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Some Results on the Symmetric Representation of the Generalized Drazin Inverse in a Banach Algebra

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Received: 1 January 2019; Accepted: 13 January 2019; Published: 17 January 2019

Abstract: Based on the conditions $ab^2 = 0$ and $b^\pi(ab) \in \mathscr{A}^{\mathrm{d}}$, we derive that $(ab)^n$, $(ba)^n$, and $ab + ba$ are all generalized Drazin invertible in a Banach algebra $\mathscr{A}$, where $n \in \mathbb{N}$ and a and b are elements of $\mathscr{A}$. By using these results, some results on the symmetry representations for the generalized Drazin inverse of $ab + ba$ are given. We also consider that additive properties for the generalized Drazin inverse of the sum $a + b$.

Keywords: generalized Drazin inverse; Banach algebra; representation

MSC: 46H05; 47A05; 15A09

1. Introduction

Let $\mathscr{A}$ be a complex unital Banach algebra with unit 1. The sets of all invertible elements and quasinilpotent elements of $\mathscr{A}$ are denoted by $\mathscr{A}^{-1}$ and $\mathscr{A}^{\mathrm{qnil}}$, respectively, where $\mathscr{A}^{-1} := \{a \in \mathscr{A} : \exists\, x \in \mathscr{A} : ax = xa = 1\}$ and $\mathscr{A}^{\mathrm{qnil}} := \{a \in \mathscr{A} : \lim\limits_{n\to+\infty} \|a^n\|^{1/n} = 0\}$. Let $a \in \mathscr{A}$ and, if there is a element $b \in \mathscr{A}$ such that

$$bab = b, \qquad ab = ba, \text{ and } \qquad a(1 - ab) \text{ is quasinilpotent,} \tag{1}$$

then b is the *generalized Drazin inverse* of a, denoted by a^{d}, and it is unique. The set of generalized Drazin-invertible elements is denoted by $\mathscr{A}^{\mathrm{d}} = \{a \in \mathscr{A} : \exists\, a^{\mathrm{d}}\}$. In particular, if $a(1 - ab) = 0$ (or $aba = a$), then b is called the *group inverse* of a. Note that aa^{d} is an idempotent element and let $a^\pi = 1 - aa^{\mathrm{d}}$. It was given, in [1] (Lemma 2.4), that a^{d} exists if and only if there is an idempotent $q \in \mathscr{A}$, such that $aq = qa$, aq is quasinilpotent, and $a + q$ is invertible.

The generalized inverse in a matrix or operator theory is very useful in scientific calculation and in various engineering technologies [2–4]. It is well known that the Drazin inverse has been applied in a few fields, such as statistics and probability [5], ordinary differential equations [6], Markov chains [7], operator matrices [8], neural network models [9,10], and the references therein. In [11], a study of the Drazin inverse for bounded linear operators in a Banach space X is given, when 0 is an isolated spectral point of the operator. In [12], some additive results on the Drazin inverse, under the condition $ab = 0$, are obtained. However, as in [12,13], this condition was not enough to derive a formula for the generalized Drazin inverse for $a + b$. In [14], authors investigated how to express the Drazin inverse of sums, differences, and products of two matrices P and Q, under the conditions $P^3Q = QP$ and $Q^3P = PQ$. The representations of the Drazin inverse for $(P + Q)$, such that $PQP = 0$ and $PQ^2 = 0$, is given in [15]. The generalized inverses in C^*-algebras has been investigated in [16] and a symmetry of the generalized Drazin inverse in a C^*-algebra has been considered in [17].

Some additive properties of the generalized Drazin inverse in a Banach algebra were investigated in [18]. Recently, the expression for the generalized Drazin inverse of the sum $a+b$ on Banach algebra has been studied, such as in the representations of the generalized Drazin inverse for $a+b$ in a Banach algebra, obtained in [19]; some new additive results for the generalized Drazin inverse in a Banach algebra, given in [20]; and additive results on the generalized Drazin inverse of a sum of two elements in a Banach algebra are derived in [21] and the references therein. In this paper, we consider the representations of the generalized Drazin inverse of the sum of two elements in a Banach algebra. By using the assumed conditions $ab^2 = 0$ and $b^\pi(ab) \in \mathscr{A}^{\mathsf{d}}$, it is implied that $(ab)^n$, $(ba)^n$, and $ab+ba \in \mathscr{A}^{\mathsf{d}}$, and a symmetry representation for the generalized Drazin inverse of $ab+ba$ is obtained, where $n \in \mathbb{N}$ and $a, b \in \mathscr{A}^{\mathsf{d}}$. We also consider the additive properties for the generalized Drazin inverse of the sum $a+b$.

In Section 2, some notation is introduced and lemmas are given. In Section 3, a symmetric representation of the generalized Drazin inverse for $ab+ba$ in a Banach algebra is derived. The additive properties of the generalized Drazin inverse of $a+b$ are investigated in Section 4.

