UNIVERSIDAD POLITÉCNICA DE VALENCIA

Departamento de Matemática Aplicada

Some compactness criteria in locally convex and Banach spaces

PhD Dissertation

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Título:

Algunos criterios de compacidad en espacios localmente convexos y de Banach.

Resumen:

Capítulo 1 Se estudian diferentes clases de conjuntos compactos. En particular, la clase de conjuntos convexo-compactos se analiza en profundidad. A partir de estas clases de conjuntos, proporcionamos criterios de compacidad mediante la verificación de un conjunto de condiciones bastante relajadas. Para asegurar que estamos realmente tratando con nociones más generales, prestamos especial atención a la separación de las clases introducidas. También proporcionamos algunos resultados sobre estabilidad de las clases de conjuntos compactos usadas. Extendemos teoremas de Valdivia y Orihuela, as como también mejoramos un teorema de Howard.

Capítulo 2 Formulamos algunos resultados sobre discos de Banach y probamos que todo subconjunto convexo y relativamente convexo-compacto de un espacio localmente convexo está contenido en un disco de Banach. Se estudian en qué casos algunas propiedades, como la separabilidad o la reflexividad, se conservan cuando se pasa a los espacios de Banach generados.

Capítulo 3 Se analizan la propiedad drop, la propiedad (α) y la condición (β). Una técnica sencilla proporciona pruebas breves de algunos resultados sobre la propiedad drop en espacios localmente convexos. Se prueba que la propiedad quasi-drop es equivalente a la propiedad drop para conjuntos numerablemente cerrados. Probamos que las propiedades drop y quasi-drop, la propiedad (α) y la condición (β) son separablemente determinadas. También estudiamos la relación entre la propiedad drop, la propiedad (α), la condición (β), la compacidad y la reflexividad.

Títol:

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Title:

Some compactness criteria in locally convex and Banach spaces.

Abstract:

Chapter 1 We study different classes of compact sets. In particular, the class of convex-compact sets is analyzed in depth. Using these classes of sets, we provide compactness criteria by checking on a quite relaxed set of conditions. In order to ensure that we are really dealing with more general notions, we pay attention to separate the classes introduced. We also provide some stability results of the classes of compact sets used. Some Valdivia and Orihuela theorems are pushed further and an extension of a theorem due to Howard is provided.

Chapter 2 We formulate some results on Banach disks and prove that every convex, relatively convex-compact subset of a locally convex space is contained in a Banach disk. We study in which cases some properties, such as separability and reflexivity, are preserved by passing to the generated Banach space.

Chapter 3 The drop property, the property (α) and the condition (β) are analyzed. A single technique provides short proofs of some results about drop properties on locally convex spaces. It is shown that the quasi-drop property is equivalent to a drop property for countably closed sets. We prove that the drop and quasi-drop properties, the property (α) and the condition (β) are separably determined. We also study the relation between drop property, property (α), condition (β), compactness and reflexivity.

Contents

0.1 Introduction

This work is presented in order to obtain the PHD degree in Mathematics. It consists of a Memoir, and it develops certain aspects of a subject that was proposed to me by my advisor, Professor Vicente Montesinos. Most of the material included we believe is new. Of course, in order to conveniently present it, we also incorporated some preliminaries, some known results (sometimes with new proofs) and some accessory material. Those different levels are carefully differentiated, in order to make it clear to the reader what is what at each stage. Certainly, not all the new material has the same importance, and we tried, as far as possible, to enhance what we think is more relevant. This is why some results wear the label of "theorems", some others of "propositions", "remarks", "lemmata" and the like, in order to give continuity and completeness to the presentation. We try to follow an order such that it makes clear what are the main results. We think that part of the purpose of this Memoir — and likewise others — is to prove that the author is able to elaborate a scientific document, one (by the way an important one) of the many that he would have to prepare along his scientific career. Then, the final coherence of the result is something to be considered, too.

The work evolves, naturally, from the scientific interest of my advisor, something that, needless to say, has been transmitted to us. This can be seen in the list of references at the end of the Memoir. Several among them are authored by the advisor and their group ([Mo78], [Mo87], [Mo91], [Mo93], [Val72-1]). The rest reflect their and our interest. We contributed already to the references with three papers on the subject of the Memoir. The first one ([MM08]) has been already published. The second one ([MM-1]) has already been accepted for publication in the Czech. Math. Journal, and the third one ([MM-2]) has been submitted already. We presented twice our contributions [MM06] and [MM07] in a congress, and delivered a talk on the subject at the Instituto de Matemática Pura y Aplicada of the Universidad Politécnica de Valencia.

Our main interest is in the theory of locally convex spaces. This is a general framework were results can be presented in a way that covers a broad spectrum. We specialize those results to particular classes of locally convex spaces (metrizable, separable, Fréchet, etc.) and, even more particularly, to Banach spaces and their duals, equipped with the norm or, quite often, with several sorts of weak topologies. Needless to say, the theory of Banach spaces provide a supply of examples and situations were our results can be checked or illustrated.

In some sense, this work must be considered unconcluded. Although we are convinced that the material presented has interest and that it is not trivial, we acknowledge that we were not able to solve some problems and we think that the work can be extended beyond the collection of results presented here. We do not think that this is a demerit. From our point of view, this is a proof, among others that have been mentioned above, that the field is alive, and we wish that we can contribute further to enlarge it in a near future. In this direction, we propose at the end of the Memoir a collection of problems that we were not able to solve, and we suggest some lines of research.

0.2 Notation, and some convergence results

A partially ordered set $(I, \leq)$ is said to be *directed upwards* if, given i_1 and i_2 in I we can find i_3 in I such that $i_1 \leq i_3$ and $i_2 \leq i_3$. We shall say that I is a *net-index set.* A *net* in a non-empty set S is a mapping $r : I \to S$, where I is a net-index set. As it is customary, we denote $(s_i)_{i\in I}$ a net in S, where $s_i := r(i)$ for every $i \in I$. If the net-index set is the set of natural numbers $\mathbb{N}$ with its natural order, we speak of a *sequence* and we write, simply, (s_n), if there is no misunderstanding.
Assume now that $(T, \mathcal{T})$ is a topological space (we shall always assume, except if explicitly stated, that a topological space is Hausdorff). Given a subset S of T, we denote by $\overline{S}$ its closure, and by $\ddot{S}$ its *sequential closure*, i.e., the set of elements in $\overline{S}$ that are limits of sequences in S. An *adherent point*—sometimes called a *cluster point*—*of a net* $(S_i)_{i\in I}$ *of subsets* of T is an element in $\bigcap_{i\in I} \overline{S_i}$. Given a subset S of T and a net-index set I, an adherent point $\overline{s}$ of the net $(S_i)_{i\in I}$, where $S_i := S$ for all $i \in I$ (i.e., $\overline{s} \in \overline{S}$), is called in short an *adherent point* (or a *cluster point*) of S. An *adherent point* —or a *cluster point*—*of a net* $(t_i)_{i\in I}$ in T is an adherent point (in the former sense) of the net of sets $(\{t_j : j \geq i\})_{i\in I}$.

The following result is almost obvious. It shows that the usual definition of a cluster point of a net coincides with the previous one. Given a net $(t_i)_{i\in I}$ let us put

$$R_{i_0} := \{t_i;\ i \geq i_0\} \text{ for all } i_0 \in I.$$

Proposition 1 *An element t in a topological space T is a cluster point of a net $(t_i)_{i\in I}$ in T if, and only if, for every neighborhood $U(t)$ of t and for every $i_0 \in I$, there exists $i \geq i_0$ such that $t_i \in U(t)$.*

Proof Let t be an adherent point of the net $(t_i)_{i\in I}$. Let $U(t)$ be a neighborhood of t and let $i_0 \in I$. Then, since $t \in \overline{R}_{i_0}$, we can find $i \geq i_0$ such that $t_i \in U(t)$.
Conversely, assume that the condition holds. Fix $i_0 \in I$ and let $U(t)$ be an arbitrary neighborhood of t. We can find $i \geq i_0$ such that $t_i \in U(t)$. This implies that $U(t) \cap R_{i_0} \neq \emptyset$. Since $U(t)$ is arbitrary, this proves that $t \in \overline{R}_{i_0}$. This is true for all $i_0 \in I$, hence $t \in \bigcap_{i\in I} \overline{R}_i$. ∎

Given a net (s_i) in a set S, a *subnet* is a mapping $\phi : J \to I$, where J is a net-index set and the mapping ϕ satisfies the following: given $i_0 \in I$ there exists $j_0 \in J$ such that $\phi(j) \geq i_0$ for every $j \in J$, $j \geq j_0$. The subnet ϕ of the net $(s_i)_{i\in I}$ will be denoted $(s_{i_j})_{j\in J}$, where $i_j := \phi(j)$ for all $j \in J$.
Let $(T, \mathcal{T})$ be a topological space. As it is well known and easy to prove, an element $t \in T$ is a cluster point of a net $(t_i)_{i\in I}$ if and only if there exists a subnet $(t_{i_j})_{j\in J}$ of $(t_i)_{i\in I}$ that converges to t.

The following statement has a very simple proof. We record it here for future references.

Proposition 2 *Given a* sequence (t_n) *in a topological space T, and letting $S := \{t_n;\ n \in \mathbb{N}\}$, every point $t \in \overline{S} \setminus S$ is an adherent point of the sequence (t_n).*

Proof Let $U(t)$ be an arbitrary neighborhood of t. Assume for a moment that $U(t) \cap S$ is a finite set. Then we can find another neighborhood $U'(t)$ of t such that $U'(t) \cap S = \emptyset$, since $t \notin S$. This is a contradiction, so $U(t) \cap S$ is indeed an infinite set. In particular, given $n_0 \in \mathbb{N}$ we can find $n \geq n_0$ such that $t_n \in U(t)$. In view of Proposition 1, t is a cluster point of the sequence (t_n). ∎

Observe that the behavior described in Proposition 2 is not true in general in the case of a net. For example, let $A := [0, 1) \cup (1, 2]$. For the net $(r)_{r\in A}$, where A is endowed with the natural order in $\mathbb{R}$, 1 is not an adherent point. However, it belongs to $\overline{A} \setminus A$.

A net in a topological vector space is called *null* if it converges to 0.

In this Dissertation, $(E, \mathcal{T})$ will denote a (real Hausdorff) locally convex space, E^* the algebraic dual of E, i.e., the space of all linear functionals on E, and E' the topological dual of E, i.e., the space of all continuous linear functionals on E. We shall always assume that E is (canonically) a subset of the space $(E')^*$. In general, if $\langle E, F\rangle$ is a dual pair, the associated weak topology on E will be denoted $\sigma(E, F)$ or, indistinctly, $w(E, F)$. The *Mackey topology*, i.e., the topology on E of the uniform convergence on the family of all absolutely convex and weakly compact subsets of F, will be denoted $\mu(E, F)$. If $F := E'$, the weak topology $w(E, E')$ will be denoted sometimes by w, if it does not lead to any misunderstanding. A word of warning: in the current literature on Banach spaces, the topological dual of a Banach space X is denoted by X^*, and the topology $w(X^*, X)$ is always denoted by w^*. It is tempting to change to this notation when speaking about Banach spaces. However, it will be quite misleading, so it seems better to stick to one consistent notation in this Memoir.

If $(E, \mathcal{T})$ is a locally convex space and a set $A \subset E$ is given, we shall write $(A, \mathcal{T})$ for the topological space A endowed with the restriction of the topology $\mathcal{T}$ if there is no misunderstanding.

If $(E, \mathcal{T})$ is a topological vector space and A is a subset of E, we denote by $\mathrm{conv}(A)$ the convex hull of A, and by $\Gamma(A)$ the absolutely convex hull of A, i.e,, the convex hull of the balanced hull of A (i.e., $\Gamma(A) := \{\sum_{i=1}^n \lambda_i a_i;\ a_i \in A,\ \sum_{i=1}^n |\lambda_i| \leq 1,\ n \in \mathbb{N}\}$). By $\overline{\mathrm{conv}}\,(A)$ and $\overline{\Gamma}(A)$ we denote the closed convex hull and the absolutely closed convex hull of A, respectively. The linear span and the closed linear span of A are denoted, respectively, by $\mathrm{span}(A)$ and $\overline{\mathrm{span}}(A)$.

The next result is simple. However, it will turn to be very useful in the rest of the Memoir. We record it here for future references.

Proposition 3 *Let $(E, \mathcal{T})$ be a locally convex space. Let (x_n) be a sequence in E. For $n \in \mathbb{N}$, put $K_n := \overline{\mathrm{conv}}\,\{x_n, x_{n+1}, \ldots\}$. Then,*
(i) *if $x \in E$ a cluster point of (x_n), we get $x \in \bigcap_{n=1}^\infty K_n$;*
(ii) *if, moreover, (x_n) converges to $x \in E$, then $\bigcap_{n=1}^\infty K_n = \{x\}$.*

Proof. (i) If x is a cluster point of (x_n), then $x \in K_n$ for all $n \in \mathbb{N}$. It follows that $x \in \bigcap_{n=1}^\infty K_n$.
(ii) Assume now that $x_n \to x$. Let $x_0 \in \bigcap_{n=1}^\infty K_n$. Fix an arbitrary closed convex neighborhood $U(x)$ of x. We can find $n_0 \in \mathbb{N}$ such that $x_n \in U(x)$ for

all $n \geq n_0$. Then $K_n \subset U(x)$ for all $n \geq n_0$. It follows that $x_0 \in U(x)$. Since the family of all closed convex neighborhoods of x is a base of neighborhoods of x in E, we get $x = x_0$. ∎

We shall need later on the following simple result.

Proposition 4 *Let $(E, \mathcal{T})$ be a locally convex space. Let (x_n) be a sequence in E and $x \in E$. Then, the following are equivalent.*
(i) *(x_n) is w-convergent to x.*
(ii) *For every subsequence (x_{n_k}) of (x_n), we have $x \in \bigcap_{k=1}^{\infty} C_k$, where $C_k := \overline{\text{conv}}\,\{x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \ldots\}$, $k \in \mathbb{N}$.*

Proof Assume first that (ii) holds but $x_n \not\to x$ in the weak topology. Therefore we can find $x' \in E'$ such that $\langle x_n, x' \rangle \not\to \langle x, x' \rangle$. This implies the existence of some $\varepsilon > 0$ and a subsequence (x_{n_k}) of (x_n) such that $|\langle x_{n_k} - x, x' \rangle| \geq \varepsilon$ for all $k \in \mathbb{N}$. Without loss of generality we may assume that $\langle x_{n_k}, x' \rangle \leq \langle x, x' \rangle - \varepsilon$ for all $k \in \mathbb{N}$. We get then that $\langle c, x' \rangle \leq \langle x, x' \rangle - \varepsilon$ for all $c \in C_k := \overline{\text{conv}}\,\{x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \ldots\}$, and for all $k \in \mathbb{N}$. Since (ii) holds, we have $x \in C_k$, for every $k \in \mathbb{N}$, and then $\langle x, x' \rangle \leq \langle x, x' \rangle - \varepsilon$, a contradiction.
Assume now that (i) holds. Then, given an arbitrary subsequence (x_{n_k}) of (x_n), we have $x_{n_k} \to x$. Apply (ii) in Proposition 3. Then $\{x\} = \bigcap C_k$, where $C_k := \overline{\text{conv}}\,\{x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \ldots\}$, $k \in \mathbb{N}$. This implies (ii) here. ∎

We refer to [Ko69] for concepts and symbols in the area of topological vector spaces that are not defined here, and to [FHHMPZ] for specific concepts and symbols in the field of Banach space theory.

0.3 Summary

0.3.1 Chapter 1

In this chapter we recall some of the most useful classical concepts around compactness, such as sequential compactness, countable compactness and (plain) compactness. There are many other more general concepts that play a role in the theory. They have been introduced along several papers and with different purposes. For example, closed subsets of a locally convex space where every functional of the dual space is bounded and attains its supremum

were considered by James in his famous characterization of weak compactness, first in Banach spaces, then in general locally convex spaces. Sets in a locally convex space E with the property that a decreasing sequence of closed and convex subsets of E that intersect the set, has an adherent point, were considered by Šmulian. Sets on which real and continuous functions are bounded were treated by Pták and Valdivia, among others. Sets interchanging limits with absolutely convex and weakly compact subsets of the dual space were considered by Pták and Grothendieck. Besides the already mentioned famous characterization of weak compactness given by James, the real classical one is what nowadays is known as the Eberlein-Šmulian Theorem, stating that the classes of weakly relatively countably compact, weakly relatively sequentially compact and weakly relatively countably compact subsets of a Banach space all coincide.
In this chapter we list those mentioned properties and several others. Our main purpose is to provide compactness criteria by checking on a quite relaxed set of conditions. In order to ensure that we are really dealing with more general notions, we pay attention to separate the classes introduced.
So, we start by proving some results on stability of the classes of compact sets used. For example, Proposition 7 analyzes stability under continuous images. Proposition 8 collects some results on the class of (relatively) pseudocompact sets. Subsection 1.3.3 deals with one of the main subjects of this Memoir, the class of (relatively) convex-compact subsets os a locally convex space. This class, introduced by Šmulian, as we mentioned earlier, although lacking some good stability properties (see, specially, Examples 39, 40 and 44, and Remark 41), is quite apt for many of the results that are important in compactness. For example, it allows to extend one of the most general results proven in the area, Theorem 77 due to Orihuela, to the class of the weakly (relatively) convex-compact subsets of a locally convex space E such that its dual, with the $w(E', E)$-topology, is a web-compact space. In Theorem 17 we prove that the circled cover of a closed convex-compact set is again convex-compact—a result that is not, at least according to the proof provided here, completely straightforward. The stability by taking closed convex covers is ensured in the framework of Krein's theorem, since in this case our set is already compact (see Corollary 19). We prove afterwards some instrumental results on convex-compactness. Related to w-strong partial compactness (w-partial compactness) we prove in Proposition 24 (respectively, Proposition 32) a more handy characterization, and then some stability results.
The existing hierarchy among all those concepts is established in Proposition

36. Careful separation of all of them is done along Examples 39, 42, 43, and Corollary 83.
Section 1.4 revisits two fundamental results in weak compactness, James' theorem and Eberlein-Šmulian-Grothendieck-Pták-Dieudonné-Schwartz-Valdivia-Pryce-De Wilde-Orihuela theorem. We mention James' example showing that completeness is a must in his norm-attaining theorem (Example 48). We formulate characterizations of the mentioned notions of compactness in a way related to the supremum-attaining condition (Propositions 28 and 34). We show how James' theorem allows to prove a compactness result for the class of relatively partially compact sets in $\mu(E, E')$-quasi-complete locally convex spaces (Corollary 51) and we provide the correct proof of a slight extension of a result of Montesinos (Theorem 59). This result will be pushed farther in Theorem 81.
The fundamental work [Ori87] considers a very general situation where angelicity is obtained. Orihuela introduced the class of web-compact topological spaces as a natural extension of the class of topological spaces X having a countable collection of relatively countably compact subsets whose union is dense in X. It was proved in [Ori87] that $C_p(X)$, the space of all real continuous functions on X endowed with the topology of the pointwise convergence, is an angelic space. Angelicity means that every relatively countably compact subset is already relatively compact; moreover, the closure of such a set is reached by sequences. The application to the locally convex setting is that a locally convex space E having a $w(E', E)$-web-compact dual is, in its weak topology, angelic. In particular, in those spaces the classes of weakly (relatively) countably compact, weakly (relatively) sequentially compact and weakly (relatively) compact coincide. We prove in Theorem 81 that in such locally convex spaces every weakly (relatively) convex-compact subset is weakly (relatively) compact. This is proved via a slight variation of the argument on interchangeable limit property used by Orihuela in his fundamental paper [Ori87]. The short Section 1.7 has two goals: first, to mention the easy observation that every weakly (relatively) partially compact subset of a locally convex space has the property that every continuous linear functional on the space attains its supremum on it (on its closure), and then to separate, even in the context of normed space, the two concepts by using James' example. Section 1.8 deals with the following observation: results in weak compactness often deal with the relative-to-relative statement, i.e., how to obtain, for example, weak relative compactness from weak relative countable compactness. Sometimes, to remove the word "relative" is not

easy/possible. An example (Example 84) is given showing that in James' Theorem this is simply not true. A result of Howard (Theorem 85) states that in the dual of a Banach space X endowed with the Mackey topology $\mu(X', X)$ every relatively sequentially compact is relatively compact. We give a sufficient condition in Theorem 86 to ensure that we can remove the word "relative" from Howard statement. Moreover, we provide a extension of Howard's result by proving that, in fact, in the same setting every relatively countably compact subset is already relatively compact. Since every relatively sequentially compact set is certainly relatively countably compact, this is indeed an extension.
In the short Section 1.9 we observe that a classical result of Klee (see, e.g., [Ko69, §24.4(3)]) can be formulated in a more precise way that has an interesting geometrical meaning: it is obvious that a quasi-reflexive locally convex space E has the property that for *every* closed hyperplane and every bounded, closed and absolutely convex subset of E there exists a parallel hyperplane supporting the set. What is surprising is that (Theorem 92), in Mackey quasi-complete space, the existence of a *single* closed hyperplane with this property ensures quasi-reflexivity. In the case of Banach spaces, we prove in Theorem 93 that a single closed hyperplane and *only* closed unit balls of equivalent norms suffice for reflexivity. The last result in this chapter is purely instrumental: it ensures that convex-compact sets in locally convex spaces are sequentially complete. This will be used later.

0.3.2 Chapter 2

This short chapter has only a preparatory character. The goal is to formulate some results on Banach disks that will be used in the next chapter. Along the way we organize some of the existing material, starting from the very basic methods for proving that a certain disk in a locally convex space is indeed a Banach disk. We recover as a consequence some of the classical results in the area. For our purposes, it is worth to ensure that every convex, relatively convex-compact subset of a locally convex space is contained in a Banach disk (Corollary 104), and that every absolutely convex (relatively) convex-compact subset of a locally convex space is indeed a Banach disk (Corollary 106). This, according to the Banach-Mackey theorem, ensures that those sets are strongly bounded. The result about Banach disks is not true for the class of absolutely convex closed and w-strongly partially compact subsets of locally convex spaces (Example 107). We are interested in elucidating which properties of a set in a locally convex space carry over the normed space they generate. In general, there is lack of stability in this sense. For instance, the most basic property, separability, is not preserved by passing to the generated Banach space (Example 109). We include a particular case where there is a positive answer (Proposition 110), and a corollary for the Fréchet case (Corollary 112). Reflexivity also lacks stability in this context (Example 113). The well-known factorization theorem of Davis, Figiel, Johnson and Pełczyński holds in the setting of Fréchet spaces (Theorem 115). In contrast with this result Valdivia proved that in every infinite-dimensional Fréchet space there exists an absolutely convex and compact subset that generates a non-reflexive Banach space. We provide a new proof that uses an interesting result of V. Pták on biorthogonal systems [Pt59]. We also give a new proof of a slight extension of Pták's result based on the well-known James' characterization of reflexivity. Pták's result lies at the origin of a fundamental result of Argyros and Mercourakis on Markushevich bases in weakly compactly generated Banach spaces [ArMe052]. In Examples 127 and 128 we show that metrizability and completeness are really needed in Valdivia's result.

