Conditional permutability of subgroups and certain classes of groups

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Abstract

Two subgroups A and B of a finite group G are said to be tcc-permutable if X permutes with Y^g for some $g \in \langle X, Y \rangle$, for all $X \leq A$ and all $Y \leq B$. Some aspects about the normal structure of a product of two tcc-permutable subgroups are analyzed. The obtained results allow to study the behaviour of such products in relation with certain classes of groups, namely the class of T-groups and some generalizations.

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1. Introduction

All groups considered in the paper are finite.

Within the theory of finite groups, the structure of subgroups and the way how they are embedded into the group influence its structure, and conversely. Then a natural approach to the subject consists in looking for decompositions of the group as product of subgroups. Direct product, and also normal products, i.e. products of normal subgroups, appear as relevant decompositions. A basic significant fact to be mentioned here is that supersoluble groups are

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closed under taking direct products but not for normal products. This has been the origin of a large and fruitful research on products of groups. Asaad and Shaalan searched in [7] for criteria of supersolubility, and introduced the corresponding concepts of totally and mutually permutable products, which can be seen as extensions of direct products and normal products, respectively: Two subgroups A and B of a group G are said to be totally permutable if every subgroup of A permutes with every subgroup of B, while A and B are called mutually permutable if A permutes with every subgroup of B, and conversely. Much is known nowadays about the structure of totally and mutually products of groups (see [1, 9, 14, 15, 17, 18, 22, 23, 34]), which turns out to be cornerstones of a huge research, continuing to present days, and extending influence into different areas, as formation theory (see [1], [11]-[14], [21], [24], [34]), Fitting classes (see [17, 18, 20, 21, 22],[28]-[30]) and classes of groups defined in terms of subgroup embeddings; we refer in particular to T-groups, i.e. groups in which normality is a transitive relation in the group, and generalizations of this class of groups given by the ascending series of classes of PT-, PST-, SC-, and SM-groups, where PT-groups and PST-groups are defined in the same way but with transitive relations given by permutability and permutability with Sylow subgroups, respectively, SM-groups are those groups where each subnormal subgroup permutes with every maximal subgroup, and following Robinson [37], SC-groups are groups whose chief factors are simple; (see [8, 9, 18, 19]). We refer to [10] for a good convenient recopilation on the topic.

Sometimes permutability turns out to be a strong hypothesis. In this direction further progress nicely show that it is still possible to obtain good information by considering conditional permutability. We recall that two subgroups X and Y of a group G are said to be *conditional permutable* (*c-permutable*, for brevity) in G if X permutes with Y^g for some $g \in G$. For instance, trivially any two Sylow p-subgroups of a group G, for a fixed prime p, are conditional permutable in G, but they are not permutable if they are different. This example shows the much wider reach of conditional permutability in comparison with permutability. Stricter, but in some respect more natural also, is the concept of *complete conditional permutability* (or *cc-permutability*): Two subgroups X and Y of a group G are said to be *cc-permutable* if X permutes with Y^g for some $g \in \langle X, Y \rangle$.

Now total permutability can be generalized to *total cc-permutability* (or tcc-permutability), i.e. the subgroups A and B of a group G are said to be

tcc-permutable if every subgroup of A is *cc*-permutable with every subgroup of B. If $G = AB$ we say that G is the *tcc-permutable product* of the subgroups A and B.

These concepts first appear in [26] and were initially used to provide new supersolubility criteria (see also [2, 27, 32, 33]). We refer to [3, 5] for surveys on these and further progress.

The present paper is mainly concerned with tcc-permutability, and we are interested in a better understanding of the inner structure of such products of subgroups, particularly in their normal structure. In this direction, we take further the study carried out in [4, 6], and are inspired by the investigation of Beidleman and Heineken in [17] on mutually permutable products. As in the case of mutually permutable products (see [17]), we prove in Section 2 that the factors of a tcc-permutable product have the cover-avoidance property in the product (Corollary 1), i.e. each factor either covers or avoids each chief factor of the product ([25, A. Definition (10.8)]). Moreover, a minimal normal subgroup of the product which is either covered or avoided simultaneously by the two factors is cyclic of prime order (Propositions 1, 2). As a consequence, Corollary 2 describes tcc-permutable products which are primitive groups of type 3, and complete, for two factors, the characterization of primitive tcc-permutable products, together with corresponding results in [4, Lemma 4, Corollary 5] for primitive groups of types 1 and 2 (for which in fact only type 1 is possible; see Lemmas 4, 5).

The results in Section 2 are then applied in Section 3 to find new classes of groups which are closed with respect to tcc-permutable products. A previous research within the framework of formation theory has been carried out in [4, 6]. Now, we take further previous developments on totally and mutually permutable products, and search for the interaction between the ascending series of classes of T-, PT-, PST-, SC-, and SM-groups and tcc- permutability. For the last two classes of groups, we prove that a tcc-permutable product belongs to the class if and only if the factors do so (Theorems 2, 3). The necessary condition holds also for T-, PT-, PST-groups (Theorem 4). The proof rests in addition on the good behaviour of the soluble residual and radical in a tcc-permutable product (Proposition 3). For the converse additional hypotheses are needed (see Remark 3 and Theorem 5), as happens for totally permutable products (see [8], [19]).

We point out that totally permutable products are tcc-permutable, and so, in many cases, the research on the last kind of products develops as

generalization of previous studies on the first ones. It is remarkable how many results on totally permutable products remain true in the more general setting in spite of the failure of significant structural properties (see [2, 4, 6] and Remark 1). As showed in this paper, even more surprising is the revealed analogy of certain results between mutually permutable products and tcc-permutable products. Mutual permutability and tcc-permutability appear to be quite different extensions of total permutability. They are concepts which are not connected to each other and basic structural properties are different (see Example 1, Remark 1, comment after Theorem 2 in relation with Part (i), and Proposition 3 (ii) with the previous comment).

We shall adhere to the notation used in [25].

2. Minimal normal subgroups and cc-permutability

The following lemma gives information about normal subgroups of tcc-permutable products, in fact for more general products by considering c-permutability instead of cc-permutability. A corresponding result for mutually permutable products was obtained in [1, Lemma 3].

