda_j = 1$. This a simple method that transforms the problem into a single-objective minimization, where the weights determine the relative importance of each objective function with respect to each other [41].



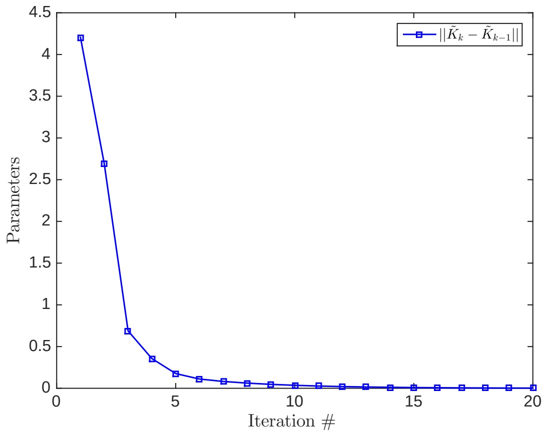
(a) Nominal response, $\beta = 0$



(b) Robust response, $\beta = 0.07$

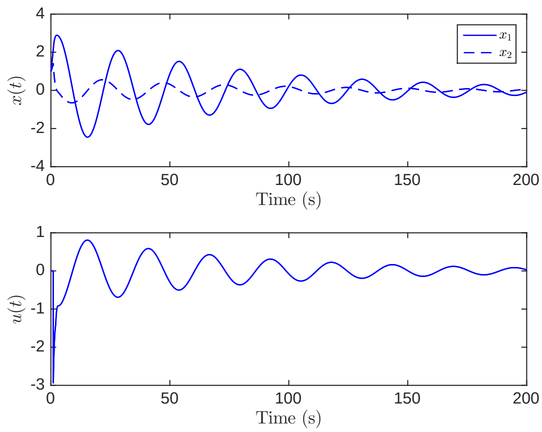


(c) Convex behavior of the cost w.r.t. the parameter ϵ

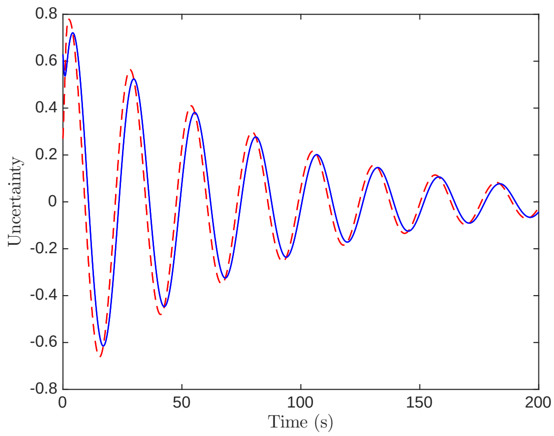


(d) Example of convergence history

Figure 1. Simulations with linear uncertainty $w(x) = \beta x_1$, the system starting at $x(0) = [1,\ 1]^T$, input delay $h = 1$ s, and controller parameters $K = [-0.27,\ -1.65]$ and $\omega_1 = 0.65$ rad/s, designed by means of Theorem 2



(a) Robust response, $\beta = 0.27$



(b) Uncertainty estimation

Figure 2. Validation of the robust stability bound $\beta = 0.27$ obtained by means of Theorem 1

stable while the solutions presented in [8], [26] cannot. Therefore, the proposed scheme can improve the robustness of the closed-loop system.

5.2. *Nonlinearity*

Let us consider now a smaller input delay $h = 0.2$ s and a nonlinearity $w(x) = \sin(x_1 + x_2)$. Its gradient is obtained as

$$\nabla w(x) = [\cos(x_1 + x_2),\ \cos(x_1 + x_2)] \leq [1,\ 1]$$

and thus Assumption 2 is satisfied globally with $H = [1,\ 1]$ and $\beta = 1$. Because of the properties of this particular nonlinearity, a controller designed by means of Theorem 2 will ensure global stability if $\beta^* > 1$, and only local stability if $\beta^* \leq 1$. Using Theorem 2 with weights $\lambda_\gamma = 0.8$, $\lambda_0 = \lambda_1 = 0.1$, yields an optimal $\beta^* = (\gamma^*)^{-1/2} = 1.17$ and controller parameters $K = [-1.19,\ -2.93]$, $\omega_1 = 1.89$ rad/s. From the reasoning above, this controller should be globally