2. Preliminaries

Let $\mathscr{B}$ be a subalgebra of the unital algebra $\mathscr{A}$. For an element $b \in \mathscr{B}^{-1}$, the inverse of b in $\mathscr{B}$ is denoted by $[b^{-1}]_{\mathscr{B}}$. As in [19], it is given that $\mathscr{B}^{-1} \not\subset \mathscr{A}^{-1}$. Let $\mathscr{P} = \{p_1, p_2, \ldots, p_n\}$ be a *total system of idempotents* in $\mathscr{A}$ if $p_i^2 = p_i$, for all i, $p_i p_j = 0$ if $i \neq j$, and $p_1 + \cdots + p_n = 1$, as in [22]. If $a \in \mathscr{A}^{\mathsf{d}}$, then

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}}, \qquad a^{\mathsf{d}} = \begin{bmatrix} [a_1{}^{-1}]_{p\mathscr{A}p} & 0 \\ 0 & 0 \end{bmatrix}_{\mathscr{P}}, \qquad a^\pi = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}}, \tag{2}$$

where $p = aa^{\mathsf{d}}$, $\mathscr{P} = \{p, 1-p\}$, $a_1 \in [p\mathscr{A}p]^{-1}$, and $a_2 \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{qnil}}$. If a has the representation given as in (2), then $a^{\mathsf{d}} = [a_1{}^{-1}]_{p\mathscr{A}p} = a_1^{\mathsf{d}}$.

The following lemmas are required in what follows.

Lemma 1 ([19]). *Let $\mathscr{P} = \{p, 1-p\}$ be a total system of idempotents in $\mathscr{A}$, and let $a, b \in \mathscr{A}$ have the following representation*

$$a = \begin{bmatrix} x & 0 \\ z & y \end{bmatrix}_{\mathscr{P}}, \qquad b = \begin{bmatrix} x & t \\ 0 & y \end{bmatrix}_{\mathscr{P}}.$$

Then there exist $(z_n)_{n=0}^\infty \subset (1-p)\mathscr{A}p$ and $(t_n)_{n=0}^\infty \subset p\mathscr{A}(1-p)$, such that

$$a^n = \begin{bmatrix} x^n & 0 \\ z_n & y^n \end{bmatrix}_{\mathscr{P}}, \qquad b^n = \begin{bmatrix} x^n & t_n \\ 0 & y^n \end{bmatrix}_{\mathscr{P}}, \qquad \forall\, n \in \mathbb{N}.$$

Lemma 2 ([22]). *Let $a, b \in \mathscr{A}$ be generalized Drazin invertible and $ab = 0$. Then, $a+b$ is generalized Drazin invertible and*

$$(a+b)^{\mathsf{d}} = b^\pi \sum_{n=0}^{\infty} b^n (a^{\mathsf{d}})^{n+1} + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+1} a^n a^\pi.$$

Lemma 3 ([22]). *Let $x, y \in \mathscr{A}$, p be an idempotent of $\mathscr{A}$, and let x and y have the representation*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_{\{p,1-p\}}, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_{\{1-p,p\}}. \tag{3}$$

(i) *If* $a \in [p\mathscr{A}p]^{\mathsf{d}}$ *and* $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, *then* $x, y \in \mathscr{A}^{\mathsf{d}}$ *and*

$$x^{\mathsf{d}} = \begin{bmatrix} a^{\mathsf{d}} & 0 \\ u & b^{\mathsf{d}} \end{bmatrix}_{\{p,1-p\}}, \qquad y^{\mathsf{d}} = \begin{bmatrix} b^{\mathsf{d}} & u \\ 0 & a^{\mathsf{d}} \end{bmatrix}_{\{1-p,p\}}, \tag{4}$$

where

$$u = \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} c a^n a^{\pi} + \sum_{n=0}^{\infty} b^{\pi} b^n c (a^{\mathsf{d}})^{n+2} - b^{\mathsf{d}} c a^{\mathsf{d}}. \tag{5}$$

(ii) *If* $x \in \mathscr{A}^{\mathsf{d}}$ *and* $a \in [p\mathscr{A}p]^{\mathsf{d}}$, *then* $b \in [(1-p)\mathscr{A}(1-p)]^{\mathsf{d}}$, *and* x^{d} *and* y^{d} *are given by* (4) and (5).

Lemma 4 ([11]). *Let* $a \in \mathscr{A}^{\mathsf{d}}$. *Then* $[(a)^n]^{\mathsf{d}} = [a^{\mathsf{d}}]^n$, *for all* $n = 1, 2, \cdots$.

Lemma 5 ([11]). *If* $a, b \in \mathscr{A}^{\mathsf{d}}$ *and* $ab = ba = 0$. *Then,* $(a+b)^{\mathsf{d}}$ *also exists and* $(a+b)^{\mathsf{d}} = a^{\mathsf{d}} + b^{\mathsf{d}}$.