Lemma 1. *Let the group $G = AB$ be the product of subgroups A and B such that every subgroup of A is c-permutable in G with every subgroup of B. If N is a normal subgroup of G, then $(A \cap N)(B \cap N)$ is a normal subgroup of G. Moreover, if N is a minimal normal subgroup of G, then either $N = (A \cap N)(B \cap N)$ or $A \cap N = B \cap N = 1$.*

Proof. By the hypothesis, there exists $g \in G$ such that $(B \cap N)A^g = A^g(B \cap N)$. But $g = ab$ for some $a \in A$ and $b \in B$. We can deduce that $B \cap N$ permutes with A, because $B \cap N$ is a normal subgroup of B. Analogously, $A \cap N$ permutes with B. We notice that $N \cap (A(B \cap N)) = (A \cap N)(B \cap N) = N \cap (B(A \cap N))$. Therefore, $(A \cap N)(B \cap N)$ is normalized by both A and B and so by G. If in addition N is a minimal normal subgroup of G, the rest of the result follows easily. □

The following lemmas are key facts in our work. A relevant structural property of a tcc-permutable product $G = AB$ of subgroups A and B is that the commutator subgroup $[A, B]$ is a normal nilpotent subgroup of G ([4, Theorem 4]). Then we can state the following easily:

Lemma 2. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B and let N be a minimal normal subgroup of G. Then $[A, B] \leq$*

$C_G(N)$ and $G/C_G(N)$ is the central product of the subgroups $AC_G(N)/C_G(N)$ and $BC_G(N)/C_G(N)$, i.e. $[AC_G(N)/C_G(N), BC_G(N)/C_G(N)] = 1$.

Proof. Let N be a minimal normal subgroup of G. From [4, Theorem 4] and [25, A. Theorem (10.6)] it follows that $[A, B] \leq F(G) \leq C_G(N)$. Consequently, $[AC_G(N)/C_G(N), BC_G(N)/C_G(N)] = 1$, as desired. □

Lemma 3. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Let N be an abelian normal subgroup of G. Then:*

(i) *$N \cap B$ and A are totally permutable subgroups.*

(ii) *If N is a p-group for a prime number p, then $O^p(A)$ normalizes $N \cap B$ and acts as a universal power automorphism group on $N \cap B$. In particular, if N is an elementary abelian p-group, $O^p(A)/C_{O^p(A)}(N \cap B)$ is a cyclic group of order dividing $p - 1$.*

(iii) *If N is a minimal normal p-subgroup of G for a prime number p and $N \leq B$, then $O^{p'}(A)$ centralizes N.*

Proof. (i) Let $n \in N \cap B$ and $a \in A$. Then there exists $x \in \langle n, a\rangle \leq N\langle a\rangle$ such that $\langle n\rangle^x\langle a\rangle = \langle a\rangle\langle n\rangle^x$. Let $x = n_0a^i$ with $n_0 \in N$ and $i \in \mathbb{Z}$. Since N is abelian, $\langle n\rangle^x\langle a\rangle = (\langle n\rangle\langle a\rangle)^{a^i}$ and so $\langle n\rangle$ permutes with $\langle a\rangle$. Therefore $N \cap B$ and A are totally permutable subgroups.

(ii) Let A_q be a Sylow q-subgroup of A with $q \neq p$ and $n \in N \cap B$. Then $[\langle n\rangle, A_q] \leq N \cap \langle n\rangle A_q = \langle n\rangle$, which means that A_q normalizes $\langle n\rangle$. Since $N \cap B$ is abelian, we deduce that $O^p(A)$ acts as a universal power automorphism group on $N \cap B$ and the result follows.

(iii) By (ii), $O^p(A)$ acts as a universal power automorphism group on $N = N \cap B$. Let A_p be a Sylow p-subgroup of A. Then $[O^p(A), A_p] \leq C := C_G(N)$, which implies that A_pC is normalized by $A = O^p(A)A_p$. Using Lemma 2, we can deduce that A_pC/C is normal in G/C, and so $A_pC/C \leq O_p(G/C)$. Since $O_p(G/C) = 1$ (cf. [25, A. Lemma (13.6)]), we have that A_p centralizes N. Therefore $O^{p'}(A)$ centralizes N as desired. □

We recall in the next two lemmas the structure of monolithic primitive groups, i.e. primitive groups with a unique minimal normal subgroup (types 1 and 2), which are product of pairwise tcc-permutable subgroups.

Lemma 4. [4, Lemma 4] *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and*

G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Assume in addition that G is a primitive group of type 1. Let N be the unique minimal normal subgroup of G and p be a prime divisor of $|N|$. Then either G is supersoluble or the following conditions are satisfied:

(i) w.l.o.g. $N \leq G_1$;

(ii) $G_2 \cdots G_r$ is a cyclic group whose order divides $p - 1$;

(iii) there exists a maximal subgroup M of G with $\mathrm{Core}_G(M) = 1$ such that $M = (M \cap G_1)(G_2 \cdots G_r)$ and $M \cap G_1$ centralizes $G_2 \cdots G_r$.

Lemma 5. [4, Corollary 5] *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \ldots, r$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Let N be a minimal normal subgroup of G. Then:*

1. *If N is non-abelian, then there exists a unique $i \in \{1, \ldots, r\}$ such that $N \leq G_i$. Moreover, G_j centralizes N and $N \cap G_j = 1$ for all $j \in \{1, \ldots, r\}$, $j \neq i$.*
2. *If G is a monolithic primitive group, then the unique minimal normal subgroup N is abelian.*

The next Theorem 1 gives information about minimal normal subgroups in relation with the factors of a tcc-permutable product. The cover-avoidance property of the factors of a tcc-permutable product will follow as an easy consequence of this result.

Theorem 1. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. If N is a minimal normal subgroup of G, then $\{A \cap N, B \cap N\} \subseteq \{1, N\}$.*

Proof. Assume that the result is not true and let the group $G = AB$ be a counterexample with minimal order. Then there exists a minimal normal subgroup N of G which does not satisfy the thesis. By Lemma 5 we have that N is an abelian p-group for some prime p. From Lemma 1 we deduce that $N \cap A \neq 1$ and $N \cap B \neq 1$. Without loss of generality we may assume that $1 \neq A \cap N \neq N$. We set $C = C_G(N)$ and note that $G \neq C$. We distinguish two cases:

Case 1: $N \cap B \neq N$.

Let q be the largest prime dividing the order of G/C. Without loss of generality we may assume that q divides the order of AC/C. Now, we split the proof into the next following steps:

Step 1.1 AC/C is a supersoluble group.

By [6, Lemma 2.7] we have that $A^{\mathcal{U}}$ is a normal subgroup of G. Since N is a minimal normal subgroup of G and $N \cap A \neq N$, we deduce that $[A^{\mathcal{U}}, N] = 1$. Therefore, we have that $A^{\mathcal{U}} \leq C$ and AC/C is a supersoluble group.

Step 1.2 Let A_q be a Sylow q-subgroup of A. Then A_qC is a normal subgroup of G. In particular, $q \neq p$.

From Step 1.1 and the choice of q, we deduce that $A_qC/C = O_q(AC/C)$. By Lemma 2 we have that AC/C is a normal subgroup of G/C. Consequently, A_qC/C is normal in G/C. Moreover, since $O_p(G/C) = 1$ (cf. [25, A. Lemma (13.6)]), it follows that $p \neq q$.

