On the maximum rank of totally nonnegative matrices ☆

Rafael Cantó[a,*], Ana M. Urbano[a]

[a] *Institut de Matemàtica Multidisciplinar, Universitat Politècnica de València, 46071 València, Spain.*

Abstract

Let $A \in \mathbb{R}^{n\times n}$ be a totally nonnegative matrix with principal rank p, that is, every minor of A is nonnegative and p is the size of the largest invertible principal submatrix of A. We introduce the sequence of the first p-indices of A as the first initial row and column indices of a $p\times p$ invertible principal submatrix of A with rank p. Then, we study the linear dependence relations between the rows and columns indexed by the sequence of the first p-indices of A and the remaining of its rows and columns. These relations, together with the irreducibility property of some submatrices of A, allow us to present an algorithm that calculates the maximum rank of A as a function of the distribution of the first p-indices. Finally, we present a method to construct $n \times n$ totally nonnegative matrices with given rank r, principal rank p and a specific sequence of the first p-indices.

Keywords: Totally nonnegative matrix, irreducible matrix, maximum rank, principal rank.
AMS classification: 15A03, 15A15, 65F40

1. Introduction

A matrix $A \in \mathbb{R}^{n\times n}$ is called totally nonnegative if all its minors are nonnegative and it is abbreviated as TN, see for instance [1]-[5]. The TN matrices have been studied by several authors due to its wide variety of applications in algebra, geometry, differential equations, economics, and others fields.

In general, given a matrix A the *principal rank* of A, denoted by p-rank(A), is the size of the largest invertible principal submatrix of A. In the class of TN matrices the principal rank provides important information about some properties of these matrices. For example, it is known that the principal rank of a TN matrix A is the number of positive eigenvalues and $n - p$ is the algebraic multiplicity of its zero eigenvalue.

☆This research was supported by the Spanish DGI grants MTM2013-43678-P and MTM2017-85669-P. e-mail: {rcanto,amurbano}@mat.upv.es
*Corresponding author

Recall that a matrix A is an *irreducible* matrix if there is no permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ O & D \end{bmatrix},$$

where O is an $(n-r) \times r$ zero matrix $(1 \leq r \leq n-1)$. If A is an irreducible TN matrix from now one we abbreviate it by IrTN matrix following the notation of [5], where one topic of interest for the authors is characterizing all the triples $(n, \text{rank}(A), p\text{-rank}(A))$, where n is the size of matrix A. We recall that a triple (n, r, p) is realizable if there exists an IrTN matrix $A \in \mathbb{R}^{n \times n}$ with $\text{rank}(A) = r$ and $p\text{-rank}(A) = p$.

For irreducible matrices Fallat, Gekhtman and Johnson [3] prove the following characterization.

Lemma 1 (Lemma 2.2 of [3]). *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be a TN matrix. Then A is irreducible if and only if $a_{ij} > 0$ for all i, j such that $|i - j| \leq 1$.*

It is known that there exists a relation between the order n of an IrTN matrix, its rank r and its principal rank p. By [4, Theorem 11] we have

Lemma 2. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an IrTN matrix with $p\text{-rank}(A) = p$ and $\text{rank}(A) = r$. Then*

$$p \leq r \leq n - \left\lceil \frac{n-p}{p} \right\rceil. \tag{1}$$

The concept of principal rank is useful in studying the dependence relations between rows and columns of an IrTN matrix. In this case, it is interesting to obtain the first principal submatrix $\bar{A}$ of A such that $\text{rank}(\bar{A}) = p\text{-rank}(A) = p$. The following definitions introduce the sequence of the first p-indices of linearly independent rows and columns of A.

Definition 1. *Let $A \in \mathbb{R}^{n \times n}$ be a matrix with $p\text{-rank}(A) = p$. We say that the sequence of integers $\alpha = \{i_1, i_2, \ldots, i_p\} \in \mathcal{Q}_{p,n}$ is the* sequence of the first p-indices of A *if for $j = 2, \ldots, p$ we have*

$$\det(A[i_1, i_2, \ldots, i_{j-1}, i_j]) \neq 0,$$
$$\det(A[i_1, i_2, \ldots, i_{j-1}, t]) = 0, \quad i_{j-1} < t < i_j.$$

We follow the notation of [1], that is, for $p \in \{1, 2, \ldots, n\}$, $\mathcal{Q}_{p,n}$ denotes the totality of strictly increasing sequences of p integers chosen from $\{1, 2, \ldots, n\}$, if A is an $m \times n$ matrix, $\alpha \in \mathcal{Q}_{k,m}$, $\beta \in \mathcal{Q}_{l,n}$ then $A[\alpha|\beta]$ is by definition the $k \times l$ submatrix of A lying in the rows numbered by α and columns numbered by β. Besides $A[\alpha] := A[\alpha|\alpha]$.

Note that if A is TN matrix without null rows or columns, then $i_1 = 1$. Taking into account this sequence, in Section 2 we study some linear dependence relations between rows or columns of a TN matrix A. From these relations we will transform A by similarity into an upper block matrix B. This matrix is not IrTN but allows us to study properties about the rank of some powers of A and we prove that the maximum rank of A associated with a realizable triple (n, r, p) can be strictly less than the upper bound of (1). In Section 3 an algorithm computes the maximum rank that A can reach when the sequence of its first p-indices is known. This fact leads us to give the following new definition of realizable triple, which generalizes the concept of triple realizable given by Fallat and Johnson in [5].

Definition 2. *A triple (n, r, p) is called $(1, i_2, \ldots, i_p)$-realizable if there exists an IrTN matrix $A \in \mathbb{R}^{n \times n}$ with $\operatorname{rank}(A) = r$, $p\text{-rank}(A) = p$, and $\{1, i_2, \ldots, i_p\}$ is the sequence of the first p-indices of A.*

If a matrix A satisfies the conditions of Definition 2, then we say that A is a matrix associated with the triple (n, r, p) $(1, i_2, \ldots, i_p)$-realizable.

In Section 4 we present a procedure to construct an IrTN matrix associated with a triple (n, r, p) $(1, i_2, \ldots, i_p)$-realizable. That procedure allows us to obtain an IrTN matrix associated with a triple (n, r, p) realizable.

2. Linear dependence relations between columns or rows of TN matrices

In this section we study some linear dependence relations between columns or rows of a TN matrix $A \in \mathbb{R}^{n \times n}$ with $p\text{-rank}(A) = p$. For these dependency relations we only need the irreducibility of one of the principal submatrices of A instead of the irreducibility of A. Applying the obtained results we can transform A by similarity into a block upper triangular matrix B, which is not a TN matrix but it allows us to prove easily the result given by Fallat and Gekhtman in [4, Theorem 10], that they prove by using a combinatorial approach based on the study of weighted planar diagrams associated with TN matrices.

Proposition 1. *Let $A \in \mathbb{R}^{n \times n}$ be a TN matrix with $p\text{-rank}(A) = p$ and let $\{1, 2, \ldots, p\}$ be the sequence of the first p-indices of A. If the principal submatrix $A[p, p+1, \ldots, n]$ is irreducible, then $\operatorname{rank}(A) = p$.*

Proof. If rows $p+1, p+2, \ldots, n$ are linear combination of the first p rows of A, then $\operatorname{rank}(A) = p$. In other case, there exists at least one row from $p+1$ to n that is not a linear combination of the first p rows. Suppose that the first row linear independent is the $(k+1)$-th row, with $k+1 \geq p+1$. Applying p steps of the Neville elimination method we obtain $A = L_p U_p$, where U_p is the following TN matrix

$$U_p = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1p} & u_{1,p+1} & \cdots & u_{1,k} & u_{1,k+1} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2p} & u_{2,p+1} & \cdots & u_{2,k} & u_{2,k+1} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{pp} & u_{p,p+1} & \cdots & u_{p,k} & u_{p,k+1} & \cdots & u_{pn} \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & u_{k+1,p+1} & \cdots & u_{k+1,k} & u_{k+1,k+1} & \cdots & u_{kn} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & u_{n,p+1} & \cdots & u_{n,k} & u_{n,k+1} & \cdots & u_{nn} \end{bmatrix}$$

with $u_{ii} > 0$, for $i = 1, 2, \ldots, p$, because $\{1, 2, \ldots, p\}$ is the sequence of the first p-indices of A. We know that $u_{k+1,k+1} = 0$. In the other case

$$\begin{aligned} &\det(A[1,2,3,\ldots,p,k+1]) = \\ &= \textstyle\sum_{\gamma\in\mathcal{Q}_{p+1,n}} \det(L_p[1,2,3,\ldots,p,k+1|\gamma]) \det(U_p[\gamma|1,2,3,\ldots,p,k+1]) \\ &= \det(L_p[1,2,\ldots,p,k+1]) \det(U_p[1,2,\ldots,p,k+1]) \\ &= u_{11}u_{22}\cdots u_{pp}u_{k+1,k+1} \neq 0, \end{aligned}$$

which contradicts that p-rank$(A) = p$.