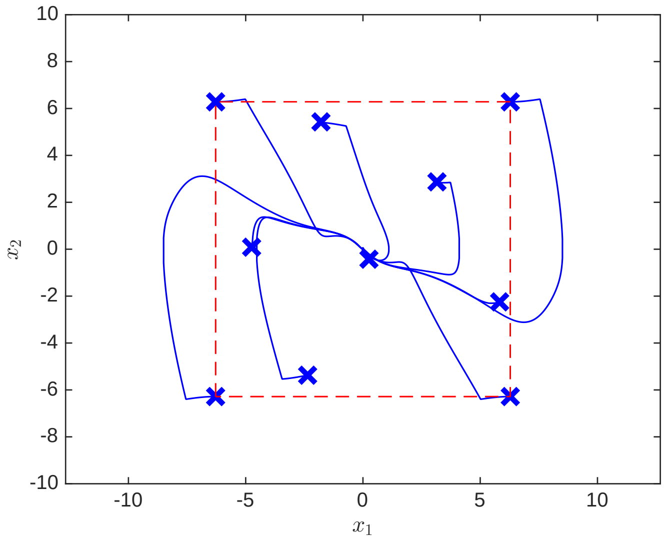
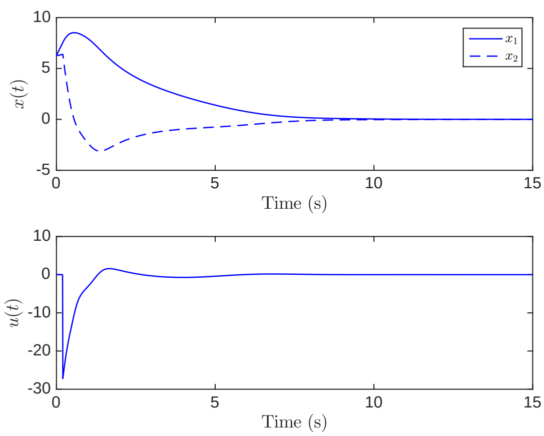
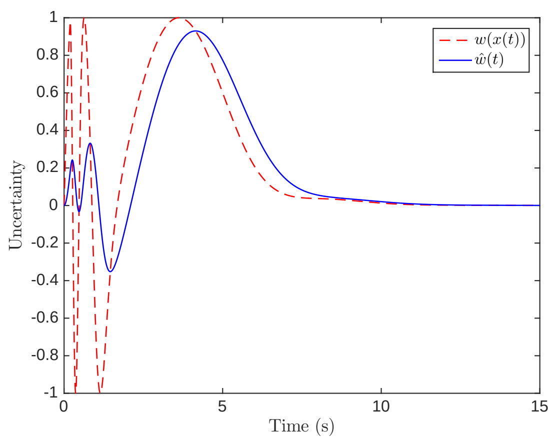


Figure 3. Validation of Theorem 2 in the case of nonlinear uncertainties



(a) State evolution and control input



(b) Uncertainty estimation

Figure 4. Simulations with a nonlinearity $w(x) = \sin(x_1 + x_2)$, the system starting at $x(0) = [2\pi,\ 2\pi]^T$, input delay $h = 0.2$ s, and controller parameters $K = [-1.19,\ -2.93]$ and $\omega_1 = 1.89$ rad/s, designed by means of Theorem 2

stabilizing in spite of the nonlinearity. The global stability is verified by running a set of ten simulations starting within $x(0) \in [-2\pi, 2\pi] \times [-2\pi, 2\pi]$ as shown in Fig. 3. To show just one example, the state trajectory starting from $x(0) = [2\pi,\, 2\pi]^T$ is depicted in Fig. 4.a and the corresponding uncertainty estimation can be seen in Fig. 4.b.

Remark 4
One of the benefits of the controller synthesis procedure proposed in this paper is that the observer bandwidths are also automatically tuned. In the delay-free case, the bandwidth is simply chosen as small as possible (only limited in practice by the measurement noise and the sampling period). However, if there is input delay, a small filter bandwidth can make the closed-loop unstable [32].

