POLYNOMIAL MAPS WITH MAXIMAL MULTIPLICITY AND THE SPECIAL CLOSURE

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Abstract. In this article we characterize the polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ for which $F^{-1}(0)$ is finite and their multiplicity $\mu(F)$ is equal to $n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$, where $\widetilde{\Gamma}_+(F)$ is the global Newton polyhedron of F. As an application, we derive a characterization of those polynomial maps whose multiplicity is maximal with respect to a fixed Newton filtration.

1. Introduction

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. We define the *multiplicity of* F as the number

$$\mu(F) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} \tag{1}$$

where $\mathbf{I}(F)$ denotes the ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the components of F. It is well-known (see for instance [**?**, p. 150]) that, when $F^{-1}(0)$ is finite and $n = p$, we have

$$\dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\mathbf{I}(F)} = \sum_{x \in F^{-1}(0)} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)} \tag{2}$$

where $\mathcal{O}_{n,x}$ is the ring of analytic function germs $(\mathbb{C}^n, x) \to \mathbb{C}$ and $\mathbf{I}_x(F)$ is the ideal of $\mathcal{O}_{n,x}$ generated by the germs at x of the components of F. We denote $\mathcal{O}_{n,0}$ simply by $\mathcal{O}_n$. Therefore, the number $\mu(F)$ gives the number of solutions of the system $F(x) = 0$ counting multiplicities.

In addition to the interest of $\mu(F)$ in the study of polynomial systems in general, the multiplicity of polynomial maps is a basic tool in singularity theory. For instance, in [**?**] Kouchnirenko obtained an expression for the total Milnor number of a polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$ in terms of the Newton polyhedron of f. We recall that if $f \in \mathbb{C}[x_1, \dots, x_n]$ has a finite number of singularities, then the total Milnor number $\mu_\infty(f)$ of f is defined as $\mu_\infty(f) = \mu(\nabla f)$, where ∇f is the polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ given by $\nabla f = (\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$, for all $x \in \mathbb{C}^n$. The total Milnor number $\mu_\infty(f)$ of f has an important connection with the topology of the generic fibres $f^{-1}(t)$, $t \in \mathbb{C}$, as can be seen in the articles [**?**, **?**, **?**, **?**]. The multiplicity $\mu(F)$ is also involved in the estimation of the Łojasiewicz exponent at infinity of F, usually denoted by $\mathcal{L}_\infty(F)$, as can be seen in [**?**, Theorem 7.3].

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Given a non-zero polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, let $\operatorname{supp}(h)$ denote the support of h (see Definition **??**). Let us fix a polynomial map $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$. By the Bernstein–Khovanskii–Kouchnirenko bound (which is stated originally for Laurent polynomial maps), we know that the number of isolated zeros of F in $(\mathbb{C} \smallsetminus \{0\})^n$, counted with multiplicities, is less than or equal to $\mathrm{MV}_n(A_1, \dots, A_n)$, where A_i denotes the convex hull in $\mathbb{R}^n$ of $\operatorname{supp}(F_i)$, for all $i, \dots, n$, and MV_n denotes mixed volume in $\mathbb{R}^n$ (see [**?**, p. 346]). We refer to the article [**?**] of Rojas for a generalization of this result. By the main result of Li-Wang shown in [**?**, Theorem 2.4] we know that if $F^{-1}(0)$ is finite, then $\mu(F) \leqslant \mathrm{MV}_n(A_1^0, \dots, A_n^0)$, where A_i^0 is the convex hull in $\mathbb{R}^n$ of $\operatorname{supp}(F_i) \cup \{0\}$, for all $i = 1, \dots, n$.

Let $\widetilde{\Gamma}_+(F)$ denote the convex hull of $\operatorname{supp}(F_1) \cup \cdots \cup \operatorname{supp}(F_n) \cup \{0\}$. We refer to this set as the *global Newton polyhedron of* F (see Definition **??**). Since $\mu(F)$ does not change when substituting each F_i by a generic $\mathbb{C}$-linear combination of $F_1, \dots, F_n$, for all $i = 1, \dots, n$, then we conclude that

$$\mu(F) \leqslant \mathrm{MV}_n(\widetilde{\Gamma}_+(F), \dots, \widetilde{\Gamma}_+(F)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F)), \tag{3}$$

where the last equality is an elementary property of mixed volumes (see for instance [**?**, p. 338]). In this article we characterize the polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ for which $F^{-1}(0)$ is finite and $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$. When this equality holds then we say that F *has maximal multiplicity*.

Our work in this article has been inspired by the results of the articles [**?**], [**?**] and [**?**]. Particularly by [**?**, Theorem 2.11], where the ideals I of $\mathcal{O}_n$ of finite colength such that $e(I) = n!\mathrm{V}_n(\mathbb{R}_{\geqslant 0} \setminus \Gamma_+(I))$ are characterized. Here $\Gamma_+(I)$ denotes the Newton polyhedron of I (see [**?**, Definition 2.1]) and $e(I)$ is the Samuel multiplicity of I. In the proof of this characterization the Rees' Multiplicity Theorem (see for instance [**?**, p. 222]) played a fundamental role.

In Section **??** we recall basic definitions and results that we will need along the article. In particular, we recall the notion of special closure of a polynomial map introduced in [**?**]. Section **??** is devoted to showing the central result of the article (Theorem **??**), where we show that a given polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ has maximal multiplicity if and only if F is Newton non-degenerate at infinity (see Definition **??**), which in turn is equivalent to saying that the special closure of F determined by the monomials x^k with $k \in \widetilde{\Gamma}_+(F)$, by the results of [**?**]. In Section **??** we introduce and characterize the notion of non-degeneracy with respect to a global Newton polyhedron. This notion generalizes simultaneously the condition of Newton non-degeneracy at infinity and the pre-weighted homogeneity of functions (see Definition **??**). As a consequence of the results of this section, we show a version for total Milnor numbers of the main result of [**?**] about Milnor numbers of analytic functions and weighted homogeneous filtrations.

2. Preliminary definitions and results

2.1. **The global Newton filtration**

We will follow the notation introduced in [**?**, Section 2]. Here we briefly recall some definitions from [**?**]. We say a that a given a subset $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ is a *global Newton polyhedron* when there exists some $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+$ is equal to the convex hull of $A \cup \{0\}$. In this case, we will also denote $\widetilde{\Gamma}_+$ by $\widetilde{\Gamma}_+(A)$.

Definition 2.1. Let us fix coordinates $x_1, \dots, x_n \in \mathbb{C}^n$. For any $k \in \mathbb{Z}^n_{\geqslant 0}$, we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. Given a polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$, if h is written as $h = \sum_k a_k x^k$, then the *support of* h is defined as the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $a_k \neq 0$. We denote this set by $\mathrm{supp}(h)$. We set $\mathrm{supp}(0) = \emptyset$.

For any $h \in \mathbb{C}[x_1, \dots, x_n]$, the *global Newton polyhedron of* h, denoted by $\widetilde{\Gamma}_+(h)$, is defined as $\widetilde{\Gamma}_+(h) = \widetilde{\Gamma}_+(\mathrm{supp}(h) \cup \{0\})$. If $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ is a polynomial map, then we define the *support of* F as $\mathrm{supp}(F) = \mathrm{supp}(F_1) \cup \cdots \cup \mathrm{supp}(F_p)$. Thus, the *global Newton polyhedron of* F, denoted by $\widetilde{\Gamma}_+(F)$ or by $\widetilde{\Gamma}_+(F_1, \dots, F_p)$, is defined as the convex hull of $\widetilde{\Gamma}_+(F_1) \cup \cdots \cup \widetilde{\Gamma}_+(F_p)$. Hence $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+(\mathrm{supp}(F) \cup \{0\})$.