Lemma 6 ([23]). *Let* $a, b \in \mathscr{A}^{\mathsf{d}}$. *Then* $(ab)^{n+1}$ *is generalized Drazin invertible, for some* $n \in \mathbb{N}$, *if and only if* ab *is generalized Drazin invertible.*

Lemma 7 ([23]). *Let* $a, b \in \mathscr{A}^{\mathsf{d}}$ *and* $(ab)^{n+1}$ *be generalized Drazin invertible for some* $n \in \mathbb{N}$. *Then,* $(ba)^n$ *is generalized Drazin invertible and* $[(ba)^n]^{\mathsf{d}} = b[(ab)^{n+1}]^{\mathsf{d}} a$.

3. The Symmetric Representation for the Generalized Drazin Inverse of $ab + ba$

Let $a, b \in \mathscr{A}^{\mathsf{d}}$. A symmetric expression of $(ab+ba)^{\mathsf{d}}$ is given, by using ab, ba, $(ab)^{\mathsf{d}}$, and $(ba)^{\mathsf{d}}$, with the following assumed conditions

$$ab^2 = 0, \;\; b^{\pi}(ab) \in \mathscr{A}^{\mathsf{d}}. \tag{6}$$

Theorem 1. *Let* $a, b \in \mathscr{A}^{\mathsf{d}}$ *satisfy* (6). *Then,* $(ab)^n, (ba)^n, ab + ba \in \mathscr{A}^{\mathsf{d}}$ $(n = 1, 2, \cdots)$, *and a representation of* $(ab+ba)^{\mathsf{d}}$ *is given as*

$$(ab+ba)^{\mathsf{d}} = (ba)^{\pi} \sum_{n=1}^{\infty} (ba)^{n-1} [(ab)^n]^{\mathsf{d}} + \sum_{n=1}^{\infty} [(ba)^n]^{\mathsf{d}} (ab)^{n-1} (ab)^{\pi}. \tag{7}$$

Proof. Let $b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}}$, where $\mathscr{P} = \{bb^{\mathsf{d}}, b^{\pi}\}$, b_1 is invertible in the subalgebra $bb^{\mathsf{d}}\mathscr{A}bb^{\mathsf{d}}$, and b_2 is quasinilpotent. Let us write $a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{\mathscr{P}}$. From $ab^2 = 0$, we have

$$a_{11} = 0, \qquad a_{21} = 0, \qquad a_{12} b_2^2 = 0, \qquad \text{and} \qquad a_{22} b_2^2 = 0. \tag{8}$$

Thus, we have $ab = \begin{bmatrix} 0 & a_{12}b_2 \\ 0 & a_{22}b_2 \end{bmatrix}$. By Lemma 3, we obtain that $ab \in \mathscr{A}^{\mathsf{d}}$ if and only if $a_{22}b_2$ is generalized Drazin invertible. Thus, $(b^{\pi}ab)^{\mathsf{d}}$ exists. By using Cline's formula, it proves that $(ab)^{\mathsf{d}}$ also is. Therefore, we obtain $(ab)^n, (ba)^n \in \mathscr{A}^{\mathsf{d}}$ by using Lemma 6 and 7. Since $ab^2 = 0$, by Lemma 2 we can prove that $ab + ba$ is generalized Drazin invertible and that (7) holds. If $n = 1$, then $(ab+ba)^{\mathsf{d}} = (ba)^{\pi}(ab)^{\mathsf{d}} + (ba)^{\mathsf{d}}(ab)^{\pi}$. By using mathematical induction, we derive that the representation can be given, as in (7). □

Remark 1. *Note that the expression given in Theorem 1 is symmetric.*

Theorem 2. *Let $a, b \in \mathscr{A}^{\mathrm{d}}$ satisfy (6) and $a^2 = 0$. Then $ab + ba \in \mathscr{A}^{\mathrm{d}}$ and $[(ab+ba)^{\mathrm{d}}]^n = [(ab)^{\mathrm{d}}]^n + [(ba)^{\mathrm{d}}]^n$, for all $n = 1, 2, \cdots$.*

Proof. Let a, b be written as in the proof of Theorem 1, and, by $ab^2 = 0$, we derive $ab = \begin{bmatrix} 0 & a_{12}b_2 \\ 0 & a_{22}b_2 \end{bmatrix}$ and $ab, ba, (ab)^n, (ba)^n \in \mathscr{A}^{\mathrm{d}}$. Since $ab^2 = 0$ and $a^2 = 0$, we have

$$(ab)^n(ba)^n = (ba)^n(ab)^n = 0, \quad (ab+ba)^n = (ab)^n + (ba)^n, \tag{9}$$

for all $n = 1, 2, \cdots$. By Lemma 4, Lemma 5, and the first equality of (9), we derive

$$[(ab+ba)^{\mathrm{d}}]^n = [(ab+ba)^n]^{\mathrm{d}} = [(ab)^n + (ba)^n]^{\mathrm{d}} = [(ab)^n]^{\mathrm{d}} + [(ba)^n]^{\mathrm{d}} = [(ab)^{\mathrm{d}}]^n + [(ba)^{\mathrm{d}}]^n.$$

□

At the end of Section 3, let $\mathscr{A}$ be a C^*-algebra, as in [17]. Then, a simple application of the generalized Drazin inverse in a C^*-algebra can be given, as follows.