Since $A[p, p+1, \ldots, n]$ is irreducible then $a_{k,k+1} \neq 0$. To obtain matrix U_p, by applying p steps of the Neville elimination method, it is necessary that there exists, at least, an entry $a_{j,k+1} \neq 0$, with $j = 1, 2, \ldots, p$. This implies that there exists, at least, an entry $u_{r,k+1} \neq 0$, with $1 \leq r \leq p$. Since U_p es TN we have that $u_{p,k+1} \neq 0$ and therefore $u_{k+1,g} = 0$, for $g = p+1, p+2, \ldots, k$.

Now, if there exists j, $k+1 < j \leq n$, such that $u_{k+1,j} > 0$, since p-rank$(A) = p$ we have that

$$\begin{aligned} &\det(A[1,2,3,\ldots,p,j]) = \\ &= \textstyle\sum_{\gamma\in\mathcal{Q}_{p+1,n}} \det(L_p[1,2,3,\ldots,p,j|\gamma]) \det(U_p[\gamma|1,2,3,\ldots,p,j]) \\ &= \det(L_p[1,2,\ldots,p,j|1,2,\ldots,p,k+1]) \det(U_p[1,2,\ldots,p,k+1|1,2,\ldots,p,j]) \\ &+ \textstyle\sum_{\gamma\in\mathcal{Q}_{p+1,n}\sim\{1,2,\ldots,p,k+1\}} \det(L_p[1,2,3,\ldots,p,j|\gamma]) \det(U_p[\gamma|1,2,3,\ldots,p,j]) \\ &= l_{j,k+1}u_{11}u_{22}\cdots u_{pp}u_{k+1,j} + S = 0, \end{aligned}$$

which implies that $S = 0$ and $l_{j,k+1} = 0$. Using that L_p is TN, we have that $l_{it} = 0$, for $i = j, j+1, \ldots, n$, $t = 1, 2, \ldots k+1$. Furthermore, since U_p is TN we obtain that $u_{hg} = 0$, for $h = k+1, k+2, \ldots, n$, $g = p+1, p+2, \ldots, j-1$.

Hence,

$$a_{j,j-1} = l^{(j)} u_{j-1} = [0\ 0\ \ldots\ 0\ l_{j,k+2}\ \ldots\ l_{j,j-1}\ 1\ 0\ \ldots\ 0] \begin{bmatrix} u_{j-1,1} \\ \vdots \\ u_{j-1,k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0,$$

which contradicts that $A[p, p+1, \ldots, n]$ is irreducible. Therefore, the $(k+1)$-th row is a linear combination of the first p rows of A. Using a similar argument we obtain that rows $k+2, k+3, \ldots, n$ will be linear combination of the first p rows of A. Then $\operatorname{rank}(A) = p$. $\square$

Remark 1. 1. *In Proposition 1, A does not need to be irreducible, but the principal submatrix $A[p, p+1, \ldots, n]$ must be irreducible as we can see with the following TN matrix*

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with $\operatorname{rank}(A) = 4$, p-$\operatorname{rank}(A) = 2$ and $\{1, 2\}$ as the sequence of the first 2-indices of A. If the principal submatrix $A[2, 3, 4, 5]$ were irreducible then by Proposition 1 we would obtain that $\operatorname{rank}(A) = 2$ and this is not true.

2. *By Proposition 1, we can describe an easy method to obtain an IrTN matrix $A \in \mathbb{R}^{n \times n}$ with $\operatorname{rank}(A) = p$-$\operatorname{rank}(A) = p$, for all p with $p = 1, 2, \ldots, n$.*
   (a) *First, we construct an IrTN matrix $A_p \in \mathbb{R}^{p \times p}$ with $\operatorname{rank}(A) = p$. For instance, let*

$$A_p = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 \\ 1 & 2 & 3 & \cdots & 3 & 3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & 3 & \cdots & p-1 & p-1 \\ 1 & 2 & 3 & \cdots & p-1 & p \end{bmatrix},$$

   or A_p can be the Vandermonde matrix corresponding to the first p positive integers.

(b) *Now, $A \in \mathbb{R}^{n\times n}$ is obtained as follows*

$$A = \left[\begin{array}{c|cccc} A_p & A_p(:,p) & A_p(:,p) & \cdots & A_p(:,p) \\ \hline A_p(p,:) & A_p(p,p) & A_p(p,p) & \cdots & A_p(p,p) \\ A_p(p,:) & A_p(p,p) & A_p(p,p) & \cdots & A_p(p,p) \\ \vdots & \vdots & \vdots & & \vdots \\ A_p(p,:) & A_p(p,p) & A_p(p,p) & \cdots & A_p(p,p) \end{array}\right],$$

where $A_p(:,p)$ denotes the last column of A_p and $A_p(p,:)$ denotes its last row.

In general, the principal rank of A is not obtained with its first p rows and columns, that is, $\{1,2,\dots,p\}$ is not always the sequence of the first p-indices of A. Nevertheless, we can also obtain conditions of linear dependence between certain rows or columns of A as the following result proves.

Proposition 2. *Let $A \in \mathbb{R}^{n\times n}$ be a TN matrix and let $\{1,2,\dots,q,q+t\}$ be the sequence of the first $q+1$-indices of A, $1 \le q < n-1$ and $1 < t$. If the submatrix $A[q,q+1,\dots,n]$ is irreducible, then each row (or column) $q+1,q+2,\dots,q+t-1$ is a linear combination of the first q rows (or columns) of A.*

Proof. Since the principal submatrix $\bar{A} = A[1,2,\dots,q,q+1,\dots,q+t-1]$, whose principal rank is q, satisfies Proposition 1 we have that each row and column indexed by $q+1,q+2,\dots,q+t-1$ is a linear combination of its first q rows and columns, respectively. As a consequence, applying q iterations of the Neville elimination process to matrix A we obtain $A = L_qU_q$, where U_q is the following TN matrix

$$U_q = \left[\begin{array}{cccc|ccc|ccc} u_{11} & u_{12} & \cdots & u_{1q} & u_{1,q+1} & \cdots & u_{1,q+t-1} & u_{1,q+t} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2q} & u_{2,q+1} & \cdots & u_{2,q+t-1} & u_{2,q+t} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{qq} & u_{q,q+1} & \cdots & u_{q,q+t-1} & u_{q,q+t} & \cdots & u_{qn} \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & u_{q+1,q+t} & \cdots & u_{q+1,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & u_{q+t-1,q+t} & \cdots & u_{q+t-1,n} \\ \hline 0 & 0 & \cdots & 0 & u_{q+t,q+1} & \cdots & u_{q+t,q+t-1} & u_{q+t,q+t} & \cdots & u_{q+t,n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & u_{n,q+1} & \cdots & u_{n,q+t-1} & u_{n,q+t} & \cdots & u_{nn} \end{array}\right]$$

Since $\det(A[1,2,\dots,q,q+t]) > 0$, then $\det(U_q[1,2,\dots,q,q+t]) > 0$. Therefore, there exits an index j, $q+1 \le j \le q+t$ such that $u_{j,q+t} \neq 0$.

If $j < q+t$, since U_q is a TN matrix, then $u_{sh} = 0$, for $s = q+t, q+t+1, \dots, n$ and $h = q+1, q+2, \dots, q+t-1$. Thus, columns $q+1, q+2, \dots, q+t-1$ are linear combination of the first q columns of A.

Otherwise, if $j = q + t$, since U_p is TN, then $u_{hg} = 0$, for $h = q+1, q+2, \dots, q+t-1$ and $h = q+t+1, q+t+2, \dots, n$. In this case, each row $q+1, q+2, \dots, q+t-1$ is a linear combination of the first q rows. $\square$

Remark 2. *Note that in Proposition 2 we would need that A to be IrTN only when $i_1 = 1$ and $i_2 > 2$, i.e., when* $\det(A[1,2]) = 0$.