6. CONCLUSIONS AND FUTURE WORK

In this paper, a robust control scheme for input-delay systems has been introduced and analyzed. A stability criterion has been derived in terms of a LMI and a design procedure has been proposed, which needs to be solved using an Iterative-LMI algorithm. This control structure is able to deal with matched and possibly nonlinear uncertainties even in the presence of input delays, as it has been shown through numerical examples. Future research will be focused on extending the analysis to prove robustness with respect to time-varying delays or modifying the proposed scheme to deal with unmatched uncertainties.

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A. PROOF OF THEOREM 1

A.1. *Auxiliary lemma: Jensen's inequality with variable kernels*

Lemma 1 ([42, 43])
Consider an $n \times n$ matrix $W > 0$, a scalar function $g : [0, \infty) \to \mathbb{R}$ belonging to $L_1[0, \infty)$ and a vector function $\phi : [0, \infty) \to \mathbb{R}^n$ such that the integrations concerned are well defined. Then, for any scalar $h > 0$, the following inequality holds

$$-\int_{-h}^{0} |g(s)| \phi^T(t+s) W \phi(t+s)\,\mathrm{d}s \leq -\frac{1}{\bar{g}} \left(\int_{-h}^{0} g(s)\phi(t+s)\,\mathrm{d}s \right)^T W \left(\int_{-h}^{0} g(s)\phi(t+s)\,\mathrm{d}s \right) \tag{26}$$

where $\bar{g} = \int_{-h}^{0} |g(s)|\,\mathrm{d}s$.

A.2. *Proof of Theorem 1*

Let us choose the Lyapunov candidate function

$$V(t) = V_1(t) + V_2(t) + V_3(t) + \sum_{i=1}^{m} \sum_{j=1}^{m} V_{4_{ij}}(t) \tag{27}$$

where

$$V_1(t) = \eta^T(t)P_1\eta(t) \qquad V_2(t) = \int_{-h}^{0}\int_{t+s}^{t} \eta^T(\tau)R\eta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$
$$V_3(t) = \int_{-h}^{0}\int_{t+s}^{t} \mu^T(\tau)S\mu(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \quad V_{4_{ij}}(t) = \int_{-h}^{0}\int_{t+s}^{t} |g_{ij}(s)|\eta^T(\tau)W_{ij}\eta(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

Using Lemma 1, the derivative of $V_{4_{ij}}(t)$ is bounded by

$$\begin{aligned}\dot V_{4_{ij}}(t) &= \eta^T(t)[\bar g_{ij}W_{ij}]\eta(t) - \int_{-h}^{0} |g_{ij}(s)|\eta^T(t+s)W_{ij}\eta(t+s)\,\mathrm{d}s \\ &\le \eta^T(t)[\bar g_{ij}W_{ij}]\eta(t) + \left(\frac{1}{\bar g_{ij}}\int_{-h}^{0} g_{ij}(s)\eta(t+s)\,\mathrm{d}s\right)^T [-\bar g_{ij}W_{ij}]\left(\frac{1}{\bar g_{ij}}\int_{-h}^{0} g_{ij}(s)\eta(t+s)\,\mathrm{d}s\right)\end{aligned} \tag{28}$$

Similarly, the derivatives of $V_2(t)$ and $V_3(t)$ are bounded (using Lemma 1 with $g(s) = 1$) by

$$\dot V_2(t) \le \eta^T(t)hR\eta(t) + \left(\frac{1}{h}\int_{-h}^{0} \eta^T(t+s)\,\mathrm{d}s\right)[-hR]\left(\frac{1}{h}\int_{-h}^{0} \eta(t+s)\,\mathrm{d}s\right) \tag{29}$$

$$\dot V_3(t) \le \mu^T(t)hS\mu(t) + \left(\frac{1}{h}\int_{-h}^{0} \mu^T(t+s)\,\mathrm{d}s\right)[-hS]\left(\frac{1}{h}\int_{-h}^{0} \mu(s)\,\mathrm{d}s\right) \tag{30}$$