Theorem 3. *Let $a, b \in \mathscr{A}$ be group invertible. If (6) is satisfied, then $(ab+ba)^{\dagger}$ exists.*

Proof. By using Theorem 1, we derive that $ab + ba$ is group invertible. As pointed out in [16], $ab + ba$ is generalized invertible. Thus, $(ab+ba)^{\dagger}$ exists. □

Theorem 4. *Let $a, b \in \mathscr{A}^{\mathrm{d}}$. If (6) is satisfied, then $(ab+ba)^{\mathrm{d}}$ is self-adjoint in a C^*-algebra.*

Proof. Note that $ab + ba$ is self-adjoint in a C^*-algebra. By Theorem 1 and using [17] (Theorem 3.2), we obtain that $(ab+ba)^{\mathrm{d}}$ is self-adjoint in a C^*-algebra. □

4. The Representation for the Generalized Drazin Inverse of $a + b$

In this section, we consider some results on the expression of $(a+b)^{\mathrm{d}}$, by using a, b, a^{d}, and b^{d}, where $a, b \in \mathscr{A}^{\mathrm{d}}$.

Lemma 8. *Let $a, b \in \mathscr{A}^{\mathrm{d}}$ satisfy $ab^2 = 0$. Then, $(a+b)^{\mathrm{d}}$ exists if and only if $b^{\pi}(a+b) \in \mathscr{A}^{\mathrm{d}}$.*

Proof. Similarly, we rewrite a, b as in the proof of Theorem 1. Since $ab^2 = 0$, we derive

$$a + b = \begin{bmatrix} b_1 & a_{12} \\ 0 & b_2 + a_{22} \end{bmatrix}_{\mathscr{P}}. \tag{10}$$

By Lemma 3, note that $(a+b)^{\mathrm{d}}$ exists if and only if $(a_{22}+b_2)^{\mathrm{d}}$ exists; that is, $(a+b)^{\mathrm{d}}$ exists if and only if $b^{\pi}(a+b)$ is generalized Drazin invertible. □

Theorem 5. *Let $a, b \in \mathscr{A}^{\mathrm{d}}$ satisfy the conditions of Theorem 2. Then*

$$\begin{aligned}(a+b)^{\mathrm{d}} = & \sum_{n=0}^{\infty} (b^{\mathrm{d}})^{2n+1} \left[b^{\mathrm{d}}(ab)^{\pi}(ab)^n a + (ab)^{\pi}(ab)^n \right] \\ & - \sum_{n=0}^{\infty} b^{\pi} b^{2n} \left\{ [(ab)^{\mathrm{d}}]^{n+1} a + b[(ab)^{\mathrm{d}}]^{n+1} \right\}.\end{aligned}$$

Proof. By Lemma 8, it also leads to (10). By Lemma 3, we can prove that $(a+b)^{\mathrm{d}}$ exists if and only if $(a_{22}+b_2)^{\mathrm{d}}$ exists; that is, $(a+b)^{\mathrm{d}}$ exists if and only if $b^{\pi}(a+b)$ is generalized Drazin invertible.

If $b^\pi ab \in \mathscr{A}^{\mathsf{d}}$, then $(a_{22}b_2)^{\mathsf{d}}$ exists. By Cline's formula, we have that $(b_2a_{22})^{\mathsf{d}}$ exists. As in the proof of Theorem 1, by Lemma 6 and 7, we also obtain that $(ab)^n, (ba)^n \in \mathscr{A}^{\mathsf{d}}$, for all $n = 1, 2, \cdots$.

By $a^2 = 0$, we get

$$a_{12}a_{22} = 0 \qquad \text{and} \qquad a_{22}^2 = 0. \tag{11}$$

By (8) and (11), we have $(b_2a_{22})(a_{22}b_2) = 0$, $(a_{22}b_2)(b_2a_{22}) = 0$. Using Lemma 5, and by Cline's formula, we derive

$$(a_{22}b_2 + b_2a_{22})^{\mathsf{d}} = (a_{22}b_2)^{\mathsf{d}} + (b_2a_{22})^{\mathsf{d}}. \tag{12}$$

By induction, let $[(a_{22}b_2)^{\mathsf{d}} + (b_2a_{22})^{\mathsf{d}}]^n = [(a_{22}b_2)^{\mathsf{d}}]^n + [(b_2a_{22})^{\mathsf{d}}]^n$ for all $n \geq 1$. Therefore, we can prove that

$$[(a_{22}b_2 + b_2a_{22})^{\mathsf{d}}][(a_{22}b_2)^{\mathsf{d}} + (b_2a_{22})^{\mathsf{d}}]^n = [(a_{22}b_2)^{\mathsf{d}}]^{n+1} + [(b_2a_{22})^{\mathsf{d}}]^{n+1}.$$