The following example shows the linear dependence structure of rows and columns of an IrTN matrix A with a given principal rank applying Propositions 1 and 2.

Example 1. Let $A \in \mathbb{R}^{11\times 11}$ be the following IrTN matrix with p-rank$(A) = 3$ and $\{i_1 = 1,\ i_2 = 5,\ i_3 = 8\}$ be the sequence of the first 3-indices of A,

$$A = \begin{bmatrix} \boxed{\mathbf{a}_{11}} & a_{12} & a_{13} & a_{14} & \boxed{\mathbf{a}_{15}} & a_{16} & a_{17} & \boxed{\mathbf{a}_{18}} & a_{19} & a_{1,10} & a_{1,11} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{2,10} & a_{2,11} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{3,10} & a_{3,11} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{4,10} & a_{4,11} \\ \boxed{\mathbf{a}_{51}} & a_{52} & a_{53} & a_{54} & \boxed{\mathbf{a}_{55}} & a_{56} & a_{57} & \boxed{\mathbf{a}_{58}} & a_{59} & a_{5,10} & a_{5,11} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{6,10} & a_{6,11} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} & a_{7,11} \\ \boxed{\mathbf{a}_{81}} & a_{82} & a_{83} & a_{84} & \boxed{\mathbf{a}_{85}} & a_{86} & a_{87} & \boxed{\mathbf{a}_{88}} & a_{89} & a_{8,10} & a_{8,11} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{9,10} & a_{9,11} \\ a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} & a_{10,11} \\ a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & a_{11,11} \end{bmatrix}$$

We study the linear dependence with respect to rows and columns 1, 5 and 8.

• First, consider the TN submatrix $A_3 = A[1,5,8,9,10,11]$. Note that, p-rank$(A_3) = 3$ and $\{1,2,3\}$ is the sequence of the first 3-indices of A_3. Since $A_3[3,4,5,6] = A[8,9,10,11]$ is irreducible, by Proposition 1 we have that rank$(A_3) = 3$ and rows and columns 4, 5 and 6 are linear combination of the first 3 rows and columns of A_3. We represent this fact in the following form

$$A_3 = A[1,5,8,9,10,11] = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{15} & \mathbf{a}_{18} & \color{green}{a_{19}} & \color{green}{a_{1,10}} & \color{green}{a_{1,11}} \\ \mathbf{a}_{51} & \mathbf{a}_{55} & \mathbf{a}_{58} & \color{green}{a_{59}} & \color{green}{a_{5,10}} & \color{green}{a_{5,11}} \\ \mathbf{a}_{81} & \mathbf{a}_{85} & \mathbf{a}_{88} & \color{green}{a_{89}} & \color{green}{a_{8,10}} & \color{green}{a_{8,11}} \\ \color{green}{a_{91}} & \color{green}{a_{9,5}} & \color{green}{a_{9,8}} & \color{green}{a_{99}} & \color{green}{a_{9,10}} & \color{green}{a_{9,11}} \\ \color{green}{a_{10,1}} & \color{green}{a_{10,5}} & \color{green}{a_{10,8}} & \color{green}{a_{10,9}} & \color{green}{a_{10,10}} & \color{green}{a_{10,11}} \\ \color{green}{a_{11,1}} & \color{green}{a_{11,5}} & \color{green}{a_{11,8}} & \color{green}{a_{11,9}} & \color{green}{a_{11,10}} & \color{green}{a_{11,11}} \end{bmatrix}$$

where the green color indicates the linear dependence with respect to the first 3 rows and columns of A_3.

• Second, consider the TN matrix $A_2 = A[1,5,6,7,8,9,10,11]$. Note that $\{1,2,5\}$ is the sequence of the first 3-indices of A_2 and the submatrix $A_2[2,3,\dots,8] = A[5,6,\dots,11]$ is irreducible, therefore A_2 satisfies Proposition 2.

Then, we have two possibilities:

1. Rows 3 and 4 are linear combination of the first 2 rows of A_2. That is,

$$
\begin{aligned}
A_{2_r} &= A[1,5,6,7,8,9,10,11] \\
&= \left[\begin{array}{cc|cc|c|ccc}
\mathbf{a}_{11} & \mathbf{a}_{15} & {\color{green}a_{16}} & {\color{green}a_{17}} & \mathbf{a}_{18} & {\color{green}a_{19}} & {\color{green}a_{1,10}} & {\color{green}a_{1,11}} \\
\mathbf{a}_{51} & \mathbf{a}_{55} & {\color{green}a_{56}} & {\color{green}a_{57}} & \mathbf{a}_{58} & {\color{green}a_{59}} & {\color{green}a_{5,10}} & {\color{green}a_{5,11}} \\ \hline
{\color{green}a_{61}} & {\color{green}a_{65}} & {\color{green}a_{66}} & {\color{green}a_{67}} & {\color{green}a_{68}} & {\color{green}a_{69}} & {\color{green}a_{6,10}} & {\color{green}a_{6,11}} \\
{\color{green}a_{71}} & {\color{green}a_{75}} & {\color{green}a_{76}} & {\color{green}a_{77}} & {\color{green}a_{78}} & {\color{green}a_{79}} & {\color{green}a_{7,10}} & {\color{green}a_{7,11}} \\ \hline
\mathbf{a}_{81} & \mathbf{a}_{85} & a_{86} & a_{87} & \mathbf{a}_{88} & {\color{green}a_{89}} & {\color{green}a_{8,10}} & {\color{green}a_{8,11}} \\
{\color{green}a_{91}} & {\color{green}a_{95}} & {\color{red}a_{96}} & {\color{red}a_{97}} & {\color{green}a_{98}} & {\color{green}a_{99}} & {\color{green}a_{9,10}} & {\color{green}a_{9,11}} \\
{\color{green}a_{10,1}} & {\color{green}a_{10,5}} & {\color{red}a_{10,6}} & {\color{red}a_{10,7}} & {\color{green}a_{10,8}} & {\color{green}a_{10,9}} & {\color{green}a_{10,10}} & {\color{green}a_{10,11}} \\
{\color{green}a_{11,1}} & {\color{green}a_{11,5}} & {\color{red}a_{11,6}} & {\color{red}a_{11,7}} & {\color{green}a_{11,8}} & {\color{green}a_{11,9}} & {\color{green}a_{11,10}} & {\color{green}a_{11,11}}
\end{array}\right]
\end{aligned}
$$

2. Columns 3 and 4 are linear combination of the first 2 columns of A_2. In this case

$$
\begin{aligned}
A_{2_c} &= A[1,5,6,7,8,9,10,11] \\
&= \left[\begin{array}{cc|cc|c|ccc}
\mathbf{a}_{11} & \mathbf{a}_{15} & {\color{green}a_{16}} & {\color{green}a_{17}} & \mathbf{a}_{18} & {\color{green}a_{19}} & {\color{green}a_{1,10}} & {\color{green}a_{1,11}} \\
\mathbf{a}_{51} & \mathbf{a}_{55} & {\color{green}a_{56}} & {\color{green}a_{57}} & \mathbf{a}_{58} & {\color{green}a_{59}} & {\color{green}a_{5,10}} & {\color{green}a_{5,11}} \\ \hline
{\color{green}a_{61}} & {\color{green}a_{65}} & {\color{green}a_{66}} & {\color{green}a_{67}} & a_{68} & {\color{red}a_{69}} & {\color{red}a_{6,10}} & {\color{red}a_{6,11}} \\
{\color{green}a_{71}} & {\color{green}a_{75}} & {\color{green}a_{76}} & {\color{green}a_{77}} & a_{78} & {\color{red}a_{79}} & {\color{red}a_{7,10}} & {\color{red}a_{7,11}} \\ \hline
\mathbf{a}_{81} & \mathbf{a}_{85} & {\color{green}a_{86}} & {\color{green}a_{87}} & \mathbf{a}_{88} & {\color{green}a_{89}} & {\color{green}a_{8,10}} & {\color{green}a_{8,11}} \\
{\color{green}a_{91}} & {\color{green}a_{95}} & {\color{green}a_{96}} & {\color{green}a_{97}} & {\color{green}a_{98}} & {\color{green}a_{99}} & {\color{green}a_{9,10}} & {\color{green}a_{9,11}} \\
{\color{green}a_{10,1}} & {\color{green}a_{10,5}} & {\color{green}a_{10,6}} & {\color{green}a_{10,7}} & {\color{green}a_{10,8}} & {\color{green}a_{10,9}} & {\color{green}a_{10,10}} & {\color{green}a_{10,11}} \\
{\color{green}a_{11,1}} & {\color{green}a_{11,5}} & {\color{green}a_{11,6}} & {\color{green}a_{11,7}} & {\color{green}a_{11,8}} & {\color{green}a_{11,9}} & {\color{green}a_{11,10}} & {\color{green}a_{11,11}}
\end{array}\right]
\end{aligned}
$$

The green color indicates the linear dependence with respect to the previous rows or columns, and the red color entries represent independent variables.