If P is a non-empty compact subset of $\mathbb{R}^n$ and $v \in \mathbb{R}^n$, then we define $\ell(v, P) = \min\{\langle v, k\rangle : k \in P\}$ and $\Delta(v, P) = \{k \in P : \langle v, k\rangle = \ell(v, P)\}$, where $\langle\, ,\rangle$ denotes the standard scalar product in $\mathbb{R}^n$. The sets of the form $\Delta(v, P)$, for some $v \in \mathbb{R}^n \smallsetminus \{0\}$ are called *faces* of P. If Δ is a face of P and $v \in \mathbb{R}^n \smallsetminus \{0\}$ verifies that $\Delta = \Delta(v, P)$, then we say that v *supports* Δ. The *dimension* of Δ, denoted by $\dim(\Delta)$, is defined as the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. The faces of P of dimension 0 are called *vertices* of P.

If $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n$ is a Newton polyhedron, then we denote by $\widetilde{\Gamma}$ the union of all faces of $\widetilde{\Gamma}_+$ not containing the origin. We will refer to $\widetilde{\Gamma}$ as the *global boundary of* $\widetilde{\Gamma}_+$. We say that $\widetilde{\Gamma}_+$ is *convenient* when $\widetilde{\Gamma}_+$ cuts any coordinate axis in a point different from the origin. Unless otherwise stated, in the remaining section we will fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_+$.

Let Δ be a face of $\widetilde{\Gamma}_+$ not containing the origin. Then, we denote by $C(\Delta)$ the *cone* over Δ, that is, the union of all half lines emanating from the origin and passing through some point of Δ. We denote by $\mathcal{R}_\Delta$ the subring of $\mathbb{C}[x_1, \dots, x_n]$ formed by those $h \in \mathbb{C}[x_1, \dots, x_n]$ such that $\mathrm{supp}(h) \subseteq C(\Delta)$.

A vector $v \in \mathbb{Z}^n$, $v \neq 0$, is called *primitive* when v is the vector of smallest length over all vectors of the form λv, where $\lambda > 0$. Let $\mathcal{F}_0(\widetilde{\Gamma}_+)$ denote the family of primitive vectors of $\mathbb{Z}^n$ supporting some face of $\widetilde{\Gamma}_+$ of dimension $n-1$ not passing through the origin (see [**?**, Section 2]). Let us write $\mathcal{F}_0(\widetilde{\Gamma}_+) = \{w^1, \dots, w^r\}$, $r \geqslant 1$. Let us denote by $M_{\widetilde{\Gamma}}$ the least common multiple of the set of positive integers $\{-\ell(w^i, \widetilde{\Gamma}_+) : i = 1, \dots, r\}$ (see [**?**, Lemma 2.3]). If $j \in \{1, \dots, r\}$, let $\phi_j : \mathbb{R}^n \to \mathbb{R}$ be the linear map defined by

$$\phi_j(k) = M_{\widetilde{\Gamma}} \frac{\langle w^j, k\rangle}{\ell(w^j, \widetilde{\Gamma}_+)}$$

for all $k \in \mathbb{R}^n$. Then, we define the map $\phi_{\widetilde{\Gamma}} : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$ by $\phi(k) = \max_{1 \leqslant j \leqslant r} \phi_j(k)$, for all $k \in \mathbb{R}^n_{\geqslant 0}$.

We will refer to $\phi_{\widetilde{\Gamma}}$ as the *filtrating map* associated to $\widetilde{\Gamma}$. If no confusion arises, then we denote $M_{\widetilde{\Gamma}}$ and $\phi_{\widetilde{\Gamma}}$ simply by M and ϕ, respectively. We observe that the restriction of ϕ to

$\widetilde{\Gamma}$ is constant and equal to M. Let us remark that

$$\widetilde{\Gamma}_+ = \big\{k \in \mathbb{R}^n_{\geqslant 0} : \langle w^j, k\rangle \geqslant \ell(w^j, \widetilde{\Gamma}_+), \text{ for all } j = 1, \dots r\big\} = \big\{k \in \mathbb{R}^n_{\geqslant 0} : \phi(k) \leqslant M\big\},$$

where the second equality follows from the fact that $\ell(w^j, \widetilde{\Gamma}_+) < 0$, for all $j = 1, \dots, r$.

Lemma 2.2. *The filtrating map ϕ satisfies the following properties:*

(a) $\phi(\mathbb{Z}^n_{\geqslant 0}) \subseteq \mathbb{Z}_{\geqslant 0}$.
(b) *If $k \in \mathbb{R}^n_{\geqslant 0}$ and $j_0 \in \{1, \dots, r\}$, then $\phi(k) = \phi_{j_0}(k)$ if and only if $k \in C(\Delta(w^{j_0}, \widetilde{\Gamma}_+))$.*
(c) *ϕ is linear on each cone $C(\Delta)$, where Δ is any face of $\widetilde{\Gamma}_+$ not passing through the origin.*
(d) *If $a, b \in \mathbb{R}^n$, then $\phi(a+b) \leqslant \phi(a)+\phi(b)$ and equality holds if and only if a and b belong to the same cone, that is, there exists a vector $w \in \mathcal{F}_0(\widetilde{\Gamma}_+)$ such that $a, b \in C(\Delta(w, \widetilde{\Gamma}_+))$.*

Proof. Let us prove (a). Since $\widetilde{\Gamma}_+$ is convenient, the line λk, $\lambda \geqslant 0$, intersects $\widetilde{\Gamma}$. Then, given a point $k \in \mathbb{Z}^n_{\geqslant 0}$, $k \neq 0$, we can write k as $k = \lambda k'$, for some $k' \in \widetilde{\Gamma}$ and some $\lambda > 0$. Since $\phi(k') = M$, there exists some $j \in \{1, \dots, r\}$ such that $\langle w^j, k'\rangle = \ell(w^j, \widetilde{\Gamma}_+) < 0$. Then $\langle w^j, k\rangle < 0$ and this implies that $\phi(k) > 0$.

Let us prove (b). As before, let us write k as $k = \lambda k'$, for some $k' \in \widetilde{\Gamma}$ and some $\lambda > 0$. By the definition of ϕ, we have $\phi(k) = \phi_{j_0}(k)$ if and only if

$$\frac{\langle k', w^j\rangle}{\ell(w^j, \widetilde{\Gamma}_+)} \leqslant \frac{\langle k', w^{j_0}\rangle}{\ell(w^{j_0}, \widetilde{\Gamma}_+)}, \tag{4}$$

for all $j \in \{1, \dots, r\}$. Since $\ell(w^{j_0}, \widetilde{\Gamma}_+) < 0$, we have $\langle k', w^{j_0}\rangle/\ell(w^{j_0}, \widetilde{\Gamma}_+) \leqslant 1$. On the other hand, the condition $k' \in \widetilde{\Gamma}$ implies the equality $\ell(w^j, \widetilde{\Gamma}_+) = \langle w^j, k'\rangle$, for some $j \in \{1, \dots, r\}$. Then (**??**) is equivalent to saying that $\langle k', w^{j_0}\rangle/\ell(w^{j_0}, \widetilde{\Gamma}_+) = 1$. In particular, $k' \in \Delta(w^{j_0}, \widetilde{\Gamma}_+)$ and the result follows. Items (c) and (d) are immediate consequences of item (b). □

Given an $h \in \mathbb{C}[x_1, \dots x_n]$, $h \neq 0$, the *degree of h with respect to* $\widetilde{\Gamma}_+$ is defined as

$$\nu_{\widetilde{\Gamma}}(h) = \max\left\{\phi(k) : k \in \operatorname{supp}(h)\right\}.$$

When $h = 0$, then we set $\nu_{\widetilde{\Gamma}}(0) = 0$. Thus, we have a map $\nu_{\widetilde{\Gamma}} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{Z}_{\geqslant 0}$. If there is no risk of confusion, then we will denote $\nu_{\widetilde{\Gamma}}$ simply by ν.