Since $(a_{22}b_2 + b_2a_{22})b_2^2 = 0$ and b_2 are quasinilpotent, by Lemma 5 and (12), we obtain

$$\begin{aligned}
[(a_{22} + b_2)^2]^{\mathsf{d}} &= (a_{22}b_2 + b_2a_{22} + b_2^2)^{\mathsf{d}} \\
&= \sum_{n=0}^{\infty} b_2^{2n}[(a_{22}b_2 + b_2a_{22})^{\mathsf{d}}]^{n+1} \\
&= \sum_{n=0}^{\infty} b_2^{2n}\left[(a_{22}b_2)^{\mathsf{d}} + (b_2a_{22})^{\mathsf{d}}\right]^{n+1}.
\end{aligned} \tag{13}$$

Then, $b^\pi(a+b) \in \mathscr{A}^{\mathsf{d}}$ implies that $(a_{22}+b_2)^{\mathsf{d}}$ exists and $(a_{22}+b_2)^{\mathsf{d}} = [(a_{22}+b_2)^2]^{\mathsf{d}}(a_{22}+b_2)$. Finally, by (13), and $(b_2a_{22})^{\mathsf{d}} = b_2\left[(a_{22}b_2)^{\mathsf{d}}\right]^2 a_{22}$, we obtain

$$\begin{aligned}
(a_{22} + b_2)^{\mathsf{d}} &= \left[(a_{22} + b_2)^{\mathsf{d}}\right]^2 (a_{22} + b_2) \\
&= \sum_{n=0}^{\infty} b_2^{2n}\left\{\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1} + \left(b_2\left[(a_{22}b_2)^{\mathsf{d}}\right]^2 a_{22}\right)^{n+1}\right\}(a_{22} + b_2) \\
&= \sum_{n=0}^{\infty} b_2^{2n}\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1} a_{22} + \sum_{n=0}^{\infty} b_2^{2n}\left[b_2\left((a_{22}b_2)^{\mathsf{d}}\right)^2 a_{22}\right]^{n+1} b_2 \\
&= \sum_{n=0}^{\infty} b_2^{2n}\left\{\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1} a_{22} + b_2\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1}\right\}
\end{aligned} \tag{14}$$

and

$$(a_{22} + b_2)^\pi = (a_{22}b_2)^\pi - \sum_{n=0}^{\infty} b_2^{2n+1}\left\{\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1} a_{22} + b_2\left[(a_{22}b_2)^{\mathsf{d}}\right]^{n+1}\right\}. \tag{15}$$

By Lemma 3, we get that $a + b \in \mathscr{A}^{\mathsf{d}}$ and

$$(a+b)^{\mathsf{d}} = \begin{bmatrix} b_1^{-1} & u \\ 0 & (a_{22}+b_2)^{\mathsf{d}} \end{bmatrix}_{\mathscr{P}}, \tag{16}$$

and

$$u = \sum_{n=0}^{\infty} (b_1^{-1})^{n+2} a_{12}(b_2 + a_{22})^n (a_{22} + b_2)^\pi - (a_{22} + b_2)^{\mathsf{d}} a_{12} b_1^{-1}. \tag{17}$$

Evidently, we have $\left[b_1^{-1}\right]_{\mathscr{P}} = b^{\mathsf{d}}$ and

$$b^{\mathsf{d}}ba = \begin{bmatrix} b_1^{-1}b_1 & 0 \\ 0 & 0 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}_{\mathscr{P}} = a_{12}.$$

One easily has (by induction and by using (8) and (11)) that, if $n \geq 1$, then

$$a_{12}(a_{22}+b_2)^n = \begin{cases} a_{12}(b_2a_{22})^{n/2} & \text{if } n \text{ is even,} \\ a_{12}(b_2a_{22})^{(n-1)/2}b_2 & \text{if } n \text{ is odd.} \end{cases} \tag{18}$$

By Lemma 1, we obtain that, for any $n \geq 1$,

$$b^{\pi}(ba)^n = \begin{bmatrix} 0 & 0 \\ 0 & b^{\pi} \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & x_n \\ 0 & (b_2a_{22})^n \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & (b_2a_{22})^n \end{bmatrix}_{\mathscr{P}} = (b_2a_{22})^n,$$

where $(x_n)_{n=0}^{\infty}$ is a sequence in $\mathscr{A}$. Furthermore, one has $b_2 = b^{\pi}b = bb^{\pi}$ and $ab^{\pi} = a(1-bb^{\mathsf{d}}) = a(1-b^2(b^{\mathsf{d}})^2) = a$. Hence, if $n \geq 1$ is even, then

$$a_{12}(a_{22}+b_2)^n = a_{12}(b_2a_{22})^{n/2} = b^{\mathsf{d}}bab^{\pi}(ba)^{n/2} = b^{\mathsf{d}}ba(ba)^{n/2} = b^{\mathsf{d}}(ba)^{(n+2)/2},$$

and if $n \geq 1$ is odd, then

$$a_{12}(a_{22}+b_2)^n = a_{12}(b_2a_{22})^{(n-1)/2}b_2 = b^{\mathsf{d}}bab^{\pi}(ba)^{(n-1)/2}b^{\pi}b = b^{\mathsf{d}}(ba)^{(n+1)/2}b.$$