• Finally, the submatrix $A_1 = A$ is IrTN and satisfies Proposition 2 because $\det(A_1[1]) > 0$, $\det(A_1[1,j]) = 0$, for $j = 2,3,4$, and $\det(A_1[1,5]) > 0$, then rows or columns $2,3$ and 4 are linear combination of the first 2 rows or columns of A_1, respectively. Therefore, depending on whether the rows or columns are linearly dependent and if we start with the matrix A_{2_r} or A_{2_c}, we have the following four matrices:

(1) $A_{1_{rr}} =$

$$
\left[\begin{array}{c|ccc|c|cc|c|ccc}
\mathbf{a}_{11} & {\color{green}a_{12}} & {\color{green}a_{13}} & {\color{green}a_{14}} & \mathbf{a}_{15} & {\color{green}a_{16}} & {\color{green}a_{17}} & \mathbf{a}_{18} & {\color{green}a_{19}} & {\color{green}a_{1,10}} & {\color{green}a_{1,11}} \\ \hline
{\color{green}a_{21}} & {\color{green}a_{22}} & {\color{green}a_{23}} & {\color{green}a_{24}} & {\color{green}a_{25}} & {\color{green}a_{26}} & {\color{green}a_{27}} & {\color{green}a_{28}} & {\color{green}a_{29}} & {\color{green}a_{2,10}} & {\color{green}a_{2,11}} \\
{\color{green}a_{31}} & {\color{green}a_{32}} & {\color{green}a_{33}} & {\color{green}a_{34}} & {\color{green}a_{35}} & {\color{green}a_{36}} & {\color{green}a_{37}} & {\color{green}a_{38}} & {\color{green}a_{39}} & {\color{green}a_{3,10}} & {\color{green}a_{3,11}} \\
{\color{green}a_{41}} & {\color{green}a_{42}} & {\color{green}a_{43}} & {\color{green}a_{44}} & {\color{green}a_{45}} & {\color{green}a_{46}} & {\color{green}a_{47}} & {\color{green}a_{48}} & {\color{green}a_{49}} & {\color{green}a_{4,10}} & {\color{green}a_{4,11}} \\ \hline
\mathbf{a}_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a}_{55} & {\color{green}a_{56}} & {\color{green}a_{57}} & \mathbf{a}_{58} & {\color{green}a_{59}} & {\color{green}a_{5,10}} & {\color{green}a_{5,11}} \\ \hline
{\color{green}a_{61}} & {\color{red}a_{62}} & {\color{red}a_{63}} & {\color{red}a_{64}} & {\color{green}a_{65}} & {\color{green}a_{66}} & {\color{green}a_{67}} & {\color{green}a_{68}} & {\color{green}a_{69}} & {\color{green}a_{6,10}} & {\color{green}a_{6,11}} \\
{\color{green}a_{71}} & {\color{red}a_{72}} & {\color{red}a_{73}} & {\color{red}a_{74}} & {\color{green}a_{75}} & {\color{green}a_{76}} & {\color{green}a_{77}} & {\color{green}a_{78}} & {\color{green}a_{79}} & {\color{green}a_{7,10}} & {\color{green}a_{7,11}} \\ \hline
\mathbf{a}_{81} & a_{82} & a_{83} & a_{84} & \mathbf{a}_{85} & a_{86} & a_{87} & \mathbf{a}_{88} & {\color{green}a_{89}} & {\color{green}a_{8,10}} & {\color{green}a_{8,11}} \\ \hline
{\color{green}a_{91}} & {\color{red}a_{92}} & {\color{red}a_{93}} & {\color{red}a_{94}} & {\color{green}a_{95}} & {\color{red}a_{96}} & {\color{red}a_{97}} & {\color{green}a_{98}} & {\color{green}a_{99}} & {\color{green}a_{9,10}} & {\color{green}a_{9,11}} \\
{\color{green}a_{10,1}} & {\color{red}a_{10,2}} & {\color{red}a_{10,3}} & {\color{red}a_{10,4}} & {\color{green}a_{10,5}} & {\color{red}a_{10,6}} & {\color{red}a_{10,7}} & {\color{green}a_{10,8}} & {\color{green}a_{10,9}} & {\color{green}a_{10,10}} & {\color{green}a_{10,11}} \\
{\color{green}a_{11,1}} & {\color{red}a_{11,2}} & {\color{red}a_{11,3}} & {\color{red}a_{11,4}} & {\color{green}a_{11,5}} & {\color{red}a_{11,6}} & {\color{red}a_{11,7}} & {\color{green}a_{11,8}} & {\color{green}a_{11,9}} & {\color{green}a_{11,10}} & {\color{green}a_{11,11}}
\end{array}\right]
$$

(2) $A_{1_{rc}} =$

$$\left[\begin{array}{c|ccc|c|cc|c|ccc}
\mathbf{a}_{11} & a_{12} & a_{13} & a_{14} & \mathbf{a}_{15} & a_{16} & a_{17} & \mathbf{a}_{18} & a_{19} & a_{1,10} & a_{1,11} \\ \hline
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & {\color{red}a_{26}} & {\color{red}a_{27}} & a_{28} & {\color{red}a_{29}} & {\color{red}a_{2,10}} & {\color{red}a_{2,11}} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & {\color{red}a_{36}} & {\color{red}a_{37}} & a_{38} & {\color{red}a_{39}} & {\color{red}a_{3,10}} & {\color{red}a_{3,11}} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & {\color{red}a_{46}} & {\color{red}a_{47}} & a_{48} & {\color{red}a_{49}} & {\color{red}a_{4,10}} & {\color{red}a_{4,11}} \\ \hline
\mathbf{a}_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a}_{55} & a_{56} & a_{57} & \mathbf{a}_{58} & a_{59} & a_{5,10} & a_{5,11} \\ \hline
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{6,10} & a_{6,11} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} & a_{7,11} \\ \hline
\mathbf{a}_{81} & a_{82} & a_{83} & a_{84} & \mathbf{a}_{85} & a_{86} & a_{87} & \mathbf{a}_{88} & a_{89} & a_{8,10} & a_{8,11} \\ \hline
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & {\color{red}a_{96}} & {\color{red}a_{97}} & a_{98} & a_{99} & a_{9,10} & a_{9,11} \\
a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & {\color{red}a_{10,6}} & {\color{red}a_{10,7}} & a_{10,8} & a_{10,9} & a_{10,10} & a_{10,11} \\
a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & {\color{red}a_{11,6}} & {\color{red}a_{11,7}} & a_{11,8} & a_{11,9} & a_{11,10} & a_{11,11}
\end{array}\right]$$