Step 1.3 Let A_q be a Sylow q-subgroup of A. Then $A_q \leq C_A(N \cap B)$.

By Lemma 3, $O^p(A)$ normalizes $N \cap B$ and $O^p(A)/C_{O^p(A)}(N \cap B)$ is a cyclic group of order dividing $p - 1$. If p divides the order of AC/C, then $p < q$ by the choice of q and Step 1.2. Then $A_q \leq C_A(N \cap B)$ and we are done. Otherwise, for a Sylow p-subgroup A_p of A, we would have that $A = O^p(A)A_p \leq N_G(N \cap B)$, which would imply that $1 \neq N \cap B$ is normal in G and so $N = N \cap B$, a contradiction.

Step 1.4 Contradiction to Case 1.

Let A_q be a Sylow q-subgroup of A. By Lemma 1 and Step 1.3, we have that $[N, A_qC] = [(N \cap B)(N \cap A), A_qC] = [N \cap A, A_qC] \leq N \cap A \neq N$. But $[N, A_qC]$ is normal in G by Step 1.2, which implies $[N, A_qC] = 1$ and $A_q \leq C$, a contradiction.

Case 2: $N \leq B$.

We split the proof into the following steps:

Step 2.1 $AC = O^p(A)C$ and it normalizes every subgroup of N.

By Lemma 3 we know that $AC = O^p(A)C$ and $O^p(A)$ acts as a universal power automorphism group on $N = N \cap B$. Hence $O^p(A)C$ normalizes every subgroup of N.

Step 2.2 $B = G$.

If K is a minimal normal subgroup of B contained in N, then K is normal in G by Step 2.1 and so we deduce that $K = N$. Assume that B is a proper

subgroup of G. Hence, we can see that $B = (A \cap B)B$ is the product of the tcc-permutable subgroups $A \cap B$ and B. The choice of G implies that either $1 = N \cap (A \cap B) = N \cap A$ or $N = N \cap (A \cap B) \leq A$. Both cases are not possible, so $G = B$.

Step 2.3 The final contradiction.

By Lemma 3(i) and Step 2.2, we have that $N \cap A$ is a permutable subgroup of G. Applying [35] (see also [10, Corollary 1.5.6]) it follows that $N \cap A \leq Z_\infty(G)$, since $\mathrm{Core}_G(N \cap A) = 1$. Therefore, $N \leq Z_\infty(G)$ and $N = N \cap A$, a contradiction which concludes the proof. □

We recall that a subgroup U of a group G is said to *cover* a chief factor H/K of G if $HU = KU$ and to *avoid* H/K if $U \cap H = U \cap K$. The subgroup U is said to have the *cover-avoidance property* in G, and it is also called CAP-subgroup in this case, if U either covers or avoids each chief factor of G ([25, A. Definition (10.8)]).

Corollary 1. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Then A and B are CAP-subgroups of G.*

Proof. If H/K is a chief factor of G, then H/K is a minimal normal subgroup of G/K, which is a tcc-permutable product of the subgroups AK/K and BK/K. Let $X \in \{A, B\}$. By Theorem 1, XK/K either covers or avoids H/K and so X does, which proves the result. □

A corresponding result to Corollary 1 for mutually permutable products was proved by Beidleman and Heineken in [17].

Propositions 1 and 2 next study the cases when a minimal normal subgroup of a tcc-permutable product is either covered or avoided simultaneously by the two factors, in which cases the minimal normal subgroup has prime order. Examples (i), (iv) in Remark 1 below show that both cases are possible (in spite of the situation when the group is a nonsupersoluble monolithic primitive group; see Lemmas 4, 5).

Proposition 1. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Let N be a minimal normal subgroup of G. Assume that $N \leq A \cap B$. Then $|N| = p$, where p is a prime number.*

Proof. From Lemma 5 we deduce that N is an abelian p-group for some prime p. By Lemma 3 (ii), (iii), we deduce that $A = O^p(A)O^{p'}(A)$ and

$B = O^p(B)O^{p'}(B)$ normalize each subgroup of $N \leq A \cap B$, which implies that $|N| = p$, since N is a minimal normal subgroup of $G = AB$. □

Nevertheless, the next example shows that it is not possible to replace tcc-permutable product by mutually permutable product in the previous result.

Example 1. Let $X = \mathrm{Sym}(3)$ be the symmetric group of degree 3. By [25, B. Corollary (10.7)], X has an irreducible and faithful module V, which has dimension 2, over $\mathbb{F}_5$, the field of 5 elements. Let $G = [V]X$ be the corresponding semidirect product, which is the mutually permutable product of $A = VX_3$ and $B = VX_2$, where X_p is a Sylow p-subgroup of X for $p \in \{2, 3\}$. We observe that V is a minimal normal subgroup of G, which is covered by both A and B, but V is not of prime order.

Next we will deal with the case when a minimal normal subgroup N in a tcc-permutable product $G = AB$ is avoided by the two factors, that is, $N \cap A = 1 = N \cap B$.

Proposition 2. *Let the group $G = AB \neq 1$ be the product of the tcc-permutable subgroups A and B. Let N be a minimal normal subgroup of G. Assume that $N \cap A = N \cap B = 1$. Then $|N| = p$, where p is a prime number.*

Proof. Assume that the result is false and let the group $G = AB$ be a counterexample with $|G|$ minimal, as in the statement. Let N be a minimal normal subgroup of G such that $N \cap A = N \cap B = 1$ but N is not of prime order. From Lemma 5 we deduce that N is an abelian p-subgroup for some prime p. (We notice that the hypotheses of the result imply that $A \neq 1$ and $B \neq 1$.) We split the proof into the following steps:

Step 1. $G/C_G(N)$ is a cyclic group.

By [4, Proposition 1] we know that A' and B' are subnormal subgroups of G. Since N is a minimal normal subgroup of G, we have that $N \leq N_G(A') \cap N_G(B')$ (cf. [25, A. Lemma (14.3)]). Then $[A', N] = [B', N] = 1$, because $N \cap A = N \cap B = 1$. On the other hand, from Lemma 2, it follows that $[A, B] \leq C_G(N)$. Consequently, $G' = A'[A, B]B' \leq C_G(N)$ and $G/C_G(N)$ is an abelian group. Now we conclude that $G/C_G(N)$ is a cyclic group (cf. [25, B. Theorem (9.8)]), as desired.

Step 2. N is not a minimal X-invariant subgroup for $X \in \{A, B\}$. In particular, $AC_G(N)$ and $BC_G(N)$ are proper subgroups of G.