Let us remark that, when $\widetilde{\Gamma}_+$ is equal to the standard n-simplex, that is, when $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(x_1, \dots, x_n)$, then $\nu_{\widetilde{\Gamma}}(h) = \max\{k_1 + \dots + k_n : k \in \operatorname{supp}(h)\}$. Therefore, in this case $\nu_{\widetilde{\Gamma}}(h)$ coincides with the usual notion of degree of h, for any $h \in \mathbb{C}[x_1, \dots, x_n]$.

Let us define, for all $r \in \mathbb{Z}_{\geqslant 0}$, the following set of polynomials:

$$\mathcal{B}_r = \big\{h \in \mathbb{C}[x_1, \dots, x_n] : \nu(h) \leqslant r\big\}. \tag{5}$$

In particular, $\mathcal{B}_0 = \mathbb{C}$ and $\mathcal{B}_M = \{f \in \mathbb{C}[x_1, \dots, x_n] : \operatorname{supp}(f) \subseteq \widetilde{\Gamma}_+\}$. By the properties of ϕ, it is immediate to check the following:

(a) $\mathcal{B}_r$ is a finite dimensional vector subspace of $\mathbb{C}[x_1, \dots, x_n]$, for all $r \geqslant 0$;
(b) $\mathcal{B}_r \subseteq \mathcal{B}_{r+1}$, for all $r \geqslant 0$;

(c) $\mathcal{B}_r\mathcal{B}_{r'} \subseteq \mathcal{B}_{r+r'}$, for all $r, r' \geqslant 0$
(d) $\widetilde{\Gamma}_+(\mathcal{B}_r) \subseteq \frac{r}{M}\widetilde{\Gamma}_+$ and equality holds if and only if $\mathrm{V}_n(\widetilde{\Gamma}_+(\mathcal{B}_r)) = (\frac{r}{M})^n \mathrm{V}_n(\widetilde{\Gamma}_+)$, where V_n denotes n-dimensional volume.

We observe that ν determines and is determined by the collection of subspaces $\{\mathcal{B}_r\}_{r\geqslant 0}$. We refer both to the map ν and the collection of subspaces $\{\mathcal{B}_r\}_{r\geqslant 0}$ as the *Newton filtration of* $\mathbb{C}[x_1,\dots,x_n]$ *induced by* $\widetilde{\Gamma}_+$.

Let us remark that we have exposed the notion of Newton filtration induced by $\widetilde{\Gamma}_+$ in a slightly different way from Kouchnirenko [**?**, Section 5.9]. That is, the filtrating map considered in [**?**, Section 5.9] equals $-\phi$ and thus in [**?**] the corresponding collection of subspaces is decreasing and indexed by $\mathbb{Z}_{\leqslant 0}$.

2.2. The special closure of a polynomial map

Let $F : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. We will say that F is *finite* when $F^{-1}(0)$ is finite. The multiplicity of F is well-defined if and only if F is finite (see [**?**, p. 39]). Let us denote $\dim_{\mathbb{C}} \mathcal{O}_{n,x}/\mathbf{I}_x(F)$ by $\mu_x(F)$, for any $x \in F^{-1}(0)$. As remarked in (**??**), it is known that $\mu(F) = \sum_{x\in F^{-1}(0)} \mu_x(F)$.

If $h \in \mathbb{C}[x_1,\dots,x_n]$, then we say that h is *special with respect to* F (see [**?**, Definition 4.1]) when there exist some positive constants C and M such that

$$|h(x)| \leqslant C\|F(x)\|$$

for all $x \in \mathbb{C}^n$ for which $\|x\| \geqslant M$. Let us denote by $\mathrm{Sp}(F)$ the set of special elements with respect to F. We refer to $\mathrm{Sp}(F)$ as the *special closure of* F. The elements of $\mathrm{Sp}(F)$ can be characterized in terms of the notion of multiplicity.

Theorem 2.3. [**?**] *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map and let $h \in \mathbb{C}[x_1,\dots,x_n]$, $h \neq 0$. Then the following conditions are equivalent:*

(a) *h is special with respect to F;*
(b) *there exists some $\delta > 0$ such that for all $\alpha \in B(0;\delta)$, the map $F + h\alpha$ is finite and $\mu(F) = \mu(F + h\alpha)$.*

Let $A \subseteq \mathbb{R}^n_{\geqslant 0}$. If $h \in \mathbb{C}[x_1,\dots,x_n]$ and h is written as $h = \sum a_k x^k$, then we denote by h_A the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap A$. If $\mathrm{supp}(h) \cap A = \emptyset$, then we set $h_A = 0$. The following definition will be fundamental for the objectives of this article.

Definition 2.4. Let $F = (F_1,\dots,F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. The map F is said to be *Newton non-degenerate at infinity* when, for any face Δ of $\widetilde{\Gamma}_+(F)$ not containing the origin, the following inclusion holds:

$$\big\{x \in \mathbb{C}^n : (F_1)_\Delta(x) = \cdots = (F_p)_\Delta(x) = 0\big\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{6}$$

Under the conditions of the above definition, we will also denote the polynomial $(F_i)_\Delta$ by $F_{i,\Delta}$, for any $i = 1,\dots,p$, and any face Δ of $\widetilde{\Gamma}_+$.

Let us denote by $\mathbf{S}(F)$ the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that $x^k \in \mathrm{Sp}(F)$. By [**?**, Lemma 3.4] we know that $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F)$. Next we recall a result from [**?**]. This characterizes the equality $\mathbf{S}(F) = \widetilde{\Gamma}_+(F)$.

Theorem 2.5. [**?**] *Let $F : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map such that $\widetilde{\Gamma}_+(F)$ is convenient. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F) = \big\{h \in \mathbb{C}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\big\}$.

As will be shown in the next section, when $p = n$ and F is finite, the condition $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$ is also equivalent to any of the conditions (a), (b) or (c) of Theorem **??** (see Corollary **??**).

3. Multiplicity of polynomial maps and convex bodies

Along this section, let us fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. Let $\{\mathcal{B}_r\}_{r \geqslant 0}$ be the Newton filtration of $\mathbb{C}[x_1, \dots, x_n]$ induced by $\widetilde{\Gamma}_+$ (see (**??**)). Therefore, we can consider the graded ring $\mathbf{R} = \bigoplus_{r \geqslant 0} R_r$, where $R_r = \mathcal{B}_r / \mathcal{B}_{r-1}$, for all $r \geqslant 0$, and we fix $\mathcal{B}_{-1} = \{0\}$. For any $f \in \mathbb{C}[x_1, \dots, x_n]$, let us denote by $\mathrm{in}(f)$ the image of f in $\mathbf{R}$, that is, $\mathrm{in}(f) = f + \mathcal{B}_{\nu(f)-1}$.

Let $\nu = \nu_{\widetilde{\Gamma}}$. We remark that the product operation in $\mathbf{R}$ is defined as follows. If $f, g \in \mathbb{C}[x_1, \dots, x_n]$, $\nu(f) = r$ and $\nu(g) = s$, then $\mathrm{in}(f)\,\mathrm{in}(g) = fg + \mathcal{B}_{r+s-1}$. By Lemma **??**, we have that this product is not zero if and only if $\nu(fg) = \nu(f) + \nu(g)$, which is to say that $\nu(f)$ and $\nu(g)$ are attained at the same cone, by Lemma **??**. We refer to $\mathbf{R}$ as the *graded ring associated to ν*.