(3) $A_{1_{cr}} =$

$$\left[\begin{array}{c|ccc|c|cc|c|ccc}
\mathbf{a}_{11} & a_{12} & a_{13} & a_{14} & \mathbf{a}_{15} & a_{16} & a_{17} & \mathbf{a}_{18} & a_{19} & a_{1,10} & a_{1,11} \\ \hline
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{2,10} & a_{2,11} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} & a_{3,10} & a_{3,11} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{4,10} & a_{4,11} \\ \hline
\mathbf{a}_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a}_{55} & a_{56} & a_{57} & \mathbf{a}_{58} & a_{59} & a_{5,10} & a_{5,11} \\ \hline
a_{61} & {\color{red}a_{62}} & {\color{red}a_{63}} & {\color{red}a_{64}} & a_{65} & a_{66} & a_{67} & a_{68} & {\color{red}a_{69}} & {\color{red}a_{6,10}} & {\color{red}a_{6,11}} \\
a_{71} & {\color{red}a_{72}} & {\color{red}a_{73}} & {\color{red}a_{74}} & a_{75} & a_{76} & a_{77} & a_{78} & {\color{red}a_{79}} & {\color{red}a_{7,10}} & {\color{red}a_{7,11}} \\ \hline
\mathbf{a}_{81} & a_{82} & a_{83} & a_{84} & \mathbf{a}_{85} & a_{86} & a_{87} & \mathbf{a}_{88} & a_{89} & a_{8,10} & a_{8,11} \\ \hline
a_{91} & {\color{red}a_{92}} & {\color{red}a_{93}} & {\color{red}a_{94}} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{9,10} & a_{9,11} \\
a_{10,1} & {\color{red}a_{10,2}} & {\color{red}a_{10,3}} & {\color{red}a_{10,4}} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} & a_{10,11} \\
a_{11,1} & {\color{red}a_{11,2}} & {\color{red}a_{11,3}} & {\color{red}a_{11,4}} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & a_{11,11}
\end{array}\right]$$

(4) $A_{1_{cc}} =$

$$\left[\begin{array}{c|ccc|c|cc|c|ccc}
\mathbf{a}_{11} & a_{12} & a_{13} & a_{14} & \mathbf{a}_{15} & a_{16} & a_{17} & \mathbf{a}_{18} & a_{19} & a_{1,10} & a_{1,11} \\ \hline
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & {\color{red}a_{26}} & {\color{red}a_{27}} & a_{28} & {\color{red}a_{29}} & {\color{red}a_{2,10}} & {\color{red}a_{2,11}} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & {\color{red}a_{36}} & {\color{red}a_{37}} & a_{38} & {\color{red}a_{39}} & {\color{red}a_{3,10}} & {\color{red}a_{3,11}} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & {\color{red}a_{46}} & {\color{red}a_{47}} & a_{48} & {\color{red}a_{49}} & {\color{red}a_{4,10}} & {\color{red}a_{4,11}} \\ \hline
\mathbf{a}_{51} & a_{52} & a_{53} & a_{54} & \mathbf{a}_{55} & a_{56} & a_{57} & \mathbf{a}_{58} & a_{59} & a_{5,10} & a_{5,11} \\ \hline
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & {\color{red}a_{69}} & {\color{red}a_{6,10}} & {\color{red}a_{6,11}} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & {\color{red}a_{79}} & {\color{red}a_{7,10}} & {\color{red}a_{7,11}} \\ \hline
\mathbf{a}_{81} & a_{82} & a_{83} & a_{84} & \mathbf{a}_{85} & a_{86} & a_{87} & \mathbf{a}_{88} & a_{89} & a_{8,10} & a_{8,11} \\ \hline
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} & a_{9,10} & a_{9,11} \\
a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} & a_{10,11} \\
a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & a_{11,11}
\end{array}\right]$$

These are all different possibilities with respect to the linear dependence structures of rows and columns of A. Since the red color entries represent independent variables and p-rank$(A) = 3$, we have that $3 \leq \text{rank}(A) \leq 8$ depending on the values that we assignee to these variables.

Using the permutation matrix $P = [1, 5, 8, 2, 3, 4, 6, 7, 9, 10, 11]$ and different similarity transformations T_{rr}, T_{rc}, T_{cr} and T_{cc}, matrices $A_{1_{rr}}$, $A_{1_{rc}}$, $A_{1_{cr}}$ and $A_{1_{cc}}$ can be transformed, respectively, into the matrices

$$B_{rr} = PT_{rr}A_{1_{rr}}T_{rr}^{-1}P^T = \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & O & O \\ O & B_{32} & O & O \\ O & B_{42} & B_{43} & O \end{array}\right] = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix},$$

$$B_{rc} = PT_{rc}A_{1_{rc}}T_{rc}^{-1}P^T = \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & B_{23} & B_{24} \\ O & O & O & O \\ O & O & B_{43} & O \end{array}\right] = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix},$$

$$B_{cr} = PT_{cr}A_{1_{cr}}T_{cr}^{-1}P^T = \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & O & O \\ O & B_{32} & O & B_{34} \\ O & B_{42} & O & O \end{array}\right] = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix},$$

$$B_{cc} = PT_{cc}A_{1_{cc}}T_{cc}^{-1}P^T = \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & B_{23} & B_{24} \\ O & O & O & B_{34} \\ O & O & O & O \end{array}\right] = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix}.$$

In all cases, these matrices are partitioned into blocks in the following form

$$p + (i_2 - i_1 - 1) + (i_3 - i_2 - 1) + (n - i_3),$$

where B_{11} denotes an invertible matrix in the four cases, and B_2 represents a nilpotent matrix with index of nilpotency less than or equal to $p = 3$, depending on the submatrices of B_2.

Note that, each matrix B_{rr}, B_{rc} and B_{cr} can be transformed by transposition or permutation similarity into a matrix of type B_{cc}.

$$B_{rr}^T = \left[\begin{array}{c|ccc} B_{11}^T & O & O & O \\ \hline O & O & B_{32}^T & B_{42}^T \\ O & O & O & B_{43}^T \\ O & O & O & O \end{array}\right] \begin{array}{c} p \\ i_2 - i_1 - 1 \\ i_3 - i_2 - 1 \\ n - i_3 \end{array}$$

$$B_{rc} \sim \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & B_{24} & B_{23} \\ O & O & O & B_{43} \\ O & O & O & O \end{array}\right] \begin{array}{c} p \\ i_2 - i_1 - 1 \\ n - i_3 \\ i_3 - i_2 - 1 \end{array},$$

$$B_{cr} \sim \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & B_{42} & B_{32} \\ O & O & O & B_{34} \\ O & O & O & O \end{array}\right] \begin{array}{c} p \\ i_3 - i_2 - 1 \\ n - i_3 \\ i_2 - i_1 - 1 \end{array}$$

Since A is similar to one of them, the index of nilpotency implies that $\operatorname{rank}(A^p) = p$. This result is given in the following Theorem and it was proved in [4, Theorem 10] by Fallat and Gekhtman using weighted planar diagrams associated with TN matrices.

Theorem 1. *Let $A \in \mathbb{R}^{n\times n}$ be an IrTN matrix with* p-rank$(A) = p$, $1 \le p < n$. *Then* $\text{rank}(A^p) = p\text{-rank}(A) = p$.

Proof. Let $\{1, i_2, i_3, \dots, i_p\}$ be the sequence of the first p-indices of A. By Propositions 1 and 2, under similarity transformation T, the permutation similarity $P = [1,\, i_2,\, i_3,\, \dots, i_p,\, 2,\, \dots, i_2 - 1,\, i_2 + 1, \dots, i_p - 1,\, i_p + 1, \dots, n]$ and after transposition or permutation similarity, A can be transformed into the matrix

$$B = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix} = \left[\begin{array}{c|ccccc} B_{11} & O & O & O & \cdots & O \\ \hline O & O & B_{23} & B_{24} & \cdots & B_{2,p+1} \\ O & O & O & B_{34} & \cdots & B_{3,p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ O & O & O & O & \cdots & B_{p,p+1} \\ O & O & O & O & \cdots & O \end{array}\right]$$

where $B_{11} \in \mathbb{R}^{p\times p}$ is invertible and B_2 is nilpotent with index of nilpotency less than or equal to p. The block partition of B_2 in rows and columns is given by a permutation of indices $i_2 - i_1 - 1,\ i_3 - i_2 - 1,\ \dots,\ i_p - i_{p-1} - 1$ and $n - i_p$.

Since A is similar to B, we have that $\text{rank}(A^p) = \text{rank}(B^p) = p$. $\square$

3. Maximum rank

In this section we recall equation (1) of Lemma 2,

$$p \le r \le n - \left\lceil \frac{n-p}{p} \right\rceil,$$

with n, r and p are the entries of a realizable triple (n, r, p).

Now we consider a triple (n, r, p) $(1, i_2, \dots, i_p)-$realizable. It is clear that the lower bound of (1), $p \le r$, holds but what happen with the upper bound? Next example shows that the upper bound of (1) is not always reachable, that is, given a sequence of the first p-indices $(1, i_2, \dots, i_p)$, it is possible that the triple $\left(n, n - \left\lceil \frac{n-p}{p} \right\rceil, p\right)$ is not $(1, i_2, \dots, i_p)$-realizable.