From (15), we have

$$\begin{aligned} a_{12}(a_{22}+b_2)^{\pi} &= a_{12}(1-b_2(a_{22}b_2)^{\mathsf{d}}a_{22}), \\ a_{22}(a_{22}+b_2)^{\pi} &= (a_{22}b_2)^{\pi}a_{22}, \\ a_{12}b_2(a_{22}+b_2)^{\pi} &= a_{12}b_2(a_{22}b_2)^{\pi}, \\ a_{22}b_2(a_{22}+b_2)^{\pi} &= a_{22}b_2(a_{22}b_2)^{\pi}. \end{aligned}$$

Thus, by using the obvious equality $(ba)^k b = b(ab)^k$, and by (14)–(16) and (18), we have

$$\begin{aligned} (a+b)^{\mathsf{d}} &= b_1^{\mathsf{d}} + u = [b_1]_{\mathscr{P}}^{-1} + \sum_{n=0}^{\infty} (\left[b_1^{-1}\right]_{\mathscr{P}})^{n+2} a_{12}(b_2+a_{22})^n (a_{22}+b_2)^{\pi} \\ &\quad - (a_{22}+b_2)^{\mathsf{d}} a_{12} b_1^{-1} + (a_{22}+b_2)^{\mathsf{d}} \\ &= \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{2n+2} b^{\pi}(ab)^n a + \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{2n+1} b^{\pi}(ab)^n \\ &\quad - \sum_{n=0}^{\infty} b^{\pi} b^{2n} \left\{ [(ab)^{\mathsf{d}}]^{n+1} a + b[(ab)^{\mathsf{d}}]^{n+1} \right\} \\ &= \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{2n+1} \left[b^{\mathsf{d}}(ab)^{\pi}(ab)^n a + (ab)^{\pi}(ab)^n \right] \\ &\quad - \sum_{n=0}^{\infty} b^{\pi} b^{2n} \left\{ [(ab)^{\mathsf{d}}]^{n+1} a + b[(ab)^{\mathsf{d}}]^{n+1} \right\}. \end{aligned}$$

The proof is completed. □

Theorem 6. *Let $a, b \in \mathscr{A}^{\mathsf{d}}$ satisfy (6) and $b^\pi a^2 = 0$. Then,*

$$(a+b)^{\mathsf{d}} = b^{\mathsf{d}} + u + v,$$

where

$$v = -\left\{ b^{\mathsf{d}} a (ba)^{\mathsf{d}} + \sum_{n=0}^{\infty} b^{\mathsf{d}} b^{2n+1} \left[((ab)^{\mathsf{d}})^{n+1} + ((ba)^{\mathsf{d}})^{n+1} \right] \right\},$$
$$u = \sum_{n=0}^{\infty} (b^{\mathsf{d}})^{n+2} a (a+b)^n + \sum_{n=0}^{\infty} (1-b^\pi) b^n a v^{n+2} - b^{\mathsf{d}} a v.$$

Proof. Let $p = bb^{\mathsf{d}}$ and $\mathscr{P} = \{p, 1-p\}$. Let a and b have the following representation

$$b = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}_{\mathscr{P}}, \qquad a = \begin{bmatrix} a_3 & a_1 \\ a_4 & a_2 \end{bmatrix}_{\mathscr{P}}, \tag{19}$$

where b_1 is invertible in $p\mathscr{A}p$ and b_2 is quasinilpotent in $(1-p)\mathscr{A}(1-p)$. Let us find the expression of $b^\pi a^2$ in the system of idempotents $\mathscr{P}$:

$$b^\pi a^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & 0 \\ 0 & a_2^2 \end{bmatrix}_{\mathscr{P}} = a_2^2.$$

Thus, $a_2^2 = 0$. On the other hand,

$$ab^2 = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1^2 & 0 \\ 0 & b_2^2 \end{bmatrix}_{\mathscr{P}} = \begin{bmatrix} 0 & a_1 b_2^2 \\ 0 & a_2 b_2^2 \end{bmatrix}_{\mathscr{P}}.$$

Therefore, $a_2 b_2^2 = 0$. By $b^\pi ab, b^\pi ba \in \mathscr{A}^d$, we obtain $(a_2 b_2), (b_2 a_2) \in \mathscr{A}^d$. We can appeal to Theorem 5, obtaining (recall that b_2 is quasinilpotent and $b_2^{\mathsf{d}} = 0$) that

$$(a_2+b_2)^{\mathsf{d}} = -a_2 (b_2 a_2)^{\mathsf{d}} - \sum_{n=0}^{\infty} b_2^{2n+1} \left[((a_2 b_2)^{\mathsf{d}})^{n+1} + ((b_2 a_2)^{\mathsf{d}})^{n+1} \right].$$