We say that a given condition depending on a parameter $x \in \mathbb{C}^n$ holds for all $\|x\| \ll 1$ when there exists some open neighbourhood U of $0 \in \mathbb{C}^n$ such that the said condition holds for all $x \in U$.

Lemma 3.1. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a finite map and let $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$. Then the map $F_\alpha = F + \alpha h$ is finite and $\mu(F) \leqslant \mu(F_\alpha)$, for all $\alpha \in \mathbb{C}^n$, $\|\alpha\| \ll 1$.*

Proof. Let $q : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ be the map given by $q(x, \alpha) = (F(x) + \alpha h, \alpha)$, for all $(x, \alpha) \in \mathbb{C}^n \times \mathbb{C}^n$. Let us define the sets

$$S = \big\{(x, \alpha) \in \mathbb{C}^n \times \mathbb{C}^n : \dim_x F_\alpha^{-1}(0) \geqslant 1\big\} \tag{7}$$

$$T = \big\{(x, \alpha) \in \mathbb{C}^n \times \mathbb{C}^n : \dim_{(x,\alpha)} q^{-1}(q(x, \alpha)) \geqslant 1\big\}. \tag{8}$$

We remark that, for a given $(x, \alpha) \in \mathbb{C}^n \times \mathbb{C}^n$, the condition $\dim_x F_\alpha^{-1}(0) \geqslant 1$ is equivalent to saying that $\mu_x(F_\alpha) = \infty$. Let us observe that

$$S = T \cap \{(x, \alpha) \in \mathbb{C}^n \times \mathbb{C}^n : F_\alpha(x) = 0\}. \tag{9}$$

By Chevalley's Theorem (see [**?**, Théorème 13.1.3, p.189]), the set T is Zariski closed. Hence S is Zariski closed. In particular F_α is finite, for all $\|\alpha\| \ll 1$, since we assume that F is finite.

By [?, Proposition 2.3(ii)], $\mu(F_\alpha)$ is a lower semi-continuous function. Hence $\mu(F) \leqslant \mu(F_\alpha)$ for all $\|\alpha\| \ll 1$. $\square$

Theorem 3.2. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $\operatorname{supp}(F_i) \subseteq \widetilde{\Gamma}_+$, for all $i = 1, \dots, n$, and F is finite. Then*

$$\mu(F) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+) \tag{10}$$

and equality holds if and only if F is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+$.

Proof. As shown in (??), inequality (??) follows as a direct application of [?, Theorem 2.4].

Let us suppose that F is Newton non-degenerate at infinity and $\widetilde{\Gamma}_+(F) = \widetilde{\Gamma}_+$. In order to prove that $\mu(F) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+)$ we will apply a series of steps which are analogous to the steps performed by Kouchnirenko in the proof of [?, Théorème AI, p. 11].

If $q \in \{0, 1, \dots, n-1\}$, then we denote by $\widetilde{\Gamma}_q$ the family of all faces of $\widetilde{\Gamma}_+$ of dimension q not containing the origin. Let $\phi = \phi_{\widetilde{\Gamma}}$ and let $\nu = \nu_{\widetilde{\Gamma}}$. Let us also denote $M_{\widetilde{\Gamma}}$ by M. Let us denote by R the ring $\mathbb{C}[x_1, \dots, x_n]$ and let $\mathbf{R}$ be the graded ring associated to ν.

Let $\mathbf{F}_i = \operatorname{in}(F_i)$, for all $i = 1, \dots, n$. Let $\mathbf{I}$ be the ideal of $\mathbf{R}$ generated by $\mathbf{F}_1, \dots, \mathbf{F}_n$. Let us consider the Koszul complex $\mathcal{K}$ associated to $\mathbf{F}_1, \dots, \mathbf{F}_n$ extended with the projection $\mathbf{R} \to \mathbf{R}/\mathbf{I}$:

$$0 \longrightarrow \mathbf{R}^{\binom{n}{n}} \longrightarrow \mathbf{R}^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}^{\binom{n}{1}} \longrightarrow \mathbf{R} \longrightarrow \mathbf{R}/\mathbf{I}. \tag{$\mathcal{K}$}$$

We claim that the complex $\mathcal{K}$ is exact in positive dimensions. Let us prove this. If Δ is any face of $\widetilde{\Gamma}_+$ not containing the origin, then let $\mathbf{R}_\Delta$ be the graded ring given by

$$\mathbf{R}_\Delta = \bigoplus_{r \geqslant 0} \mathbf{R}_{\Delta, r} \quad \text{with} \quad \mathbf{R}_{\Delta,r} = \frac{\mathcal{B}_r \cap \mathcal{R}_\Delta}{\mathcal{B}_{r-1} \cap \mathcal{R}_\Delta} \quad \text{for all } r \geqslant 1.$$

Let $\mathbf{F}_{i,\Delta}$ denote the image of $F_{i,\Delta}$ in $\mathbf{R}_\Delta$, for all $i = 1, \dots, n$. Let $\mathcal{K}_\Delta$ be the Koszul complex of the elements $\mathbf{F}_{1,\Delta}, \dots, \mathbf{F}_{n,\Delta}$ in $\mathbf{R}_\Delta$:

$$0 \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n}} \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}_\Delta^{\binom{n}{1}} \longrightarrow \mathbf{R}_\Delta. \tag{$\mathcal{K}_\Delta$}$$

Given any integer $q = 0, 1, \dots, n-1$, let us denote by C_q the direct sum of all graded rings $\mathbf{R}_\Delta$, where Δ varies in $\widetilde{\Gamma}_q$. We denote by $\mathcal{K}_q$ the direct sum of the complexes $\mathcal{K}_\Delta$ over all faces $\Delta \in \widetilde{\Gamma}_q$. Hence, for any $q = 0, 1, \dots, n-1$, we obtain a complex

$$0 \longrightarrow C_q^{\binom{n}{n}} \longrightarrow C_q^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow C_q^{\binom{n}{1}} \longrightarrow C_q. \tag{$\mathcal{K}_q$}$$

By [?, Proposition 2.6], there exists an exact sequence of $\mathbf{R}$-modules respecting the graduations

$$0 \longrightarrow \mathbf{R} \longrightarrow C_{n-1} \longrightarrow C_{n-2} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0 \tag{$\mathcal{C}$}$$

Therefore we can construct the commutative diagram shown in Figure 1, where each row is formed by $\binom{n}{j}$ copies of the complex $\mathcal{C}$, for $j = 1, \dots, n$, and the columns are given by the complexes $\mathcal{K}, \mathcal{K}_{n-1}, \dots, \mathcal{K}_1, \mathcal{K}_0$, respectively.

$$
\begin{tikzcd}
 & \mathbf{R}/\mathbf{I} & & & & & \\
0 \arrow[r] & \mathbf{R} \arrow[r] \arrow[u] & C_{n-1} \arrow[r] & C_{n-2} \arrow[r] & \cdots \arrow[r] & C_0 \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{1}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{1}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{1}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{2}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{2}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{2}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{2}} \arrow[r] \arrow[u] & 0 \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & & \vdots \arrow[u] & \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{n-1}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{n-1}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{n-1}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \mathbf{R}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-1}^{\binom{n}{n}} \arrow[r] \arrow[u] & C_{n-2}^{\binom{n}{n}} \arrow[r] \arrow[u] & \cdots \arrow[r] & C_0^{\binom{n}{n}} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}
$$

FIGURE 1.