0.3.3 Chapter 3

This chapter deals with a property that plays an important role in optimization theory, and that it is closely related to variational problems. It was motivated by a theorem of Daneš [Da72] in the context of Banach spaces and, because of the shape of the geometrical object involved (the convex hull of a non-empty closed convex subset and a point not in the set) it is known as the *drop property*. As far as we know, the property as it is was defined by S. Rolewicz [Ro85] (again in the context of Banach spaces), and it has been extensively studied since then by many authors (S. Rolewicz himself, J. Daneš, J. Penot, V. Montesinos, D. Kutzarova, A. Maaden, J. H. Qiu, J. R. Giles, B. Sims, A. C. Yorke, L. Cheng, Y. Zhou, F. Zhang, among others). Daneš himself proved in [Da72] that his theorem is equivalent to many other results appeared afterwards (as the so-called Penot's Petal Theorem, [Pe86]). Variants of the drop theorem, obtained by modifying the basic object—adding smoothness, for example, what amounts to what has been called accordingly the *smooth drop property* in [Maad95b] and [GKM96]—have been considered. The theory has been also developed in the more general context of locally convex spaces. There is a fairly big amount of results that are collected in the literature at the end of this Memoir. Reviewing the references, we realized

that many of the results in this more general ambiance can be deduced from classical results in Banach space theory, as the original Daneš Theorem. Part of our goal here was then to clarify this situation and, certainly, to use our approach to provide new statements in the area.
We start this chapter by providing the basic definitions and recording some of the previous results given. This is done in Section 3.1. At the beginning of Section 3.2 we unify the nomenclature given to properties considered by several authors in the context of topological vector spaces. Then we extend several results given in [Q03a] and [Q04] to the more general setting of a locally convex space endowed with a topology that is not necessarily the weak topology. This is not just an rephrasing of former results in a wider context; instead, we provide a different technique. Basically, it consists in embedding certain sets in a single Banach space by using the procedures developed in Chapter 2. In this way we prove that in a locally convex space, closed convex and sequentially compact subsets have the drop property (Theorem 141) and that closed convex and countably compact subsets have the quasi-drop property (Theorem 142). With the same technique we prove that, under some separation property, the drop property holds for locally closed bounded and convex subsets of a locally complete locally convex space as "base sets" and locally closed sets as "target sets" (Theorem 143). Theorem 146 is a neat improvement of known results: we are able to ensure that closed convex convex-compact subsets of locally convex spaces satisfy the drop property for countably closed subsets disjoints from them. Essentially we proved that a convenient embedding in a Banach space provides the result by noticing that, in the case treated, the two sets are at positive distance each other. It will be enough then to use the classical Daneš Theorem. Along the proof we note that, apparently, non-empty closed convex countably compact subsets of a locally convex space enjoy a stronger property than the quasi-drop, precisely the possibility to find "drop points" for every non-empty countably closed disjoint subset. We prove in Theorem 149 that this two seemingly different concepts in fact coincide. This is not just a curiosity; in fact, it is used to prove, for example, that the quasi-drop and drop properties are separably determined, something that is not obvious from the very definition (Corollaries 150 and 151).
We pass then to study the drop and quasi-drop properties for closed convex and unbounded subsets of a locally convex space. This has been done in the context of Banach spaces (see the references at the end of the Memoir). Again, the techniques used in the more general setting of locally convex

spaces are not just a mere transposition of the ones used in Banach spaces. We believe that our approach here is again original. We introduce a function g_a that measures naturally the position of a point in a drop. It is interesting to note that this function is strictly decreasing on streams (Lemma 152). Lemma 153 is also a very useful tool: although a drop defined by a convex unbounded set is not necessarily closed (something that can be seen easily), it contains the closures of all "smaller drops". This two tools play a very important role in the proof of Theorem 154, extending Theorem 149 to the unbounded situation. Proposition 156 proves that the quasi-drop property of a closed convex subset of a locally convex space implies that all sections defined by bounded-above continuous linear functionals—the single ones that can define sections—are bounded.
Section 3.3 deals with another property that is closely related to the drop property, namely what has been called *property* (α). This was introduced again by Rolewicz ([Ro87]) for Banach spaces, and was studied also by V. Montesinos, D. Kutzarova and P. L. Papini, among others. The difficulty to deal with the Kuratowski index of non-compactness in this context is solved by using translates of neighborhoods of zero to cover sections. Of course, the notion is interesting when using topologies other than the weak topology, since every bounded subset of a locally convex space is weakly precompact. Most of the sets having property (α) have bounded slices.
We prove in Theorem 162 that every closed convex subset with the quasi-drop property in a locally convex space has property (α). To prove this result we use some ideas of Rolewicz. Certainly, the second property is strictly more general than the first one. We provide an example in normed spaces by using differentiability theory (Example 164). An example separating property (α) and the quasi-drop property in a locally convex space (Example 165) by using a result of Qiu (Theorem 166) is given by exhibiting and adequate closed convex subset that is not weakly compact. In the context of Banach spaces, and providing a certain unbounded closed and convex subset, we separate the drop property and property (α). The example is somehow involved. It is convenient to quote what we say in Remark 168: "This example clarifies the main result in [Mo93]. It was proved there that if a Banach space X contains an unbounded closed convex subset B with property (α) *and such that* $\operatorname{int}(B) \neq \emptyset$, then X is reflexive. No example was given to ensure that this last property was, indeed, necessary. One can think that, as soon as such an unbounded closed convex set B with property (α) should lie inside a closed hyperplane, it will obviously had an empty interior and this will

provide the desired example. However, it was proved in [Mo93, Corollary 3.3] that if B is such a set, then $\overline{\Gamma(B)} = X$, so this procedure turns out to be inadequate. The previous example gives an unbounded closed convex set B having property (α); moreover, it is obviously not compact nor it has a non-empty interior. By [KR91, Theorem 3], it does not have drop property. The space X is not reflexive, and this proves that the condition about the existence of a non-empty interior in [Mo93, Proposition 3.4] is unavoidable."

The use of James's compactness theorem allows us to prove that closed convex and bounded subsets with property (α) of a quasi-complete locally convex spaces are weakly compact (Theorem 169). We provide two proofs of this result. The first one we believe has an intrinsic interest because the treatment of nets in sets with small Kuratowski index of non-compactness. The second proof uses just an argument about precompactness, and its interest lies in the fact that maximizing sequences are shown to be precompact (Proposition 170). As a corollary we obtain a characterization of reflexivity in quasi-complete locally convex spaces (Corollary 172). We provide examples to ensure that a weakening of the quasi-completeness hypothesis to local completeness (Example 173) or to sequential completeness (Example 174) is not possible.

Again, we prove that property (α) is separably determined (Theorem 176).

Since boundedness of sections of a certain closed convex subsets of a Banach space is a subject that appears recursively in studies of property (α) is Banach spaces, we analyze this behavior in full in Proposition 181 to help to clarify several statements appearing in the literature ([Mo91], [Mo93], [KR91]). In Example 182 we exhibit an unbounded closed and convex subset of a Banach space such that no section defined by bounded-above continuous functionals is precompact.

Section 3.4 considers, finally, another property that has been treated in the context of Banach spaces, the so-called *condition* (β). This was also introduced by Rolewicz ([Ro87]). It is more restrictive than property (α) (Theorem 186). Still, it does not implies boundedness (Example 187). However, it has bounded slices (Proposition 188). We prove in Theorem 189 that, in quasi-complete locally convex spaces, condition (β) implies the quasi-drop property. We prove, too, that condition (β) is separably determined (Theorem 191).

0.4 A matter of contrast

Parts of this Memoir have been submitted to standard verification procedures. We contributed with three papers on the subject of the Memoir. The first one (*Drop property in locally convex spaces*, [MM08]) has been already published in Studia Math. In this paper it is shown that the quasi-drop property is equivalent to a drop property for countably closed sets. As a byproduct, we prove that the drop and quasi-drop properties are separably determined. The second one (*Compact-convex sets and Banach disks*, [MM-1]) has already been accepted for publication in the Czech. Math. Journal. In this paper we present some of the results that we develop in this Memoir about compactness and Banach disks. The third one (*Property (α) in locally convex spaces*, [MM-2]) has been submitted already to Studia Math. In this paper, we study the relation of property (α) with the drop property, compactness and semi-reflexivity.
Moreover, we presented twice our contributions at mathematical meetings, namely *Compacidad convexa*, [MM06], and *Drop and α properties on locally convex spaces*, [MM07]), in the VII and VIII Jornadas de Matemática Aplicada of the Universidad Politécnica de Valencia, respectively, and delivered a talk on the subject at the Instituto de Matemática Pura y Aplicada of the Universidad Politécnica de Valencia.

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Chapter 1
Some results on weak compactness

1.1 Introduction

One of the most important theorems about weak compactness in Banach spaces says that the classes of weakly compact sets, weakly countably compact sets and weakly sequentially compact sets agree, and the same happens for the corresponding relative concepts (Theorem of Šmulyan-Eberlein). This is not always the case in some other important classes of locally convex spaces. A good deal of information is available in these more general situations, specially in cases when some sort of completeness or metrizability is present. In this chapter we provide some results which allow to check compactness with a minimum of requirements.

We first present the definitions of compactness used and give a brief description of the relations between them. Good references are [Ko69] and [Fl80]. The concepts and symbols not defined here can be found in [Ko69].

The following definition collects several concepts in compactness. The three first ones are classical and they are related to the Eberlein-Šmulyan Theorem. To the fourth one we shall devote in this chapter some attention. The sixth will be used to push further Šmulyan's Theorem (see Theorem 59 and Corollary 63). The fifth and seventh are treated in Köthe's book [Ko69]. One extra concept will be considered, too: weakly closed subsets of a locally convex space such that every continuous linear functional attains its supremum, naturally related to the famous James' theorem. They form a natural extension of the class of closed convex-compact subsets of a locally convex space in its weak topology when the decreasing sequence of convex sets involved are slices defined by a continuous linear funcional (see Propositions 28, 34 and Remark 49).

1.2 Definitions

Definition 5 *Let A be a subset of a locally convex space $(E, \mathcal{T})$. Then A is said to be*

1. (Relatively) compact (in short (R)K) *if every net in A has a subnet that converges to some point in A (respectively, that converges).*

2. (Relatively) countably compact (in short (R)NK) *if every sequence in A has a subnet that converges to some point in A (respectively, that converges) or, equivalently, if every sequence in A has an adherent point in A (respectively, in E).*

3. (Relatively) sequentially compact (in short (R)SK) *if every sequence in A has a subsequence that converges to some point in A (respectively, that converges).*

4. (Relatively) convex-compact (in short (R)CK) *if the following holds: suppose that $K_1 \supset K_2 \supset \ldots$ is a decreasing sequence of closed convex subsets of E such that all the intersections $K_n \cap A$ are non-empty; then the sequence $(K_n \cap A)$ has an adherent point in A (respectively, has an adherent point).*

5. (Relatively) pseudo-compact (in short (R)ΨK) *if every continuous real-valued function f defined on A is bounded (respectively, if for every unbounded continuous real-valued function f defined on A, there exists a point a in $\overline{A}$, such that f is unbounded in all its neighborhoods).*

6. Weakly (relatively) strongly partially compact (in short w-(R)ΞK) *if it is bounded and the following holds: given a sequence (a_n) in A and a $w(E'^*, E')$-adherent point a'^* of (a_n) in E'^*, then, for every sequence (x'_n) in E', there exists a point $a \in A$ (respectively, $a \in \overline{A}^w$) such that* $\lim_n \langle a'^* - a, x'_n \rangle = 0$ *for all $n \in \mathbb{N}$.*

7. Weakly (relatively) partially compact (in short w-(R)∂K) *if it is bounded and the following holds: given a sequence (a_n) in A and a $w(E'^*, E')$-adherent point a'^* of (a_n) in E'^*, then, for every sequence (x'_n) contained in an absolutely convex and $w(E', E)$-compact subset of E', there exists a point $a \in A$ (respectively, $a \in \overline{A}^w$) such that* $\lim_n \langle a'^* - a, x'_n \rangle = 0$.

The concepts (R)K, (R)NK, (R)SK and (R)ΨK make sense in every topological space. A set in a locally convex space $(E, \mathcal{T})$ is said to be *weakly (relatively) compact* (in short w-(R)K) if it is (relatively) compact in the weak topology. Analogous definitions apply to w-(R)NK, w-(R)SK, w-(R)CK and w-(R)ΨK. The concepts (R)ΞK and (R)∂K, from the very definition, are only considered in the weak topology.

Remark 6 Boundedness was a must for the concepts w-(R)ΞK and w-(R)∂K. That it was not included in the definition of the other concepts is due to the fact that all of them imply already boundedness (see Proposition 8 and 9). See also Remark 30.

1.3 Basic results

Some preliminary information about those concepts is given in the next propositions. Some of them appear in [Ko69, §24.3(3)]; we include here the statements and proofs for the sake of completeness.

1.3.1 Continuous images

Proposition 7 *Let ♡ be one of the symbols* S, N, *or just no symbol at all, and ◇ one of the symbols* C, Ξ, *or* ∂. *Then,*
(i) *If S and T are topological spaces, $f : S \to T$ is a continuous mapping, and A is a* (R)♡K *subset of S, then $f(A)$ is* (R)♡K.
(ii) *If E and F are locally convex spaces, $T : E \to F$ is a linear continuous mapping, and A is a* (R)◇K *subset of S, then $T(A)$ is* (R)◇K. *(In the case of ◇ being Ξ or ∂, the topology considered on both E and F is the weak topology.)*
(iii) *If S and T are topological spaces, $f : S \to T$ is a continuous mapping, and A is a* ΨK *subset of S, then $f(A)$ is* ΨK. *If S and T are metrizable, T is moreover complete, and f is uniformly continuous, then $f(A)$ is* RΨK *if A is* RΨK.

Proof (i) If A is K, then obviously $f(A)$ is compact. If A is RK, we have $f(A) \subset f(\overline{A}) \subset \overline{f(A)}$ and $f(\overline{A})$ is K, hence $f(\overline{A}) = \overline{f(A)}$. This will be used later on. In particular, $f(A)$ is RK.

If A is (R)NK, and $(f(a_n))$ is a sequence in $f(A)$, there exists a subnet (a_{n_i}) of (a_n) that converges to some $a \in A$ (that converges). Then $f(a_{n_i})$ converges to $f(a) \in f(A)$ (converges). This proves that $f(A)$ is (R)NK.

If A is (R)SK, and $(f(a_n))$ is a sequence in $f(A)$, there exists a subsequence (a_{n_k}) of (a_n) that converges to some $a \in A$ (that converges). Then $f(a_{n_k})$ converges to $f(a) \in f(A)$ (converges). This proves that $f(A)$ is (R)SK.

(ii) Assume now that A is (R)CK. Let (K_n) be a decreasing sequence of closed convex subsets of F such that $K_n \cap T(A) \neq \emptyset$ for every $n \in \mathbb{N}$. First of all, for all $n \in \mathbb{N}$ the set $T^{-1}(K_n)$ is convex and closed in E, and $T^{-1}(K_n) \cap A \neq \emptyset$. Observe, too, that the sequence $(T^{-1}(K_n))$ is also decreasing. By the fact that A is (R)CK we get a point $x \in A \cap \bigcap_n \overline{T^{-1}(K_n) \cap A}$ ($x \in \bigcap_n \overline{T^{-1}(K_n) \cap A}$). It follows that $T(x) \in T(A) \cap \bigcap_n \overline{K_n \cap T(A)}$ ($T(x) \in \bigcap_n \overline{K_n \cap T(A)}$). This proves that $T(A)$ is (R)CK.

Let A be a w-(R)∂K subset of E. Let (f'_n) be a sequence in F' that is contained in an absolutely convex and $w(F', F)$-K subset M of F'. The mapping T' is $w(F', F)$-$w(E', E)$-continuous, hence $T'(M)$ is (absolutely convex and) $w(E', E)$-K. The mapping T'' is $w(E'', E')$-$w(F'', F')$-continuous. Since A is bounded, $\overline{A}^{w(E'',E')}$ is $w(E'', E')$-K. Then $T''(\overline{A}^{w(E'',E')}) = \overline{T(A)}^{w(F'',F')}$ (see the first paragraph in this proof). Let $f'' \in \overline{A}^{w(F'',F')}$. Then we can find $e'' \in \overline{A}^{w(E'',E')}$ such that $T''(e'') = f''$. There exists $a \in A$ ($a \in \overline{A}^{w}$) such that $\lim_n \langle e'' - a, T'(f_n) \rangle = 0$. Then, $\lim_n \langle T''(e'') - T(a), f'_n \rangle = 0$, i.e., $\lim_n \langle f'' - T(a), f'_n \rangle = 0$. Since $T(a) \in T(A)$ ($T(a) \in \overline{T(A)}^{w}$), this proves that $T(A)$ is w-(R)∂K.

The proof of the w-(R)ΞK case is similar. We do not need now to ensure that the sequence of functionals is contained in an absolutely convex and weakly compact set.

(iii) Assume that A is ΨK. Let $g : f(A) \to \mathbb{R}$ be a continuous map. Then $g \circ f : A \to \mathbb{R}$ is continuous, hence bounded. This implies that g is bounded, so $f(A)$ is ΨK.

If S and T are metrizable, and T is moreover complete, every uniformly continuous map $f : A \to f(A)$ can be extended to a (uniformly) continuous map $\overline{f} : \overline{A} \to T$. Assume that $g : f(A) \to \mathbb{R}$ is a continuous unbounded function. Then $g \circ f : A \to \mathbb{R}$ is also continuous and unbounded. By the fact that A is RΨK we get an element $\overline{a} \in \overline{A}$ such that f is unbounded on $U(\overline{a}) \cap A$ for every neighborhood $U(\overline{a})$ of $\overline{a}$. The element $\overline{f}(\overline{a})$ belongs to

$\overline{f(A)}$. Let $V := V(\overline{f}(\overline{a}))$ be a neighborhood of $\overline{f}(\overline{a})$. Then $(\overline{f})^{-1}(V)$ is a neighborhood of $\overline{a}$, hence $g \circ f$ is unbounded on $(\overline{f})^{-1}(V) \cap A$. This implies that g is unbounded on V. Since V was chosen arbitrarily, this proves that $f(A)$ is RΨK. ∎

1.3.2 (R)ΨK

Proposition 8 *Let $(E, \mathcal{T})$ be a locally convex space and A a subset of E.*
(i) *If A is* ΨK *then it is* RΨK.
(ii) *If A is* RΨK *then $\overline{A}$ is* ΨK.
(iii) *If A is* (R)ΨK *and E is a subspace of the locally convex space $(F, \mathcal{T})$, then A is* (R)ΨK *in F.*
(iv) *If A is* (R)ΨK *and $\mathcal{S}$ is a locally convex topology in E coarser than $\mathcal{T}$, then A is* (R)ΨK *in $(E, \mathcal{S})$.*
(v) *If A is* (R)ΨK *then it is bounded.*

Proof (i) is trivial, since in this case there is no continuous unbounded real function defined on A.
(ii) Assume that $f : \overline{A} \to \mathbb{R}$ is continuous and unbounded. Then $f|_A$ is also continuous and unbounded. This implies that there exists $\overline{a} \in \overline{A}$ such that f is unbounded on $U(\overline{a}) \cap A$, where $U(\overline{a})$ is an arbitrary neighborhood of $\overline{a}$. Hence, for a given $n \in \mathbb{N}$ there exists a net (a_i) in A that converges to $\overline{a}$ and such that $|f(a_i)| \geq n$ for all i. By continuity, $|f(\overline{a})| \geq n$. This happens for all $n \in \mathbb{N}$, and we reach a contradiction.
(iii) Assume that $f : A \to \mathbb{R}$ is continuous and unbounded. Then we can find $\overline{a} \in \overline{A}^E$ such that f is unbounded on $U(\overline{a}) \cap A$ for all neighborhood $U(\overline{a})$ of $\overline{a}$ in E. Certainly, $\overline{a} \in \overline{A}^F$. Given a neighborhood $V(\overline{a})$ of $\overline{a}$ in F, $V(\overline{a}) \cap E$ is a neighborhood of $\overline{a}$ in E, and f is unbounded on $V(\overline{a}) \cap E \cap A$ $(= V(\overline{a}) \cap A)$. This proves the assertion for the RΨK case. The ΨK case is trivial (also a trivial consequence of (iii) in Proposition 7).
(iv) Every $\mathcal{S}$-neighborhood is a $\mathcal{T}$-neighborhood, and as a consequence, every $\mathcal{S}$-continuous function $f : A \to \mathbb{R}$ is also $\mathcal{T}$-continuous. The ΨK case follows, too, from (iii) in Proposition 7.
(v) If A is (R)ΨK then, by (i) and (ii), $\overline{A}$ is ΨK. It follows that $|e'|$ is bounded on $\overline{A}$ for all $e' \in E'$. This concludes that $\overline{A}$ (and hence A) is bounded. ∎

1.3.3 (R)CK

We start by collecting some easy results about (relatively) convex-compact sets.

Proposition 9 *Every RCK subset of a locally convex space is bounded.*

Proof Let A be a RCK subset of the locally convex space E. Let $f \in E'$. Assume that $|f|$ is unbounded on A. We may assume that there exists a sequence (a_n) in A such that $f(a_n) \geq n$ for all $n \in \mathbb{N}$. Put $K_n := \{x \in E;\ f(x) \geq n\}$, for $n \in \mathbb{N}$. Each K_n is a closed convex subset of E such that $K_n \cap A \neq \emptyset$, and the sequence (K_n) is decreasing. It follows that there exists $a \in \overline{A}$ in $\bigcap_n \overline{A \cap K_n}$. In particular, $f(a) \geq n$ for all $n \in \mathbb{N}$. This is impossible. This proves that $|f|$ is bounded on A for every $f \in E'$. Therefore, A is bounded. ∎

Remark 10 Let $(E, \mathcal{T})$ be a locally convex space. Since the identity mapping $I : (E, \mathcal{T}) \to (E, w)$ is continuous, it follows from (ii) in Proposition 7 that every $\mathcal{T}$-RCK subset A of E is w-RCK; apply now Proposition 9 to conclude that A is bounded.

Proposition 11 *A closed RCK subset of a locally convex space is CK.*

Proof. Assume that A is a RCK subset of a locally convex space $(E, \mathcal{T})$. Let (K_n) be a decreasing sequence of closed convex subsets of E such that $K_n \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Then, if A is closed,

$$\left(\bigcap_{n \in \mathbb{N}} (K_n \cap A) = \right) \bigcap_{n \in \mathbb{N}} \overline{K_n \cap A} \neq \emptyset.$$

This proves that A is CK. ∎

Recall that $\ddot{A}$ denotes the sequential closure of a subset A of a topological space T, i.e., the subset of $\overline{A}$ consisting of all limits of convergent sequences lying in A.

Proposition 12 *Let F be a locally convex space, and let E be a subspace of F. Let $A \subset E$ be a (R)CK subset of E. Then,*

$$\dddot{A}^F = A, \quad \left(\dddot{A}^F \subset E \right).$$

In particular, if A is CK in E, then A is sequentially closed in E.