Example 2. Consider the IrTN matrix A,

$$A = \begin{bmatrix}
\boxed{\mathbf{a}_{11}} & a_{12} & \boxed{\mathbf{a}_{13}} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & \boxed{\mathbf{a}_{19}} & a_{1,10} & a_{1,11} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} & a_{2,10} & a_{2,11} \\
\boxed{\mathbf{a}_{31}} & a_{32} & \boxed{\mathbf{a}_{33}} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & \boxed{\mathbf{a}_{39}} & a_{3,10} & a_{3,11} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} & a_{4,10} & a_{4,11} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} & a_{5,10} & a_{5,11} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} & a_{6,10} & a_{6,11} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} & a_{7,10} & a_{7,11} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} & a_{8,10} & a_{8,11} \\
\boxed{\mathbf{a}_{91}} & a_{92} & \boxed{\mathbf{a}_{93}} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & \boxed{\mathbf{a}_{99}} & a_{9,10} & a_{9,11} \\
a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & a_{10,9} & a_{10,10} & a_{10,11} \\
a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & a_{11,9} & a_{11,10} & a_{11,11}
\end{bmatrix}$$

with p-rank$(A) = 3$ and $\{i_1 = 1, i_2 = 3, i_3 = 9\}$ as the sequence of the first 3-indices of A.

By Propositions 1 and 2, and using similarity transformation we transform A into the following block matrix

$$B = \left[\begin{array}{c|ccc} B_{11} & O & O & O \\ \hline O & O & B_{23} & B_{24} \\ O & O & O & B_{34} \\ O & O & O & O \end{array}\right] = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix}$$

where $B_{11} \in \mathbb{R}^{3\times 3}$ is invertible and its entries are known, and $B_2 \in \mathbb{R}^{8\times 8}$ is nilpotent with index of nilpotency less than or equal to 3 and partitioned into blocks in the three following possible forms.

$$\begin{bmatrix} O & B_{23} & B_{24} \\ O & O & B_{34} \\ O & O & O \end{bmatrix} \begin{matrix} 1 \\ 5 \\ 2 \end{matrix} \quad \begin{bmatrix} O & B_{23} & B_{24} \\ O & O & B_{34} \\ O & O & O \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 5 \end{matrix} \quad \begin{bmatrix} O & B_{23} & B_{24} \\ O & O & B_{34} \\ O & O & O \end{bmatrix} \begin{matrix} 2 \\ 5 \\ 1 \end{matrix}$$

Since the entries of blocks B_{23}, B_{24} and B_{34} are variables to be determined depending on the rank that we want to obtain, it is not difficult to see, in all cases, that the maximum rank that B can reach is 6. Since rank$(A) =$ rank(B) we have that $3 \leq$ rank$(A) \leq 6$, that is, the maximum rank of A is strictly less than 8, which is the upper bound of (1). Then, the triple $(11, 8, 3)$ is not $(1, 3, 9)$-realizable.

However in Example 1, where A is an IrTN matrix with p-rank$(A) = 3$ and $\{i_1 = 1, i_2 = 5, i_3 = 8\}$ is the sequence of the first 3-indices of A, the maximum rank of A reaches the upper bound of (1).

These two examples show that the maximum rank of an IrTN matrix with p-rank$(A) = p$ depend on the sequence of the first p-indices of A. In this section we present an algorithm that calculates the maximum rank of IrTN matrix A, represented by r_{max}, as a function of the sequence of its first p-indices. This algorithm is based in the following procedure.

Procedure 1. *This process obtains the maximum rank, denoted by* $r_{\max}$, *of a block matrix* B *given as follows,*

$$B = \begin{bmatrix} B_{11} & O \\ O & B_2 \end{bmatrix} = \left[\begin{array}{c|cccccc} B_{11} & O & O & O & \cdots & O & O \\ \hline O & O & B_{23} & B_{24} & \cdots & B_{2p} & B_{2,p+1} \\ O & O & O & B_{34} & \cdots & B_{3p} & B_{3,p+1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ O & O & O & O & \cdots & O & B_{p,p+1} \\ O & O & O & O & \cdots & O & O \end{array}\right] \begin{matrix} n_1 \\ n_2 \\ n_3 \\ \vdots \\ n_p \\ n_{p+1} \end{matrix}$$

where $p = n_1$, $B_{11} \in \mathbb{R}^{n_1 \times n_1}$ *is invertible and its entries are known, while the entries of the remaining nonzero blocks are variables that will be determined to achieve the maximum possible rank of* B

- *We denote by* $r_{(p)_{\max}}$ *the maximum rank of* $B_{p,p+1}$*, then*

$$r_{(p)_{\max}} = \min\{n_p, n_{p+1}\}$$

If $n_p < n_{p+1}$*, we define* $s_p = n_{p+1} - n_p$*. In other case* $s_p = 0$*.*

(Note that, s_p *is the number of columns of the submatrix* B_{pp+1} *that we do not use for increasing the rank of* B*.)*

- *For each* $j = p-1, p-2, \ldots, 2$*, consider the submatrix*

$$S_j = \begin{bmatrix} B_{j,j+1} & B_{j,j+2} & \ldots & B_{j,p+1} \\ O & B_{j+1,j+2} & \ldots & B_{j+1,p+1} \\ \vdots & \vdots & & \vdots \\ O & O & \ldots & B_{p,p+1} \end{bmatrix}$$

and we denote by $r_{(j)_{\max}}$ *its maximum rank. Then,*

$$r_{(j)_{\max}} = r_{(j+1)_{\max}} + K,$$

where

$$K = \min\{n_j,\ n_{j+1} + s_{j+1}\}.$$

If $n_j \leq n_{j+1} + s_{j+1}$*, then*

$$\begin{cases} K = n_j \\ s_j = s_{j+1} + n_{j+1} - n_j \end{cases}$$

If $n_j > n_{j+1} + s_{j+1}$*, then*

$$\begin{cases} K = n_{j+1} + s_{j+1} \\ s_j = 0 \end{cases}$$

- *Finally,* $r_{\max} = n_1 + r_{(2)_{\max}}$*.*

The result given in Procedure 1 is the same if we have a similar block partition, but we change the size of blocks of B_2 by a permutation of indices $n_2,\ n_3,\ \ldots,\ n_{p+1}$. Thus, from now on, and without loss of generality, we can consider IrTN matrices with principal rank p, any sequence of the first p-indices, and with linearly dependent columns instead of rows. Under this assumption and by Procedure 1 we give an algorithm to compute the maximum rank of an IrTN matrix A, with p-rank$(A) = p$ and $1 < i_2 < i_3 < \ldots < i_p \leq n$, as its sequence of the first p-indices and $i_2 > 0$. Note that,

1. If $i_p = p$ and the submatrix $A[p, p+1, \ldots, n]$ is irreducible, by Proposition1 we have that $r_{\max} = p$.

2. If $i_j = j$, but $i_{j+1} > j+1$, with $1 < j < p$, and the principal submatrix $A[j, j+1, \ldots, n]$ is irreducible, then

$$r_{\max}(A) = (j-1) + r_{\max}(B),$$

where $B = A[j, j+1, \ldots, n]$ is IrTN with p-rank$(B) = p - (j-1)$ and $1 < i_{t_2} < i_{t_3} < \cdots < i_{t_{p-(j-1)}}$, as sequence of the $(p-j+1)$-first indices of B, with $i_{t_h} = i_{j+h-1} - (j-1)$, $h = 2, 3, \ldots, p-(j-1)$.

Therefore, in the following algorithm we assume without lost of generality that $i_2 > 2$ and the TN matrix A is irreducible.