By a simple diagram chase argument, we conclude that the complex $\mathcal{K}$ is exact provided that the columns of the diagram of Figure 1 are exact under the dotted line. That is, for any $q \in \{0, 1, \dots, n-1\}$, the complexes $\mathcal{K}_q$ are exact in dimensions $\geqslant n - q$.

The latter condition is equivalent to saying that the following part $\mathcal{K}'_\Delta$ of the complex $\mathcal{K}_\Delta$ is exact

$$
(\mathcal{K}'_\Delta) \qquad 0 \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n}} \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow \mathbf{R}_\Delta^{\binom{n}{n-q}}
$$

for any face $\Delta \in \widetilde{\Gamma}_q$ and for all $q = 0, 1, \dots, n-1$.

Let us fix any $q \in \{0, 1, \dots, n-1\}$ and let us fix a face $\Delta \in \widetilde{\Gamma}_q$. Let $\mathbf{I}_\Delta$ be the ideal of $\mathbf{R}_\Delta$ generated by $\mathbf{F}_{1,\Delta}, \dots, \mathbf{F}_{n,\Delta}$. The ring $\mathbf{R}_\Delta$ is Cohen-Macaulay ring of dimension $q+1$ (see [**?**] or [**?**, Théorème 5.6]). Thus, since $\mathbf{I}_\Delta$ has finite colength, the depth in $\mathbf{R}_\Delta$ of $\mathbf{I}_\Delta$ is $q+1$. By [**?**, Theorem 16.8], which is also known as the grade-sensitivity of the Koszul complex (see also [**?**, Proposition 5.2]), the homology of $\mathcal{K}_\Delta$ is zero in dimensions $\geqslant n-q$. Therefore the complex $\mathcal{K}$ is exact.

The exactness of $\mathcal{K}$ implies that the Hilbert series of $\mathbf{R}/\mathbf{I}$ is expressed as

$$H_{\mathbf{R}/\mathbf{I}}(t) = (1-t^M)^n H_{\mathbf{R}}(t). \tag{11}$$

Moreover, the exactness of $\mathcal{C}$ leads to the following expression for $H_{\mathbf{R}}(t)$:

$$H_{\mathbf{R}}(t) = \sum_{q=0}^{n-1}(-1)^{n+q+1}H_{C_q}(t) = \sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q}H_{\mathbf{R}_\Delta}(t) + \sum_{\Delta\in\widetilde{\Gamma}_{n-1}}H_{\mathbf{R}_\Delta}(t). \tag{12}$$

From [**?**, Lemme 2.9] we know that $H_{\mathbf{R}_\Delta}(t)$ is a rational function and that $t=1$ is a pole of $H_{\mathbf{R}_\Delta}(t)$ of order $q+1$, for any $\Delta\in\widetilde{\Gamma}_q$ and any $q=0,\dots,n-1$. Moreover, if $\Delta\in\widetilde{\Gamma}_{n-1}$, then $\lim_{t\to 1}(1-t^M)^n H_{\mathbf{R}_\Delta}(t) = n!\mathrm{V}_n(P(\Delta))$, where $P(\Delta)$ denotes the pyramid with vertex at 0 and basis equal to Δ. Applying this result and (**??**) and (**??**) we obtain

$$\begin{aligned}
\dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} &= \lim_{t\to 1}(1-t^M)^n H_{\mathbf{R}}(t)\\
&= \lim_{t\to 1}(1-t^M)^n\left(\sum_{\Delta\in\widetilde{\Gamma}_{n-1}}H_{\mathbf{R}_\Delta}(t) + \sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q}H_{\mathbf{R}_\Delta}(t)\right)\\
&= \lim_{t\to 1}\sum_{\Delta\in\widetilde{\Gamma}_{n-1}}(1-t^M)^n H_{\mathbf{R}_\Delta}(t) + \lim_{t\to 1}\left(\sum_{q=0}^{n-2}(-1)^{n+q+1}\sum_{\Delta\in\widetilde{\Gamma}_q}\left((1-t^M)^n H_{\mathbf{R}_\Delta}(t)\right)\right)\\
&= \sum_{\Delta\in\widetilde{\Gamma}_{n-1}} n!\mathrm{V}_n(P(\Delta)) = n!\mathrm{V}_n(\widetilde{\Gamma}_+). \qquad (13)
\end{aligned}$$

By [**?**, Théorème 4.1(i)], the exactness of **??** in dimension 1 implies the following isomorphism of graded $\mathbb{C}$-modules:

$$\bigoplus_{r\geqslant 0}\frac{\mathcal{B}_r+I}{\mathcal{B}_{r-1}+I}\cong\frac{\mathbf{R}}{\mathbf{I}}. \tag{14}$$

Since the ring $\mathbf{R}/\mathbf{I}$ has finite length, the above isomorphism implies that there exists some $s\in\mathbb{Z}_{\geqslant 0}$ such that $\mathcal{B}_s+I=\mathcal{B}_{s+1}+I=\cdots=R$. In particular

$$\dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} = \dim_{\mathbb{C}}\left(\bigoplus_{r\geqslant 0}\frac{\mathcal{B}_r+I}{\mathcal{B}_{r-1}+I}\right) = \sum_{\ell=0}^{s}\dim_{\mathbb{C}}\frac{\mathcal{B}_\ell+I}{\mathcal{B}_{\ell-1}+I} = \dim_{\mathbb{C}}\frac{R}{I}. \tag{15}$$

By joining (**??**) and (**??**) we finally obtain that

$$\mu(F) = \dim_{\mathbb{C}}\frac{R}{I} = \dim_{\mathbb{C}}\frac{\mathbf{R}}{\mathbf{I}} = n!\mathrm{V}_n(\widetilde{\Gamma}_+).$$

Let us see the converse. Let us suppose that $\mu(F)=n!\mathrm{V}_n(\widetilde{\Gamma}_+)$ and that F is not Newton non-degenerate at infinity. By Theorem **??**, there exists some $k\in\mathbf{v}(\widetilde{\Gamma}_+)$ such that $x^k\notin\mathrm{Sp}(F)$. By Lemma **??**, there exists some $\varepsilon>0$ such that $\mu(F+\alpha x^k)$ is finite and $\mu(F)\leqslant\mu(F+\alpha x^k)$, for all $\alpha\in B(0;\varepsilon)$.

The condition $x^k \notin \mathrm{Sp}(F)$, implies, by Theorem **??**, that there exists some $\alpha_0 \in B(0;\varepsilon)$ such that

$$\mu(F) < \mu(F + \alpha_0 x^k) \leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+)$$

where we have applied (**??**) in the last inequality. Hence $\mu(F) < n!\mathrm{V}_n(\widetilde{\Gamma}_+)$, which is a contradiction. Thus the result follows. □

As an immediate application of Theorem **??** and Theorem **??** we obtain the following result.

Corollary 3.3. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F) = \left\{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\right\}$
(d) $\mu(F) = n!\mathrm{V}_n(\widetilde{\Gamma}_+(F))$.

4. Non-degeneracy with respect to a global Newton polyhedron

The objective of this section is to obtain a characterization of an important class of polynomial maps $\mathbb{C}^n \to \mathbb{C}^n$ that extends the class of pre-weighted homogeneous maps (see Definition **??**) and the maps which are Newton non-degeneracy at infinity. In particular, we obtain a version of [**?**, Theorem 3.3] in the ring of polynomials $\mathbb{C}[x_1, \dots, x_n]$ and, in turn, a version for total Milnor numbers of the main result of [**?**].

Motivated by [**?**, Section 3] we introduce the following concept.