Assume that N is a minimal A-invariant subgroup. If $AN < G$, since $AN = A(AN \cap B) \neq 1$ is the product of the tcc-permutable subgroups A and $AN \cap B$, and N is a minimal normal subgroup of AN, which is avoided by both A and $AN \cap B$, the choice of G implies that $|N| = p$, a contradiction. Consequently, $G = AN$ and A is a maximal subgroup of G. Therefore, if we set $L = \mathrm{Core}_G(A)$, we have that $G/L = (A/L)(NL/L)$ is a primitive group with a unique minimal normal subgroup $NL/L \cong N$, since $A \cap N = 1$. Moreover, $G/L = (A/L)(BL/L)$ is the product of the tcc-permutable subgroups A/L and BL/L. By Lemma 4, either G/L is supersoluble or A/L is a cyclic group whose order divides $p - 1$. In both cases we can conclude that $|N| = p$, a contradiction. Therefore, N is not a minimal A-invariant subgroup, and the same is true for B. In particular, $AC_G(N)$ and $BC_G(N)$ are proper subgroups of G.

Step 3. The final contradiction.

By Step 2 there exists M a maximal subgroup of G containing $AC_G(N)$. Observe that $M = A(M \cap B) \neq 1$ is the product of the tcc-permutable subgroups A and $B \cap M$, and $N \leq M$. The minimal choice of G implies that N is not a minimal normal subgroup of M. Moreover, any minimal normal subgroup of M contained in N has order p. On the other hand, from Step 1 we have that $G/C_G(N)$ is abelian, so $M \trianglelefteq G$ and $|G : M| = q$ where q is a prime. We consider now N as an irreducible $\mathbb{F}_pG$-module, $\mathbb{F}_p$ the field of p-elements, and apply Clifford's theorem (cf. [25, B. Theorem (7.3)]). Let W be an irreducible $\mathbb{F}_pM$-submodule of N_M and denote $H(W)$ the homogeneous component of N belonging to W and $I_G(W) = N_G(H(W))$ the inertia subgroup of W. Then either $I_G(W) = M$ or $I_G(W) = G$. In the first case, since $H(W)$ is irreducible as $\mathbb{F}_pI_G(W)$-module, it follows that $W = H(W)$ has order p and $|N| = p^q$, because N is a direct sum of its homogeneous components. If this were the case for any maximal subgroup of G containing either $AC_G(N)$ or $BC_G(N)$, we would deduce that $G/AC_G(N)$ and $G/BC_G(N)$ would be q-groups, and then $G/(AC_G(N) \cap BC_G(N))$ would be a cyclic q-group. Therefore, since $G = AB$, it would follow that either $G = A(AC_G(N) \cap BC_G(N)) = AC_G(N)$ or $G = B(AC_G(N) \cap BC_G(N)) = BC_G(N)$, a contradiction with Step 2. Hence we may assume that $I_G(W) = G$ and then N is a homogeneous $\mathbb{F}_pM$-module with $|W| = p$. But this implies that M acts as a universal power automorphism group on N, in particular, M normalizes every subgroup of N, and then N is a minimal B-invariant subgroup, the final contradiction. □

As mentioned in the Introduction (see also Lemmas 4, 5), the structure of tcc-permutable products which are monolithic primitive groups has been described in [4, Lemma 4, Corollary 5]. As an application of our results on the cover-avoidance property, the structure of a tcc-permutable product of two factors, which is a non-monolithic primitive group, is also clarified.

Corollary 2. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B with $A \neq 1$ and $B \neq 1$. Assume in addition that G is a primitive group of type 3, i.e. with two minimal normal subgroups. Let N_1 and N_2 be the minimal normal subgroups of G. Then $G = A \times B$ where $A = N_1$, $B = N_2$ and $N_1 \cong N_2$ are nonabelian simple groups.*

Proof. From [25, A. Theorem (15.2)] we have that G is a primitive group with stabilizer M, $G = MN_1 = MN_2$, $N_1 = C_G(N_2)$ and $N_2 = C_G(N_1)$. Using Lemma 2 we deduce that $[A, B] \leq C_G(N_1) \cap C_G(N_2) = N_2 \cap N_1 = 1$. By Proposition 1, Proposition 2 and Corollary 1 we have w.o.l.g that $N_1 \leq A$ and $B \cap N_1 = 1$. So, we obtain that $[N_1, B] = 1$ and $1 \neq B \leq C_G(N_1) = N_2$. Using again Corollary 1 we have that $B = N_2$ and $[A, N_2] = 1$. We conclude that $A = N_1$ and $G = A \times B = N_1 \times N_2$ with $N_1 \cong N_2$ nonabelian simple groups. □

Remark 1. We refer to [2, Examples 2, 3] (also [6, Examples 3.5, 3.6]) and to [4, Example 1] for examples showing the failure of significant structural properties and results of products of totally permutable subgroups when considering instead tcc-permutability.

When comparing totally permutable products and mutually permutable products, the intersection of the factors has played a key role. We gather next some significant previous results in this direction as well as basic structural properties of mutually permutable products in relation with minimal normal subgroups. Also the statement 4 below is a key fact within the study of mutually permutable products in the framework of Fitting classes. Then we provide examples showing that they are missed when considering tcc-permutability. These examples together with Example 1, comment after Theorem 2 in relation with Part (i), and Proposition 3 (ii) with the previous comment, show that the concepts of tcc-permutability and mutual permutability are not connected to each other and basic structural properties of corresponding products are different, as mentioned in the Introduction.

Known results on mutually and totally permutable products of subgroups:

1. ([34, Lemma 2]) If $G = AB$ is the product of the totally permutable subgroups A and B, then $A \cap B$ is a nilpotent subnormal subgroup of G.

   For statements $2 - 7$, let $G = AB$ be the product of the mutually permutable subgroups A and B. Then:
2. ([23, Proposition 3.5(b)]) If $A \cap B = 1$, then A and B are totally permutable subgroups.
3. ([17, Lemma 1(v)], [23]) $\langle (A \cap B)^G \rangle / Core_G(A \cap B)$ is nilpotent; in particular, $A \cap B$ is subnormal in G.
4. ([22, Theorem 2]) There exist subnormal subgroups L and M of G, with $A' \leq L \leq A$ and $B' \leq M \leq B$, such that $A \cap B \leq L \cap M$ and $G' \leq LM$.
5. ([10, Lemma 4.3.3(3)]) If N is a minimal normal subgroup of G, then either $N \leq A \cap B$ or $[N, A \cap B] = 1$.
6. ([17, Lemma 1 (viii)], [10, Lemma 4.3.3(5)]) If N is a minimal normal subgroup of G with $N \leq A$ and $N \cap B = 1$, then either $[N, A] = 1$ or $[N, B] = 1$.
7. ([17, Lemma 2)], [10, Lemma 4.3.9)]) If N is a minimal normal subgroup of G with $N \cap A = 1 = N \cap B$, then $N \cong Z_p$, where p is a prime, and either $[N, A] = 1$ or $[N, B] = 1$

Examples on tcc-permutable products of subgroups:

(i) *(Failure of 1. when considering tcc-permutability instead of total permutability.)*
Let $G = \mathrm{Sym}(3)$ be the symmetric group of degree 3 and consider $G = AB$ the trivial factorization with $G = A = B$. Then A and B are tcc-permutable subgroups in G, but $A \cap B = G = \mathrm{Sym}(3) \notin \mathcal{N}$. (See also (iii)).