Algorithm 1 (*Maximum rank of A*) Let $A \in \mathbb{R}^{n \times n}$ be an IrTN matrix with p-rank$(A) = p$. Let $1 < i_2 < i_3 < \ldots < i_p \leq n$, the first p-indices of A with $i_2 > 2$. This algorithm obtains the maximum rank of A, $r_{\max}$.

```
Require: n, p, i_1 = 1, i_2, i_3, ..., i_p, i_{p+1} = n + 1, A ∈ R^{n×n}
k = p, s = 0
for j = p to 2 do
    f = i_j − i_{j−1} − 1
    c = i_{j+1} − i_j − 1 + s
    if f ≤ c then
        k = k + f
        s = c − f
    else
        k = k + c
        s = 0
    end if
end for
return r_max = k, p ≤ rank(A) ≤ r_max
```

Remark 3. *If the final value of s in Algorithm 1 (s_2 in Procedure 1) is greater than 0, then the maximum rank of A is $n - (i_2 - 2) - s = n + 2 - (i_2 + s)$, that is, the number $i_2 + s - 2$ is the total number of columns that we can not use to increase the rank. Therefore, the maximum rank of A will be the highest possible when $s = 0$. It is not difficult to see that this occurs when the indices i_j are distributed along the matrix A equidistantly (although it is not the only case, it is the most obvious). Let's see what is the maximum rank of A when the indices i_j are equidistant.*

Suppose that n is a multiple of p. In this case, the best way to have the indices i_j distributed is

$$i_1 = 1, \; i_2 = i_1 + \frac{n}{p}, \; i_3 = i_1 + 2\,\frac{n}{p}, \; \ldots, \; i_p = i_1 + (p-1)\,\frac{n}{p}$$

and the maximum rank of A is equal to

$$p + (p-1)\left(\frac{n}{p} - 1\right) = \frac{n(p-1)}{p} - 1 = n - \frac{n-p}{p}$$

If n is not a multiple of p, the best way to choose the maximum number of columns to increase the rank is to follow the previous pattern but with $\left\lceil \frac{n}{p} \right\rceil$. Suppose that $n = kp + s$, with $s < p$, then $\left\lceil \frac{n}{p} \right\rceil = k + 1$, with the indices i_j distributed as follows

$$i_1 = 1,\ i_2 = i_1 + \left\lceil \frac{n}{p} \right\rceil = 1 + (k+1),\ \ldots,\ i_p = i_1 + (p-1)\left\lceil \frac{n}{p} \right\rceil = 1 + (p-1)(k+1)$$

and the maximum rank of A is equal to

$$\begin{aligned} p + (p-2)\left(\left\lceil \frac{n}{p} \right\rceil - 1\right) + n - i_p &= p + (p-2)k + n - (1 + (p-1)(k+1)) \\ &= n - k = n - \left\lceil \frac{n-p}{p} \right\rceil. \end{aligned}$$

Note that in both cases we get the same maximum rank. As a consequence, if A is an IrTN matrix with p-rank$(A) = p$ and if we do not consider any condition on the sequence of the first p-indices of A, we obtain the result given in Lemma 2.

4. Triple (n, r, p) $(1, i_2, \ldots, i_p)$-realizable

We have seen in the previous section that the maximum rank of an IrTN matrix A with p-rank$(A) = p$ depends on the sequence of its first p-indices. So, in this section we consider Definition 2 of triple $(1, i_2, \ldots, i_p)$-realizable and we give a method to construct IrTN matrices associated with these realizables triples. As a consequence, an IrTN matrix associated with a triple (n, r, p) realizable is obtained for all r, with $p \leq r \leq n - \left\lceil \frac{n-p}{p} \right\rceil$.

Procedure 2. *We consider a triple (n, r, p) $(1, i_2, \ldots, i_p)$-realizable. This process constructs an IrTN matrix $A \in \mathbb{R}^{n \times n}$ with* rank$(A) = r$*,* p-rank$(A) = p$*, and $\{1, i_2, \ldots, i_p\}$ as the sequence of the first p-indices of A, with $i_2 > 2$.*

1. *We construct a TN matrix $U \in \mathbb{R}^{n \times n}$ with* rank$(U) = r$ *in the following form: U is in upper block form with p echelons of width $i_j - i_{j-1}$, for $j = 2, 3, \ldots, p+1$, $i_{p+1} = n + 1$, and height $i_j - i_{j-1}$, $j = 1, 2, \ldots, p$,*

$i_0 = 0$, *and the submatrix* $U[i_p+1, i_p+2, \ldots, n|1, 2, \ldots, n] = O$, *that is,*

$$U = \left[\begin{array}{c|ccc|c|ccc|c|c|ccc}
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \hline
0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\ \hline
0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\ \hline
 & & & & & & & & \ddots & & & & \\ \hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\ \hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\ \hline
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]
\begin{array}{c}
1 \\ 2 \\ 3 \\ \vdots \\ i_2 - 1 \\ i_2 \\ \vdots \\ i_{p-1}+1 \\ i_{p-1}+2 \\ \vdots \\ i_p - 1 \\ i_p \\ i_p + 1 \\ \vdots \\ n
\end{array}$$

where $\star$ *denotes a positive number such that* U *will be TN and* $\operatorname{rank}(U) = r$. *Obviously,* $p\text{-rank}(U) = p$.

2. *Now, we construct a lower triangular TN matrix* $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ *with* $l_{ij} = 1$, *for* $i \geq j$, $1 \leq i, j \leq n$. *Using MatLab notation,* $L = tril(ones(n, n))$.

3. *Let* $A = LU$. *Note that* A *is TN because* U *and* L *are TN,* $\operatorname{rank}(A) = \operatorname{rank}(U) = r$, *and by Lemma 1* A *is irreducible. We only need to prove that* $p\text{-rank}(A) = p$.

Proposition 3. *Let* A *be a matrix given by Procedure 2, then* $p\text{-rank}(A) = p$.

Proof. Note that $\det(A[1]) \neq 0$. For $j = 2, 3, \ldots, p$, we have

$$\det(A[1, i_2, \ldots, i_j]) = \sum_{\gamma\in\mathcal{Q}_{j,n}} \det(L[1, i_2, \ldots, i_j|\gamma]) \det(U[\gamma|1, i_2, \ldots, i_j])$$

$$= \det(L[1, i_2, \ldots, i_j]) \det(U[1, i_2, \ldots, i_j]) +$$

$$+ \sum_{\gamma\in\mathcal{Q}_{j,n}\sim\{1,i_2,\ldots,i_j\}} \det(L[1, i_2, \ldots, i_j|\gamma]) \det(U[\gamma|1, i_2, \ldots, i_j]) > 0,$$

then $p\text{-rank}(A) \geq p$.

Now, for $j = 2, 3, \ldots, p$ and for any t, with $i_{j-1} < t < i_j$, we have

$$\det(A[1, i_2, \ldots, i_{j-1}, t]) = \sum_{\gamma\in\mathcal{Q}_{j,n}} \det(L[1, i_2, \ldots, i_{j-1}, t|\gamma]) \det(U[\gamma|1, i_2, \ldots, i_{j-1}, t])$$

Since L is a lower triangular matrix, we have that

$$\det(L[1, i_2, \ldots, i_{j-1}, t|\gamma_1, \gamma_2, \ldots, \gamma_{j-1}, \gamma_j]) \neq 0,$$

if the following relation holds

$$\gamma_1 = 1 < \gamma_2 \leq i_2 < \gamma_3 \leq i_3 < \cdots \leq i_{j-1} < \gamma_j \leq t,$$

but in this case, since U is an upper echelon matrix, it is verified that

$$\det(U[1, \gamma_2, \ldots, \gamma_{j-1}, \gamma_j|1, i_2, \ldots, i_{j-1}, t]) = 0.$$

Thus, for $j = 2, 3, \ldots, p$, and for any t, such that $i_{j-1} < t < i_j$, we have

$$\det(A[1, i_2, \ldots, i_{j-1}, t]) = 0, \quad i_{j-1} < t < i_j$$

Finally, if $t > i_p$, applying a similar reasoning as in the previous case to obtain $\det(A[1, i_2, \ldots, i_p, t]) = 0$. Therefore p-rank$(A) = p$. □

Remark 4. *If $i_2 = 2$ to obtain matrix A we construc U in the following way:*

1. *If $i_p = p$ matrix U is given by*

$$U = \left[\begin{array}{ccccc|ccc} 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & \star & \star & \star & \ldots & \star \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 & \star & \star & \ldots & \star \\ 0 & 0 & \ldots & 0 & 1 & \star & \ldots & \star \\ \hline 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ \vdots \\ p-1 \\ p \\ p+1 \\ \vdots \\ n \end{array}$$

where $\star$ denotes a positive number such that U will be TN.