The derivative of the first contribution is obtained as $\dot V_1(t) = 2\eta^T(t)P_1\dot\eta(t)$. Using (17)-(18) and the inequalities (28)-(30), it follows that

$$\begin{aligned}\dot V(t) &\le 2\eta^T(t)P_1\mu(t) + 2[\eta^T(t)N_2^T + \mu^T(t)N_3^T]\cdot[\text{RHS of (18)}] + (28) + (29) + (30) \\ &\le q^T(t)\begin{bmatrix} (1,1) & (1,2) & hN_2^TA_2 & hN_2^TA_3 & \bar g_{11}N_2^TA_{4_{11}} & \dots & \bar g_{mm}N_2^TA_{4_{mm}} & N_2^T\Gamma \\ (*) & (2,2) & hN_3^TA_2 & hN_3^TA_3 & \bar g_{11}N_3^TA_{4_{11}} & \dots & \bar g_{mm}N_3^TA_{4_{mm}} & N_3^T\Gamma \\ (*) & (*) & -hR & 0 & 0 & \dots & 0 & 0 \\ (*) & (*) & (*) & -hS & 0 & \dots & 0 & 0 \\ (*) & (*) & (*) & (*) & -\bar g_{11}W_{11} & \dots & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\bar g_{mm}W_{mm} & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & 0 \end{bmatrix} q(t)\end{aligned} \tag{31}$$

where the matrices N_2, N_3 help in reducing conservatism by introducing additional freedom [21], and $(1,1) = N_2^TA_1 + A_1^TN_2 + hR + \sum_{i=1}^{m}\sum_{j=1}^{m} \bar g_{ij}W_{ij}$, $(1,2) = P_1 - N_2^T + A_1^TN_3$, $(2,2) = -N_3 - N_3^T + hS$ and

$$q_0(t) \triangleq \text{col}\left\{\eta(t),\, \mu(t),\, \frac{1}{h}\int_{-h}^{0}\eta(t+s)\,\mathrm{d}s,\, \frac{1}{h}\int_{-h}^{0}\mu(t+s)\,\mathrm{d}s,\, \frac{1}{\bar g_{11}}\int_{-h}^{0} g_{11}(s)\eta(t+s)\,\mathrm{d}s,\, \dots,\right. \\ \left. \frac{1}{\bar g_{mm}}\int_{-h}^{0} g_{mm}(s)\eta(t+s)\,\mathrm{d}s,\, \dot w_x(x)\right\}$$

Now, by Assumption 2, one has that $\dot w^T(x)\dot w(x) \le \dot x^T(t)H^T\beta^2H\dot x(t)$, which is equivalent to the following quadratic inequality

$$\begin{bmatrix}\mu(t) & \dot w(x)\end{bmatrix}\begin{bmatrix}(H\Phi)^T\beta^2H\Phi & 0 \\ 0 & -I_m\end{bmatrix}\begin{bmatrix}\mu(t) \\ \dot w(x)\end{bmatrix} \ge 0 \tag{32}$$

where $\Phi = [I_n, 0_{n\times m}, 0_{n\times m}]$ is defined such that $\dot x(t) = \Phi\mu(t)$. Using (31), (32) and the S-procedure [44], it follows that $\dot V < 0$ is implied by the existence of a positive scalar $\tau > 0$ such that

$$\begin{bmatrix} (1,1) & (1,2) & hN_2^TA_2 & hN_2^TA_3 & \bar{g}_{11}N_2^TA_{4_{11}} & \dots & \bar{g}_{mm}N_2^TA_{4_{mm}} & N_2^T\Gamma \\ (*) & (2,2)^* & hN_3^TA_2 & hN_3^TA_3 & \bar{g}_{11}N_3^TA_{4_{11}} & \dots & \bar{g}_{mm}N_3^TA_{4_{mm}} & N_3^T\Gamma \\ (*) & (*) & -hR & 0 & 0 & \dots & 0 & 0 \\ (*) & (*) & (*) & -hS & 0 & \dots & 0 & 0 \\ (*) & (*) & (*) & (*) & -\bar{g}_{11}W_{11} & \dots & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\bar{g}_{mm}W_{mm} & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & -\tau I_m \end{bmatrix} < 0 \tag{33}$$

where $(2,2)^* = (2,2) + \tau(H\Phi)^T\beta^2H\Phi$. Dividing (33) by τ, redefining $[P_1,\, N_2,\, N_3,\, R,\, S,\, W] = \tau^{-1}[P_1,\, N_2,\, N_3,\, R,\, S,\, W]$, applying Schur complement to the quadratic term $(H\Phi)^T\beta^2(H\Phi)$ and defining $\gamma = \beta^{-2}$ leads to