The next examples show the failure of statements $2-7$ when considering tcc-permutability instead of mutual permutability

(ii) *(Failure of 2, 6.)* We consider the group constructed in [2, Example 3]: Let $V = \langle a, b \rangle \cong Z_5 \times Z_5$ and $Z_6 \cong C = \langle \alpha, \beta \rangle \leq \mathrm{Aut}(V)$ given by $a^\alpha = a^{-1},\ b^\alpha = b^{-1};\quad a^\beta = b,\ b^\beta = a^{-1}b^{-1}$. Let $G = [V]C$ the corresponding semidirect product of V with C. Set $A = \langle \alpha \rangle$ and $B = V\langle \beta \rangle$. Then $G = AB$ is the tcc-permutable product of the subgroups A and B. But:

Obviously $A \cap B = 1$ but A and B are not totally permutable.

V is the unique minimal normal subgroup of G, $V \leq B$ and $A \cap V = 1$, but $[V, A] \neq 1$ and $[V, B] \neq 1$.

(iii) *(Failure of 3, 4, 5.)* ([2, Example 2]) Let again $G = \mathrm{Sym}(3)$ be the symmetric group of degree 3 and consider the factorization $G = AB$ with $A = G$ and B a Sylow 2-subgroup of G. Then A and B are tcc-permutable in G. But:

$A \cap B = B$ is not a subnormal subgroup of G;

$\langle (A \cap B)^G \rangle / \mathrm{Core}_G(A \cap B) \cong G = \mathrm{Sym}(3)$ is not nilpotent.

Since $A \cap B = B$ and the unique subnormal subgroup M of G contained in B is the trivial one, there exists no subnormal subgroup L of G, with $A' \leq L \leq A$, such that $A \cap B \leq L \cap M$.

Let N be the Sylow 3-subgroup of G. Then N is the unique minimal normal subgroup of G, $N \not\leq A \cap B$ and $[N, A \cap B] = [N, B] \neq 1$.

(iv) *(Failure of 7.)* Let $V = \langle a, b \rangle \cong Z_3 \times Z_3$ and $Z_2 \cong C = \langle \alpha \rangle \leq \mathrm{Aut}(V)$ given by $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$. Let $G = [V]C$ the corresponding semidirect product of V with C. Set $A = [\langle a \rangle]\langle \alpha \rangle$ and $B = [\langle b \rangle]\langle \alpha \rangle$. Then $G = AB$ and A and B are tcc-permutable subgroups.
Let D be the diagonal subgroup of V. Then D is a minimal normal subgroup of G, $A \cap D = B \cap D = 1$, but $[D, A] \neq 1$ and $[D, B] \neq 1$.
(Proposition 2 shows that the first part of statement 7 holds for tcc-permutable products.)

Nevertheless, the intersection $A \cap B$ of the factors of a tcc-permutable product $G = AB$ of subgroups A and B still enjoys some nice properties. As a direct consequence of Proposition 1, by considering the trivial factorization $A \cap B = (A \cap B)(A \cap B)$, it follows that $A \cap B$ is supersoluble, since its chief factors have prime order (Corollary 3(i) below). Though it should be mentioned that this fact is also a consequence of a stronger result [26, Theorem 3.8] involving c-permutability, which shows in particular that a group is supersoluble if and only if every maximal subgroup is c-permutable in the group. The second part of the next Corollary 3 is a consequence of a significant property of tcc-permutable products, obtained in [4, Theorem 3], which states that the nilpotent residuals of the corresponding factors are normal subgroups in the whole group. With the previous notation, since $A = A(A \cap B)$ and $B = B(A \cap B)$ are tcc-permutable products of the subgroups A and $A \cap B$, and B and $A \cap B$, respectively, we can deduce that $G = AB$ normalizes $(A \cap B)^{\mathcal{N}}$. Hence, the following result follows.

Corollary 3. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Then,*

(i) *$A \cap B$ is a supersoluble group.*

(ii) *$(A \cap B)^{\mathcal{N}}$ is a normal subgroup of G. In particular, $(A \cap B)/\mathrm{Core}_G(A \cap B)$ is a nilpotent group.*

Corollary 3(i) will be applied in the next section to prove Theorem 3.

3. On T-, PT-, PST-, SC- and SM-groups and conditional permutability

First, we collect here the permutability notions considered in this section: a subgroup of a group G is called *permutable* if it permutes with every subgroup of G, and a subgroup of G is called *S-permutable* if it permutes with all Sylow subgroups of G. A group G is called a *T-group* if normality is a transitive relation in G, that is, if all subnormal subgroups of G are normal in G. A group G is called a *PT-group* if permutability is a transitive relation in G. As a consequence of [36, 13.2.1], PT-groups are exactly those groups where all subnormal subgroups are permutable. A group G is called a *PST-group* if S-permutability is a transitive relation in G. From [31, Satz 1], PST-groups are exactly those groups in which all subnormal subgroups are S-permutable. Robinson in [37] introduced and classified *SC-groups* as groups whose chief factors are simple. Finally, Beidleman and Heineken in [16] began the study of the so-called SM-groups, which are those groups where each subnormal subgroup permutes with every maximal subgroup. The classes of all finite T-, PT-, PST-, SC- and SM-groups will be denoted by $\mathcal{T}$, $\mathcal{PT}$, $\mathcal{PST}$, $\mathcal{SC}$ and $\mathcal{SM}$, respectively. They form the following ascending series $\mathcal{T} \subset \mathcal{PT} \subset \mathcal{PST} \subset \mathcal{SC} \subset \mathcal{SM}$. (We refer to [16, Theorem A] for the last containment and to [10] for an overview of the rest.)

We start the study with the class $\mathcal{SC}$. We note that this class is a formation closed under taking normal subgroups, but it is neither subgroup closed nor saturated (see [10, Theorem 1.6.3]). As a consequence of Corollary 1 we obtain the following result:

Theorem 2. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Then:*

(i) *If A and $B \in \mathcal{SC}$, then $G \in \mathcal{SC}$.*

(ii) *If $G \in \mathcal{SC}$, then A and $B \in \mathcal{SC}$.*

Proof. (i) We argue by induction on the order of G. We notice that, for any minimal normal subgroup N of G, the factor group $G/N = (AN/N)(BN/N)$ is the product of the tcc-permutable SC-subgroups AN/N and BN/N. Then, by induction we have that G/N is an SC-group. Using Corollary 1 we have that A and B either covers or avoids N. If either $N \leq A \cap B$ or $N \cap A = N \cap B = 1$, then by Proposition 1 and Proposition 2 it follows that $|N| = p$, where p is a prime. Therefore, we have that G is an SC-group. Hence, w.o.l.g. we may assume that $N \cap A = 1$ and $N \leq B$. We take N_0 a minimal normal subgroup of B contained in N. By hypothesis $AN_0 \leq G$. We notice that A normalizes N_0, because $N_0 = N_0(N \cap A) = N \cap AN_0$. So, we deduce that $N = N_0$ and N is a minimal normal subgroup of B. Since B is an SC-group, we obtain that N is a simple group. Consequently, G is an SC-group, which proves (i).