Proof Let $f \in \overset{\cdots}{A}^{F}$. Then we can find a sequence (a_n) in A that converges to f. Put $K_n := \overline{\text{conv}}^{F}\{a_n, a_{n+1}, \ldots\}$, for $n \in \mathbb{N}$. We get a decreasing sequence of closed convex sets in F. The sequence $(K_n \cap E)$ is a decreasing sequence of closed convex sets in E, and $A \cap (K_n \cap E) \neq \emptyset$ for all $n \in \mathbb{N}$. Since A is (R)CK in E, we can find $a \in A$ ($a \in E$) such that

$$a \in A \cap \bigcap_{n=1}^{\infty} \overline{A \cap (K_n \cap E)}^{E} \quad \left(a \in \overline{A} \cap \bigcap_{n=1}^{\infty} \overline{A \cap (K_n \cap E)}^{E} \right).$$

In both cases, we get $a \in \bigcap_{n=1}^{\infty} K_n$. Proposition 3 says, in particular, that $\bigcap_{n=1}^{\infty} K_n = \{f\}$. This implies that $f = a$, so $f \in A$ ($f \in E$). ∎

In particular, we get the following result.

Corollary 13 *Let $(E, \mathcal{T})$ be a locally convex space. Let A be a w-(R)CK subset of E such that $\overset{\cdots}{A}^{w(E'^*,E')} = \overline{A}^{w(E'^*,E')}$. Then A is w-(R)K.*

Proof Since A is certainly bounded (see Proposition 9), it will be enough to prove that $\overline{A}^{(E'^*,w(E'^*,E'))} = A$ ($\overline{A}^{(E'^*,w(E'^*,E'))} \subset E$). By considering the space $(E'^*, w(E'^*, E'))$ and its subspace $(E, w(E, E'))$ in Proposition 12, we obtain the conclusion. ∎

Recall that the *circled cover* of A consists of all $\lambda\, x$, where $x \in A$ and $|\lambda| \leq 1$. In order to prove a stability result about the circled cover of a CK subset of a locally convex space, we will formulate and prove some previous results.

Lemma 14 *Let K be a non-empty closed subset of a topological vector space E. Let a and b be two real numbers such that $a < b$. Put $M := \bigcup_{a \leq \lambda \leq b} \lambda K$. Then,*
(i) *If $0 \notin [a, b]$, the set M is closed. Moreover, if K is convex then M is convex, too.*
(ii) *If K is bounded, the set M is closed.*

Proof Let $(\lambda_i k_i)_{i \in I}$ be a net in M that converges to some $x \in E$, where $\lambda_i \in [a, b]$ and $k_i \in K$ for all $i \in I$. By passing to a subnet if necessary, we may assume, without loss of generality, that the net (λ_i) converges to some λ ($\in [a, b]$).
(i) If $0 \notin [a, b]$ then $\lambda \neq 0$ and we may assume that $\lambda_i \neq 0$ for all $i \in I$. Since $\lambda_i^{-1} \to \lambda^{-1}$ we get $k_i = (\lambda_i^{-1})(\lambda_i k_i) \to \lambda^{-1} x$, hence $\lambda^{-1} x \in K$, since K is closed. It follows that $x \in \lambda K$ ($\subset M$), so M is closed.

Assume now that K is convex. Take two elements $\lambda_1 k_1$ and $\lambda_2 k_2$ in M, where $\lambda_i \in [a,b]$ and $k_i \in K$ for $i = 1,2$. Let $\alpha \in [0,1]$. Then

$$\alpha\lambda_1 k_1 + (1-\alpha)\lambda_2 k_2 = [\alpha\lambda_1 + (1-\alpha)\lambda_2]\frac{\alpha\lambda_1 k_1 + (1-\alpha)\lambda_2 k_2}{\alpha\lambda_1 + (1-\alpha)\lambda_2} = \lambda k,$$

where

$$\lambda := \alpha\lambda_1 + (1-\alpha)\lambda_2 \in [a,b]$$

and

$$k := \frac{\alpha\lambda_1 k_1 + (1-\alpha)\lambda_2 k_2}{\alpha\lambda_1 + (1-\alpha)\lambda_2}. \tag{1.1}$$

If $0 < a < b$, $k \in K$ since K is convex and the expression (1.1) is a convex combination of elements in K. If, on the contrary, $a < b < 0$, write

$$k := \frac{\alpha(-\lambda_1)k_1 + (1-\alpha)(-\lambda_2)k_2}{\alpha(-\lambda_1) + (1-\alpha)(-\lambda_2)}. \tag{1.2}$$

This time (1.2) is a convex combination of two elements in K, so again $k \in K$. Finally, in both cases, $\alpha\lambda_1 k_1 + (1-\alpha)\lambda_2 k_2 \in M$, as we wanted to show.
(ii) The first part of the proof covers the case $\lambda \neq 0$. Assume now that $\lambda = 0$ $(\in [a,b])$. Since K is supposed now to be bounded, the net $(\lambda_i k_i)$ converges to 0 $(\in M)$, so we obtain also in this case that M is closed. ∎

Remark 15 The boundedness condition in Lemma 14 (ii) cannot be omitted in general. For example, consider the set $K := \{(x,y) \in \mathbb{R}^2 : x = 1\}$. The set $M := \bigcup_{0 \le \lambda \le 1} \lambda K$ is not closed in $\mathbb{R}^2$ with the usual topology.

Lemma 16 *Let A be a closed CK subset of a locally convex space $(E, \mathcal{T})$, K a convex closed set and define $I := \{\lambda \in [-1,1] : K \cap \lambda A \neq \emptyset\}$. Then I is closed.*

Proof. We shall prove that for every sequence (λ_n) in I that converges in $\mathbb{R}$, its limit λ is actually in I. For $n \in \mathbb{N}$ we have $\lambda_n \in I$, so we can find $x_n \in K \cap \lambda_n A$; put $x_n := \lambda_n a_n$, where $a_n \in A$.
Assume first that $\lambda = 0$. The set A is bounded (see Remark 10). Then, $x_n = \lambda_n a_n \to 0$. Since $x_n \in K$ and K is closed, we get $0 \in K$. It follows that $0 \in K \cap 0A$, hence $0 \in I$.

Assume now that $\lambda \neq 0$. Without loss of generality we can suppose $\lambda > 0$. Obviously, it is enough to deal with monotone sequences. First, let (λ_n) be an increasing sequence of non-zero elements in I which tends to λ. For any $n \in \mathbb{N}$, $a_n = \lambda_n^{-1} x_n \in (\lambda_n^{-1} K) \cap A$. Put $K_n := \bigcup_{\lambda^{-1} \leq \lambda \leq \lambda_n^{-1}} \lambda K$, $n \in \mathbb{N}$. Use Lemma 14 (i) to conclude that (K_n) is a decreasing sequence of closed convex sets, each of them intersecting A, so, by the fact that A is CK, there exists

$$x \in \bigcap_{n=1}^{\infty} \overline{K_n \cap A} = \bigcap_{n=1}^{\infty} K_n \cap A.$$

We `Claim` that $\bigcap_{n \in \mathbb{N}} K_n = \lambda^{-1} K$. To prove the Claim let $y \in \bigcap_{n \in \mathbb{N}} K_n$. Then $y = \delta_n k_n$, where $\delta_n \in [\lambda^{-1}, \lambda_n^{-1}]$ and $k_n \in K$ for all $n \in \mathbb{N}$. Then $k_n = \delta_n^{-1} y \to \lambda y$, so $\lambda y \in K$, since K is closed. Therefore, $y \in \lambda^{-1} K$ and the Claim is proved, since the other inclusion is trivial.
It follows that $x \in (\lambda^{-1} K) \cap A$, so $\lambda \in I$.
On the other hand, if (λ_n) is a decreasing sequence in I with limit $\lambda \neq 0$, we can consider sets

$$K_n := \bigcup_{\lambda \leq t \leq \lambda_n} t^{-1} K, \quad n \in \mathbb{N}.$$

Again by Lemma 14 (i), we get that $\{K_n\}$ is a decreasing sequence of closed convex sets which intersect A, so there exists

$$x \in \bigcap_{1}^{\infty} \overline{K_n \cap A} = \bigcap_{1}^{\infty} K_n \cap A.$$

As before, it follows that $x \in (\lambda^{-1} K) \cap A$ and so $\lambda \in I$.
This proves that I is a closed set. ∎

Theorem 17 *Let A be a closed CK subset of a locally convex space $(E, \mathcal{T})$. Then, the circled cover of A is CK.*

Proof Let us put M for the circled cover of A, i.e., $M := \bigcup_{|\lambda| \leq 1} \lambda A$.
Let us prove first that M is closed. To this end, take $(\lambda_i a_i)$ a net in M that converges to some $x \in E$. Here, (λ_i) is a net in $[-1, 1]$ and (a_i) a net in A. By passing to a subnet if necessary, we may assume without loss of generality that (λ_i) converges to some $\lambda \in [-1, 1]$. If $\lambda \neq 0$, we get that $a_i \to a := x/\lambda$. Since A is closed it follows that $a \in A$, hence $x \in \lambda A \subset M$.

Assume now that $\lambda = 0$. Then, since A is bounded (see Remark 10), it follows that $x = 0$ ($\in M$). We got in all situations that $x \in M$.
Let (K_n) be any decreasing sequence of closed convex sets which intersects M and let $I_n := \{\lambda \in [-1,1] : K_n \cap \lambda A \neq \emptyset\}$. Certainly, for all $n \in \mathbb{N}$, $I_n \neq \emptyset$. Moreover, (I_n) is a decreasing sequence of subsets of $[-1,1]$. By Lemma 16, each I_n is closed, so the sequence (I_n) has a non-empty intersection. Take $\lambda \in \bigcap_{n\in\mathbb{N}} I_n$. It follows that every K_n intersects λA (a CK set) and so there exists an adherent point of the sequence $(K_n \cap M)$ in $\lambda A \subset M$. ∎

Under some requirement of completeness (see Example 40 to ensure that indeed we need some conditions for a positive result), the closed convex cover of a w-CK set is again w-CK. To see this, we need a similar statement for compact sets (see [Ko69, §24.5]) due to Krein. Recall that a topological vector space is called *quasi-complete* if every closed bounded subset is complete.

Theorem 18 (Krein) *The closed convex cover $\overline{\text{conv}}\,(A)$ of a compact subset A of a locally convex space $(E, \mathcal{T})$ is compact if and only if $\overline{\text{conv}}\,(A)$ is μ-complete.*
This is always the case if $(E, \mathcal{T})$ is μ-quasi-complete.

Later (Corollary 51), we will see that w-RCK sets and w-RK sets coincide in μ-quasi-complete spaces. Therefore we obtain

Corollary 19 *Let A be a weakly CK set of a locally convex space E. Suppose that $\overline{\text{conv}}\,(A)$ is Mackey complete (in particular if E is Mackey quasi-complete). Then $\overline{\text{conv}}\,(A)$ is weakly CK.*

The following simple result will be used several times.

Proposition 20 *Let $(E, \mathcal{T}_1)$ be a locally convex space. Let $\mathcal{T}_2$ be a compatible topology on E. Let A be a subset of E.*
(i) *If A is simultaneously $\mathcal{T}_1$-closed and $\mathcal{T}_1$-CK. then A is $\mathcal{T}_2$-CK.*
(ii) *If A is $\mathcal{T}_1$-(R)CK and $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$, then A is $\mathcal{T}_2$-(R)CK.*
(iii) *If A is convex, A is $\mathcal{T}_1$-(R)CK if and only if A is $\mathcal{T}_2$-(R)CK.*

Proof (i) Let $(K_n)_1^\infty$ be a decreasing sequence of $\mathcal{T}_2$-closed convex subsets of E such that $K_n \cap A \neq \emptyset$, for all $n \in \mathbb{N}$. By Mazur's Theorem (see, e.g., [Ko69, §20.7(6)]), the sets K_n are also $\mathcal{T}_1$-closed; hence, each set $K_n \cap A$ is $\mathcal{T}_1$-closed, since A is $\mathcal{T}_1$-closed, too. By the fact that the set A is $\mathcal{T}_1$-CK, we get $\bigcap_n (K_n \cap A) = \bigcap_n \overline{K_n \cap A}^{\mathcal{T}_1} \neq \emptyset$. Then $\bigcap_n \overline{K_n \cap A}^{\mathcal{T}_2} \cap A \neq \emptyset$. It follows that A is $\mathcal{T}_2$-CK.

(ii) It follows from Proposition 7.(ii), since the identity mapping I from $(E, \mathcal{T}_1)$ onto $(E, \mathcal{T}_2)$ is obviously continuous.

(iii) is again a consequence of Mazur's Theorem. Indeed, given a sequence (K_n) as in (i), we have $\overline{A \cap K_n}^{\mathcal{T}_1} = \overline{A \cap K_n}^{\mathcal{T}_2}$ for all $n \in \mathbb{N}$. This, obviously, implies the assertion. ∎

It is easy to check that RCK is hereditary by passing to closed convex subsets (however, we shall see in Example 44 that the property of being CK is not hereditary by taking arbitrary closed subsets).

Proposition 21 *Let A be a RCK subset of a locally convex space $(E, \mathcal{T})$. Then*
(i) *if C is a closed convex subset of E, then $C \cap A$ is RCK.*
(ii) *Every closed convex subset of A is CK.*
(iii) *If A is CK and C is a closed convex subset of E, then $C \cap A$ is CK.*

Proof. (i) Let (K_n) be a decreasing sequence of closed convex subsets of E such that $K_n \cap (C \cap A)(= (K_n \cap C) \cap A) \neq \emptyset$ for every $n \in \mathbb{N}$. Observe that $(K_n \cap C)$ is a decreasing sequence of closed convex subsets of E: Since A is RCK, there exists $a \in \bigcap_{n\in\mathbb{N}} \overline{(K_n \cap C) \cap A}$ $(= \bigcap_{n\in\mathbb{N}} \overline{K_n \cap (C \cap A)}$. Hence $C \cap A$ is RCK.
(ii) This is a consequence of part (i) and Proposition 11.
(iii) Proceeding as in (i), it is enough to observe that $a \in \overline{K_n \cap C} = K_n \cap C \subset C$ for every $n \in \mathbb{N}$. Moreover, if A is CK, then $a \in A$, so finally $a \in C \cap A$ and $C \cap A$ is CK. ∎

Lemma 22 *Let $E(\mathcal{T})$ be a locally convex space. Let $F \subset E$ be a closed subspace. Let A be a subset of E. If A is* (R)CK*, then $A \cap F$ is* (R)CK *in F.*

Proof This follows from Proposition 21, since a subset of F that is (R)CK with respect to E is, obviously, (R)CK with respect to F. ∎

Remark 23 Let A be a CK subset of a metrizable locally convex space $(E, \mathcal{T})$. Then A is closed. This is clear from the fact that every point x in E has a countable base $U_1(x) \supset U_2(x) \supset \ldots$ of closed convex neighborhoods. Assume that $x \in \overline{A}$. Then, $U_n(x) \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. There exists $a \in A$ such that $a \in \bigcap_{n=1}^{\infty} \overline{A \cap U_n(x)}$. This certainly implies $x = a$, so A is closed. In general, this is not the case, as Example 39 shows. Indeed, the set A there is a SK, hence a CK, subset of a locally convex space, and $\overline{A}$ ($\neq A$) is not CK.

1.3.4 w-(R)ΞK

Proposition 24 *A bounded subset A of a locally convex space E is w-*(R)ΞK *if and only if, given a sequence (a_n) in A and a $w(E'^*, E')$-adherent point a'^* of (a_n) in E'^*, then, for every sequence (x'_n) in E', there exists a point $a \in A$ $(\in \overline{A}^w)$ such that $\langle a'^* - a, x'_n \rangle = 0$ for every $n \in \mathbb{N}$.*

Proof The sufficient condition is clear. To prove the necessary condition, take an arbitrary sequence (x'_n) in E'. Consider the sequence

$$(x'_1, x'_1, x'_2, x'_1, x'_2, x'_3, x'_1, x'_2, x'_3, x'_4, \ldots)$$

and apply the definition of w-(R)ΞK. ∎

Proposition 25 *Let A be a w-RΞK subset of a locally convex space $(E, \mathcal{T})$. Assume that for all $\overline{a} \in \overline{A}^w$ there exists a countable subset N of A such that $\overline{a} \in \overline{N}^w$. Then $\overline{A}$ is w-ΞK.*

Proof. Let a'^* $(\in E'^*)$ be a w^*-cluster point of a certain sequence $(\overline{a}_n)$ in $\overline{A}^w$. By assumption, there exists, for all $n \in \mathbb{N}$, a countable set $M_n \subset A$ such that $\overline{a}_n \in \overline{M_n}^w$. Put $M := \bigcup_{n \in \mathbb{N}} M_n$ $(\subset A)$, a countable set. Let $M = \{a_m;\ m \in \mathbb{N}\}$. Certainly, $a'^* \in \overline{M}^{w^*}$. We have two possibilities:
a) Assume that $a'^* \in M$. Then there is nothing to prove; given a sequence (x'_n) in E', we find an element $\overline{a}$ $(= a'^*)$ in $\overline{A}^w$ such that $\langle a'^* - \overline{a}, x'_n \rangle = 0$ for all $n \in \mathbb{N}$.
b) Assume, on the contrary, that $a'^* \in \overline{M}^{w^*} \setminus M$. Then, a'^* is a w^*-adherent point to the sequence (a_m) (see Proposition 2). Let (x'_n) be a sequence in E'. Since A is w-RΞK, there exists an element $\overline{a} \in \overline{A}^w$ such that $\langle a'^* - \overline{a}, x_n \rangle = 0$ for all $n \in \mathbb{N}$ (see Proposition 24).

This implies that $\overline{A}^w$ is w-ΞK. ∎

In particular, we have the following

Corollary 26 *Let $(E, \mathcal{T})$ be a locally convex space such that $E' = \bigcup_{n=1}^{\infty} M_n$, where, for all $n \in \mathbb{N}$, M_n is a $w(E', E)$-K subset of E'. Then, the w-closure of every w-RΞK in E is w-ΞK.*

Proof. The corollary is a consequence of Kaplansky's theorem (see [Ko69, Thm. §24.1(6)], since in this case the space $(E, w(E, E'))$ has *countable tightness*, i.e., every point in $\overline{A}^w$ is in the w-closure of a countable subset of A, for every subset A of E. ∎

Remark 27 Spaces satisfying the condition in Corollary 26 are, for example, the metrizable locally convex spaces. The result also applies to the strict LF-spaces (it is enough to use the argument in [Ko69, §24.1(4)]).

The following proposition gives some geometrical characterizations of being w-(R)ΞK.

Proposition 28 *Let $(E, \mathcal{T})$ be a locally convex space and let A be a non-empty subset of E. Then, the following statements are equivalent.*
(i) *Given a decreasing sequence (K_n) of sets in E such that every K_n is a finite intersection of closed halfspaces and such that $A \cap K_n \neq \emptyset$ for every $n \in \mathbb{N}$, we have $A \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ ($\overline{A}^{w(E,E')} \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$).*
(ii) *A is bounded and, for every sequence (e'_n) in E' and every point a'^* in $\overline{A}^{w(E'^*,E')}$, there exists $a \in A$ ($a \in \overline{A}^{w(E,E')}$) such that $\langle a'^* - a, e'_n \rangle = 0$ for every $n \in \mathbb{N}$.*
(iii) *A is w(R)ΞK.*

Proof. (i)⇒(ii) If A is not bounded then, by a theorem of Mackey (see, e.g., [Ko69, §20.11(7)]), we can find $e' \in E'$ unbounded on A, and so we may assume that there exists a sequence (a_n) in A such that $\langle a_n, e' \rangle \geq n$ for all $n \in \mathbb{N}$. Then, applying the stated property for the (decreasing) sequence (K_n), where $K_n := \{x \in E; \langle x, e' \rangle \geq n\}$, we can find $a \in A \cap \bigcap_{n\in\mathbb{N}} K_n$ ($a \in \overline{A}^{w(E,E')} \cap \bigcap_{n\in\mathbb{N}} K_n$), something impossible. We reach a contradiction, so A is bounded.

For $n \in \mathbb{N}$ put $K_n := \{x \in E; |\langle x - a'^*, e'_i\rangle| \leq 1/n,\ i = 1, 2, \ldots, n\}$. Each of the sets K_n is a finite intersection of closed halfspaces, and, since K_n is a $w(E'^*, E')$-neighborhood of a'^* in E', we have $K_n \cap A \neq \emptyset$. Moreover, the sequence (K_n) is decreasing. It follows that there exists an element $a \in A \cap \bigcap_{n\in\mathbb{N}} K_n$ ($a \in \overline{A}^{w(E,E')} \cap \bigcap_{n\in\mathbb{N}} K_n$). This element satisfies, obviously, that $\langle a'^* - a, e'_i\rangle = 0$ for all $i \in \mathbb{N}$.
(ii)$\Rightarrow$(iii) is obvious.
(iii) $\Rightarrow$ (i) Assume that A is w(R)ΞK. Let (K_m) be a sequence as in (i). We can find a sequence (e'_n) in E', a sequence (α_n) in $\mathbb{R}$ and a sequence $n_0 := 0$, $n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that $K_m := \{x \in E; \langle x, e'_i\rangle \leq \alpha_i,\ i = n_{m-1} + 1, \ldots, n_m\}$. Choose $a_m \in A \cap K_m$, $m \in \mathbb{N}$. The set A is bounded, so the sequence (a_m) has a $w(E'^*, E')$-cluster point $a'^* \in E'^*$. Observe that, by the fact that the sequence (K_m) is decreasing, we get $\langle a'^*, e'_i\rangle \leq \alpha_i$ for all $i \in \mathbb{N}$. By the very definition of w(R)ΞK, there exists $a \in A$ ($a \in \overline{A}^{w(E,E')}$) such that $\langle a'^* - a, e'_i\rangle = 0$ for all $i \in \mathbb{N}$. This implies, in particular, that $\langle a, e'_i\rangle \leq \alpha_i$ for all $i \in \mathbb{N}$, and this proves (i). ∎

Remark 29 Of course, $a \in A \cap \bigcap_{n\in\mathbb{N}} K_n$ is the same statement that $a \in \bigcap_{n\in\mathbb{N}}(A \cap K_n)$, the condition used in the definition of CK for a decreasing sequence (K_n) of closed convex sets. However, the statement $a \in \overline{A}^{w(E,E')} \cap \bigcap_{n\in\mathbb{N}} K_n$ is *not* the same as $a \in \bigcap_{n\in\mathbb{N}} \overline{A \cap K_n}^{w(E,E')}$, even in the case of a decreasing sequence (K_n) of convex closed subsets of E each of them a finite intersection of closed halfspaces, as it has been used in the (equivalent) description of w-(R)ΞK sets in Proposition 28.
This can be seen even in $\mathbb{R}^2$. Indeed, let $x_n := (0, 1/n)$ and $y_n = (1, 1/n)$ for every $n \in \mathbb{N}$ and set $A := \{x_n;\ n \in \mathbb{N}\} \cup \{y_n;\ n \in \mathbb{N}\}$. Let $(K_n) := \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x/n\}$. Then $\bigcup_n \overline{A \cap K_n} = \{y\}$, while $\overline{A} \cap \bigcap_n K_n = \{x, y\}$.

