On a theorem of Kang and Liu on factorised groups

A. Ballester-Bolinches and M.C. Pedraza-Aguilera

Abstract

Kang and Liu ['On supersolvability of factorized finite groups', Bull. Math. Sci. 3 (2013), 205-210] investigate the structure of finite groups that are products of two supersoluble groups. The goal of this note is to give a correct proof of their main theorem.
Mathematics Subject Classification (2010): 20D10, 20D20, 20D40
Keywords: finite group, factorisations, supersolubility.

1 Introduction

All groups considered in this paper are finite.

We recall that two subgroups A and B of a group G are said to permute if AB is a subgroup of G. Further, A and B are called mutually permutable if every subgroup of A permutes with B and every subgroup of B permutes with A.

Products of mutually permutable subgroups have been widely studied in the last twenty-five years and receive a full discussion in [3]. The emphasis is on how the structure of the factors A and B affects the structure of the factorised group $G = AB$ and vice versa.

The goal of the present paper is to give a correct proof of the main result of the paper [5]. Therefore this paper had best be read in conjunction with [5].

First we recall the main theorem of that paper.

Theorem A ([5, Theorem C])**.** *Let the group $G = HK$ be the product of the subgroups H and K. Assume that H permutes with every maximal subgroup of K and K permutes with every maximal subgroup of H. If H is supersoluble, K is nilpotent and K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H, then G is supersoluble.*

The statement of the Theorem A resembles that of a theorem of Cossey and the authors [2, Theorem 3], which asserts the same conclusion under the stronger assumption that K permutes with *every* Sylow subgroup of H. The proof of Theorem A presented in [5] is not new, but it is an exact copy of the proof of [2, Theorem 3]. However, it is abundantly clear that this proof does not hold if K is δ-permutable in H. In fact, in the last paragraph of the proof (as in [2, Theorem 3]), the authors wrote:

"By hypothesis, K is δ-permutable in H, where δ is a complete set of Sylow subgroups of H. Hence we can easily deduce that K permutes with every Hall p'-subgroup of H".

There are many examples showing that this claim is false in general (consider, for example, the symmetric group of degree four which is a product of the alternating group and a transposition). We need K to be permutable with *all* Sylow subgroups of H to ensure that K permutes with every Hall p'-subgroup of H.

2 Proof of Theorem A

Proof. Assume the result is false and let G be a counterexample of minimal order. By [2, Theorem 1], G is soluble. Let $1 \neq N$ be a normal subgroup of G. It is clear that the hypotheses of the theorem hold in G/N. By minimality of G, G/N is supersoluble. Consequently, G has a unique minimal normal subgroup N which is abelian and complemented in G by a core-free maximal subgroup M of G. Let p be the prime dividing $|N|$ and let q be the largest prime dividing $|G|$.

Assume that $p \neq q$. Let H_q be a Sylow q-subgroup of H. Then H_q is a normal subgroup of H because H is supersoluble. Moreover K has a unique Sylow q-subgroup because K is nilpotent. Applying [1, Lemma 2.4.2], we see that H_q permutes with ${K_q}^g$ for each $g \in G$. Since $O_q(G) = 1$, it follows that $[{H_q}^G, {K_q}^G] = 1$ by [1, Lemma 2.5.1]. It is quite clear that we can assume that either ${H_q}^G \neq 1$ or ${K_q}^G \neq 1$ because, otherwise, G would be a q'-group.

Suppose that ${H_q}^G \neq 1$ (the case ${K_q}^G \neq 1$ is analogous). Then N is contained in ${H_q}^G$. Therefore $[N, {K_q}^G] = 1$ and ${K_q}^G \leq C_G(N) = N$. Hence ${K_q}^G = 1$ and K is a q'-group. Since every Sylow q-subgroup of M is a Sylow q-subgroup of G, we may assume that H_q is contained in M. Since M is supersoluble, it follows that H_q is normalised by M. If $G = N_G(H_q)$, then N is contained in H_q, which is a contradiction. Thus $M = N_G(H_q)$. This implies that H is contained in M. Therefore $M = H(M \cap K)$. Hence $M \cap K$ is a maximal subgroup of K. Applying [4, Lemma 2.3], we deduce that K is a Sylow q-subgroup of G with $|K| = q$. Moreover $H = M$ and $|G : H| = q$.

Then $N \leq M$, which is a contradiction.

Suppose now that p is the largest prime dividing $|G|$. Since M is supersoluble and $O_p(M) = 1$, we see that M is a p'-group and so N is a Sylow p-subgroup of G. In particular G is a Sylow tower group of supersoluble type. Let $K_{p'}$ be the Hall p'-subgroup of K. Assume that $(H \cap K)K_{p'}$ is a proper subgroup of K and let K_0 be a maximal subgroup of K containing $(H \cap K)K_{p'}$. Then HK_0 is a proper subgroup of G. Write $S = HK_0$. If $\mathrm{Core}_G(S) = 1$, then $S \cap N = 1$ and $G = SN$. Hence $|N| = |G : S| = |HK : HK_0| = |K : K_0| = p$, which is a contradiction. Suppose that $\mathrm{Core}_G(S) \neq 1$. Then N is contained in HK_0 and so N is a Sylow p-subgroup of HK_0. Since $H \cap K = H \cap K_0$, it follows that K_0 contains a Sylow p-subgroup of K. This contradiction shows that $K = (H \cap K)K_{p'}$ and so N is contained in H. In particular, $H = N(H \cap M)$.

Let $q \neq p$ be a prime and let H_q be a Sylow q-subgroup of H permuting with K. Then $X = KH_q = (H \cap K)K_{p'}H_q$ is a subgroup of G. Since X is a Sylow tower group of supersoluble type, it follows that the Sylow p-subgroup A of $H \cap K$ is normal in X. Hence A is normalised by H_q. This implies that A is a normal subgroup of H and so A is normal in G. Consequently $A = N$ or $A = 1$. Assume that $A = N$ so that $K = NK_{p'}$. Let N_1 denote a minimal normal subgroup of H with $N_1 \leq N$. Then $|N_1| = p$ and K normalises N_1. Therefore $N = N_1$, which is a contradiction. Thus we may assume that $A = 1$. In this case, K is contained in M, $M = K(M \cap H)$ and $M \cap H$ is a maximal subgroup of H. Therefore $p = |H : M \cap H| = |N|$, which is the final contradiction.

□

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A. Ballester-Bolinches
Departament de Matemàtiques, Universitat de València
Dr. Moliner 50, 46100 Burjassot, València (Spain)
e-mail: Adolfo.Ballester@uv.es

M.C. Pedraza-Aguilera
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València,
Camino de Vera, 46022, Valencia (Spain)
e-mail: mpedraza@mat.upv.es