(ii) We prove that A and B are SC-groups by induction on the order of G. We notice that, for any minimal normal subgroup N of G, the factor group $G/N = (AN/N)(BN/N)$ satisfies the hypothesis and, by induction, we have that AN/N and BN/N are SC-groups. Since G is an SC-group, N is a simple group. From Corollary 1 we have that A either covers or avoids N. Therefore $N \leq A$ or $N \cap A = 1$, and in both cases it follows that A is an SC-group. Analogously, B is an SC-group, and (ii) holds. □

We recall that soluble SC-groups are exactly supersoluble groups. Therefore the above result can be seen as a natural extension of the well-known criterion of supersolubility due to Asaad and Shaalan, which states that a totally permutable product of supersoluble subgroups is supersoluble ([7, Theorem 3.1]). More exactly, the result of Asaad and Shaalan appears as a consequence of Theorem 2 together with [2, Remark], which provides the solubility of a product of totally c-permutable soluble subgroups.

Also in relation with the study of tcc-permutable products in the framework of formation theory, it may be of interest to point out here that $\mathcal{SC}$ is a formation containing all supersoluble groups, and closed with respect to tcc-permutable products, which is not saturated (see [6, Theorem 1.4], [4, Theorem 5, Example 1]).

A result corresponding to Theorem 2 for totally permutable products was obtained in [8, Theorem A] for two factors, and in [19, Theorem A] for an arbitrary number of factors, though also the arguments used for the proofs

here are different. Regarding mutually permutable products of subgroups, a corresponding result to Part (ii) in Theorem 2 was obtained in [9, Theorem 3], also in [18, Theorem 3] using a different approach, for two factors, and in [10, Theorem 4.5.11] for an arbitrary number of factors. The corresponding result to Part (i) for this kind of products has been obtained under the additional hypothesis that the intersection of the factors has trivial core ([9, Theorem 2], [18, Theorem 3]).

We continue the study with the class $\mathcal{SM}$. This class is closed under taking factor groups and normal subgroups. We will need the following structural result on SM-groups:

Lemma 6. ([16, Theorem A]) *Let G be a group. Then the following are equivalent:*

(i) *G is an SM-group;*

(ii) *$G/\phi(G)$ is an SC-group;*

(iii) (a) *all soluble quotients of G are supersoluble,*

(b) *all perfect subnormal subgroups of G are normal,*

(c) *$G/\phi(G)$ is an extension of a direct product of non-abelian simple groups by a supersoluble group.*

Beidleman, Hauck and Heineken proved in [19, Theorem B] that the class $\mathcal{SM}$ is closed with respect to products of pairwise totally permutable subgroups. Next, we obtain a corresponding result for products of two tcc-permutable subgroups, using also a different approach, as application of Corollary 1.

Theorem 3. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Then:*

(i) *If A and $B \in \mathcal{SM}$, then $G \in \mathcal{SM}$.*

(ii) *If $G \in \mathcal{SM}$, then A and $B \in \mathcal{SM}$.*

Proof. (i) Assume that the result is false and let the group $G = AB$ be a counterexample with $|G|$ minimal, and subgroups A and B as in the statement. Hence $A \neq 1$ and $B \neq 1$. Let H be a subnormal subgroup of G of

minimal order such that H does not permute with some maximal subgroup of G, say M. We split the proof into the following steps:

Step 1. G is a primitive group of type 1 with $G = NM$, $\mathrm{Core}_G(M) = 1$, N the unique minimal normal subgroup of G, $C_G(N) = N$, and $N \cap M = 1$. Moreover, G is not a supersoluble group.

Assume that $C := \mathrm{Core}_G(M) \neq 1$. Since the factor group $G/C = (AC/C)(BC/C)$ satisfies the hypotheses of the result, the choice of G implies that G/C is an SM-group. Hence, $HM = (HC)M = M(HC) = MH$, a contradiction. Therefore, M is a maximal subgroup of G with $\mathrm{Core}_G(M) = 1$, and G is a primitive group. From Lemma 5, G is a primitive group of type either 1 or 3. If G were a primitive group of type 3, then Corollary 2 implies that $G = A \times B$ with $A \cong B$ nonabelian simple groups. By Theorem 2, G would be an SM-group, a contradiction. Therefore, G is a primitive group of type 1. If $G \in \mathcal{U}$, then G would be an SC-group and, consequently, an SM-group, a contradiction which proves Step 1.

Step 2. $H \leq N$.

If $1 < N \cap H < H$, the minimal choice of H implies that $(H \cap N)M = M(H \cap N)$. Since M is a maximal subgroup of G, we have that $G = (N \cap H)M$ and $G = HM$, a contradiction. Hence, either $N \cap H = H$ or $N \cap H = 1$. Assume that $N \cap H = 1$. We notice that N normalizes H, because H is a subnormal subgroup of G (cf. [25, A. Lemma (14.3)]). So, it follows that $[N, H] \leq N \cap H = 1$. Since $C_G(N) = N$, we have that $H \leq N$, a contradiction. Consequently, $N \cap H = H$, and $H \leq N$, as desired.

Step 3. Final contradiction.

From Steps 1, 2 and Lemma 4, we may assume w.l.o.g. that $H \leq N \leq A = N(M \cap A)$ and $M = (M \cap A)B$. By Lemma 3, B normalizes every subgroup of N. In particular, B permutes with H, and also N is a minimal subgroup of A, which implies that $M \cap A$ is a maximal subgroup of A. Since A is an SM-group, the subnormal subgroup H of A permutes with the maximal subgroup $M \cap A$ of A. Consequently, H permutes with $M = B(M \cap A)$, the final contradiction.

(ii) Assume that the result is false, and let the group $G = AB$ be a counterexample with $|G| + |A| + |B|$ minimal, where A and B are subgroups of G as in the statement. From Lemma 6 we have that $G/\phi(G)$ is an SC-group. If $\phi(G) = 1$, it follows from Theorem 2 that A and B are SC-groups and, consequently, A and B are SM-groups. Hence $\phi(G) \neq 1$. Let N be

a minimal normal subgroup of G such that $N \leq \phi(G)$. We notice that $G/N = (AN/N)(BN/N)$ satisfies the hypotheses of the result. The choice of G implies that AN/N and BN/N are SM-groups.