Remark 30 The boundedness assumption in Proposition 28 (ii) is necessary. Indeed, let $(E, \mathcal{T}) := (\mathbb{R}^{\mathbb{N}}, \mathcal{T}_p)$, where $\mathcal{T}_p$ denotes the topology of the pointwise convergence (topology that coincides with $w(E, E')$), and let $A := \mathbb{R}^{\mathbb{N}}$. We have, in this case, $E'^* = E$ and so the condition about halfspaces in (ii) trivially holds. However, the set $\mathbb{R}^{\mathbb{N}}$ is, obviously, not w-ΞK.

Remark 31 Observe that, from the previous proposition, some of the conditions which appear in the definition of w-(R)Ξ-compactness are superfluous.

To be precise, 6. in Definition 5 refers to elements in $\overline{A}^{w(E'^*,E')}$ *that are* w'^**-cluster points of sequences in* A. However, (ii) in Proposition 8 refers to *all* elements in $\overline{A}^{w(E'^*,E')}$. We proved that both descriptions of what a w(R)Ξ-compact set is are indeed equivalent.

Unfortunately, the concept of w-(R)Ξcompactness is not stable by taking intersections with closed subspaces, even in the case that the dual space of the subspace is separable in the topology of the pointwise convergence on the elements in the subspace. In order to give an instance of this situation, we need some ingredients provided in Example 39, so will postpone it to Example 60.

1.3.5 w-(R)∂K

The concept of w-(R)∂K given in Definition 5 comes from [Ko69]. It is an easy observation that this concept has a more simple description. This is done in the following proposition.

Proposition 32 *A bounded subset* A *of a locally convex space* E *is* w-(R)∂K *if and only if, given a sequence* (a_n) *in* A *and a* $w(E'^*, E')$*-adherent point* a'^* *of* (a_n) *in* E'^**, then, for every sequence* (x'_n) *contained in an absolutely convex and* $w(E', E)$*-compact subset of* E'*, there exists a point* $a \in A$ $(\in \overline{A}^w)$ *such that* $\langle a'^* - a, x'_n\rangle = 0$ *for every* $n \in \mathbb{N}$.

Proof The proof follows the pattern used in the proof of Proposition 24. Now we shall take an arbitrary sequence (x'_n) in an absolutely convex and $w(E', E)$-compact subset of E' and proceed as we did there. ∎

The following proposition gives a sufficient condition for the w-closure of a w-R∂K to be w-∂K. The proof is a reproduction of the one of Proposition 25 (it will be enough now to take a sequence (x'_n) in an absolutely convex and $w(E', E)$-K subset of E'), hence it will be omitted. Corollary 26 and Remark 27 adapted to this situation also apply.

Proposition 33 *Let* A *be a* w*-R∂K subset of a locally convex space* $(E, \mathcal{T})$*. Assume that for all* $\overline{a} \in \overline{A}^w$ *there exists a countable subset* N *of* A *such that* $\overline{a} \in \overline{N}^w$*. Then* $\overline{A}$ *is* w*-∂K.*

The following result is similar to Proposition 28, now for the class of w-(R)∂K sets. It provides a geometric characterization in terms of decreasing sequences of convex sets that can be written as finite intersection of half-spaces. Precisely, we have

Proposition 34 *Let $(E, \mathcal{T})$ be a locally convex space and let A a non-empty subset of E. The following are equivalent.*
(i) *Given a decreasing sequence (K_n) of sets in E such that every K_n is a finite intersection of closed halfspaces defined by functionals in an absolutely convex and $w(E', E)$-compact subset of E', and such that $A \cap K_n \neq \emptyset$ for every $n \in \mathbb{N}$, we have $A \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ ($\overline{A}^{w(E,E')} \cap \bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$).*
(ii) *A is bounded and, for every sequence (e'_n) in an absolutely convex and $w(E', E)$-compact subset of E' and every point a'^* in $\overline{A}^{w(E'^*,E')}$, there exists $a \in A$ ($a \in \overline{A}^{w(E,E')}$) such that $\langle a'^* - a, e'_n\rangle = 0$ for every $n \in \mathbb{N}$.*
(iii) *A is w(R)∂K.*

The proof is similar to the one given to Proposition 28 (the sequence (e'_n) there, lies now in an absolutely convex and $w(E', E)$-K subset of E'), so we will omit it.

Remark 35 Observations similar to Remarks 29, 30 and 31 can be done for the case of w-(R)∂-compactness.

1.3.6 Hierarchy

The following list of implications, for a set in a locally convex space, can be found, for example, in [Ko69, §24.3], and can be considered as standard.

$$\begin{aligned} \text{(R)K} &\Rightarrow \text{(R)NK} \Rightarrow \text{(R)}\Psi\text{K} \Rightarrow w\text{-(R)}\partial\text{K} \\ \text{(R)SK} &\Rightarrow \text{(R)NK} \Rightarrow \text{(R)CK} \Rightarrow w\text{-(R)}\partial\text{K} \end{aligned}$$

In order to complete the picture, those implications together with some additions are presented in Proposition 36. Those concerning the intermediate concept (R)ΞK are new.

Proposition 36 *Let $(E, \mathcal{T})$ be a locally convex space. Let A be a subset of E. Then we have the following implications.*
(R)K $\Rightarrow$ (R)NK $\Rightarrow$ (R)ΨK $\Rightarrow$ w-(R)ΨK $\Rightarrow$ w-(R)ΞK $\Rightarrow$ w-(R)∂K.
(R)SK $\Rightarrow$ (R)NK $\Rightarrow$ (R)CK $\Rightarrow$ w-(R)CK $\Rightarrow$ w-(R)ΞK $\Rightarrow$ w-(R)∂K.

Proof

(R)K $\Rightarrow$ (R)NK is clear from the definition.

(R)NK $\Rightarrow$ (R)ΨK: Let $f : A \to \mathbb{R}$ be a continuous mapping. If f is not bounded, there exists a sequence (a_n) in A with $|f(a_n)| \geq n$ for all n. Let $a \in A$ be the limit of a subnet of (a_n); we reach a contradiction. (R): Let $f : A \to \mathbb{R}$ be an unbounded continuous mapping. Let $a \in \overline{A}$ be the limit of a convergent subnet of (a_n). It is clear that f is unbounded on $U(a) \cap A$, for every neighborhood $U(a)$ of a.

(R)ΨK $\Rightarrow$ w-(R)ΨK: This follows from (iv) in Proposition 8.

w-(R)ΨK $\Rightarrow$ w-(R)ΞK: Let (e'_n) be a sequence in E', and let $a'^* \in \overline{A}^{w(E'^*,E')}$. Put $f_n(\overline{a}) := |\langle a'^* - \overline{a}, e'_n\rangle|$ for $\overline{a} \in \overline{A}^w$ and $n \in \mathbb{N}$. All of them are w-continuous functions on $\overline{A}^w$, a w-ΨK subset of E (see (ii) in Proposition 8). By (v) in Proposition 8, there exists some $k_n > 0$ such that $f_n(\overline{a}) \leq k_n$, for $\overline{a} \in \overline{A}$ and $n \in \mathbb{N}$. The function $f : \overline{A}^w \to \mathbb{R}$ given by

$$f := \sum_{n=1}^{\infty} \frac{1}{2^n k_n} f_n$$

is the sum of a uniform convergent series of real continuous functions in $(\overline{A}^w, w)$, hence a w-continuous function. Observe, too, that $\inf f(A) = 0$. Assume for a moment that f does not vanishes on A (on $\overline{A}^w$). Then $g := 1/f$ is a w-continuous function on A (on $\overline{A}^w$). Since g is unbounded, we reach a contradiction. Therefore, we can find $a \in A$ ($a \in \overline{A}^w$) such that $\langle a, f_n\rangle = 0$ for all $n \in \mathbb{N}$. This proves the statement.

w-(R)ΞK $\Rightarrow$ w-(R)∂K is obvious.

(R)SK $\Rightarrow$ (R)NK is also obvious.

(R)NK $\Rightarrow$ R(C)K: Assume that (K_n) is a decreasing sequence of closed convex subsets of E such that $K_n \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $a_n \in K_n \cap A$ for $n \in \mathbb{N}$. Then (a_n) has an adherent point $a \in A$ ($\in \overline{A}$), so the sequence $(K_n \cap A)$ has an adherent point in A (in $\overline{A}$).

(R)CK $\Rightarrow$ w-(R)CK: By Mazur's Theorem, a w-closed convex subset of E is also closed. Obviously, $A \cap \bigcap_{n\in\mathbb{N}} \overline{A \cap K_n}^{\mathcal{T}} \subset A \cap \bigcap_{n\in\mathbb{N}} \overline{A \cap K_n}^{w}$ ($\overline{A}^{\mathcal{T}} \cap \bigcap_{n\in\mathbb{N}} \overline{A \cap K_n}^{\mathcal{T}} \subset \overline{A}^{w} \cap \bigcap_{n\in\mathbb{N}} \overline{A \cap K_n}^{w}$). For a more precise statement, see Proposition 20.

w-(R)CK $\Rightarrow$ w-(R)ΞK: Let a'^* be an element in the closure $\overline{A}^{w(E'^*,E')}$ of A in $(E'^*, w(E'^*, E'))$. Let (e'_i) be a sequence in E'. Put

$$K_n := \{x \in E : |\langle a'^* - x, e'_i\rangle| \leq 1/n,\ i = 1, 2, \ldots, n\},\ n \in \mathbb{N}.$$

Then (K_n) is a decreasing sequence of closed convex subsets of E. Clearly, $K_n \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. Therefore, there exists a point $a \in A$ $(\in \overline{A}^{w(E,E')})$ such that $a \in \bigcap_{n=1}^{\infty} \overline{(A \cap K_n)}^{w(E,E')}$. This proves that $\langle a'^* - a, e'_n\rangle = 0$ for all $n \in \mathbb{N}$, so A is w-(R)ΞK. ∎

Remark 37 From the proof of w-(R)ΨK $\Rightarrow$ w-(R)ΞK and w-(R)CK $\Rightarrow$ w-(R)ΞK above, it seems that we are indeed proving something more precise: that both w-(R)ΨK and w-(R)CK imply a stronger statement than being w-(R)ΞK, since a certain behavior is true *for every element in the* $w(E'^*, E')$-*closure of* A, and not only for cluster points in $(E'^*, w(E'^*, E'))$ of sequences in A. That this is not the case was proved in Proposition 28, (ii).

Remark 38 Another proof of the implication w(R)CK$\Rightarrow$ w(R)ΞK can be done using the geometric characterization of this last property provided in Proposition 28. Using this characterization, the proof of the implication becomes now straightforward. Of course, the same could be say for the implication w-(R)ΞK$\Rightarrow$ w-(R)∂K, although this time this followed already from the very definition.

1.3.7 Separation (and some negative results)

The closure of a RΨK (RK) set is ΨK (respectively, K), see Proposition 8 (respectively, apply the definition). However, in [Fl80, p.9], an example of an absolutely convex sequentially compact subset A of a locally convex space $(E, \mathcal{T})$ such that $\overline{A}$ is not countably compact (hence, not sequentially compact) is given. We prove here that, in fact, $\overline{A}$ is not even convex-compact, a slight improvement of the quoted result. This provides, in particular, an example of a convex-compact subset of a locally convex space which is not closed, and whose closure is not convex-compact.

Example 39 *There exists a locally convex space* $(E, \mathcal{T})$ *and an absolutely convex subset* A *of* E *with the following properties:*

1. A *is w-SK (and then w-NK and so w-CK),*
2. $\overline{A}$ *is not w-CK.*
3. $\overline{A}$ *is w-*ΨK.

Proof: Let (X_n) be a disjoint sequence of uncountable sets and define $X := \bigcup_{n=1}^{\infty} X_n$. For $f : X \to \mathbb{R}$, the *support of* f is defined as supp $(f) := \{x \in X; f(x) \neq 0\}$. Let the vector space

$$E := \left\{ f : X \to \mathbb{R};\ \exists\, n \in \mathbb{N} \text{ such that supp } (f) \cap \bigcup_{m=n}^{\infty} X_m \text{ is countable} \right\}$$

be endowed with the restriction of the topology $\mathcal{T}_p$ in $\mathbb{R}^X$ of the pointwise convergence on X, denoted again by $\mathcal{T}_p$. Clearly, $(E, \mathcal{T}_p)$ turns out to be a locally convex space.
The topology $\mathcal{T}_p$ on E coincides with the weak topology $w(E, E')$. Indeed, E is a dense subspace of $(\mathbb{R}^X, \mathcal{T}_p)$, as it is simple to prove (given any $f \in \mathbb{R}^X$ and a finite set $F \subset X$, the element $f.\chi_F$ is in E and coincides on F with f). It follows that E' coincides with the topological dual of $(\mathbb{R}^X, \mathcal{T}_p)$, i.e., $\varphi(X)$ $(:= \{f : X \to \mathbb{R};$ supp$f$ finite$\})$. As a consequence, $E'^{*} = \mathbb{R}^X$, and the topology $w(E'^{*}, E')$ is, obviously, $\mathcal{T}_p$.

1. By using a diagonal procedure, it is easy to see that the set

$$A := \{f : X \to \mathbb{R},\ \text{supp } (f) \text{ is countable}, \|\mathrm{f}\|_\infty \leq 1\} \,(\subset E).$$

   is sequentially compact in $(E, \mathcal{T}_p)$, where $\|f\|_\infty := \sup\{|f(x)|; x \in X\}$ if this supremum is finite. Indeed, given a sequence (f_n) in A, it is enough to consider the countable set $S := \bigcup_{n\in\mathbb{N}}$ supp (f_n) and select a subsequence (f_{n_k}) of (f_n) which converges pointwise on S. The limit is obviously an element in A and the convergence is, in fact, in $\mathcal{T}_p$.

2. It is also easy to see that

$$\overline{A}^{(E,\mathcal{T}_p)} = \{f \in E; \|f\|_\infty \leq 1\}\,.$$

   Indeed, given an element $f \in \{h \in E; \|h\|_\infty \leq 1\}$ and a finite set $I \subset X$, there exists an element $g \in A$ such that $g(x) = f(x)$ for all $x \in I$.

   We shall prove that $\overline{A}^{(E,\mathcal{T}_p)}$ is not CK. To see this, let f_n be the characteristic function of $\bigcup_{i=1}^{n} X_i$, $n \in \mathbb{N}$. The sequence (f_n) is in $\overline{A}^{(E,\mathcal{T}_p)}$ and

$\mathcal{T}_p$-converges to $f \in \mathbb{R}^X$, the characteristic function of X, which is not in E (so, in particular, $\overline{A}^{(E,\mathcal{T}_p)}$ is not countably compact in $(E, \mathcal{T}_p)$). Consider now the sets

$$K_n := \overline{\text{conv}}^{(E,\mathcal{T}_p)}\{f_i\}_{i=n}^{\infty},\ n \in \mathbb{N}.$$

They form a decreasing sequence of closed convex sets in $(E, \mathcal{T}_p)$ such that $K_n \cap \overline{A}^{(E,\mathcal{T}_p)} \neq \emptyset$ for all $n \in \mathbb{N}$. If $g \in K_n$ then $g(x) = 1$ for all $x \in \bigcup_{k=1}^{n} X_k$. Then the sequence $K_n \cap \overline{A}^{(E,\mathcal{T}_p)}$ has no adherent point in $(E, \mathcal{T}_p)$.

3. The set A is w-NK, hence w-ΨK (see Proposition 36). Now, by using (i) and (ii) in Proposition 8 we have that $\overline{A}^{(E,\mathcal{T}_p)}$ is w-ΨK (in particular, it is w-ΞK).

∎

Example 39 shows, in particular, that a CK subset of a locally convex space does not need to be closed, and that the closure of a CK set does not need to be CK. See, however, Proposition 11.

In [Ko69, §20.6], an example of a compact set such that its closed convex cover is not compact is provided. Actually, it is not even CK, as it is proved in Example 40 below. So the convex cover (or the closed convex cover) of a CK set is not always CK. However, the circled cover of a closed CK subset of a locally convex space is CK. This was proved in Theorem 17.

Example 40 *There exists a compact (and then CK) subset of a normed space whose convex cover (or even closed convex cover) is not CK.*

Proof Let φ be the normed space of all finitely-supported vectors of the Hilbert space $(l^2, \|\cdot\|_2)$. All topological and uniform concepts in this example refer to the norm topology. The set

$$A := \left\{0, e_1, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4, ...\right\}$$

is compact in φ. Consider a sequence of real positive number (α_n), such that $\sum_{n=1}^{\infty} \alpha_n = 1$. The sequence

$$y_k := \frac{1}{\sum_{n=1}^{k} \alpha_n} \sum_{n=1}^{k} \alpha_n \frac{1}{n} e_n$$

is a Cauchy sequence in $\mathrm{conv}(A)$, whose limit y does not belong to φ. Let

$$K_n := (y + (1/n)B_{l^2}) \cap \varphi,\ n \in \mathbb{N},$$

where B_{l^2} is the closed unit ball of l^2. Each K_n is closed, convex, and intersects $\mathrm{conv}(A)$ (and $\overline{\mathrm{conv}}^{\varphi}(A)$). However $\bigcap_{n=1}^{\infty}(K_n \cap \overline{\mathrm{conv}}^{\varphi}(A)) = \emptyset$. Then, nor $\mathrm{conv}(A)$ neither $\overline{\mathrm{conv}}^{\varphi}(A)$ are CK in φ. ∎

Remark 41 Example 40 proves, too, that even the closed convex hull of a w-K subset of a normed space fails to be w-CK in general. Indeed, the set A there is $\|\cdot\|_2$-K, hence w-compact. We proved that $\overline{\mathrm{conv}}^{\|\cdot\|_2}(A)$ is not $\|\cdot\|_2$-CK in φ. This last set is convex, so we can apply (iii) in Proposition 20 to conclude that it is not w-CK either.

Example 39 separates the class of w-ΞK sets from the class of w-CK sets. The following example separates the first class from the class of the w-∂K sets.

Example 42 *There exists a locally convex space $(E, \mathcal{T})$ with separable dual and a bounded subset $A \subset E$ that is w-∂K, but is not w-ΞK.*

Proof. Let $(E, \mathcal{T}) := (\ell_1, w(\ell_1, \varphi))$, where $\varphi \subset \ell_\infty$ is the linear subspace of all eventually zero sequences. The topology $w(\ell_1, \varphi)$ coincides with the restriction to ℓ_1 of the topology $\mathcal{T}_p = w(\mathbb{R}^{\mathbb{N}}, \varphi)$ of the pointwise convergence on $\mathbb{R}^{\mathbb{N}}$, and so it is metrizable. By [Ko69, §21.5(3)], $w(\ell_1, \varphi) = \mu(\ell_1, \varphi)$. In other words, every absolutely convex and $w(\varphi, \ell_1)$-K subset of φ is finite-dimensional.
Let $A := \prod_{n=1}^{\infty}[-n, n] \cap \ell_1$, an absolutely convex and bounded subset of $(\ell_1, w(\ell_1, \varphi))$.
We claim that A is w-∂K. Indeed, given a sequence (a_n) of points of A, let $a'^* \in E'^*$ be an adherent point of (a_n) in $(\mathbb{R}^{\mathbb{N}}, w(\mathbb{R}^{\mathbb{N}}, \varphi))$. Observe that $a'^* \in \prod_{n\in\mathbb{N}}[-n, n]$. Take a sequence (f'_n) contained in an absolutely convex and $w(\varphi, \ell_1)$-compact subset U of φ. Then, since U lies in a finite-dimensional subspace of φ, we have that $\bigcup_{n=1}^{\infty} \mathrm{supp}\,(f'_n)$ is finite. We can find then an element a in A such that $\langle a, f'_n\rangle = \langle a'^*, f'_n\rangle$ for all $n \in \mathbb{N}$, so A is w-∂K in E.
On the other hand, A is not w-ΞK. To see this, take $f'_n := e_n$, where e_n is the n-th element of the canonical basis of φ, $n \in \mathbb{N}$. Let $a'^* \in E'^*$ be an adherent point of a sequence (a_n) of points in A such that $a'^* \notin E$. Assume

that there exists a point $a \in A$ such that $\langle a, e_n \rangle = \langle a'^*, e_n \rangle$ for all $n \in \mathbb{N}$. Then $a = a'^* \in E$, a contradiction. ∎

We can provide examples showing that convex-compactness does not imply countable compactness.

Example 43 *There exists a locally convex space with an absolutely convex, CK subset, which is not NK.*

Proof. Let $(X, \|\cdot\|)$ be any infinite-dimensional reflexive Banach space. The closed unit ball B_X is w-K, so it is w-closed and w-CK. By Proposition 20 (i), B_X is $\|\cdot\|$-CK; however, it is not $\|\cdot\|$-compact (hence it is not $\|\cdot\|$-countably compact), since X is infinite-dimensional. ∎

We have seen in Example 39 and Example 40 that the concept w-CK is rather unstable. We can add now that it is not hereditary by passing to closed subsets, something rather unusual in compactness. We provide an example of this pathology.

Example 44 *There exists a closed subset of the closed unit ball B_{ℓ_2} of the Hilbert space ℓ_2 (itself a $\|\cdot\|_2$-CK set) which is not $\|\cdot\|_2$-CK.*

Proof. As a consequence of Proposition 20, it follows that the closed unit ball of ℓ_2 is, in the norm topology, a CK set. The set in question is S_{ℓ_2}. Indeed, the topology $w(\ell_2, \ell_2)$ on B_{ℓ_2} is metrizable. Let $(K_n)_{n\in\mathbb{N}}$ be a fundamental system of neighborhoods of 0 in $(B_{\ell_2}, w(\ell_2, \ell_2))$. We may assume that the sequence (K_n) is decreasing and that every K_n is closed and convex. Obviously, $K_n \cap S_{\ell_2} \neq \emptyset$, for every $n \in \mathbb{N}$. However, $\bigcap_{n\in\mathbb{N}} \overline{K_n \cap S_{\ell_2}}^{\|\cdot\|_2} = (\bigcap_{n\in\mathbb{N}} K_n) \cap S_{\ell_2} = \emptyset$. This concludes that S_{ℓ_2}, in the norm topology, is not CK. ∎

Observe that the set S_{ℓ_2} in Example 44 is, indeed, wRCK, as it is wRK.

1.4 Two cornerstones in weak compactness

The purpose of introducing weakened forms of compactness is the possibility of checking compactness (specially weak compactness) in familiar settings with a minimum of requirements. Typically in the literature two scenarios appear: some kind of completeness on one side and some sort of metrizabilty on the other.
In the first case, the fundamental James' Theorem plays the ultimate role.

Theorem 45 (James, [Jam64]) *Let E be a $\mu(E,E')$-quasi-complete locally convex space. Let $A \subset E$ be a $w(E,E')$-closed set such that every $e' \in E'$ attains its supremum on A. Then A is $w(E,E')$-compact.*

In the second, the pioneering work was done by Šmulyan, pushed further by Grothendieck, Dieudonné and Schwartz and later by Valdivia, De Wilde, Pryce and Fremlin. The following is already a quite general statement. It can be found, for example, in [Fl80]. Recall that a topological space $(T,\mathcal{T})$ is called *angelic* if 1) every RNK subset of T is RK, and 2) for such a set, its closure coincides with its sequential closure (i.e., every element in the closure is the limit of a sequence in the set).