$$\begin{bmatrix} (1,1) & (1,2) & hN_2^TA_2 & hN_2^TA_3 & \bar{g}_{11}N_2^TA_{4_{11}} & \dots & \bar{g}_{mm}N_2^TA_{4_{mm}} & N_2^T\Gamma & 0 \\ (*) & (2,2) & hN_3^TA_2 & hN_3^TA_3 & \bar{g}_{11}N_3^TA_{4_{11}} & \dots & \bar{g}_{mm}N_3^TA_{4_{mm}} & N_3^T\Gamma & (H\Phi)^T \\ (*) & (*) & -hR & 0 & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & -hS & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & -\bar{g}_{11}W_{11} & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\bar{g}_{mm}W_{mm} & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & -I_m & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & -\gamma I_m \end{bmatrix} < 0 \tag{34}$$

The LMI (34), which ensures asymptotic stability, is convex in h: if it holds for $\bar{h} > 0$ then it is also feasible for all $0 \leq h \leq \bar{h}$; and thus the theorem follows.

B. PROOF OF THEOREM 2

Let us define $Y_2 = N_2^{-1}$, $[Y_1, Y_R, Y_S, Y_{W_{ij}}] = Y_2^T\,[P_1, R, S, W_{ij}]\,Y_2$, and assume that $N_3 = \epsilon N_2$, with a tuning scalar ϵ [22]. Multiplying (34) by $\mathrm{diag}\left\{Y_2^T,\, \dots,\, Y_2^T,\, I_m,\, I_m\right\}$ and its transpose from the left and right, respectively, results in

$$\begin{bmatrix} (1,1) & (1,2) & hA_2Y_2 & hA_3Y_2 & \bar{g}_{11}A_{4_{11}}Y_2 & \dots & \bar{g}_{mm}A_{4_{mm}}Y_2 & \Gamma & 0 \\ (*) & (2,2) & \epsilon hA_2Y_2 & \epsilon hA_3Y_2 & \epsilon\bar{g}_{11}A_{4_{11}}Y_2 & \dots & \epsilon\bar{g}_{mm}A_{4_{mm}}Y_2 & \epsilon\Gamma & Y_2^T(H\Phi)^T \\ (*) & (*) & -hY_R & 0 & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & -hY_S & 0 & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & -\bar{g}_{11}Y_{W_{11}} & \dots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & \ddots & 0 & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & -\bar{g}_{mm}Y_{W_{mm}} & 0 & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & -I_m & 0 \\ (*) & (*) & (*) & (*) & (*) & (*) & (*) & (*) & -\gamma I_m \end{bmatrix} < 0 \tag{35}$$

with $(1,1) = A_1Y_2 + Y_2^TA_1^T + hY_R + \sum_{i=1}^m\sum_{j=1}^m \bar{g}_{ij}Y_{W_{ij}}$, $(1,2) = Y_1 - Y_2 + \epsilon Y_2^TA_1^T$ and $(2,2) = -\epsilon(Y_2^T - Y_2) + hS$. Now, the following decomposition is taken

$$A_1 = A_{10} + B_{10}\tilde{K} \qquad A_2 = \tilde{K}^TB_{20}\tilde{K} \qquad A_3 = B_{30}\tilde{K} \tag{36}$$

where $\tilde{K} = \mathrm{diag}\left\{K, \Omega, \Omega\right\}$ is an augmented gain matrix and the other matrices involved are defined in (25). Plugging (36) into (35), and defining $\tilde{K}Y_2 = X$, the LMI (24) follows. Finally, as suggested in [45], the conditions $X^TX < \kappa_xI$ and $Y^{-1} < \kappa_yI$, which are equivalent to

$$\begin{bmatrix} -\kappa_xI & X^T \\ (*) & -I \end{bmatrix} < 0 \qquad \text{and} \qquad \begin{bmatrix} Y_2 & I \\ (*) & \kappa_yI \end{bmatrix} > 0 \tag{37}$$

respectively, would ensure that $\left\|\tilde{K}\right\|^2 = Y_2^{-1}X^TXY_2^{-1} < \kappa_x\kappa_y^2$. The theorem follows by defining the submatrices $Y_2^{(j)}$, $X^{(j)}$ and imposing conditions like (37) to every submatrix.

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