We claim that A is an SM-group. Using Corollary 1 we have that A either covers or avoids N. If $N \cap A = 1$, then A is an SM-group. Assume now that $N \leq A$. If $N \leq \phi(A)$, since A/N is an SM-group, it follows that $A/\phi(A)$ is an SC-group by Lemma 6. But, again Lemma 6 implies that A is an SM-group. Consider now the case that N is not contained in $\phi(A)$, and let M be a maximal subgroup of A such that $A = NM$. Since A and B are tcc-permutable subgroups and N is normal in G, we may assume that M permutes with B. Then $G = AB = NMB = MB$ is the product of the tcc-permutable subgroups M and B, because $N \leq \phi(G)$. The choice of (G, A, B) implies that M and B are SM-groups. On the other hand, $A = A \cap MB = M(A \cap B)$ is the product of the tcc-permutable subgroups M and $A \cap B$. By Corollary 3(i), $A \cap B$ is a supersoluble group. Consequently, $A \cap B$ is an SC-group and so also an SM-group. Hence, Part (i) implies now that A is an SM-group, and the claim is proved.

Analogously, B is an SM-group, which provides the final contradiction and concludes the proof. □

Now we consider the relationship between tcc-permutability and the classes $\mathcal{T}$, $\mathcal{PT}$ and $\mathcal{PST}$. We recall that these three classes are closed under taking quotient groups and normal subgroups.

Again as application of the cover-avoidance property in tcc-permutable products of groups, obtained in Corollary 1, we will show first that the property of being a T-group, a PT-group or a PST-group is inherited by the factors in a tcc-permutable product (Theorem 4 below). Corresponding results have been obtained for pairwise totally permutable products of subgroups ([19, Theorem C]; see also [8, Theorem B]) as well as for mutually permutable products of two factors ([18, Theorem 5]). In fact, in order to prove Theorem 4 it is possible to argue as in the proof of [18, Theorem 5], taking into account Corollary 1, and after stating, as in the referred paper, the results about the soluble radical and residual in Part (i) and first statement of Part (ii) of the next Proposition 3.

Regarding Part (i) of Proposition 3, the behaviour of the $\mathcal{F}$-residual in a tcc-permutable product, for a saturated formation $\mathcal{F}$ containing all supersoluble groups, has been studied in [6] (under the additional hypothesis that $\mathcal{F} \subseteq \mathcal{S}$) and [4] (in the general case). The required result here, for the soluble

residual, is a particular case of [6, Corollary 1.5] (see also [4, Corollary 7]) when considering the formation of all soluble groups.

On the other hand, a corresponding result to the first statement in the next Proposition 3 (ii), for mutually permutable products, has been obtained in [18, Theorem 4]. Taking into account [6, Lemma 2.3], Lemma 5 and Lemma 1, it is also possible to adapt the arguments in the proof of [18, Theorem 4], to prove that $X_{\mathcal{S}} = X \cap G_{\mathcal{S}}$ for $X \in \{A, B\}$ and a tcc-permutable product $G = AB$ of subgroups A and B.

It is observed in [18, Example] that in general $G_{\mathcal{S}} \neq A_{\mathcal{S}} B_{\mathcal{S}}$ in a mutually permutable product $G = AB$ of subgroups A and B. However, as proved next, this result holds for tcc-permutable products.

We state now the following result:

Proposition 3. *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then:*

(i) *([6, Corollary 1.5], [4, Corollary 7]) $A^{\mathcal{S}}$ and $B^{\mathcal{S}}$ are normal subgroups in G and $G^{\mathcal{S}} = A^{\mathcal{S}} B^{\mathcal{S}}$.*

(ii) *$A_{\mathcal{S}} = A \cap G_{\mathcal{S}}$, $B_{\mathcal{S}} = B \cap G_{\mathcal{S}}$, and $G_{\mathcal{S}} = A_{\mathcal{S}} B_{\mathcal{S}}$.*

Proof. By the previous comments, we need only to prove that $G_{\mathcal{S}} = A_{\mathcal{S}} B_{\mathcal{S}}$. Assume that this result is false and let the group $G = AB$ be a counterexample with $|G|$ minimal, and A and B subgroups as in the statement. By Lemma 1, we have that $A_{\mathcal{S}} B_{\mathcal{S}} = (A \cap G_{\mathcal{S}})(B \cap G_{\mathcal{S}})$ is a normal subgroup of G and $A_{\mathcal{S}} B_{\mathcal{S}} \leq G_{\mathcal{S}}$. Assume first that $A_{\mathcal{S}} B_{\mathcal{S}} \neq 1$. We notice that $G/A_{\mathcal{S}} B_{\mathcal{S}} = (AB_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})(BA_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})$ is the tcc-permutable product of the subgroups $AB_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}$ y $BA_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}$. Moreover, $(G/A_{\mathcal{S}} B_{\mathcal{S}})_{\mathcal{S}} = G_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}$. Whence, the choice of G implies that:

$$\begin{aligned} 1 \neq G_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}} &= (G/A_{\mathcal{S}} B_{\mathcal{S}})_{\mathcal{S}} = (AB_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})_{\mathcal{S}} (BA_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})_{\mathcal{S}} \\ &= ((G_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}) \cap (AB_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}))((G_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}) \cap (BA_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})) \\ &= ((G_{\mathcal{S}} \cap A)B_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}})((G_{\mathcal{S}} \cap B)A_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}) = A_{\mathcal{S}} B_{\mathcal{S}}/A_{\mathcal{S}} B_{\mathcal{S}}, \end{aligned}$$

a contradiction.

Hence, $A_{\mathcal{S}} B_{\mathcal{S}} = 1$ and $S := G_{\mathcal{S}} \neq 1$. Also, $A \neq 1$ and $B \neq 1$. We claim that AS is a proper subgroup of G. Assume that $AS = G$. We notice that $S < BS$, and so $BS = BS \cap G = BS \cap AS = (BS \cap A)S$, which implies that $BS \cap A \neq 1$. Since $[A, B] \leq S$, by [4, Theorem 4], we have that

$[BS \cap A, A] \leq S \cap A = A_{\mathcal{S}} = 1$, which implies that $1 \neq BS \cap A \leq Z(A) \leq A_{\mathcal{S}} = 1$, a contradiction which proves the claim.

Now, on the one hand, $AS = AS \cap G = AS \cap AB = A(AS \cap B)$ is the tcc-permutable product of the subgroups A and $AS \cap B$. The choice of G implies that $(AS)_{\mathcal{S}} = A_{\mathcal{S}}(AS \cap B)_{\mathcal{S}} = (AS \cap B)_{\mathcal{S}} \leq B$.

On the other hand, $S \leq (AS)_{\mathcal{S}} = S((AS)_{\mathcal{S}} \cap A) = SA_{\mathcal{S}} = S$.