2. *If $i_j = j$, but $i_{j+1} > j+1$, with $1 < j < p$, then U is given by*

$$U = \left[\begin{array}{cccc|ccc|c|ccc|c|c|ccc}
1 & 1 & \dots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
0 & 1 & \dots & \star & \star & \cdots & \star & \star & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & \dots & 1 & \star & \cdots & \star & \star & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & \star & \cdots & \star & \cdots & \star & \star & \cdots & \star \\
\hline
 & & & & & & & & & & & \ddots & & & & \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & \star & \cdots & \star \\
\hline
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0
\end{array}\right]
\begin{array}{c}
1 \\ 2 \\ \vdots \\ i_j \\ i_j+1 \\ i_j+2 \\ \vdots \\ i_{j+1}-1 \\ i_{j+1} \\ \vdots \\ i_{p-1}+1 \\ i_{p-1}+2 \\ \vdots \\ i_p-1 \\ i_p \\ i_p+1 \\ \vdots \\ n
\end{array}$$

Example 3. *Applying Procedure 2, we obtain a matrix associated with the triple* $(11,7,3)$ $(1,5,8)$*-realizable.*

1. First, we construct a TN matrix $U \in \mathbb{R}^{11\times 11}$ in upper block form with 3 steps of width 4, 3, and 4, and height 1, 4 and 3. The submatrix $U[9,10,11|1,2,\dots,11] = O$, and $\mathrm{rank}(U) = 7$. Note that with these conditions p-$\mathrm{rank}(U) = 3$ and $\{1,5,8\}$ is the sequence of the first 3-indices of U.

$$U = \left[\begin{array}{c|ccc|c|cc|c|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right].$$

2. Now, we construct a lower triangular TN matrix $L = tril(ones(11,11)$.

3. Finally, $A = LU$ is an IrTN matrix associated with the triple $(11, 7, 3)$ $(1, 5, 8)$-realizable:

$$A = LU = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 4 & 6 & 7 & 7 & 7 & 7 & 7 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 11 & 11 & 11 & 11 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 12 & 13 & 13 & 13 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 13 & 15 & 16 & 16 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 14 & 17 & 19 & 19 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 14 & 17 & 19 & 19 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 14 & 17 & 19 & 19 \\ 1 & 1 & 1 & 1 & 5 & 8 & 10 & 14 & 17 & 19 & 19 \end{bmatrix}.$$

If we consider a realizable triple (n, r, p) but we do not know the sequence of the first p-indices, then the relations between n, r and p are given by equation (1). Now, we construct an IrTN matrix $A \in \mathbb{R}^{n \times n}$ associated with a triple (n, r, p) realizable for all r, with $p \leq r \leq n - \left\lceil \frac{n-p}{p} \right\rceil$.

By Remark 3, if the sequence of the first p-indices is equidistantly distributed along the matrix A, then for any integer r the equation (1) holds and we can apply Procedure 2 to obtain an IrTN matrix A associated with the triple (n, r, p) realizable.

Example 4. *Construct an IrTN matrix associated with the triple* $(11, r, 3)$ *realizable with* $3 \leq r \leq 8$.

By Remark 3, the sequence of the first 3-indices is $\{i_1 = 1,\ i_2 = 5,\ i_3 = 9\}$. Therefore, we construct the following matrix

$$U_r = \left[\begin{array}{c|ccc|c|ccc|c|cc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & u_{36} & u_{37} & u_{38} & u_{39} & u_{3,10} & u_{3,11} \\ 0 & 0 & 0 & 0 & 1 & u_{46} & u_{47} & u_{48} & u_{49} & u_{4,10} & u_{4,11} \\ \hline 0 & 0 & 0 & 0 & 1 & u_{56} & u_{57} & u_{58} & u_{59} & u_{5,10} & u_{5,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u_{6,10} & u_{6,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u_{7,10} & u_{7,11} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u_{8,10} & u_{8,11} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & u_{9,10} & u_{9,11} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. If $r = 3$, we consider

$$u_{ij} = \begin{cases} 1, & i = 3, 4, 5, \quad j = 6, 7, \ldots, 11 \\ 1, & i = 6, 7, 8, 9, \quad j = 10, 11 \end{cases}$$

then, $\text{rank}(U_3) = 3$ and $A_3 = LU_3$, with $L = \text{tril}(\text{ones}(11, 11))$, is an IrTN matrix associated with the triple $(11, 3, 3)$.

2. If $r = 4$, we consider

$$u_{ij} = \begin{cases} 1, & i = 3, 4, 5, \quad j = 6, 7, \ldots, 11 \\ 1, & i = 6, 7, 8, \quad j = 10, 11 \\ 2, & i = 9, \quad j = 10, 11 \end{cases}$$

then, $\text{rank}(U_4) = 4$ and $A_4 = LU_4$ is an IrTN matrix associated with the triple $(11, 4, 3)$.

3. If $r = 5$, we consider

$$u_{ij} = \begin{cases} 1, & i = 3, 4, 5, \quad j = 6, 7, \ldots, 11 \\ 1, & i = 6, 7, \quad j = 10, 11 \\ 2, & i = 8, \quad j = 10, 11 \\ 2, & i = 9, \quad j = 10 \\ 3, & i = 9, \quad j = 11 \end{cases}$$

then $\text{rank}(U_5) = 5$ and $A_5 = LU_5$ is an IrTN matrix associated with the triple $(11, 5, 3)$.

4. If $r = 6$, we consider

$$u_{ij} = \begin{cases} 1, & i = 3, 4, \quad j = 6, 7, \ldots, 11 \\ 2, & i = 5, \quad j = 6, 7, \ldots, 11 \\ 1, & i = 6, 7, \quad j = 10, 11 \\ 2, & i = 8, \quad j = 10, 11 \\ 2, & i = 9, \quad j = 10 \\ 3, & i = 9, \quad j = 11 \end{cases}$$

then $\text{rank}(U_6) = 6$ and $A_6 = LU_6$ is an IrTN matrix associated with the triple $(11, 6, 3)$.

5. If $r = 7$, we consider

$$u_{ij} = \begin{cases} 1, & i = 3, \quad j = 6, 7, \ldots, 11 \\ 2, & i = 4, \quad j = 6, 7, \ldots, 11 \\ 2, & i = 5, \quad j = 6 \\ 3, & i = 5, \quad j = 7, 8, \ldots, 11 \\ 1, & i = 6, 7, \quad j = 10, 11 \\ 2, & i = 8, \quad j = 10, 11 \\ 2, & i = 9, \quad j = 10 \\ 3, & i = 9, \quad j = 11 \end{cases}$$

then $\text{rank}(U_7) = 7$ and $A_7 = LU_7$ is an IrTN matrix associated with the triple $(11, 7, 3)$.

6. If $r = 8$, we consider

$$u_{ij} = \begin{cases} 2, & i = 3, \quad j = 6, 7, \ldots, 11 \\ 2, & i = 4, \quad j = 6 \\ 3, & i = 4, \quad j = 7, 8, \ldots, 11 \\ 2, & i = 5, \quad j = 6 \\ 3, & i = 5, \quad j = 7 \\ 4, & i = 5, \quad j = 8, \ldots, 11 \\ 1, & i = 6, 7, \quad j = 10, 11 \\ 2, & i = 8, \quad j = 10, 11 \\ 2, & i = 9, \quad j = 10 \\ 3, & i = 9, \quad j = 11 \end{cases}$$

then $\text{rank}(U_8) = 8$ and $A_8 = LU_8$ is an IrTN matrix associated with the triple $(11, 8, 3)$.

References

[1] T. Ando, *Totally positive matrices*. Linear Algebra and its Applications, Vol. 90, pp. 165–219 (1987).

[2] S. M. Fallat, *Bidiagonal factorizations of totally nonnegative matrices*. Amer. Math. Monthly, Vol. 109, pp. 697–712 (2001).

[3] S. M. Fallat, M. I. Gekhtman and C. R. Johnson, *Spectral structures of irreducible totally nonnegative matrices*. SIAM J. Matrix Anal. Appl., Vol. 22, No. 2, pp. 627–645 (2000).

[4] S. M. Fallat and M. I. Gekhtman *Jordan structure of Totally Nonnegative Matrices*. Canad. Journal of Math., Vol. 57, No. 1, pp. 82-98 (2005).

[5] S. M. Fallat and C. R. Johnson, *Totally Nonnegative Matrices*. Princenton Series in Applies Mathematics (2011).