From Lemma 3 and the representation of $a+b$ in (16), we have

$$\begin{aligned} (a+b)^{\mathsf{d}} &= \left[b_1^{-1} \right]_{\mathscr{P}} + (a_2+b_2)^{\mathsf{d}} + u \\ &= \left[b_1^{-1} \right]_{\mathscr{P}} + u - \left\{ a_2 (b_2 a_2)^d + \sum_{n=0}^{\infty} b_2^{2n+1} \left[((a_2 b_2)^d)^{n+1} + ((b_2 a_2)^d)^{n+1} \right] \right\}, \end{aligned} \tag{20}$$

where

$$\begin{aligned} u &= \sum_{n=0}^{\infty} (\left[b_1^{-1} \right]_{\mathscr{P}})^{n+2} a_1 (a_2+b_2)^n (a_2+b_2)^\pi \\ &+ \sum_{n=0}^{\infty} b_1^\pi b_1^n a_1 ((a_2+b_2)^{\mathsf{d}})^{n+2} - \left[b_1^{-1} \right]_{\mathscr{P}} a_1 (a_2+b_2)^{\mathsf{d}} \\ &= \sum_{n=0}^{\infty} (b_1^{\mathsf{d}})^{n+2} a_1 (a_2+b_2)^n. \end{aligned}$$

Observe that $\left[b_1^{-1}\right]_{\mathscr{P}} = b^{\mathrm{d}}$, and

$$\begin{aligned}
(b^{\mathrm{d}})^{n+2}a(a+b)^n &= \begin{bmatrix} (b_1^{\mathrm{d}})^{n+2} & 0 \\ 0 & 0 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \end{bmatrix}_{\mathscr{P}} \begin{bmatrix} b_1^n & x_n \\ 0 & (a_2+b_2)^n \end{bmatrix}_{\mathscr{P}} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (b_1^{\mathrm{d}})^{n+2}a_1(a_2+b_2)^n \end{bmatrix}_{\mathscr{P}} = (b_1^{\mathrm{d}})^{n+2}a_1(a_2+b_2)^n, \\
v = b^{\pi}(a+b)^{\mathrm{d}} &= (a_2+b_2)^{\mathrm{d}} = -\left\{ b^{\mathrm{d}}a(ba)^{\mathrm{d}} + \sum_{n=0}^{\infty} b^{\mathrm{d}}b^{2n+1}\left[((ab)^{\mathrm{d}})^{n+1} + ((ba)^{\mathrm{d}})^{n+1}\right] \right\}.
\end{aligned}$$

Thus, the above expression of u reduces to

$$u = \sum_{n=0}^{\infty}(b^{\mathrm{d}})^{n+2}a(a+b)^n + \sum_{n=0}^{\infty}(1-b^{\pi})b^n a(v)^{n+2} - b^{\mathrm{d}}av. \quad (21)$$

Expressions (20) and (21) finish the proof. □

5. Conclusions

In this paper, we have proved that the multiplications $(ab)^n$ and $(ba)^n$ of elements $a, b \in \mathscr{A}^{\mathrm{d}}$ in a Banach algebra are both generalized Drazin invertible with the conditions (6). A symmetry representation of the generalized Drazin inverse for $ab + ba$ has been derived. The expression given in Theorem 1 is symmetric, as in Remark 1. In the other words, if the result is applied in the computation of $(ab + ba)^{\mathrm{d}}$, maybe it will improve the corresponding computational effectiveness and reduce its complexity. The additive properties of $(a + b)^{\mathrm{d}}$ have been investigated under the conditions $ab^2 = 0$, $b^{\pi}ab \in \mathscr{A}^{\mathrm{d}}$, and $a^2 = 0$. With similar conditions, but $a^2 = 0$ being replaced by $b^{\pi}a^2 = 0$, we have also given a resulting expression of $(a + b)^{\mathrm{d}}$.

In fact, as pointed out as in [19], it is still an interesting and open problem to express the generalized Drazin inverse of $a + b$ as a function of a, b, and their respective generalized Drazin inverses. In the future, we plan to consider the representations of the generalized Drazin inverse for $a \pm b$ by using a, b, and their generalized Drazin inverses, without side conditions.

Author Contributions: Funding acquisition, Y.Q. and X.L.; Methodology, X.L.; Supervision, J.B.; Writing-review and editing, Y.Q.

Funding: This work was supported by the National Natural Science Foundation of China (grant number: 11361009, 61772006,11561015), the Special Fund for Science and Technological Bases and Talents of Guangxi (grant number: 2016AD05050, 2018AD19051), the Special Fund for Bagui Scholars of Guangxi (grant number: 2016A17), the High level innovation teams and distinguished scholars in Guangxi Universities (grant number: GUIJIAOREN201642HAO), the Natural Science Foundation of Guangxi(grant number: 2017GXNSFBA198053, 2018JJD110003), and the open fund of Guangxi Key laboratory of hybrid computation and IC design analysis (grant number: HCIC201607).