Theorem 46 *Let X be a topological space such that $X = \overline{\bigcup_{n=1}^{\infty} K_n}$, where K_n is relatively countably compact for all $n \in \mathbb{N}$. Let Z be a metric topological space. Then $(C(X,Z),\mathcal{T}_p)$, the space of continuous functions from X into Z, endowed with the topology of the pointwise convergence, is angelic.*

In particular, the following result holds.

Corollary 47 (Valdivia, [Val72-1]) *Let $(E,\mathcal{T})$ be a locally convex space such that there is a sequence (K_n) of $w(E',E)$-RNK subsets of E' with the property $\overline{\bigcup_{n\in\mathbb{N}} K_n}^{w(E',E)} = E'$. Then $(E,w(E,E'))$ is angelic. In particular, the classes (R)K, (R)NK and (R)SK coincide in $(E,w(E,E'))$.*

This result has been pushed farther by Orihuela in [Ori87]. We shall return to this later on (see Section 1.6).

A general class of sets involved in James' Theorem are the weakly closed ones where every element of the dual space attains the supremum on them. In the second situation, the classical theorems deal with the subclass of the w-(R)NK ones.
Unfortunately, the larger class of sets with the properties that they are weakly closed and that all linear continuous functionals attain their supremum on them is not suitable for getting weak compactness in the second setting—even in the case that the space is metrizable—since we have the following example.

Example 48 ([Jam71]) *There exists a normed (non-complete, and therefore non-reflexive) space X such that every element of the dual space attains its supremum on the closed unit ball of X (which is not w-compact).*

Sketch of the proof. The (weakly compact) unit ball B_X of a reflexive Banach space X is, by the Krein-Milman Theorem, the closed convex hull of its extreme points:

$$B_X = \overline{\text{conv}}\,(\text{ext}(B_X)).$$

Therefore $E := \text{span}(\text{ext}(B_X))$ is dense in X. Every $f' \in E' = X'$ attains its supremum at an extreme point of B_X, i.e., in B_E, the unit ball of E. It is enough to produce X such that $E \neq X$ to obtain the announced example. For this, take the finite-dimensional Banach spaces $X_n = (\mathbb{R}^n, \|\cdot\|_\infty)$. $n \in \mathbb{N}$ and consider the Banach space

$$X = \ell_2(X_1, X_2, \ldots) := \left\{ (x_n) \in \prod_{n=1}^{\infty} X_n : \ \|(x_n)\| < \infty \right\},$$

where $\|(x_n)\| := \left(\sum_{n=1}^{\infty} (\|x_n\|_\infty)^2\right)^{1/2}$. Then, X is reflexive and it can be proved that $E \neq X$. ∎

Using this example we will separate in Corollary 83 the class of weakly closed subsets where every element of X' attains the supremum and the smaller class of w-∂K subsets, even in the setting of normed spaces.

We saw in Propositions 28 and 34 how the concepts w-(R)ΞK and w-(R)∂K are described geometrically. This gives immediately the implications w-(R)CK⇒ w-(R)ΞK⇒ w-(R)∂K, and also the connection with the class of bounded subsets of E where every element of E' attains the supremum. This stress the fact that the different intermediate compactness classes are natural when we consider the Eberlein-Šmulyan Theorem at one side and James's Theorem at the other.

Remark 49 A bounded set $A \subset E$ of the locally convex space $(E, \mathcal{T})$ has the property that an element $f \in E'$ attains its supremum on it precisely when $\bigcap K_n \cap A \neq \emptyset$ for the sequence (K_n) of closed halfspaces $K_n := \{x \in E; \langle x, f\rangle \geq s - (1/n)\}$, where $s := \sup_A f$.

It should be clear now, from Proposition 28, Proposition 34 and Remark 49, that w(R)CK⇒w(R)ΞK⇒w(R)∂K. Moreover, a set A in the last class has the property that every element $f \in E'$ attains its supremum on A (on $\overline{A}^w$). We formulate the last statement as a proposition and provide a direct proof.

Proposition 50 *Let $(E, \mathcal{T})$ be a locally convex space. Then every element $e' \in E'$ attains its supremum on every w-∂K subset of E. If a set A is only w-R∂K, then every element $e' \in E'$ attains its supremum on A at an element in $\overline{A}^w$.*

Proof. Let $A \subset E$ be a w-∂K subset of E. Let $e' \in E'$. By definition, A is bounded. Let $s := \sup\langle A, e'\rangle$, the supremum of e' on the set A. There exists a sequence (a_n) in A such that $\langle a_n, e'\rangle \to s$. Let a'^* be a $w(E'^*, E')$-cluster point of (a_n) in E'^*. Obviously, $\langle a'^*, e'\rangle = s$. By the very definition, there exists an element $a \in A$ such that $\langle a'^* - a, e'\rangle = 0$. Then $\langle a, e'\rangle = s$.
Assume now that a set A is w-R∂K and that e'^* is an element in E'. Then the first part of the proof gives an element $e \in \overline{A}^w$ where e' attains its supremum on A. ∎

This result, together with James' supremum-attaining Theorem 45, provides the following result (see [Ko69, Theorem §24.3(6)]).

Corollary 51 *In a $\mu(E, E')$-quasi-complete locally convex space, every w-R∂K is w-RK.*

The previous result holds for the weak topology. The original Eberlein's Theorem holds for any compatible topology of a $\mu(E, E')$-quasi-complete locally convex space. This is not the case when using $\mathcal{T}$-CK subsets of such a class of locally convex spaces. Precisely, the closed unit ball B_{ℓ_2} of the reflexive Banach space $(\ell_2, \|\cdot\|_2)$ is $\|\cdot\|_2$-CK (see Proposition 20) and the space $(\ell_2, \|\cdot\|_2)$ is, certainly, $\|\cdot\|_2 = \mu(\ell_2, \ell_2)$-complete. However, B_{ℓ_2} is not $\|\cdot\|_2$-compact.

Definition 52 *Let $(E, \mathcal{T})$ be a locally convex space. A set $A \subset E$ has* property (*) *if it is bounded and the following holds: given a sequence (a_n) in A, a $w(E'^*, E')$-adherent point a'^* of (a_n) in E'^*, and a sequence (e'_n) in E', there exists $e \in E$ such that $\langle a'^* - e, e'_n\rangle = 0$ for all $n \in \mathbb{N}$.*

The following result characterizes this property in a geometrical language. Moreover, the equivalence between (ii) and (iii) shows that we can avoid considering $w(E'^*, E')$-adherent points of sequences in A, taking arbitrary elements in $\overline{A}^{w(E'^*, E')}$ instead. The proof follows closely the one provided for Proposition 28.

Proposition 53 *Let $(E,\mathcal{T})$ be a locally convex space and let A be a non-empty subset of E. Then, the following statements are equivalent.*
(i) *Given a decreasing sequence (K_n) of sets in E such that every K_n is a finite intersection of closed halfspaces and such that $A \cap K_n \neq \emptyset$ for every $n \in \mathbb{N}$, we have $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$.*
(ii) *A is bounded and, for every sequence (e'_n) in E' and every point a'^* in $\overline{A}^{w(E'^*,E')}$, there exists $e \in E$ such that $\langle a'^* - e, e'_n\rangle = 0$ for every $n \in \mathbb{N}$.*
(iii) *A has property (*).*

Proof. (i)⇒(ii) If A is not bounded then, by a theorem of Mackey (see, e.g., [Ko69, §20.11(7)]), we can find $e' \in E'$ unbounded on A, and so we may assume that there exists a sequence (a_n) in A such that $\langle a_n, e'\rangle \geq n$ for all $n \in \mathbb{N}$. Then, applying the stated property for the (decreasing) sequence (K_n), where $K_n := \{x \in E; \langle x, e'\rangle \geq n\}$, we can find $e \in \bigcap_{n\in\mathbb{N}} K_n$, something impossible. We reach a contradiction, so A is bounded.
For $n \in \mathbb{N}$ put $K_n := \{x \in E; |\langle x - a'^*, e'_i\rangle| \leq 1/n,\ i = 1, 2, \ldots, n\}$. Each of the sets K_n is a finite intersection of closed halfspaces, and, since K_n is a $w(E'^*, E')$-neighborhood of a'^* in E', we have $K_n \cap A \neq \emptyset$. Moreover, the sequence (K_n) is decreasing. It follows that there exists an element $e \in \bigcap_{n\in\mathbb{N}} K_n$. This element satisfies, obviously, that $\langle a'^* - e, e'_i\rangle = 0$ for all $i \in \mathbb{N}$.
(ii)⇒(iii) is obvious.
(iii) ⇒ (i) Assume that A has property (*). Let (K_m) be a sequence as in (i). We can find a sequence (e'_n) in E', a sequence (α_n) in $\mathbb{R}$ and a sequence $n_0 := 0,\ n_1 < n_2 < \ldots$ in $\mathbb{N}$ such that $K_m := \{x \in E; \langle x, e'_i\rangle \leq \alpha_i,\ i = n_{m-1} + 1, \ldots, n_m\}$. Choose $a_m \in A \cap K_m$, $m \in \mathbb{N}$. The set A is bounded, so the sequence (a_m) has a $w(E'^*, E')$-cluster point $a'^* \in E'^*$. Observe that, by the fact that the sequence (K_m) is decreasing, we get $\langle a'^*, e'_i\rangle \leq \alpha_i$ for all $i \in \mathbb{N}$. By the very definition of property (*), there exists $e \in E$ such that $\langle a'^* - e, e'_i\rangle = 0$ for all $i \in \mathbb{N}$. This implies, in particular, that $\langle e, e'_i\rangle \leq \alpha_i$ for all $i \in \mathbb{N}$, and this proves (i). ∎

Definition 54 *Let $(T,\mathcal{T})$ be a topological space. A subset B of T is called* $\mathcal{T}$-bounding *if every real continuous function defined on T is bounded on B.*

Bounding sets are considered, for example, in [Fl80, §2], and have been considered by Valdivia in [Val77].

The following result holds.

Proposition 55 *Let $(E, \mathcal{T})$ be a locally convex space. Then*
(i) *every w-RΨK subset of E is w-bounding.*
(ii) *Every w-bounding subset B of E is bounded and has property* (*).

Proof (i) Let $A \subset E$ be a w-RΨK subset of E. Let $f : (E, w) \to \mathbb{R}$ be a continuous function. We shall prove that $f|_A$ is bounded. Assume not. Then, there exists $\overline{a} \in \overline{A}^w$ such that $f|_{N(\overline{a})\cap A}$ is unbounded, for every w-neighborhood $N(\overline{a})$ of $\overline{a}$. Fix $M > 0$. Then we can find a net $(a_i)_{i\in I}$ in A such that $a_i \xrightarrow{w} \overline{a}$ and $|f(a_i)| \geq M$ for all $i \in I$. Since f is w-continuous, we get $|f(\overline{a})| \geq M$. This is true for every $M > 0$, and this is impossible.
Let now B be a w-bounding subset of E. Let (e'_n) be a sequence in E' and b'^* an element in $\overline{B}^{w(E'^*,E')}$. Put

$$f(e) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\langle b'^* - e, e'_n\rangle|}{1 + |\langle b'^* - e, e'_n\rangle|}, \text{ for all } e \in E. \tag{1.3}$$

Since this is a w-uniformly convergent series of real w-continuous functions on E, f is w-continuous. Moreover, $\inf_A f = 0$. Assume that there exists $e \in E$ such that $f(e) = 0$. Then, $\langle b'^* - e, e'_n\rangle = 0$ for all $n \in \mathbb{N}$, and we are done. If, on the contrary, f does not vanishes on E, the mapping $g := 1/f$ is well defined and w-continuous on E, and $f|_B$ is unbounded, a contradiction with the fact that B is w-bounding. ∎

Theorem 56 *Let $E(\mathcal{T})$ be a locally convex space such that $(E', w(E', E))$ is separable. Let A be a w-ΞK subset of E (a subset of E with property* (*)*). Then, A is w-*(R)K.

Proof Let a'^* be an element in $\overline{A}^{w(E'^*,E')}$. Thanks to the equivalence between (ii) and (iii) in Proposition 28, given a countable subset $N \subset E'$, there exists a point $a_N \in A$ ($a_N \in E$) such that $a_N \restriction_N = a'^* \restriction_N$. Let $D \subset E'$ be a countable and $w(E', E)$-dense subset of E'. Let $x' \in E'$ be arbitrary. Consider the points $a_{D\cup\{x'\}}$ and a_D in E. They coincide on D, so $a_{D\cup\{x'\}} = a_D$ by the $w(E', E)$-density of D. At the same time, we have $\langle a'^*, x'\rangle = \langle a_{D\cup\{x'\}}, x'\rangle$ $(= \langle a_D, x'\rangle)$. Since $x' \in E'$ was arbitrary, we get $a'^* = a_D \in A$ ($\in E$). This is true for every $a'^* \in \overline{A}^{w(E'^*,E')}$. This implies that A is w-(R)K, since it is bounded. ∎

Corollary 57 (Valdivia, [Val77]) *Let E be a locally convex space such that $(E', w(E', E))$ is separable. Then every w-bounding subset of E is w-RK.*

Proof Just combine Proposition 55 and Theorem 56. ∎

Remark 58 Note that, in order to prove our extension of Valdivia's result (Theorem 56), we did not make use of a lemma due again to Valdivia (see [Fl80, 2.3] and the proof of this lemma by W. Govaerts there).

In [Mo78], it is shown that every w-(R)CK set in a locally convex space $(E, \mathcal{T})$ is w-(R)K if there exists a sequence (M_n) of $w(E', E)$-RK subsets of E' such that $\bigcup_{n\in\mathbb{N}} M_n$ is $w(E', E)$-dense in E' (in particular, if $(E', w(E', E))$ is separable). We believe that there is a flaw in the proof there. In Theorem 59 we shall provide a correct proof of an extension of this result.

Theorem 59 *Let $E(\mathcal{T})$ be a locally convex space such that in the dual there is a sequence (M_n) of $w(E', E)$-RNK subsets such that $\bigcup_{n\in\mathbb{N}} M_n$ is $w(E', E)$-dense in E'. Let A be a bounded subset of E with the following property: given a sequence (a_n) in A and a $w(E'^*, E')$-adherent point $a'^* \in E'^*$, and a sequence (e'_n) in E', there exists $a \in A \cap \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$ ($e \in \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$) such that $\langle a'^* - e, e'_n\rangle = 0$ for all $n \in \mathbb{N}$. Then A is w-(R)K.*

Proof We shall prove that A is w-(R)NK. To do this, take a sequence (a_n) in A. Consider $F := \overline{\text{span}}\{a_n\}_{n\in\mathbb{N}}$. F is a separable locally convex space. Its dual is $F' = q(E') = E'/F^\perp$, where $q : E' \to E'/F^\perp$ is the canonical quotient mapping. It follows from Proposition 7 that, for every $n \in \mathbb{N}$, $q(M_n)$ is $w(F', F)$-RNK; moreover, $\bigcup_{n\in\mathbb{N}} q(M_n)$ is dense in $(F', w(F', F))$. On the other hand, the space $(F', w(F', F))$ has a separable dual F, so we can apply Theorem 56 to conclude that, for every $n \in \mathbb{N}$, $q(M_n)$ is $w(F', F)$-RK. Therefore, $\overline{q(M_n)}^{w(F',F)}$ is $w(F', F)$-K. It follows that $\overline{q(M_n)}^{w(F',F)}$ is $w(F', F)$-metrizable. We conclude that $\overline{q(M_n)}^{w(F',F)}$ is $w(F', F)$-separable (and so it is $q(M_n)$, due to the $w(F', F)$-metrizability of $\overline{q(M_n)}^{w(F',F)}$). This holds for every $n \in \mathbb{N}$. Since $F' = \overline{\bigcup_n q(M_n)}^{w(F',F)}$, the space $F'(w(F', F))$ is separable, too.

We Claim now that $A \cap F$ is w-ΞK (it has property (*)). Indeed, let f'^* be a $w(F'^*, F')$-adherent point of a given sequence (x_n) in $A \cap F$. The element

$f'^* \circ q$ ($\in E'^*$) is a $w(E'^*, E')$-adherent point to (x_n), and there exists a sequence (e'_n) in E' such that $q(e'_n) = f'_n$ for all $n \in \mathbb{N}$. By the assumption, we can find $a \in A \cap \overline{\text{span}}\{x_n\}_{n\in\mathbb{N}} \subset A \cap F$ ($a \in \overline{\text{span}}\{x_n\}_{n\in\mathbb{N}} \subset F$) such that $\langle e'^* - a, e'_n \rangle = 0$, i.e., $\langle f'^* - a, f'_n \rangle = 0$, for all $n \in \mathbb{N}$. This proves the Claim.

Therefore, we can apply Theorem 56 to conclude that $A \cap F$ is a w(R)K subset of F.
Since this holds for every sequence (a_n) in A, this proves that A is w(R)NK in E. [Fl80, Theorem 3.10] shows that (E, w) is angelic. This proves that A is w-(R)K. ∎

The class of w-(R)ΞK subsets of a locally convex does not appear to be suitable for a result as Theorem 59. The reason is that it is not true in general that the intersection of a w-(R)ΞK subset of a locally convex space with a closed subspace (even with a closed subspace F such that $(F', w(F', F))$ is separable) is itself w-(R)ΞK. In order to see that, consider the following example.

Example 60 *There is a locally convex space E, a w-ΞK subset A of E, and a closed subspace F of E with $(F', w(F', F))$ separable, such that $A \cap F$ is not w-ΞK.*

Proof To show this—and we follow the notation in Example 39—let $f_n := \chi_{\bigcup_{k=1}^n X_k}$, $n \in \mathbb{N}$, where χ_S denotes the characteristic function of a set S. Certainly $f_n \in E$ for all $n \in \mathbb{N}$. Let

$$G := \text{span}\{f_n;\ n \in \mathbb{N}\} \subset E.$$

We `Claim` that G is a closed subspace.
To prove this, take $g \in G$, say $g := \sum_{k=1}^n \lambda_k f_k$, where $\lambda_k \in \mathbb{R}$ for all $k = 1, 2, \ldots, n$, and $n \in \mathbb{N}$. Then, $g(x) = \sum_{k=m}^n \lambda_k$ for $x \in X_m$, $m = 1, 2, \ldots, n$, and $g(x) = 0$ for $x \in X_m$, $m = n+1, n+2, \ldots$. What is relevant in this description is that g is constant on each X_m, $m \in \mathbb{N}$.
Let $(g_i)_{i\in I}$ be a net in G that converges to some element $e \in E$. Since the topology on E is the topology of the pointwise convergence, it is clear that e is a constant function on each X_m, $m \in \mathbb{N}$. Recall now that there exists $n_0 \in \mathbb{N}$ such that supp $(e) \cap \bigcup_{n_0}^{\infty} X_n$ is countable. Since each X_m is uncountable, we get that $e(x) = 0$ for every $x \in X_m$, $m \geq n_0$. Moreover, e is constant on each X_m, $m = 1, 2, \ldots, n_0 - 1$ (a different constant, in general,

on each X_m). This implies that $e \in G$; indeed, assume that $e(x) = a_k$ if $x \in X_k$, $k = 1, 2, \ldots, n_0 - 1$. Then $e = \sum_{k=1}^{n_0} \lambda_k f_k$, where $a_i - a_{i+1} = \lambda_i$, $i = 1, 2, \ldots, n_0 - 2$, and $a_{n_0-1} = \lambda_{n_0-1}$. It follows that G is a closed subset of E.

We `Claim` now that $(G', w(G', G))$ is topologically isomorphic to $(\varphi(\mathbb{N}), \mathcal{T}_p)$, where $\varphi(\mathbb{N})$ denotes the vector space of all the finitely supported elements in $\mathbb{R}^{\mathbb{N}}$, and $\mathcal{T}_p$ denotes the topology of the pointwise convergence on $\varphi(\mathbb{N})$. Indeed, the isomorphism $\phi : G' \to \varphi(\mathbb{N})$ is defined as $\phi(g') = (\langle f_n, g' \rangle)_{n \in \mathbb{N}}$, where $g' \in G'$. It is simple to prove that ϕ is indeed an algebraic isomorphism, due to the fact that $\{f_n;\ n \in \mathbb{N}\}$ is a Hamel basis of G. This show, too, that ϕ is a topological isomorphism whenever G' is endowed with the topology $w(G', G)$ and $\varphi(\mathbb{N})$ is endowed with the topology of the pointwise convergence.
Since $(\varphi(\mathbb{N}), \mathcal{T}_p)$ is separable, we get that $(G', w(G', G))$ is separable, too.

Recall that $\overline{A}$ is a w-ΞK subset of E. We shall prove now that $\overline{A} \cap G$ is not a w-ΞK subset of G. In order to see this, first consider that given $e \in E$ and $\varphi \in \varphi(X)$ $(= E')$, we have $\langle e, \varphi \rangle = \sum_{x \in X} e(x)\varphi(x)$. In particular, $\langle e, \delta_x \rangle = e(x)$ for all $x \in X$, where $\delta_x := \chi_{\{x\}}$. The sequence (f_n) is in $\overline{A} \cap G$. Choose a single element $x_n \in X_n$, for $n \in \mathbb{N}$. Then $\delta_{x_n} \in E'$, for $n \in \mathbb{N}$. The space G' can be identified to $q(E')$ $(= E'/G^{\perp})$, where $q : E' \to E'/G^{\perp}$ is the canonical quotient mapping. Then, $\langle f_m, \delta_{x_n} \rangle = \langle f_m, q(\delta_{x_n}) \rangle = 1$ for all $m \geq n$. Let $f'^* \in G'^*$ be a $w(G'^*, G')$-adherent point of the sequence (f_n). It follows that $\langle f'^*, q(\delta_{x_n}) \rangle = 1$ for all $n \in \mathbb{N}$. Assume for a moment that $\overline{A} \cap G$ is a w-ΞK subset of G. Then we can find $a \in \overline{A} \cap G$ such that $\langle a, q(\delta_{x_n}) \rangle = \langle f'^*, q(\delta_{x_n}) \rangle$ $(= 1)$ for all $n \in \mathbb{N}$. Certainly, $\langle a, q(\delta_{x_n}) \rangle = \langle a, \delta_{x_n} \rangle$ $(= a(x_n))$ for all $n \in \mathbb{N}$. This contradicts the fact that $a \in G$, since every element in G vanishes on $\bigcup_{k=n}^{\infty} X_k$ for some $n \in \mathbb{N}$. ∎

We shall prove now that Theorem 59 applies to the class of the w-(R)CK subsets of a locally convex space.