Definition 4.1. Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a convenient global Newton polyhedron. Let $\phi = \phi_{\widetilde{\Gamma}}$ and $\nu = \nu_{\widetilde{\Gamma}}$. Let $h \in \mathbb{C}[x_1, \dots, x_n]$, $h \neq 0$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let Δ be a face of $\widetilde{\Gamma}_+$ not passing through the origin. The *initial* or *principal part of h over* Δ is the polynomial obtained as the sum of all terms $a_k x^k$ such that $k \in C(\Delta)$ and $\phi(k) = \nu(h)$. We will denote this polynomial by $\mathrm{q}_{\widetilde{\Gamma},\Delta}(h)$. If no such terms exist or $h = 0$, then we set $\mathrm{q}_\Delta(h) = 0$. We observe that, if $h_\Delta \neq 0$, then $h_\Delta = \mathrm{q}_\Delta(h)$ if and only if $\nu(h) = M$. If there is no risk of confusion, then we will denote $\mathrm{q}_{\widetilde{\Gamma},\Delta}(h)$ simply by $\mathrm{q}_\Delta(h)$.

Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map. We say that F is *non-degenerate with respect to* $\widetilde{\Gamma}_+$ when

$$\left\{x \in \mathbb{C}^n : \mathrm{q}_\Delta(F_1)(x) = \cdots = \mathrm{q}_\Delta(F_p)(x) = 0\right\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}, \tag{16}$$

for any face Δ of $\widetilde{\Gamma}_+$ not containing the origin.

The definition of non-degeneracy with respect to $\widetilde{\Gamma}_+$ is specially significant when $p = n$ and constitutes a generalization of the notion of pre-weighted homogeneity of maps $\mathbb{C}^n \to \mathbb{C}^p$, which we now recall.

Definition 4.2. Let $w \in \mathbb{Z}^n_{\geqslant 1}$ be a primitive vector and let $h \in \mathbb{C}[x_1, \dots, x_n]$. Let us suppose that h is written as $h = \sum_k a_k x^k$. Let $F = (F_1, \dots, F_p) : \mathbb{C}^n \to \mathbb{C}^p$ be a polynomial map.

(a) We will denote the integer $\max\{\langle w, k\rangle : k \in \mathrm{supp}(h)\}$ by $\mathrm{d}_w(h)$. Let us define the *principal part of h at infinity with respect to w*, denoted by $\mathrm{q}_w(h)$, as the sum of those terms $a_k x^k$ such that $\langle w, k\rangle = \mathrm{d}_w(h)$. If $h = 0$, then we set $\mathrm{d}_w(h) = 0$ and $\mathrm{q}_w(h) = 0$. We define $\mathrm{q}_w(F) = (\mathrm{q}_w(F_1), \dots, \mathrm{q}_w(F_p))$ and $\mathrm{d}_w(F) = (\mathrm{d}_w(F_1), \dots, \mathrm{d}_w(F_p))$.

(b) Let $d = (d_1, \dots, d_p) \in \mathbb{Z}^p_{\geqslant 1}$. If F_i is weighted homogeneous of degree d_i, for all $i = 1, \dots, p$, then F is called *weighted-homogeneous with respect to w with vector of degrees d*. If $p \geqslant n$ and $(\mathrm{q}_w(F))^{-1}(0) = \{0\}$ then we say that F is *pre-weighted homogeneous with respect to w*.

(c) Let $d \in \mathbb{Z}_{\geqslant 1}$. We say that h is *weighted-homogeneous of degree d with respect to w* when $h \neq 0$ and $\mathrm{supp}(h)$ is contained in the hyperplane of equation $\langle w, k\rangle = d$. That is, when $\mathrm{q}_w(h) = h$ and $\mathrm{d}_w(h) = d$. We say that h is *pre-weighted homogeneous* when $\mathrm{q}_w(h)$ has at most a finite number of singularities, or equivalently, when the gradient map $\nabla \mathrm{q}_w(h) : \mathbb{C}^n \to \mathbb{C}^n$ is finite.

We refer to [?, ?] for interesting properties of pre-weighted homogeneous maps. Let us remark that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is weighted homogeneous with respect to w, then $F^{-1}(0)$ is finite if and only if $F^{-1}(0) = \{0\}$.

Let $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$ be a primitive vector. Let us denote by $\widetilde{\Gamma}^w_+$ the global Newton polyhedron $\widetilde{\Gamma}_+(x_1^{w_1 \cdots w_n / w_1}, \dots, x_n^{w_1 \cdots w_n / w_n})$ and by $\widetilde{\Gamma}^w$ the global boundary of $\widetilde{\Gamma}^w_+$. We remark that $\widetilde{\Gamma}^w$ equals the unique face of $\widetilde{\Gamma}^w_+$ of dimension $n-1$. This face is supported by $-w$ and is equal to the convex hull of the points belonging to the intersection of the hyperplane of equation $w_1 k_1 + \cdots + w_n k_n = w_1 \cdots w_n$ with the union of the coordinate axis.

We will apply the following well-known result of Kouchnirenko [?] in Corollary **??**, which in turn is applied in the proof of Corollary **??**.

Theorem 4.3. [?, Théorème 6.2, p. 26] *Let $\Delta \subseteq \mathbb{R}^n_{\geqslant 0}$ be a lattice polytope of dimension $q \in \{0, 1, \dots, n-1\}$. Let us suppose that Δ is not contained in any linear subspace of dimension q. Let $g_1, \dots, g_s \in \mathbb{C}[x_1, \dots, x_n]$ such that $\mathrm{supp}(g_i) \subseteq \Delta$, for all $i = 1, \dots, s$. Then the following conditions are equivalent:*

(a) *the ideal of $\mathcal{R}_\Delta$ generated by $g_1, \dots, g_s$ has finite colength in $\mathcal{R}_\Delta$;*

(b) *for all faces $\Delta' \subseteq \Delta$, the set of common zeros of $(g_1)_{\Delta'}, \dots, (g_s)_{\Delta'}$ is contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.*

Let us fix a subset $\mathtt{I} \subseteq \{1, \dots, n\}$, $\mathtt{I} \neq \emptyset$. We define

$$\mathbb{K}^n_{\mathtt{I}} = \{x \in \mathbb{K}^n : x_i = 0, \text{ for all } i \notin \mathtt{I}\}.$$

If S is any subset of $\mathbb{K}^n$, then we denote the intersection $S \cap \mathbb{K}^n_{\mathtt{I}}$ by $S^{\mathtt{I}}$. Given a polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$, if we suppose that h is written as $h = \sum_k a_k x^k$, then we denote by $h^{\mathtt{I}}$ the sum of all terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap \mathbb{R}^n_{\mathtt{I}}$.

Corollary 4.4. *Let $w \in \mathbb{Z}^n_{\geqslant 1}$ be a primitive vector. Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that F is weighted homogeneous with respect to w. Then $F^{-1}(0) = \{0\}$*

if and only if, for all $\mathtt{I} \subseteq \{1,\dots,n\}$, $\mathtt{I} \neq \emptyset$, *we have* $\{x \in \mathbb{C}^n : F_1^{\mathtt{I}}(x) = \cdots = F_n^{\mathtt{I}}(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$.