Conflicts of Interest: The authors declare that they have no conflict of interest.

References

1. Castro González, N. Additive perturbation results for the Drazin inverse. *Linear Algebra Appl.* **2005**, *397*, 279–297. [CrossRef]
2. Ben-Israel, A.; Greville, T.N.E. *Generalized Inverses: Theory and Applications*; John Wiley & Sons: New York, NY, USA; London, UK; Sydney, Australia, 1974.
3. Djordjević, D.S.; Rakočević, V. *Lectures on Generalized Inverses*; Faculty of Sciences and Mathematics, University of Niš: Niš, Serbia, 2008.
4. Wang, G.R.; Wei, Y.M.; Qiao, S.Z. *Generalized Inverses: Theory and Computations*, 2nd ed.; Developments in Mathematics; Springer: Singapore; Science Press: Beijing, China, 2018; Volume 53.

5. Campbell, S.L.; Meyer, C.D. *Generalized Inverses of Linear Transformations*; Classics in Applied Mathematics; Society for Industrial and Applied Mathematics (SIAM): Philadelphia, PA, USA, 2009; Volume 56.
6. Bulatova, A.A. Numerical solution of degenerate systems of second-order ordinary differential equations using the Drazin inverse matrix. In *Algebrodifferential Systems and Methods for Their Solution (Russian)*; VO Nauka: Novosibirsk, Russia, 1993; Volume 90, pp. 28–43.
7. Zhang, X.Y.; Chen, G.L. The computation of Drazin inverse and its application in Markov chains. *Appl. Math. Comput.* **2006**, *183*, 292–300. [CrossRef]
8. Castro-González, N.; Dopazo, E.; Martínez-Serrano, M.F. On the Drazin inverse of the sum of two operators and its application to operator matrices. *J. Math. Anal. Appl.* **2009**, *350*, 207–215. [CrossRef]
9. Qiao, S.Z.; Wang, X.Z.; Wei, Y.M. Two finite-time convergent Zhang neural network models for time-varying complex matrix Drazin inverse. *Linear Algebra Appl.* **2018**, *542*, 101–117. [CrossRef]
10. Stanimirović, P.S.; Živković, I.S.; Wei, Y.M. Recurrent neural network for computing the Drazin inverse. *IEEE Trans. Neural Netw. Learn. Syst.* **2015**, *26*, 2830–2843. [CrossRef] [PubMed]
11. Koliha, J.J. A generalized Drazin inverse. *Glasg. Math. J.* **1996**, *38*, 367–381. [CrossRef]
12. Hartwig, R.E.; Wang, G.R.; Wei, Y.M. Some additive results on Drazin inverse. *Linear Algebra Appl.* **2001**, *322*, 207–217. [CrossRef]
13. Djordjević, D.S.; Wei, Y.M. Additive results for the generalized Drazin inverse. *J. Aust. Math. Soc.* **2002**, *73*, 115–125. [CrossRef]
14. Liu, X.J.; Xu, L.; Yu, Y.M. The representations of the Drazin inverse of differences of two matrices. *Appl. Math. Comput.* **2010**, *216*, 3652–3661. [CrossRef]
15. Yang, H.; Liu, X.F. The Drazin inverse of the sum of two matrices and its applications. *J. Comput. Appl. Math.* **2011**, *235*, 1412–1417. [CrossRef]
16. Harte, R.; Mbekhta, M. On generalized inverses in C^*-algebras. *Stud. Math.* **1992**, *103*, 71–77. [CrossRef]
17. Djordjević, D.S.; Stanimirović, P.S. On the generalized Drazin inverse and generalized resolvent. *Czechoslov. Math. J.* **2001**, *51*, 617–634. [CrossRef]
18. Cvetković-Ilić, D.S.; Djordjević, D.S.; Wei, Y. Additive results for the generalized Drazin inverse in a Banach algebra. *Linear Algebra Appl.* **2006**, *418*, 53–61. [CrossRef]
19. Benítez, J.; Liu, X.; Qin, Y. Representations for the generalized Drazin inverse in a Banach algebra. *Bull. Math. Anal. Appl.* **2013**, *5*, 53–64.
20. Liu, X.J.; Qin, X.L.; Benítez, J. New additive results for the generalized Drazin inverse in a Banach algebra. *Filomat* **2016**, *30*, 2289–2294. [CrossRef]
21. Mosić, D.; Zou, H.L.; Chen, J.L. The generalized Drazin inverse of the sum in a Banach algebra. *Ann. Funct. Anal.* **2017**, *8*, 90–105. [CrossRef]
22. Castro González, N.; Koliha, J.J. New additive results for the g-Drazin inverse. *Proc. R. Soc. Edinb. Sect. A* **2004**, *134*, 1085–1097. [CrossRef]
23. Mosić, D. A note on Cline's formula for the generalized Drazin inverse. *Linear Multilinear Algebra* **2015**, *63*, 1106–1110. [CrossRef]