Proposition 61 *Let E be a locally convex space. Then, every w-(R)CK subset A of E is bounded and has the following property: given a sequence (a_n) in A and a $w(E'^*, E')$-adherent point $a'^* \in E'^*$, and a sequence (e'_n) in E', there exists $a \in A \cap \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$ ($a \in \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$) such that $\langle a'^* - a, e'_n \rangle = 0$ for all $n \in \mathbb{N}$.*

Proof Obviously, A is bounded. Let (a_n), a'^* and (e'_n) be as in the statement. Put

$$K_n := \{x \in \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\} :\ |\langle a'^* - x, e'_i \rangle| \leq 1/n,\ i = 1, 2, \ldots, n\},\ n \in \mathbb{N}.$$

Certainly (K_n) is a decreasing sequence of closed convex sets and, since a'^* is $w(E'^*, E')$-adherent to (a_n), we get $K_n \cap A \neq \emptyset$ for all $n \in \mathbb{N}$. From the very definition of w-(R)C-compactness we can find an element $a \in \bigcap_{n=1}^{\infty} \overline{K_n \cap A}^w \cap A$ ($a \in \bigcap_{n=1}^{\infty} \overline{K_n \cap A}^w$). We get $a \in A \cap \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$ ($a \in \overline{\text{span}}\{a_n;\ n \in \mathbb{N}\}$). Obviously, $\langle a'^* - a, e'_n \rangle = 0$ for all $n \in \mathbb{N}$. ∎

As a result, Theorem 59 is an extension of Theorem 2 in [Mo78]. In turn, this result extends a theorem of Valdivia in [Val77], an improvement of the theorem of Dieudonné and Schwartz (see, for example, [Ko69, §24.1(3)]) extending the classic theorem of Šmulyan (see, for example, [Ko69, §24.1(2)]).

Definition 62 *A locally convex space $(E, \mathcal{T})$ (or its topology $\mathcal{T}$) is called* submetrizable *if there exists on E a locally convex topology coarser than $\mathcal{T}$ and metrizable.*

Corollary 63 *Let $(E, \mathcal{T})$ be a submetrizable locally convex space (in particular, a locally convex space $(E, \mathcal{T})$ such that $(E', w(E', E))$ is separable). Then, every w-(R)ΞK set is w-(R)K.*

Corollary 63 applies, for example, to $(X', w(X', X))$ in the case of a separable Banach space X or, more generally, in the case that X is a weakly compactly generated (in short, WCG) Banach space, i.e., a Banach space with a weakly compact and linearly dense subset K, for in this case $(X', \mu(X', X))$ is submetrizable.

Remark 64 Observe that Theorem 59 is not true for w-∂K sets even in spaces with separable dual (see Example 42).

Remark 65 Set considered in Theorem 59 form a class strictly smaller than the class of w-ΞK subsets of a locally convex space. In order to prove this, it is enough to have a close look at Example 39. Indeed, we shall prove that the set $\overline{A}^E$ has not the property stated in Theorem 59. We elaborate on the argument in Example 60. Take $f_n := \chi_{\bigcup_{k=1}^n K_k}$ for $n \in \mathbb{N}$. Then $f_n \in \overline{A}^E$ for all $n \in \mathbb{N}$.

Observe that $G := \text{span}\{f_n;\ n \in \mathbb{N}\}$ is closed, hence $G = \overline{\text{span}}\{f_n;\ n \in \mathbb{N}\}$, so $f_n \in \overline{A}^E \cap G$ for all $n \in \mathbb{N}$. Choose an element $x_n \in X_n$, $n \in \mathbb{N}$. The sequence (δ_{x_n}) is in E'. Note that $\langle f_m, \delta_{x_n} \rangle = 1$ for all $m \geq n$. Let $e'^* \in E'^*$ be a $w(E'^*, E')$-adherent point of (f_n). Then $\langle e'^*, \delta_{x_n} \rangle = 1$ for all $n \in \mathbb{N}$. We can find $a \in \overline{A}^E$ such that $\langle a, \delta_{x_n} \rangle = \langle e'^*, \delta_{x_n} \rangle = 1$ for all $n \in \mathbb{N}$. We get $a(x_n) = 1$ for all $n \in \mathbb{N}$. Then $a \notin \overline{\text{span}}\{f_n;\ n \in \mathbb{N}\}$ $(= \text{span}\{f_n;\ n \in \mathbb{N}\})$ $(= G)$, since every element in G vanishes on some $\bigcup_{k=n}^{\infty} X_k$ (for some $n \in \mathbb{N}$).

1.5 Interchange of limits

Further criteria for weak compactness use the interchangeable limit condition, as in [Pt63] and [Gr52].

Definition 66 *Let Z, A and B be three non-empty sets, with $A \subset Z^X$ and $B \subset X$. We say that* A and B interchange limits in Z *(and we write $A \sim_Z B$) if given two arbitrary sequences (a_n) in A and (b_n) is B such that both limits*

$$\lim_n \lim_m a_n(b_m),\ \lim_m \lim_n a_n(b_m)$$

exists, then

$$\lim_n \lim_m a_n(b_m) = \lim_m \lim_n a_n(b_m).$$

If $Z := \mathbb{R}$ or $Z := \mathbb{C}$, and $A \sim_Z B$, we shall write simply $A \sim B$ if there is no misunderstanding.

Remark 67 A simple observation, that will be used later on, is that $A \sim B$ implies that $A' \sim B'$ for every couple of non-empty sets $A' \subset A$ and $B' \subset B$.

A central result in [Gr52] is that *a bounded subset of a $\mu(E, E')$-quasi-complete locally convex space E is $w(E, E')$-RK if and only if it interchanges limits with every absolutely convex $w(E', E)$-K subset of E'.*
A more complete form of this result appears in [Fl80, 1.6] as the Eberlein-Grothendieck Theorem. We reproduce it here for future references.

Theorem 68 (Eberlein-Grothendieck) *Let $(E, \mathcal{T})$ be a locally convex space, B a convex and $\mathcal{T}$-complete subset of E. For a subset A of B, the following are equivalent:*

(i) *A is* $w(E, E')$*-RNK.*
(ii) *A is* $w(E, E')$*-RK.*
(iii) *A is bounded and interchanges limits with all* $\mathcal{T}$*-equicontinuous subsets of* E'.

Proposition 69 *Let* $(E, \mathcal{T})$ *be a locally convex space. If* $A \subset E$ *is* w*-R∂K, then* $A \sim B$ *for every absolutely convex and* $w(E', E)$*-K subset of* E'.

Proof. Take a sequence (x_n) in A and a sequence (x'_n) in B and suppose that both limits $\lim_n \lim_m \langle x_n, x'_m \rangle$ and $\lim_m \lim_n \langle x_n, x'_m \rangle$ exist. Let x' be a $w(E', E)$-adherent point of x'_n in B (it exists since B is $w(E', E)$-RK) and a'^* a $w(E'^*, E')$-adherent point of x_n in E'^* (it exists since A is bounded, hence $w(E'^*, E')$-RK in E'^*). Then

$$\lim_n \lim_m \langle x_n, x'_m \rangle = \lim_n \langle x_n, x' \rangle = \langle a'^*, x' \rangle.$$

On the other hand,

$$\lim_m \lim_n \langle x_n, x'_m \rangle = \lim_m \langle a'^*, x'_m \rangle.$$

Since the sequence (x'_n) is contained in an absolutely convex and $w(E', E)$-K subset of E', there exists, by Proposition 32, an element $a \in E$ such that $\langle a'^*, x'_m \rangle = \langle a, x'_m \rangle$ for every $m \in \mathbb{N}$ and $\langle a'^*, x' \rangle = \langle a, x' \rangle$. Then, $\lim_m \langle a'^*, x'_m \rangle = \lim_m \langle a, x'_m \rangle = \langle a, x' \rangle = \langle a'^*, x' \rangle$. ∎

Remark 70 As a consequence of Proposition 69 and the Grothendieck's theorem mentioned at the beginning of this section, every w-R∂K subset of a $\mu(E, E')$-quasi-complete locally convex space E is w-RK. The same statement follows also from James' theorem once we shall prove that w-R∂K sets have the property that continuous linear functionals attain their supremum on its weak closure. This will be elaborate at the beginning of section 1.7.

Proposition 71 *Let* $(E, \mathcal{T})$ *be a locally convex space. If* $A \subset E$ *is bounded and has the following property: given a sequence* (a_n) *in* A*, a* $w(E'^*, E')$*-adherent point* $a'^* \in E'^*$*, there exists* $x \in E$ *such that* $\langle a'^* - x, x'_n \rangle = 0$ *for all* $n \in \mathbb{N}$*. Then* $A \sim B$ *for every* $w(E', E)$*-RNK subset* B *of* E'.

Proof The proof is similar to the one provided for Proposition 69, with slight changes. We give the details for the sake of completeness. Take a sequence (x_n) in A and a sequence (x'_n) in B and suppose that both limits $\lim_n \lim_m \langle x_n, x'_m \rangle$ and $\lim_m \lim_n \langle x_n, x'_m \rangle$ exist. Let x' be a $w(E', E)$-adherent point of (x'_n) in B (it exists since B is $w(E', E)$-RNK) and a'^* a $w(E'^*, E')$-adherent point of (x_n) in E'^* (it exists since A is bounded, hence $w(E'^*, E')$-RK in E'^*). Then

$$\lim_n \lim_m \langle x_n, x'_m \rangle = \lim_n \langle x_n, x' \rangle = \langle a'^*, x' \rangle.$$

On the other hand,

$$\lim_m \lim_n \langle x_n, x'_m \rangle = \lim_m \langle a'^*, x'_m \rangle.$$

There exists, by Proposition 24, an element $a \in E$ such that $\langle a'^*, x'_m \rangle = \langle a, x'_m \rangle$ for every $m \in \mathbb{N}$ and $\langle a'^*, x' \rangle = \langle a, x' \rangle$. Then, $\lim_m \langle a'^*, x'_m \rangle = \lim_m \langle a, x'_m \rangle = \langle a, x' \rangle = \langle a'^*, x' \rangle$. ∎

Remark 72 Note that the proof of Proposition 71 shows that every sequence in a w-RΞK subset of E interchanges limits with every sequence (x'_n) in E' having a $w(E', E)$-cluster point in E'. This will be used in the proof of Proposition 78.

1.6 Web-compact spaces

J. Orihuela introduced, in [Ori87], a class of locally convex spaces that includes most of the spaces used in Functional Analysis in connection with compactness and angelicity. First of all, he considers a non-void set X and a non-void subset Σ of $\mathbb{N}^{\mathbb{N}}$. Let

$$S := \{(a_1, a_2, \ldots, a_n) : \text{ there exists } \alpha \in \Sigma;\ \alpha = (a_1, a_2, \ldots),\ n \in \mathbb{N}\}. \quad (1.4)$$

Assume that there is a covering $\{A_\alpha;\ \alpha \in \Sigma\}$ of X consisting of non-void subsets of X. Given $\alpha = (a_m) \in \Sigma$ and $n \in \mathbb{N}$, put

$$C_{a_1, a_2, \ldots, a_n} := \bigcup \{A_\beta;\ \beta \in \Sigma,\ \beta = (b_m),\ b_j = a_j,\ j = 1, 2, \ldots, n\}. \quad (1.5)$$

The family of sets defined this way is clearly countable.
In this Memoir, and given a set Z, the product space Z^X will be endowed with the topology $\mathcal{T}_p$ of the pointwise convergence. Given a subset A of a topological space S, we denote by $\ddot{A}$ the *sequential closure* of A, i.e., the set of all limit points of sequences in A that converge in S. The following result holds. We follow the notation introduced in the previous paragraph.

Theorem 73 (Orihuela, [Ori87]) *Let (Z, d) be a compact metric space and let $A \subset Z^X$ be a non-empty set. Assume that, for every $\alpha = (a_m) \in \Sigma$ and every sequence (x_n) in X eventually in every set $C_{a_1,a_2,\ldots,a_m}$, $m \in \mathbb{N}$, we have that (x_n) interchanges limits in Z with every sequence in A. Then, $\ddot{A} = \overline{A}$.*

To apply this theorem we need to ensure the interchangeable double-limit property of sequences in our set A with adequate sequences in the base space. This is accomplished by introducing the following class of topological spaces.

Definition 74 (Orihuela, [Ori87]) *A topological space X is called a* web-compact space *if there exists a subset Σ of $\mathbb{N}^{\mathbb{N}}$ and a family $\{A_\alpha;\ \alpha \in \Sigma\}$ of subsets of X such that the following two conditions are satisfied:*
(i) $\overline{\bigcup\{A_\alpha;\ \alpha \in \Sigma\}} = X$*;*
(ii) *if $\alpha = (a_n) \in \Sigma$ and $x_n \in C_{a_1,a_2,\ldots,a_n}$, $n \in \mathbb{N}$, then the sequence (x_n) has an adherent point in X.*

Remark 75 Each of the sets A_α in the former definition is RNK. This is so due to the following. Fix $\alpha = (a_1, a_2, \ldots) \in \Sigma$; let (x_n) be a sequence in A_α. Certainly the set A_α belongs to $C_{a_1,a_2,\ldots,a_n}$ for every $n \in \mathbb{N}$. This ensures that $x_n \in C_{a_1,a_2,\ldots,a_n}$ for every $n \in \mathbb{N}$. Therefore, (x_n) has an adherent point, and this implies that A_α is indeed RNK.

The following result is one of the key theorems in [Ori87].

Theorem 76 (Orihuela, [Ori87]) *Let X be a web-compact topological space. Then, $C_p(X)$, the space of real continuous functions on X endowed with the topology of the pointwise convergence, is angelic.*

As an application of the former result to the theory of locally convex spaces, we have

Theorem 77 (Orihuela, [Ori87]) *Let E be a locally convex space such that $(E', w(E', E))$ is web-compact. Then, $(E, w(E, E'))$ is angelic.*

In particular, under the conditions in Theorem 77, and in view of part (1) in the theorem in Section 3.3 in [Fl80], the following classes of subsets of E coincide: $w(E, E')$-(R)NK, $w(E, E')$-(R)SK and $w(E, E')$-(R)K. One significative part (in the spirit of the original Eberlein's theorem) is that every $w(E, E')$-(R)NK subset of E is $w(E, E')$-(R)K. We can extend this to the class of all $w(E, E')$-(R)CK subsets of E. This will be done in Theorem 81.

In order to prove our extension we present first a proposition whose proof follows closely the original one of Theorem 73 [Ori87, Theorem 1], with some small changes. We provide a detailed proof to emphasize the differences and to present the technique for proving the next compactness results (Theorems 79 and 81). We follow the notation introduced in the first paragraph of this section (equations (1.4) and (1.5)).

Proposition 78 *Let $(E, \mathcal{T})$ be a locally convex space. Let A be a subset of a w-(R)ΞK subset of E. Assume that there exists a covering $\{A_\alpha;\ \alpha \in \Sigma\}$ of E' such that if $\alpha = (a_n) \in \Sigma$ and $x'_n \in C_{a_1, a_2, \ldots, a_n}$, $n \in \mathbb{N}$, then the sequence (x'_n) has a $w(E', E)$-adherent point in E'. Then $\dddot{A} = \overline{A}$ in the space $(E'^*, w(E'^*, E'))$.*

Proof Put $D_{(a_1, \ldots, a_n)} := C_{(a_1, \ldots, a_n)} \cap A^\circ$, for $\alpha := (a_1, a_2, \ldots) \in \Sigma$ and $n \in \mathbb{N}$. The set S is countable. In particular, there exists a one-to-one and onto mapping $\phi : \mathbb{N} \to S$. Take $f_1 \in \overline{A}^{w(E'^*, E')}$. Let $G_1 : E' \to \mathbb{R}$ defined by $G_1(x') := f_1(x')$ for all $x' \in E'$. Since $G_1(D_{\phi(1)}) \subset [-1, 1]$, there exists a finite subset L_1^1 of $D_{\phi(1)}$ such that

$$\min_{y' \in L_1^1} |f_1(x') - f_1(y')| \leq 1, \quad \text{for all } x' \in D_{\phi(1)}.$$

We can then find $f_2 \in A$ such that

$$\max_{y' \in L_1^1} |f_1(y') - f_2(y')| \leq 1/2.$$

Define now a mapping $G_2 : E' \to \mathbb{R}^2$ as $G_2(x') := (f_1(x'), f_2(x'))$ for all $x' \in E$, and observe that $G_2(D_{\phi(i)}) \subset [-1, 1] \times [-1, 1]$ for $i = 1, 2$. Therefore we can find a finite subset $L_2^i \subset D_{\phi(i)}$ for $i = 1, 2$, such that

$$\min_{y' \in L_2^i} \max_{k=1,2} |f_k(y') - f_k(x')| \leq 1/2, \text{ for all } x' \in D_{\phi(i)},\ i = 1, 2.$$

Proceeding recursively, we can find finite subsets $L_n^i \subset D_{\phi(i)}$ for $1 \leq i \leq n$ and a sequence $(f_n)_{n=2}^{\infty}$ in A, such that

$$\min_{y' \in L_n^i} \max_{k=1,2,\ldots,n} |f_k(y') - f_k(x')| \leq 1/n, \text{ for all } x' \in D_{\phi(i)},\ 1 \leq i \leq n, \tag{1.6}$$

$$\max_{y' \in \bigcup_{1 \leq i \leq j \leq n} L_j^i} |f_{n+1}(y') - f_1(y')| \leq 1/(n+1). \tag{1.7}$$

Obviously, $f_n \to f_1$ pointwise on $\bigcup_{1 \leq i \leq j} L_j^i$.
Take an arbitrary element $x' \in A^\circ$. There exists $\alpha := (a_1, a_2, \ldots) \in \Sigma$, such that $x' \in A_\alpha$. Find a sequence (p_n) of distinct points in $\mathbb{N}$ such that $\phi(p_m) = (a_1, \ldots, a_m)$ for all $m \in \mathbb{N}$. Certainly, $x' \in D_{\phi(p_m)}$ for all $m \in \mathbb{N}$. Let $m \in \mathbb{N}$. According to (1.6), there exists, for all $n \geq p_m$, an element $y_n^{p_m} \in L_n^{p_m}$ such that

$$\max_{1 \leq k \leq n} |f_k(y_n^{p_m}) - f_k(x')| \leq 1/n. \tag{1.8}$$

For $m \in \mathbb{N}$, put $y'_m := y_{p_m}^{p_m}$, and observe that $y'_m \in L_{p_m}^{p_m} \subset C_{\phi(p_m)} = C_{a_1,\ldots,a_m}$. This implies, by hypothesis, that the sequence (y'_m) has a $w(E', E)$-cluster point.
We shall prove that the sequence $(f_k(x'))_k$, a sequence in $[-1, 1]$, has $f_1(x')$ as its only cluster point. This will imply that, in fact, $(f_k(x'))_k$ converges to $f_1(x')$, so $f_k \to f_1$ pointwise on A°, and so on E', since A is bounded.
Let $r \in [-1, 1]$ be a cluster point of $(f_k(x'))_k$. There exists then an increasing sequence (k_n) of natural numbers such that $f_{k_n}(x') \to_n r$. Then we have

$$\begin{aligned} r &= \lim_n f_{k_n}(x') = \lim_n \lim_m f_{k_n}(y'_m) \\ &= \lim_m \lim_n f_{k_n}(y'_m) = \lim_m f_1(y'_m) = f_1(x'). \end{aligned} \tag{1.9}$$

The second equality in (1.9) comes from (1.8) and the fact that the set $\{p_m;\ m \in \mathbb{N}\}$ is infinite. The third one, i.e., the interchange of limits, comes from the fact that (y'_m) has a $w(E', E)$-cluster point, that (f_{k_n}) is a sequence in A and that A is contained in a $w(E, E')$-RΞK subset of E; then we can use Remark 72. The fifth equality comes from (1.8) again. This proves the uniqueness of the cluster point and its coincidence with $f_1(x')$. This concludes the proof. ∎

Theorem 79 *Let E be a locally convex space such that, for some index set $\Sigma \subset \mathbb{N}^{\mathbb{N}}$, there exists a covering $(A_\alpha)_{\alpha\in\Sigma}$ of E', with the following property: given $\alpha := (a_1, a_2, \dots) \in \Sigma$, and letting $C_{a_1,\dots,a_n} := \bigcup_{\beta\in\Sigma,\beta=(a_1,\dots,a_n,\dots)} A_\beta$ for all $n \in \mathbb{N}$, every sequence (y_n) in E' such that $y_n \in C_{a_1,\dots,a_n}$ for all $n \in \mathbb{N}$ has a $w(E', E)$-cluster point in E'. Then, every $w(E, E')$-(R)CK subset of E is $w(E, E')$-(R)K.*

Proof It is enough to apply Proposition 36, then Proposition 78 and finally Corollary 13. ∎

To extend Theorem 79 to the case where $(E', w(E', E))$ is web-compact we need the following intermediate result.