Consequently, $S = (AS)_{\mathcal{S}} \leq B$, which implies $S \leq B_{\mathcal{S}} = 1$, the final contradiction which proves the result. □

Remark 2. 1. Example (iii) in Remark 1 shows that the soluble radicals of the factors in a tcc-permutable product are not necessarily normal subgroups in the whole group.

2. Totally permutable products in the context of Fitting classes have been studied in [28]-[30]. Particularly, a corresponding result to Proposition 3 (ii) for a general Fischer class containing all supersoluble groups, instead of the class $\mathcal{S}$ of soluble groups, and a pairwise totally permutable product of subgroups appears in [30, Theorem 1].

We still follow the notation introduced in [18], to prove the corresponding above-mentioned Theorem 5 there. Let $\Theta \in \{\mathcal{T}, \mathcal{PT}, \mathcal{PST}\}$. A subnormal subgroup H of a group G is said to be Θ-well embedded in G if

(a) H is a normal subgroup of G for $\Theta = \mathcal{T}$,

(b) H is a permutable subgroup in G for $\Theta = \mathcal{PT}$,

(c) H is an S-permutable subgroup in G for $\Theta = \mathcal{PST}$,

Consequently, the following result holds:

Theorem 4. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. If $G \in \Theta$, then A, $B \in \Theta$.*

Remark 3. The converse of Theorem 4 does not hold in general, not even if the group $G = AB$ is a product of totally permutable subgroups A and B, as stated in [19], as well as in [8]; the direct product of the symmetric group of degree three with a cyclic group of order three is an example. Though Theorem C in [8] shows that a positive result is possible for totally permutable products of two subgroups under the additional hypothesis that the indices of the factors are coprime. We prove next that a corresponding result for

tcc-permutable products still holds. Using the above notation we formulate and prove the following theorem. We notice that some arguments used here appear in [8, Theorem C].

Theorem 5. *Let the group $G = AB$ be the product of the tcc-permutable subgroups A and B. Assume that $(|G : A|, |G : B|) = 1$. If A, $B \in \Theta$, then $G \in \Theta$.*

Proof. Assume that the result is false and let $G = AB$ be a counterexample with $|G|$ is minimal and A and B tcc-permutable. Let H be a subnormal subgroup of G of minimal order such that H is not Θ-well embedded in G. We split the proof into the following steps:

Step 1. $\mathrm{Core}_G(H) = 1$ and H has a unique maximal normal subgroup.

Set $L = \mathrm{Core}_G(H)$ and assume that $L \neq 1$. Since the factor group $G/L = (AL/L)(BL/L)$ satisfies the hypotheses of the result, the choice of G implies that H/L is Θ-well embedded in G/L. Therefore, H is Θ-well embedded in G, a contradiction. Then $L = 1$. Let M_1 and M_2 be two maximal normal subgroups of H. The choice of H implies now that M_1 and M_2 are Θ-well embedded in G. Consequently, $H = M_1M_2$ is Θ-well embedded in G, a contradiction which completes the proof of Step 1.

Step 2. H is a soluble group.

Let $T = H^{\mathcal{S}}$ denote the soluble residual of H. We notice that A and B are SM-groups, and so is G, by Theorem 3. Therefore, Lemma 6 implies that T is a normal subgroup of G, because T is a perfect subnormal subgroup of G. Consequently, $T \leq \mathrm{Core}_G(H) = 1$ and H is a soluble group, as claimed.

Step 3. The final contradiction.

Now, we distinguish the following cases:

Case $\Theta = \mathcal{T}$. From Step 1, Step 2 and the minimal choice of H, we deduce that H is a p-group for some prime p. W.l.o.g. we may assume that p does not divide $|G : A|$, because $(|G : A|, |G : B|) = 1$. Hence, $H \leq O_p(G) \leq A$ and H is a normal subgroup of A, because A is a T-group. We claim that H is normalized by B. We notice first that $|B| = |A \cap B||G : A|$, and consequently a Sylow p-subgroup of $A \cap B$ is a Sylow p-subgroup of B which normalizes H. Moreover, by hypothesis, for each prime $q \neq p$ dividing the order of B, there exists a Sylow q-subgroup B_q of B which permutes with H. But then H is normalized by B_q since H is a subnormal Sylow p-subgroup

of HB_q, and the claim follows. Consequently, H is a normal subgroup of G, a contradiction.

Case $\Theta = \mathcal{PT}$ or $\Theta = \mathcal{PTS}$. Since H is not Θ-well embedded in G, we claim that there is a p-subgroup P of G, for some prime p, which does not permute with H, and which is a Sylow p-subgroup of G in the case $\Theta = \mathcal{PTS}$. If $\Theta = \mathcal{PT}$, there exists a subgroup W of G which does not permutes with H. Therefore, for some prime p, there exists a Sylow p-subgroup P of W such that P does not permute with H. The case $\Theta = \mathcal{PTS}$ is obvious, and the claim is proved. W.l.o.g. we may assume that p does not divide $|G : A|$. For any $g \in G$ it is well-known that $G = A^g B$, and also A^g and B are tcc-permutable subgroups ([6, Lemma 2.1]). Therefore, we may also assume that $P \leq A$. By Step 1 and Step 2, there exists a unique maximal normal subgroup M of H and H/M is a cyclic group of prime order, q say. The minimal choice of H implies that $MP = PM$. Assume first that q does not divide $|G : A|$. Then a Sylow q-subgroup A_q of A is a Sylow q-subgroup of G and $A_q \cap H$ is a Sylow q-subgroup of H, since H is subnormal in G. Consequently, $H = M(H \cap A)$. But $H \cap A$ is a subnormal subgroup of $A \in \Theta$, which implies that $H \cap A$ permutes with $P \leq A$, and so H permutes with P, a contradiction. Consequently, q divides $|G : A|$ and so q does not divide $|G : B|$. As before, if B_q is a Sylow q-subgroup of B, then $H_q := B_q \cap H$ is a Sylow q-subgroup of H, which is contained in B, and $H = MH_q$. Let N be a minimal normal subgroup of G. We observe that N is simple, because G is an SC-group by Theorem 2. Moreover, since N normalizes H and $\mathrm{Core}_G(H) = 1$ by Step 1, it follows that $[N, H] \leq N \cap H = 1$, i.e. N centralizes H. Now, the choice of G implies that HN/N is Θ-well embedded in G/N and so HNP is a subgroup of G. By hypothesis there exists $x \in \langle H_q, P\rangle \leq HNP$ such that $H_q^x P = P H_q^x$, where $x = hny$ with $h \in H$, $n \in N$ and $y \in P$. Then $H_q^h P = P H_q^h$, because N centralizes H. Therefore, P permutes $H = MH_q^h$, the final contradiction which concludes the proof. □

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