Proof. Let $a_i = \mathrm{d}_w(F_i)$, for all $i = 1,\dots,n$, and let $a = a_1 \cdots a_n$. Let us consider the function $G_i = F_i^{a/a_i}$, for all $i = 1,\dots,n$, and the map $G = (G_1,\dots,G_n)$. It is clear that $G^{-1}(0) = \{0\}$ if and only if $F^{-1}(0) = \{0\}$. Let $\Delta = \{k \in \mathbb{R}^n_{\geqslant 0} : |k| = a\}$. Then, we can apply Theorem **??** to Δ and $G_1,\dots,G_n$. Let us remark that $\mathcal{R}_\Delta = \mathcal{O}_n$. The set of faces of Δ is given by $\{\Delta^{\mathtt{I}} : \mathtt{I} \subseteq \{1,\dots,n\}, |\mathtt{I}| \neq \emptyset\}$. Moreover, we have

$$(G_i)_{\Delta^{\mathtt{I}}} = (F_i^{a/a_i})_{\Delta^{\mathtt{I}}} = ((F_i)_{\Delta^{\mathtt{I}}})^{a/a_i} = (F_i^{\mathtt{I}})^{a/a_i}$$

for all $i = 1,\dots,n$. Then the result follows as an immediate application of Theorem **??** to Δ and $G_1,\dots,G_n$. □

As we will see in the following two results, non-degeneracy of maps with respect to a fixed convenient global Newton polyhedron is a condition that includes both Newton non-degeneracy at infinity and pre-weighted homogeneity of maps.

Corollary 4.5. *Let* $F = (F_1,\dots,F_p) : \mathbb{C}^n \to \mathbb{C}^p$ *be a polynomial map. Let* $w \in \mathbb{Z}^n_{\geqslant 1}$ *be a primitive vector and let* $d = (d_1,\dots,d_p) \in \mathbb{Z}^p_{\geqslant 1}$. *Then the following conditions are equivalent:*

(a) *F is pre-weighted homogeneous with respect to w and $d = \mathrm{d}_w(F)$.*
(b) *F is non-degenerate with respect to $\widetilde{\Gamma}^w_+$ and $\nu_{\widetilde{\Gamma}^w}(F_i) = d_i$, for all $i = 1,\dots,p$.*

Proof. Let $\Delta = \Delta(-w, \widetilde{\Gamma}_+)$. Since $\mathcal{F}_0(\widetilde{\Gamma}^w_+) = \{-w\}$, the filtrating map $\tau : \mathbb{R}^n_{\geqslant 0} \to \mathbb{R}$ associated to $\widetilde{\Gamma}^w_+$ is given by $\tau(k) = \langle w, k \rangle$, for all $k \in \mathbb{R}^n_{\geqslant 0}$. Therefore $\mathrm{q}_w(F_i) = \mathrm{q}_\Delta(F_i)$, for all $i = 1,\dots,p$. Then the result follows as direct application of Corollary **??**. □

Remark 4.6. It is immediate to deduce that if $F : \mathbb{C}^n \to \mathbb{C}^p$ is a polynomial map such that F is Newton non-degenerate at infinity, then F is non-degenerate with respect to $\widetilde{\Gamma}_+(F)$. An easy example showing that the converse is not true is given by the map $F : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $F(x,y) = (x + 2y, x^2 - y^2)$. In the next result we will see when the equivalence between both concepts holds, in the case $n = p$.

Proposition 4.7. *Let* $F : \mathbb{C}^n \to \mathbb{C}^n$ *be a polynomial map such that* $F^{-1}(0)$ *is finite and* $F(0) = 0$. *Let* $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(F)$ *and let* $\nu = \nu_{\widetilde{\Gamma}(F)}$. *Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity*
(b) *F is non-degenerate with respect to $\widetilde{\Gamma}_+(F)$ and $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$.*

Proof. Let $e_1,\dots,e_n$ denote the canonical basis in $\mathbb{C}^n$. Let us suppose that F is not convenient. Then there exists some $i \in \{1,\dots,n\}$ such that $\mathrm{supp}(F)$ does not contain any vector of the form re_i, for some $r > 0$. In particular, we conclude that $F(\alpha e_i) = 0$, for all $\alpha \in \mathbb{C}$, since $F(0) = 0$. This contradicts the condition of finiteness of $F^{-1}(0)$. Therefore $\widetilde{\Gamma}_+(F)$ is convenient.

Let us prove (a) ⇒ (b). Let Δ be a face of $\widetilde{\Gamma}_+$ of dimension $n-1$ such that $0 \notin \Delta$. It is known that $\mathcal{R}_\Delta$ is a Cohen-Macaulay ring of dimension n (see [**?**] or [**?**, Théorème 5.6]). Since F is

Newton non-degenerate at infinity, the solutions of the system $(F_1)_{\Delta'}(x) = \cdots = (F_n)_{\Delta'}(x) = 0$ are contained in $\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}$, for any face Δ' of $\widetilde{\Gamma}_+$ such that $\Delta' \subseteq \Delta$. In particular, the ideal I generated by $\{(F_1)_\Delta, \dots, (F_n)_\Delta\}$ in $\mathcal{R}_\Delta$ has finite colength in $\mathcal{R}_\Delta$ (see [**?**, Théorème 6.2]), which implies that I is generated by al least n non-zero elements of $\mathcal{R}_\Delta$. Then $(F_i)_\Delta \neq 0$, for all $i = 1, \dots, n$. In particular we have $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$ and thus (b) follows.

The implication (b) $\Rightarrow$ (a) is immediate since the condition $\nu_{\widetilde{\Gamma}}(F_1) = \cdots = \nu_{\widetilde{\Gamma}}(F_n)$ implies that $\mathrm{q}_\Delta(F_i) = (F_i)_\Delta$, for all $i = 1, \dots, n$ and all faces Δ of $\widetilde{\Gamma}_+$ such that $0 \notin \Delta$. $\square$

In the remaining section, let us fix a convenient global Newton polyhedron $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. Let $\phi = \phi_{\widetilde{\Gamma}}$, $\nu = \nu_{\widetilde{\Gamma}}$ and $M = M_{\widetilde{\Gamma}}$. Let $\{\mathcal{B}_r\}_{r \geqslant 0}$ be the corresponding family of subspaces defined in (**??**).

Proposition 4.8. *Let $F = (F_1, \dots, F_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a finite polynomial map. Let $d_i = \nu_{\widetilde{\Gamma}}(F_i)$, for all $i = 1, \dots, n$, and let $d = d_1 \cdots d_n$. Then the following conditions are equivalent:*

(a) *F is non-degenerate with respect to $\widetilde{\Gamma}_+$.*
(b) *The map $(F_1^{d/d_1}, \dots, F_n^{d/d_n})$ is Newton non-degenerate at infinity and its global Newton polyhedron is equal to $\frac{d}{M}\widetilde{\Gamma}_+$.*
(c) *There exists a vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ such that the map $F^a = (F_1^{a_1}, \dots, F_n^{a_n}) : \mathbb{C}^n \to \mathbb{C}^n$ verifies that $\widetilde{\Gamma}_+(F^a)$ is homothetic to $\widetilde{\Gamma}_+$ and F^a is Newton non-degenerate at infinity.*

Proof. Let $\nu = \nu_{\widetilde{\Gamma}}$. Let us prove (a) $\Rightarrow$ (b). Let $a_i = d/d_i$, for all $i = 1, \dots, n$, and $a = (a_1, \dots, a_n)$. Clearly we have the inclusions

$$\widetilde{\Gamma}_+(F_i^{a_i}) \subseteq \widetilde{\Gamma}_+(F^a) \subseteq \widetilde{\Gamma}_+(\mathcal{B}_d) \subseteq \frac{d}{M}\widetilde{\Gamma}_+ \tag{17}$$

for all $i = 1, \dots, n$.