Proposition 80 *Let $(E, \mathcal{T})$ be a locally convex space such that $(E', w(E', E))$ is web-compact. Then, the web-compact structure $(A_\alpha)_{\alpha\in\Sigma}$ in E' can be assumed to have the property that $\bigcup_{\alpha\in\Sigma} A_\alpha$ is a linear subspace of E'.*

Proof. For $\alpha \in \Sigma$, put $B_\alpha := [0, 1]A_\alpha := \bigcup_{r\in[0,1]} rA_\alpha$. It is clear that $\{B_\alpha;\ \alpha \in \Sigma\}$ is a web compact structure in $(E', w(E', E))$.
Indeed, observe first that $\bigcup_{\alpha\in\Sigma} A_\alpha \subset \bigcup_{\alpha\in\Sigma} B_\alpha$, so $\bigcup_{\alpha\in\Sigma} B_\alpha$ is $w(E', E)$-dense in E'.
Second, note that, for $\alpha := (a_1, a_2, \dots) \in \Sigma$ and for all $n \in \mathbb{N}$,

$$\bigcup_{(a_1,\dots,a_n,\dots)\in\Sigma} B_{(a_1,\dots,a_n,\dots)} = [0, 1] \bigcup_{(a_1,\dots,a_n,\dots)\in\Sigma} A_{(a_1,\dots,a_n,\dots)}.$$

Therefore, as soon as a sequence (y'_n) in E' is given such that

$$y'_n \in \bigcup_{(a_1,\dots,a_n,\dots)\in\Sigma} B_{(a_1,\dots,a_n,\dots)}$$

for all $n \in \mathbb{N}$, the sequence (y'_n) has a $w(E', E)$-cluster point in E'. So we may assume from the very beginning that $[0, 1]A_\alpha = A_\alpha$ for all $\alpha \in \Sigma$.
The set $\bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ is countable, hence there exists a one-to-one and onto mapping $\phi : \mathbb{N} \to \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$. We define a subset Σ' of $\mathbb{N}^{\mathbb{N}}$ consisting precisely of all elements $\alpha' := (a_1, a_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ with the following properties: a_1 is an arbitrary natural number. Let

$$\phi(a_1) := (b_1, b_2, \dots, b_k).$$

Then we necessarily have

$$\begin{aligned}\psi(1) &:= (a_2, a_{k+2}, a_{2k+2}, \ldots) \in \Sigma,\\ \psi(2) &:= (a_3, a_{k+3}, a_{2k+3}, \ldots) \in \Sigma,\\ &\ddots\\ \psi(k) &:= (a_{k+1}, a_{2k+1}, a_{3k+1}, \ldots) \in \Sigma.\end{aligned}$$

Now we define a family $\{B_{\alpha'};\ \alpha' \in \Sigma'\}$ in the following way: For $\alpha' \in \Sigma'$,

$$B_{\alpha'} := \sum_{i=1}^{k} b_i A_{\psi(i)}.$$

It is again easy to show that $\{B_{\alpha'};\ \alpha' \in \Sigma'\}$ is a web-compact structure in $(E', w(E', E))$. Indeed, that $\bigcup_{\alpha'\in\Sigma'} B_{\alpha'}$ is $w(E', E)$-dense in E' is clear. The second property holds, too. It is enough to remark that given $\alpha' := (a_1, a_2, \ldots) \in \Sigma'$ and a sequence (y_n) in E' such that $y_n \in C'_{(a_1,\ldots,a_n)}$ for all $n \in \mathbb{N}$ (where, for all $n \in \mathbb{N}$, $C'_{(a_1,\ldots,a_n)} := \bigcup_{(a_1,\ldots,a_n,\ldots)\in\Sigma} B_{(a_1,\ldots,a_n,\ldots)}$), then, since a_1 is the common first element of all finite sequences $(a_1, a_2, \ldots, a_n)$ involved, (y_n) has a $w(E', E)$-cluster point in E'.
It is clear now that $\bigcup_{\alpha'\in\Sigma'} B_{\alpha'}$ is a linear subspace of E'. ∎

Theorem 81 *Let $(E, \mathcal{T})$ be a locally convex space such that $(E', w(E', E))$ is web-compact. Then, every w-(R)CK subset of E is w-(R)K.*

Proof. Let $\{A_\alpha;\ \alpha \in \Sigma\}$ be the family of subsets of E' giving its web-compact structure. According to Proposition 80 we may assume that $F := \bigcup_{\alpha\in\Sigma} A_\alpha$ is a linear subspace of E'. Let us consider the locally convex space $(E, w(E, F))$. Let A be a w-RCK subset of E. The identity mapping

$$\mathrm{Id} : (E, w(E, E')) \to (E, w(E, F))$$

is continuous. We can use Proposition 7 to ensure that A is a $w(E, F)$-(R)CK subset of E. Fix $x'^* \in \overline{A}^{(E'^*, w(E'^*, E'))}$. Certainly, $x'^*|_F$ ($\subset F^*$) is an element in $\overline{A}^{(F^*, w(F^*, F))}$. Since $\{A_\alpha;\ \alpha \in \Sigma\}$ is a covering of F and it defines a web-compact structure in $(F, w(F, E))$, we get from Proposition 78 that there exists a sequence (a_n) in A such that $a_n|_F \to x'^*|_F$ pointwise on F.
Let (a_{n_k}) be an arbitrary subsequence of (a_n). Put

$$C_k := \overline{\mathrm{conv}}^{(E, w(E, E'))}\{a_{n_k}, a_{n_{k+1}}, \ldots\}, \quad \text{for } k \in \mathbb{N}.$$

Certainly, C_k is a closed convex subset of E and $C_k \cap A \neq \emptyset$ for all $k \in \mathbb{N}$. Since A is $w(E, E')$-(R)CK, there exists $a \in A$ ($a \in E$) such that

$$a \in \bigcap_{k \in \mathbb{N}} \overline{C_k \cap A}^{(E, w(E,E'))}.$$

Fix $f \in F$ and $\varepsilon > 0$. Since $a_n \to x'^*$ pointwise on F, we can find $n_0 \in \mathbb{N}$ such that

$$|\langle x'^* - a_n, f \rangle| \leq \varepsilon, \quad \text{for all } n \geq n_0.$$

Find $k_0 \in \mathbb{N}$ such that $n_k \geq n_0$ for all $k \geq k_0$. It follows that

$$|\langle x'^* - x, f \rangle| \leq \varepsilon, \quad \text{for all } x \in \text{conv}\{a_{n_k}, a_{n_{k+1}}, \ldots\}, \text{ for all } k \geq k_0,$$

and so

$$|\langle x'^* - x, f \rangle| \leq \varepsilon, \quad \text{for all } x \in C_k, \text{ for all } k \geq k_0.$$

In particular,

$$|\langle x'^* - a, f \rangle| \leq \varepsilon.$$

This holds for all $\varepsilon > 0$, and for all $f \in F$. Therefore, $a|_F = x'^*|_F$.
Assume now that we choose a different subsequence of (a_n) and we repeat the procedure above to find another (in principle, different) element $a' \in A$ ($a' \in E$). We obtain again $a'|_F = x'^*|_F$. Since both a and a' belong to E and they agree on F, a $w(E', E)$-dense subspace of E', we get $a = a'$. We can apply now Proposition 4 to obtain that $a_n \to a$ pointwise on E'.
Fix now $x' \in E'$. Put $B_\alpha := A_\alpha \cup \{x'\}$ for all $\alpha \in \Sigma$. This defines again, as it is elementary to prove, a web-compact structure $\{B_\alpha;\ \alpha \in \Sigma\}$. By proceeding as in Proposition 80 we may assume that $\bigcup_{\alpha \in \Sigma} B_\alpha$ is the linear subspace span$\{F \cup \{x'\}\}$. Repeat the former argument to find a sequence (a'_n) in A such that

$$a'_n \to x'^* \text{ pointwise on } F \cup \{x'\}. \tag{1.10}$$

As before, we can find $a' \in E$ such that

$$a'_n \to a' \text{ pointwise on } E'. \tag{1.11}$$

Note that a agrees with x'^* on F and a' agrees with x'^* on F, hence a and a' agree on F and so they agree on E', i.e.,

$$a = a'. \tag{1.12}$$

Now we have

$$\langle a'_n, x'\rangle \rightarrow \langle x'^*, x'\rangle \tag{1.13}$$
$$\langle a'_n, x'\rangle \rightarrow \langle a', x'\rangle \ (= \langle a, x'\rangle). \tag{1.14}$$

Indeed, the first line follows from (1.10), while the second one follows from (1.11) and (1.12). From (1.13) and (1.14) we get, finally,

$$\langle x'^*, x'\rangle = \langle a, x'\rangle. \tag{1.15}$$

This is true for all $x' \in E'$ (and a is independent of the particular $x' \in E'$) chosen. This proves that $x'^* \in A$ ($x'^* \in E$) and so A is $w(E, E')$-(R)K, as we wanted to prove. ∎

Remark 82 As a consequence of the Theorem 81 and Example 39, we can conclude that the space $(E, \mathcal{T})$ introduced in that example has a dual space E' that, endowed with the $w(E', E)$-topology, is not web-compact. Indeed, the set A there is certainly $w(E, E')$-NK. Should $(E', w(E', E))$ be web-compact, A will be $w(E, E')$-K, in particular $w(E, E')$-closed. This is not the case (recall that the space $(E, \mathcal{T})$, in this particular example, carries its weak topology).

1.7 Supremum-attaining functionals

It is a simple observation that *a w-(R)∂K subset A of a locally convex space $(E, \mathcal{T})$ has the property that every $x' \in E'$ attains the supremum on A (on $\overline{A}^w$).* Indeed, A is bounded, so $\overline{A}^{w(E'^*, E')}$ is $w(E'^*, E')$-K, and then certainly x' attains the supremum s on it (say at a'^*). Consider the sequence (x'_n) in E' where $x'_n := x'$ for all $n \in \mathbb{N}$. By Proposition 32 there exists $a \in A$ ($\in \overline{A}^w$) such that ($s =$) $\langle a'^*, x'\rangle = \langle a, x'\rangle$, so x' attains the supremum on A (on $\overline{A}^w$). As a consequence of this observation and Theorem 45 it follows that *every w(R)∂K subset of a $\mu(E, E')$-quasi-complete locally convex space E is w-(R)K.*

Example 39 provides a locally convex space $(E, \mathcal{T})$ and a closed absolutely convex subset that is w-∂K and not w-CK. In particular, every element $x' \in E'$ attains its supremum on it and still the set is not weakly compact.

Thanks to James' Example 48 it is possible to provide an instance of this behavior even in the normed case, so we can separate in this context the property w-∂K from the property of being closed and that every continuous linear functional attains its supremum.

Corollary 83 *There exists a normed space X and a closed convex subset of X which is not w-∂K (and so it is not weakly CK either) and such that every element of the dual attains its supremum on it.*

Proof. Let X and E be as in Example 48. We follow the notations there. Define $S := \bigcup_1^\infty X_n$ (disjoint union). Every element $s \in S$ can be written $s := (r_{1,1}, r_{2,1}, r_{2,2}, r_{3,1}, r_{3,2}, r_{3,3}, \ldots)$, and so, by renumbering, $r = (r_1, r_2, \ldots)$. Clearly,

$$E' = \ell_2((X_1, \|\cdot\|_1), (X_2, \|\cdot\|_1), \ldots)$$

and

$$E'' = X := \ell_2((X_1, \|\cdot\|_\infty), (X_2, \|\cdot\|_\infty), \ldots).$$

Take a point $x \in B_X$ such that $x \neq E$. There exists a sequence (x_n) of points of B_E such that $x_n \to x$. Consider the sequence (f'_n) of functionals of E', defined by $f'_n := e_n$, where e_n is a sequence with value 1 in the n-th coordinate of S, and 0 elsewhere, for $n \in \mathbb{N}$. (f'_n) is contained in $B_{E'}$, an absolutely convex and $w(E', E)$-compact set. Suppose that there exists $x_0 \in E$ such that $\langle x_0, f'_n\rangle = \langle x, f'_n\rangle$. Then $x_0 = x$, a contradiction. Therefore, B_E is not w-∂K. However, B_E has the property that every element of E' attains the supremum on B_E. ∎

Summing up some results and examples provided in this chapter, we get the following implications (here SA denotes the class of subsets of a locally convex space $(E, \mathcal{T})$ such that every element in E' attains its supremum on them):

$$\text{(R)NK} \Rightarrow \text{(R)CK} \Rightarrow w\text{-(R)}\Xi\text{K} \Rightarrow w\text{-(R)}\partial\text{K} \Rightarrow SA.$$

None of the previous implications can be reversed.

1.8 (R)SK, (R)NK, (R)K

It is worth mentioning that removing the adjective "relatively" in weak compactness statements is not always easy nor possible. Corollary 47 extends non-trivially in this direction the Dieudonné-Schwartz Theorem.

The "non-relatively" statement is not true in James' Theorem. More precisely,

Example 84 *There exists a $\mu(E,E')$-quasi-complete locally convex space E and a set $A \subset E$ such that every element $e' \in E'$ attains its supremum on A although A is not weakly compact (of course, A is weakly relatively compact).*

In order to present the example, recall the following statement (see, e.g., [Ko69, §21.6.4]):
Let E be a Fréchet locally convex space. Then $(E,' \mu(E',E))$ is complete.
Here, and given a locally convex space X, $\mu(X',X)$ denotes the *dual Mackey topology*, i.e., the topology on X' of the uniform convergence on the family of all absolutely convex and weakly compact subsets of X.
In particular, if X is a Banach space, then $(X',\mu(X',X))$ is complete.
Now, for some uncountable set Γ, let

$$(E,\mathcal{T}) := (\ell_\infty(\Gamma), w(\ell_\infty(\Gamma),\ell_1(\Gamma)))$$

(a $\mu(\ell_\infty(\Gamma),\ell_1(\Gamma))$-complete locally convex space), and let

$$A := \{a \in B_{\ell_\infty(\Gamma)}; \#\text{supp }(a) \le \omega\},$$

where supp (a) denotes the support of an element $a \in \ell_\infty(\Gamma)$. Then *$A$ is $w(\ell_\infty(\Gamma),\ell_1(\Gamma))$-NK* (indeed, given a sequence (a_n) in A, there exists a $w(\ell_\infty(\Gamma),\ell_1(\Gamma))$-cluster point a in $\ell_\infty(\Gamma)$. Since $\bigcup_{n\in\mathbb{N}} \text{supp }(a_n)$ is countable, we get $a \in A$). In particular, every element $x \in \ell_1(\Gamma)$ attains its supremum on A.
However, *A is not $w(\ell_\infty(\Gamma),\ell_1(\Gamma))$-closed in $(E,\mathcal{T})$*. Indeed, it is dense in $(B_{\ell_\infty(\Gamma)},\mathcal{T})$; to see this, let $x' \in B_{\ell_\infty(\Gamma)}$. Given $\varepsilon > 0$ and a finite number of vectors $x_1,\dots,x_n$ in $\ell_1(\Gamma)$, we have that $\bigcup_{i=1}^n \text{supp } x_i$ is countable. Let $a \in A$ be defined as $a(\gamma) := x'(\gamma)$ for $\gamma \in \bigcup_{n\in\mathbb{N}} \text{supp } x_i$ and $a(\gamma) := 0$ otherwise. Then a belongs to the $w(\ell_\infty(\Gamma),\ell_1(\Gamma))$-neighborhood of x' defined by ε and $x_1,\dots,x_n$.

Another example of similar relatively versus non-relatively statements is the following result.

Theorem 85 (Howard [How73]) *Let X be a Banach space. If $A \subset X'$ is $\mu(X',X)$-relatively sequentially compact, then A is $\mu(X',X)$-relatively compact. The converse does not hold true.*

In order to obtain the non-relative statement, we request some extra conditions. Observe that WCG Banach spaces or, more generally, subspaces of WCG Banach spaces, satisfy the condition in the following theorem; indeed, a Banach space X is a subspace of a WCG Banach space if and only if $(B_{X'}, w(X', X))$ is an Eberlein compact, and it is well known that every Eberlein compact is angelic. Even more generally, every weakly Lindelöf determined (WLD, in short) Banach space X has a $w(X', X)$-angelic dual closed unit ball. Recall that a Banach space X is WLD if the dual closed unit ball, endowed with the $w(X', X)$-topology, is a *Corson* compactum, i.e., it is homeomorphic to a compact subspace of a product of lines, and each of its elements has a countable support.
We propose the following result, providing a direct proof. However, the statement can be deduced (even in a stronger form) from a result in the theory of angelic spaces (see Remark 87).

Theorem 86 *Let X be a Banach space such that $(B_{X'}, w(X', X))$ is angelic. Then every $\mu(X', X)$-(R)SK subset of E' is $\mu(X', X)$-(R)K.*

Proof. The relative statement is a particular case of Theorem 85. For the non-relative one, assume that A is a $\mu(X', X)$-SK subset of E' (in particular, it is $w(X', X)$-bounded, and we may, without loss of generality, assume that it is a subset of the closed dual unit ball. From Theorem 85, we know that A is $\mu(X', X)$-RK. This implies that $\overline{A}^{\mu(X',X)}$ is $\mu(X', X)$-K. It follows that $(\overline{A}^{\mu(X',X)}, \mu(X', X)) = (\overline{A}^{\mu(X',X)}, w(X', X))$, in particular $\overline{A}^{\mu(X',X)} = \overline{A}^{w(X',X)}$ and the topological space $(\overline{A}^{\mu(X',X)}, \mu(X', X))$ is angelic. In angelic spaces, we have (R)NK=(R)SK=(R)K (see, e.g., [Fl80, (1) in Theorem in 3.3]). ∎

Remark 87 Another related way to obtain Theorem 86 is to use the so-called "angelic lemma" ([Fl80, Lemma in 3.1]) and again one of its consequences [Fl80, (1) in Theorem in 3.3]. Indeed, the identity mapping $j : (B_{X'}, \mu(X', X)) \to (B_{X'}, w(X', X))$ is one-to-one, continuous, and certainly $(B_{X'}, \mu(X', X))$ is a regular space. Then, if $(B_{X'}, w(X', X))$ is angelic so it is $(B_{X'}, \mu(X', X))$, and the result follows. Indeed, if this is the case, we have $\mu(X', X)$-(R)K=$\mu(X', X)$-(R)SK=$\mu(X', X)$-(R)K for a bounded subset of X'.

Let $\langle E, F\rangle$ be a dual pair. Recall that a subset A of E is called F*-limited* if a sequence (f_n) in F converges to 0 uniformly on A as soon as (f_n) is a $w(F,E)$-null sequence.
The following characterization of limited sets is well known. We refer to [HMVZ08] for a proof.

Proposition 88 (Grothendieck, [Gr53]) *Let X be a Banach space. Then a bounded set $A' \subset X'$ is $\mu(X',X)$-RK if and only if it is X-limited.*

We may improve now Howard's result.

Theorem 89 *Let X be a Banach space. Then, every $\mu(X',X)$-RNK subset of X' is $\mu(X',X)$-RK.*

Proof Let A' be a $\mu(X',X)$-RNK subset of X'. Let (x_n) be a w-null sequence in X. Let (a'_n) be a sequence in A'. We
`Claim` that $\langle x_n, a'_n\rangle \to 0$.
To prove the Claim, let $Y := \overline{\text{span}}\{x_n;\ n \in \mathbb{N}\}$. Obviously, $M := \overline{\Gamma}\{x_n;\ n \in \mathbb{N}\}$ is a w-K subset of Y (use Krein's theorem, see, e.g., [Ko69, §24.5.(4)]). Let $q : X' \to Y'$ be the canonical quotient mapping. The quotient topology of $\mu(X',X)$ is $\mu(Y',Y)$. This is a consequence of [Ko69, §22.2.(4)]. In particular, $q : (X', \mu(X',X)) \to (Y', \mu(Y',Y))$ is continuous. By Proposition 7 we get that $q(A')$ is a $\mu(Y',Y)$-RNK subset of Y'.
On Y' we consider the topology $\mathcal{T}_M$ of the uniform convergence on the set M. This is a Hausdorff metrizable locally convex topology, due to the fact that M is linearly dense in Y, and $\mathcal{T}_M \preceq \mu(Y',Y)$. Again by Proposition 7 we get that $q(A')$ is $\mathcal{T}_M$-RNK. Since $\mathcal{T}_M$ is metrizable, we get that $q(A')$ is $\mathcal{T}_M$-RSK. In particular, the sequence (a'_n) has a subsequence (a'_{n_k}) such that $(q(a'_{n_k}))$ is $\mathcal{T}_M$-convergent to $q(x')$, for some $x' \in X'$.
Given $\varepsilon > 0$ we can find $n_0 \in \mathbb{N}$ such that

$$|\langle x_n, x'\rangle| < \varepsilon, \text{ for all } n \geq n_0.$$

On the other hand, there exists $k_0 \in \mathbb{N}$ such that

$$|\langle x_n, q(a'_{n_k}) - q(x')\rangle| < \varepsilon, \text{ for all } k \geq k_0 \text{ and for all } n \in \mathbb{N}.$$

We can find $k_1 \geq k_0$ such that $n_{k_1} \geq n_0$. Then, for all $k \geq k_1$,

$$|\langle x_{n_k}, q(a'_{n_k})\rangle| \leq |\langle x_{n_k}, q(a'_{n_k}) - q(x')\rangle| + |\langle x_{n_k}, q(x')\rangle| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

Observe now that the previous argument can be carried over every a priory chosen subsequence of the sequence of the natural numbers as indices. This concludes that $\langle x_n, a'_n \rangle \to 0$, as we claimed.

Certainly, the set A' is bounded, since it is $\mu(X', X)$-RNK and boundedness, by the Banach-Mackey theorem (see, e.g., [Ko69, §20.11(3) and (8)]), depends only on the dual pair. We shall prove now that A' is a X-limited set. Assume not. Then we can find a w-null sequence (x_n) is X that does not converges to 0 uniformly on A'. This implies the existence of a subsequence (x_{n_k}) of (x_n), a sequence (a'_k) in A', and some $\varepsilon > 0$ such that $|\langle x_{n_k}, a'_k \rangle| \geq \varepsilon$ for all $k \in \mathbb{N}$. This certainly contradicts the previous claim. The set A' is then limited.
To finish the proof it is enough to apply Proposition 88. ∎

1.9 Reflexivity

The bidual E'' of $(E, \mathcal{T})$ is the topological dual of $(E', \beta(E', E))$, where $\beta(E', E)$ is the strong topology on E', i.e., the topology on E' of the uniform convergence on the family of all bounded subsets of E. When $E'' = E$, the space E is called *semi-reflexive.* If, additionally, the strong topology $\beta(E'', E')$ on E'' induces on E the original topology $\mathcal{T}$, then E is called *reflexive.*

It is easy to obtain the following characterization of semi-reflexivity (see [Ko69, §23.3]).

Proposition 90 *For a locally convex space E the following are equivalent:*
(a) *E is semi-reflexive.*
(b) *Every bounded subset of E is weakly relatively compact.*
(c) *The topologies $\mu(E', E)$ and $\beta(E', E)$ on E' coincide.*
(d) *E is weakly quasi-complete.*

In [Ko69, §24.4(3)] a further criterion for semi-reflexivity is provided.

Theorem 91 *A $\mu(E', E)$-quasi-complete locally convex space $(E, \mathcal{T})$ is semi-reflexive if and only if every bounded closed absolutely convex subset A has a supporting hyperplane parallel to each closed hyperplane H.*

Actually, the argument goes like that: first, it is assumed that the space $(E, \mathcal{T})$ is not semi-reflexive. It follows that every closed hyperplane is not semi-reflexive. We select one, given as the kernel of a certain $f \in E'$, and use Corollary 51 to produce a decreasing sequence of closed convex subsets of the hyperplane with empty intersection. Adequate shifts in the direction of a one-dimensional complement produce a set whose closed absolutely convex hull has the property that f does not attain on it the supremum. This proof is due to Klee, see [Ko69, §24.4(3)]. A careful analysis of this proof shows that in fact a more precise statement holds. We write it as a theorem.

Theorem 92 *A $\mu(E', E)$-quasi-complete locally convex space $(E, \mathcal{T})$ is semi-reflexive if and only if there exists a closed hyperplane H such that every bounded closed absolutely convex subset A has a supporting hyperplane parallel to H. Equivalently, if and only if there exists $f \in E'$ which attains it supremum on every absolutely convex closed and bounded subset of E.*

This result cannot be obtained from James's Theorem.

A similar construction permits us to slightly improve this theorem for Banach spaces.

Theorem 93 *Let $(X, \|\cdot\|)$ be a Banach space. Then X is reflexive if and only if there exists a closed hyperplane H such that the closed unit ball $B_{(X,|||\cdot|||)}$ of X in every equivalent norm $|||\cdot|||$ has a supporting hyperplane parallel to H.*

Proof. One implication is immediate, since if X is reflexive then, for every equivalent norm $\|\cdot\|$ on X, $B_{(X,\|\cdot\|)}$ is weakly compact and the condition is satisfied.
Suppose on the contrary that X is not reflexive. Take any hyperplane H with $0 \in H$. We shall construct an equivalent norm $|||\cdot|||$ such that $B_{(X,|||\cdot|||)}$ has no supporting hyperplane parallel to H.
It is easy to check that H is not reflexive. Then $B_{(H,\|\cdot\|)}$ is not weakly compact. By Corollary 51 it is not weakly CK either. Therefore we can find a decreasing sequence of bounded non-empty convex closed subset C_n ($\subset B_{(H,\|\cdot\|)}$) ($n = 1, 2, \ldots$) of H with empty intersection. Take $x_0 \in E$, not lying in H and let $C_0 = B_{(H,\|\cdot\|)}$. Define

$$K := \overline{\Gamma}\left(\bigcup_{n=0}^{\infty}\left(1 - \frac{1}{n+1}\right)x_0 + C_n\right),$$

where Γ denotes the absolutely convex cover. As in the proof of Theorem 91, K does not have any supporting hyperplane parallel to H. It suffices now to prove that K is the closed unit ball of an equivalent norm $|||\cdot|||$. It is immediate that $K \subset (1+\|x_0\|)B_X$. On the other hand, fix $c_1 \in C_1$ and let $P : X \to H$ be the projection from X onto H parallel to $(1/2)x_0 + c_1$. Both P and $Q := Id_X - P$, where Id_X is the identity mapping on X, are continuous. Now it is easy to see that there exists some $\varepsilon > 0$ such that $\varepsilon.B_{(X,\|\cdot\|)} \subset \overline{\Gamma}(C_0 \cup [(1/2)x_0 + C_1])$, and this proves that $\varepsilon.B_{(X,\|\cdot\|)} \subset K$. ∎

1.10 Completeness

Compactness is strongly related with completeness and boundedness. Actually, a set A is compact if and only if it is complete and totally bounded (i.e., given any neighbourhood U of the origin, then A can be covered by a finite number of translates of U). Moreover, countable compactness (and so sequential compactness too) implies sequential completeness and total boundedness.