Let k be a vertex of $\widetilde{\Gamma}_+$. By condition (a), there exists some $i \in \{1, \dots, n\}$ such that $\mathrm{q}_{\{k\}}(F_i) \neq 0$. This means that there exists some $k' \in \mathrm{supp}(F_i)$ such that $\phi(k') = d_i$ and there exists some $\lambda > 0$ such that $k' = \lambda k$. Since $d_i = \phi(k') = \phi(\lambda k) = \lambda\phi(k) = \lambda M$, we obtain $\lambda = \frac{d_i}{M}$. In particular $a_i k' = a_i \frac{d_i}{M} k = \frac{d}{M} k$. Then, for any vertex k of $\widetilde{\Gamma}_+$, we have $\frac{d}{M}k$ belongs to $\mathrm{supp}(F_i^{a_i})$, for some $i \in \{1, \dots, n\}$. This fact together with (**??**) shows that

$$\widetilde{\Gamma}_+(F^a) = \frac{d}{M}\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(\mathcal{B}_d). \tag{18}$$

Let Δ be a face of $\widetilde{\Gamma}_+(F^a)$. By (**??**) there exists a face Δ' of $\widetilde{\Gamma}_+$ such that $\Delta = \frac{d}{M}\Delta'$. Using Definition **??**, it is immediate to see that $(\mathrm{q}_{\Delta'}(F_i))^{a_i} = (F_i^{a_i})_\Delta$, for all $i = 1, \dots, n$. Thus condition (b) follows.

The implication (b) $\Rightarrow$ (c) is obvious. Let us prove that (c) $\Rightarrow$ (a). Let $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geqslant 1}$ and $\mu > 0$ such that $\widetilde{\Gamma}_+(F^a) = \mu\widetilde{\Gamma}_+$. Hence, if $\Delta \subseteq \widetilde{\Gamma}$, then Δ is a face of $\widetilde{\Gamma}_+$ if and only if $\mu\Delta$ is a face of $\widetilde{\Gamma}_+(F^a)$. Then, the implication follows by observing that, if Δ is a face of $\widetilde{\Gamma}_+$ not passing through the origin, then $(\mathrm{q}_\Delta(F_i))^{a_i} = (F_i^{a_i})_{\mu\Delta}$, for all $i = 1, \dots, n$. $\square$

Theorem 4.9. *Let $F=(F_1,\dots,F_n):\mathbb{C}^n\to\mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. Let $d_i=\nu(F_i)$, for all $i=1,\dots,n$. Then*

$$\mu(F)\leqslant\frac{d_1\cdots d_n}{M^n}n!\mathrm{V}_n(\widetilde{\Gamma}_+). \tag{19}$$

and equality holds if and only if F is non-degenerate with respect to $\widetilde{\Gamma}_+$.

Proof. Let $d=d_1\cdots d_n$. Let us consider the map $G=(G_1,\dots,G_n):\mathbb{C}^n\to\mathbb{C}^n$ given by $G_i=F_i^{d/d_i}$, for all $i=1,\dots,n$. Then G has also finite multiplicity and this is given by

$$\mu(G)=\dim_{\mathbb{C}}\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{I}(G)}=\frac{d}{d_1}\cdots\frac{d}{d_n}\dim_{\mathbb{C}}\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{I}(F)}=d^{n-1}\mu(F). \tag{20}$$

Let us observe that $\nu(G_i)=d$, for all $i=1,\dots,n$. Then $\widetilde{\Gamma}_+(G)\subseteq\widetilde{\Gamma}_+(\mathcal{B}_d)\subseteq\frac{d}{M}\widetilde{\Gamma}_+$. Therefore, applying inequality (**??**), we obtain that

$$\mu(G)\leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(G))\leqslant n!\mathrm{V}_n(\widetilde{\Gamma}_+(\mathcal{B}_d))\leqslant\frac{d^n}{M^n}n!\mathrm{V}_n(\widetilde{\Gamma}_+). \tag{21}$$

Inequality (**??**) follows by joining (**??**) and (**??**). By relation (**??**), we have that equality holds in (**??**) if and only if $\mu(G)=\frac{d^n}{M^n}n!\mathrm{V}_n(\widetilde{\Gamma}_+)$, which is equivalent to saying that all inequalities of (**??**) become equalities. In turn, this is equivalent to saying that the following holds: $\widetilde{\Gamma}_+(G)=\widetilde{\Gamma}_+(\mathcal{B}_d)=\frac{d}{M}\widetilde{\Gamma}_+$ and G is Newton non-degenerate (by Theorem **??**). Thus, by Proposition **??**, we obtain the desired equivalence. □

When equality holds in (**??**), then we also say that F *has maximal multiplicity with respect to ν*. The particularization to weighted homogeneous filtrations of the previous result is shown in the following result.

Corollary 4.10. *Let $F=(F_1,\dots,F_n):\mathbb{C}^n\to\mathbb{C}^n$ be a finite polynomial map. Let us fix a primitive vector $w=(w_1,\dots,w_n)\in\mathbb{Z}^n_{\geqslant 1}$ and let $d_i=d_w(F_i)$, for all $i=1,\dots,n$. Then*

$$\mu(F)\leqslant\frac{d_1\cdots d_n}{w_1\cdots w_n} \tag{22}$$

and equality holds if and only if $(\mathrm{q}_w(F))^{-1}(0)=\{0\}$.

Proof. Inequality (**??**) follows by applying Theorem **??** to F and $\widetilde{\Gamma}^w_+$. Equality holds in (**??**) if and only if F is Newton non-degenerate with respect to $\widetilde{\Gamma}^w_+$, which is equivalent to saying that $(\mathrm{q}_w(F))^{-1}(0)=\{0\}$, by Corollary **??**. □

The application of Corollary **??** to gradient maps leads to the following result, which is the version for total Milnor numbers of the main result of Furuya-Tomari in [**?**].

Corollary 4.11. *Let $f\in\mathbb{C}[x_1,\dots,x_n]$ with a finite number of singularities and let us fix a primitive vector $w=(w_1,\dots,w_n)\in\mathbb{Z}^n_{\geqslant 1}$. Let $d=\mathrm{d}_w(f)$. Then*

$$\mu_\infty(f)\leqslant\frac{(d-w_1)\cdots(d-w_n)}{w_1\cdots w_n}. \tag{23}$$

Moreover, the following conditions are equivalent:

(a) f *is pre-weighted homogeneous with respect to* w.
(b) $(\mathrm{q}_w(\nabla f))^{-1}(0) = \{0\}$ *and* $\mathrm{d}_w(f_{x_i}) = \mathrm{d}_w(f) - w_i$, *for all* $i = 1, \dots, n$.
(c) *equality holds in* (**??**).

Proof. Let $f_{x_i} = \partial f / \partial x_i$, for all $i = 1, \dots, n$, and let $d = \mathrm{d}_w(f)$. Since f has a finite number of singularities, given an index $i \in \{1, \dots, n\}$, then $f_{x_i} \neq 0$ and thus, there exists some $k \in \mathrm{supp}(f)$ such that $k_i > 0$ and $k - e_i \in \mathrm{supp}(\mathrm{q}_w(f_{x_i}))$. In particular $\mathrm{d}_w(f_{x_i}) = \langle k, w \rangle - w_i \leqslant \mathrm{d}_w(f) - w_i$. Therefore

$$\mu_\infty(f) \leqslant \frac{\mathrm{d}_w(f_{x_1}) \cdots \mathrm{d}_w(f_{x_n})}{w_1 \cdots w_n} \leqslant \frac{(d - w_1) \cdots (d - w_n)}{w_1 \cdots w_n} \tag{24}$$

where the first inequality is a direct application of (**??**). Hence (**??**) is proven.

The equivalence between (a) and (b) easily follows by observing that, under the conditions of any of both items, we have

$$\frac{\partial\, \mathrm{q}_w(f)}{\partial x_i} = \mathrm{q}_w\left(\frac{\partial f}{\partial x_i}\right)$$

for all $i = 1, \dots, n$. The equivalence between (b) and (c) follows by a direct application of (**??**) and Corollary **??**. $\square$

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