Proposition 94 *Let E be a locally convex space. Let A be a* CK *subset of E. Then A is sequentially complete and bounded. If A is RCK, then every Cauchy sequence in A converges in E.*

Proof. Boundedness comes from Proposition 28. Let $\widetilde{E}$ be the completion of E. The space E is a subspace of $\widetilde{E}$. Take a Cauchy sequence (a_n) in A. It converges to an element $\widetilde{a}$ in $\widetilde{E}$. Assume that A is CK. Then, by Proposition 12, $\widetilde{a} \in A$. This implies that A is sequentially complete in E. If, on the other hand, A is RCK, we get, again from Proposition 12, that $\widetilde{a} \in E$. ∎

Remark 95 Convex-compactness does not imply total boundedness. For an example, it is enough to take the closed unit ball of $(\ell_2, \|\cdot\|_2)$ in the norm $\|\cdot\|_2$.

Chapter 2
On Banach disks

2.1 Introduction

Let $(E, \mathcal{T})$ be a locally convex space and let $S \subset E$ be a non-empty set. It is natural to ask whether a certain property of S can be checked/translated to the corresponding property of S seen as a subspace of a certain Banach space, or if the Banach space itself enjoys the same or a related property.

The regular way to embed S as a subset of a Banach space is to build a *disk* (i.e., a bounded absolutely convex set) $D \subset E$ containing S and to consider the space E_D endowed with the Minkowski functional $\|\cdot\|_D$ (here, E_D is the linear hull of D, and the Minkowski functional $\|\cdot\|_D$ is defined as $\|x\|_D := \inf\{\lambda;\ \lambda > 0,\ x \in \lambda D\}$ for all $x \in E_D$). The space $(E_D, \|\cdot\|_D)$ is a normed space. In some cases it is in fact a Banach space (the disk D is then called a *Banach disk*). It is so, for example, if D is sequentially complete (see, for example, [Ko69, §20.11.(2)]).

There are essentially two ways to prove that a certain disk is a Banach disk. The first one derives from the well-known property saying that completeness for a coarser topology implies completeness for a finer topology *as soon as the later has a neighborhood basis formed by sets closed in the former*. For details, see, e.g., [Ko69, §18.4.(4)] and the proof of Theorem §11.20.(2) again in [Ko69]. The second one consists of defining a mapping from $l_1(\Gamma)$ into $(\widetilde{E}, \widetilde{\mathcal{T}})$ (where $(\widetilde{E}, \widetilde{\mathcal{T}})$ denotes de completion of $(E, \mathcal{T})$) in such a way that $S \subset TB_{l_1(\Gamma)}$ and to ensure that $TB_{l_1(\Gamma)}$ lies, in fact, in E. This second procedure provides in fact the sought Banach disk, while the first one checks that a certain disk is a Banach disk.

Now, we can precise the kind of questions we want to address. Consider a certain property $\mathcal{P}$. Assume that $S \subset E$ has property $\mathcal{P}$. Is it true that

there exists a (Banach) disk $D \supset S$ in E such that $(E_D, \|\cdot\|_D)$ has property $\mathcal{P}$ (or a closely related property)?
Without extra assumptions, even simple instances deny the possibility to obtain a positive answer. For example, consider separability. Assume that S is separable. Does it imply that S is contained in a (Banach) disk D such that $(E_D, \|\cdot\|_D)$ is separable? The simple Example 109 proves that in general this is not the case.
The second procedure for getting a Banach disk allows to derive some properties of the generated Banach space from properties of $\ell_1(\Gamma)$. Before giving a precise result we need some general statements. They are spread in the literature. We collect them here for future reference.

Along this chapter, $(\widetilde{E}, \widetilde{\mathcal{T}})$ will denote the completion of the topological vector space $(E, \mathcal{T})$.

Proposition 96 *Let $(E, \mathcal{T})$ be a locally convex space. Let $S \subset E$ be a bounded set. Then the mapping*

$$T : \ell_1(S) \to \widetilde{E} \tag{2.1}$$

given by $T((a_s)_{s\in S}) := \sum_{s\in S} a_s s$ for all $(a_s)_{s\in S} \in \ell_1(S)$ is well defined, linear, and continuous.

Proof. We shall prove that the series $\sum_{s\in S} a_s s$ is Cauchy (note that this series has only a countable number of non-zero summands). To that end, let U be an absolutely convex neighborhood of 0 in $(E, \mathcal{T})$. There exists $m \in \mathbb{N}$ such that $S \subset mU$. We can find a finite set $F \subset S$ such that $\sum_{s\in G} |a_s| \leq 1/m$ for every finite set $G \subset S$ such that $G \cap F = \emptyset$. Now, $m \sum_{s\in G} a_s s \in mU$, hence $\sum_{s\in G} a_s s \in U$ and so the series $\sum_{s\in S} a_s s$ is Cauchy. It follows that it converges to an element in $(\widetilde{E}, \widetilde{\mathcal{T}})$ and the mapping T is well defined. It is clear that T is linear. In order to see that it is continuous, proceed as in the first part of the proof: given an absolutely convex neighborhood U of 0 in $(E, \mathcal{T})$, find $m \in \mathbb{N}$ such that $S \subset mU$. Then $TB_{\ell_1} \subset m\widetilde{U}$, where $\widetilde{U}$ denotes the closure of U in $(\widetilde{E}, \widetilde{\mathcal{T}})$ (the family $\{\widetilde{U}; U \in \mathcal{U}\}$ is a basis of neighborhoods of 0 in $(\widetilde{E}, \widetilde{\mathcal{T}})$, were $\mathcal{U}$ denotes a basis of neighborhoods of 0 in $(E, \mathcal{T})$, see, e.g., [Ko69, §15.3(1)]). We get $(1/m)TB_{\ell_1} \subset \widetilde{U}$ and so T is continuous. ∎

In the context of Proposition 96, the normed space $(\widetilde{E}_{TB_{\ell_1(S)}}, \|\cdot\|_{TB_{\ell_1(S)}})$ is complete. This will be proved in Proposition 98. We need the following result.

Proposition 97 ([BP87], Prop. 3.2.3) *Let B be a disk in a topological vector space E such that, for every sequence (b_n) in B, the series $\sum_{n=1}^{\infty} 2^{-n} b_n$ converges in E to an element of B. Then E_B is a Banach space.*

Proposition 98 *Let $(E, \mathcal{T})$ be a locally convex space. Let S be a bounded subset of E, and let $T : \ell_1(S) \to \widetilde{E}$ be the mapping defined in (2.1). Then, the normed space $(\widetilde{E}_{TB_{\ell_1(S)}}, \| \cdot \|_{TB_{\ell_1(S)}})$ is complete.*

Proof. Let $T(b_n)$ be a sequence in $TB_{\ell_1(S)}$. The series $\sum_{n=1}^{\infty} 2^{-n} b_n$ in $\ell_1(S)$ is clearly Cauchy, hence it converges in $\ell_1(S)$ (to an element in $B_{\ell_1(S)}$, since each partial sum $\sum_{k=1}^{n} 2^{-k} b_k$ belongs to $B_{\ell_1(S)}$). Since T is linear and continuous, the series $\sum_{n=1}^{\infty} 2^{-n} T(b_n)$ converges to an element in $TB_{\ell_1(S)}$. By Proposition 97, the normed space $(\widetilde{E}_{TB_{\ell_1(S)}}, \| \cdot \|_{TB_{\ell_1(S)}})$ is complete. ∎

Observe that, since S is bounded, the topology $\| \cdot \|_{TB_{\ell_1(S)}}$ is finer than $\widetilde{\mathcal{T}}$. Therefore, for complete spaces we obtain the following result.

Theorem 99 *Let A be a bounded subset of the complete locally convex space $(E, \mathcal{T})$. Consider the mapping*

$$T : \ell_1(A) \longrightarrow E$$

given by

$$(\alpha_a)_{a \in A} \xrightarrow{T} \sum_{a \in A} \alpha_a \, a.$$

Then $D := T(B_{\ell_1(A)})$ is a Banach disk and the topology induced on E_D by $\mathcal{T}$ is coarser than the norm topology.

Remark 100 At first glance, it seems that completeness is not used in the proof of Proposition 98. This is not the case. The very existence of the mapping T is based on the completeness of the target space.
For non complete locally convex spaces, we can still consider the map $T : \ell_1(A) \to \widetilde{E}$, where $\widetilde{E}$ is the completion of E. If a bounded subset A is such that $D := T(B_{\ell_1(A)}) \subset E$, then A is contained in the Banach disk D.

Corollary 101 ([Ko69], §20.11.(2)) *Let $(E, \mathcal{T})$ be a locally convex space. Assume that A is an absolutely convex bounded closed and sequentially complete subset of E. Then A is a Banach disk.*

Proof. The sequential completeness of A implies that $D := T(B_{\ell_1(A)}) \subset E$. Consider now Remark 100. ∎

Another consequence appears in [Fl80, p.17].

Corollary 102 *Let $(E, \mathcal{T})$ be a locally convex space. Let A be a convex and relatively countably compact subset of E. Then A is contained in a Banach disk $D \subset E$.*

Proof. The argument consists again is proving that $T(B_{\ell_1(A)})$ is in fact in E. It is enough to use Remark 100, the relative countable compactness of A and a simple convexity argument. For details, see the reference given. ∎

Corollaries 101 and 102 are extended in the following result.

Theorem 103 *Let A be a bounded convex subset of a locally convex space $(E, \mathcal{T})$, such that*

$$\overset{\dots(\widetilde{E},\widetilde{\mathcal{T}})}{A} \subset E.$$

Then A is contained in a Banach disk.

Proof. By Remark 100 it is enough to prove that $T(B_{\ell_1(A)})$ is a subset of E, where $T : \ell_1(A) \to \widetilde{E}$ is the mapping defined in (2.1).
Every element $\widetilde{a} \in T(B_{\ell_1(A)})$ can be written $\widetilde{a} = \sum_{i=1}^{\infty} \alpha_i \ a_i \in \widetilde{E}$, where $a_i \in A$ and $\sum_{i=1}^{\infty} |\alpha_i| \leq 1$. Looking for positive and negative coefficients, we can split the sum into two parts,

$$\widetilde{a} = \underbrace{\sum_{i=1}^{\infty} \beta_i b_i}_{b} - \underbrace{\sum_{i=1}^{\infty} \gamma_i c_i}_{c},$$

where $\beta_i > 0$, $\gamma_i > 0$, $b_i \in A$ and $c_i \in A$ (we just renamed coefficients and vectors). Let $s_n := \sum_{i=1}^{n} \beta_i$, $s := \sum_{i=1}^{\infty} \beta_i$ and $x_n := (1/s_n) \sum_{i=1}^{n} \beta_i \, b_i \in A$ for $n \in \mathbb{N}$. Take any $\varepsilon > 0$. There exists N such that, for all $n \geq N$,

$$(1/s)b - x_n = (1/s - 1/s_n) \sum_{i=1}^{\infty} \beta_i b_i + (1/s_n) \sum_{i=n+1}^{\infty} \beta_i b_i \in \varepsilon \widetilde{A},$$

where $\widetilde{A}$ denotes the closure of A in $\widetilde{E}$ (a bounded set in $\widetilde{E}$). This proves that x_n tends to $(1/s)b$ in any compatible topology on $\widetilde{E}$. By the assumption,

$b \in E$. Analogously $c \in E$ and so does $\widetilde{a}$. If all coefficients in the sum that defines $\widetilde{a}$ have the same sign, only one of the two steps is needed. ∎

Theorem 103 can be applied to the class of convex relatively convex-compact sets. This is done in the next result.

Corollary 104 *Every convex, RCK subset A of a locally convex space $(E, \mathcal{T})$ is contained in a Banach disk.*

Proof. In Proposition 12, take $\widehat{E} := \widetilde{E}$. We get $\overset{\cdots}{A}^{(\widetilde{E},\widetilde{\mathcal{T}})} \subset E$. It is enough now to apply Theorem 103. ∎

If A is absolutely convex and convex-compact, we can be a little bit more precise.

Corollary 105 *Every absolutely convex, CK subset A of a locally convex space $(E, \mathcal{T})$ is a Banach disk.*

Proof. Let $T : \ell_1(A) \to \widetilde{E}$ the mapping defined in (2.1). Let B be the closed unit ball of $\ell_1(A)$. Obviously, $A \subset TB$. Let $\tilde{y} \in TB$. Then we can find a sequence (a_n) of elements in A and a sequence (α_n) of scalars such that $\sum_{n=1}^{\infty} |\alpha_n| \leq 1$ and $\tilde{y} := \sum_{n=1}^{\infty} \alpha_n a_n$. Obviously, $s_n := \sum_{k=1}^{n} \alpha_n a_n \in A$ for all $n \in \mathbb{N}$ and the sequence (s_n) converges to $\tilde{y}$. Proposition 12 ensures that $\overset{\cdots}{A}^{(\widetilde{E},\widetilde{\mathcal{T}})} \subset A$, so $\tilde{y} \in A$. This proves that $TB = A$. The conclusion follows from Theorem 99. ∎

Since, according to Corollary 104, convex RCK sets in locally convex spaces are contained in a Banach disk, we can use, for example, [Ko69, 20.11(3)] to conclude the following result.

Corollary 106 *Every convex, RCK subset A of a locally convex space $(E, \mathcal{T})$ is* strongly bounded*, i.e.,* $\sup_{u \in B,\ x \in A} |u(x)| < \infty$*, for each $w(E', E)$-bounded set $B \subset E'$.*

We give a bound for the class of sets in $(E, \mathcal{T})$ which are included in a Banach disk.

Example 107 *There exists a locally convex space $(E, \mathcal{T})$ and an absolutely convex, closed and w-ΞK subset of E and yet not included in a Banach disk.*

Proof. Let $(E,\mathcal{T})$ the locally convex space defined in Example 39, and let A be the subset of E defined there. We follow notations used there. It is clear that A is absolutely convex. Consider any disk $D \supset \overline{A}$ in E. We shall prove that $(E_D, \|\cdot\|_D)$ is not a Banach space. For this, put $f_n := \sum_{k=1}^{n} \frac{1}{k}\chi_{X_k}$, $n \in \mathbb{N}$, and consider the sequence (f_n) in E. This sequence satisfies that, for $n < m$ in $\mathbb{N}$, $\|f_n - f_m\| \leq \frac{1}{n}$, hence $f_n - f_m \in \frac{1}{n}\overline{A} \subset D$ and so it is a Cauchy sequence in E_D. Assume for a moment that there exists $f \in E_D$ such that

$$f_n \xrightarrow{\|\cdot\|_D} f.$$

Then $f_n \xrightarrow{\mathcal{T}_p} f$. This implies that $f \notin E$, a contradiction. This proves that $(E_D, \|\cdot\|_D)$ is not complete. ∎

The following example is a remark on the previous one, and shows the lack of stability of the concept *Banach disk*, or *contained in a Banach disk*.

Example 108 *The closure of a Banach disk does not need to be a Banach disk nor to be contained in a Banach disk.*

Proof. The set A in Example 39 is absolutely convex and $\mathcal{T}$-CK. Corollary 105 ensures that A is a Banach disk. However, we proved in Example 107 that $\overline{A}$ is not a Banach disk nor it is contained in a Banach disk. ∎

2.2 Separability

We mentioned at the Introduction that simple examples prove that separability is not preserved by passing to normed spaces generated by separable subsets of a locally convex space. Here we provide one such example.

Example 109 *There exists a locally convex space $(E,\mathcal{T})$ and an absolutely convex compact subset K such that $(K,\mathcal{T})$ is metrizable, hence separable, and yet, if K is contained in a disk $D \subset E$, the space $(E_D, \|\cdot\|_D)$ is not separable.*

Proof. Take $(E,\mathcal{T}) := (\ell_\infty, w(\ell_\infty, \ell_1))$, and let $K := B_{\ell_\infty}$. Then K is compact and metrizable (hence separable). Assume that D is a disk in $(E,\mathcal{T})$ such that $K \subset D$. The fact that $(\ell_1, \|\cdot\|_1)$ is a barrelled space proves that D is $\|\cdot\|_\infty$-bounded. Then, the set E_D is ℓ_∞ and the space $(E_D, \|\cdot\|_D)$ is isomorphic to $(\ell_\infty, \|\cdot\|_\infty)$, a nonseparable space. It follows that $(E_D, \|\cdot\|_D)$ is also nonseparable. ∎

The following statement provides an instance when we can conclude separability of the Banach space generated by a certain subset of a locally convex space.

Proposition 110 *Let $(E, \mathcal{T})$ be a $\mu(E, E')$-quasi-complete locally convex space. Let (x_n) be a weakly null sequence in E. Then the weakly compact set $D := \overline{\Gamma}\{x_n; n \in \mathbb{N}\}$ is a Banach disk and $(E_D, \|\cdot\|_D)$ is separable.*

Proof. It is enough to use [Ko69, §20.9.(6)] (indeed, $\mu(E, E')$-completeness of the space E is enough in this result). The set $\overline{\Gamma}\{x_n; n \in \mathbb{N}\}$ coincides with $T(B_{\ell_1})$, where T is the mapping defined in Proposition 96 by taking $S := \{x_n; n \in \mathbb{N}\}$. The continuity of the map T gives that $(E_D, \|\cdot\|_D)$ is a separable Banach space. ∎

Remark 111 From the proof of [Ko69, §20.9.(6)], in order to get the conclusion of Proposition 110 it is enough to suppose that the sequence (x_n) lies in a $\mu(E, E')$-sequentially complete absolutely convex subset of E.

Note that, as a consequence of Proposition 110 and Example 109, the closed unit ball of ℓ_∞ is not contained in the absolutely convex and $w(\ell_\infty, \ell_1)$-closed hull of a $w(\ell_\infty, \ell_1)$-null sequence in ℓ_∞.

From Proposition 110, we have in particular the following result.

Corollary 112 *Let $(E, \mathcal{T})$ be a Fréchet locally convex space. Then every precompact set $P \subset E$ is contained in a Banach disk D such that $(E_D, \|\cdot\|_D)$ is separable.*

Proof. As it is well known (see, for example, [Ko69, §21.10.(3)]), every precompact subset of a metrizable locally convex space $(E, \mathcal{T})$ lies in the closed absolutely convex hull of a $\mathcal{T}$-null sequence in E. It is now enough to apply Proposition 110. ∎

2.3 Reflexivity

Example 109 showed a separable Banach disk D such that $(E_D, \|\cdot\|_D)$ is not separable. The same example provides a semi-reflexive locally convex space and a weakly compact Banach disk D such $(E_D, \|\cdot\|_D)$ is not reflexive.

Example 113 *There exists a semi-reflexive locally convex space $(E, \mathcal{T})$ and an absolutely convex compact subset K and yet, if K is contained in a Banach disk $D \subset E$, the space $(E_D, \|\cdot\|_D)$ is not reflexive.*

Under certain conditions we can conclude, however, reflexivity. We need a previous lemma, which mimics a well known Grothendieck's result.

Lemma 114 *Let $(E, \mathcal{T})$ be a complete locally convex space. Let $K \subset E$ such that for every $U \in \mathcal{U}(0)$ (a fundamental system of absolutely convex and closed neighbourhoods of 0), there exists a weakly compact subset $K_U \subset E$ such that $K \subset K_U + U$. Then K is w-RK.*

Proof. Obviously K is bounded, so $\overline{K}^{E''[w(E'',E')]}$ is $w(E'', E')$-compact in E'', the topological bidual of E. Recall that on E'' the natural topology $\mathcal{T}_n$ has a basis of neighbourhoods of 0 given by the closures of the sets $U \in \mathcal{U}(0)$ in $E''[w(E'', E')]$, and that $\mathcal{T}_n$ on E'' induces on E the original topology [Ko69, §23.4]. We have

$$\overline{K}^{E''[w(E'',E')]} \subset \overline{K_U}^{E''[w(E'',E')]} + \overline{U}^{E''[w(E'',E')]} =$$

$$= K_U + \overline{U}^{E''[w(E'',E')]} \subset E + \overline{U}^{E''[w(E'',E')]}.$$

Then

$$\overline{K}^{E''[w(E'',E')]} \subset \bigcap_{U \in \mathcal{U}(0)} E + \overline{U}^{E''[w(E'',E')]} = E,$$

since E is closed in $E''[\mathcal{T}_n]$, since $E[\mathcal{T}]$ is complete. ∎

The following result is inspired in a well-known interpolation theorem of Davis, Figiel, Johnson and Pełczyńsky [DFJP74] in the setting of Banach spaces.

Theorem 115 *Let $(E, \mathcal{T})$ a Fréchet locally convex space. Then every weakly compact subset of E is contained in a Banach disk D such that $(E_D, \|\cdot\|_D)$ is reflexive.*

Proof. Let K be weakly compact subset of E. We can suppose that K is absolutely convex (otherwise, take the absolutely convex hull of K). Let $\mathcal{U}(0) = \{U_n\}_{n=1}^{\infty}$ be a basis of closed absolutely convex neighbourhoods of 0 for $E[\mathcal{T}]$. $C_n := 2^n K + U_n$ is an absolutely convex and closed (it is the sum of a weakly compact and a weakly closed subset of E, hence a weakly closed convex subset of E) subset of E. It is not in general bounded, hence its Minkowski functional $\|\cdot\|_n := \|\cdot\|_{C_n}$ is a seminorm. $\|\cdot\|_n$ is defined in E, since C_n is absorbing. Put $L := \ell_2\left((E, \|\cdot\|_1), (E, \|\cdot\|_2), \ldots\right)$ and equip L with the $\|\cdot\|_2$-seminorm

$$\|x\|_2^2 = \sum_{n=1}^{\infty} \|x_n\|_n^2, \quad \text{if } \ x = (x_1, x_2, \ldots).$$

In general, $\|\cdot\|_2$ is not a norm.
Let $C := \{x \in E : \sum_{n=1}^{\infty} \|x\|_n^2 \leq 1\}$. Observe that $C = \bigcap_{n=1}^{\infty} \{x \in E : \sum_{k=1}^{n} \|x\|_k^2 \leq 1\}$, hence C is closed. Moreover $C \subset C_n$, for every n. By Lemma 114, C is weakly compact. Given $k \in K$, $2^n k \in C_n$, hence $\|2^n k\|_n \leq 1$. We get $\|k\|_n \leq 2^{-n}$, $n = 1, 2, \ldots$, hence $K \subset C$.
We shall prove that $(E_C, \|\cdot\|_C)$ is reflexive. In order to show this, let us define $T : E_C \to L$ by $T(x) = (x, x, x, \ldots)$. It is well defined, linear and, in fact, an isometry into if E_C carries the $\|\cdot\|_C$-topology and L the $\|\cdot\|_2$-topology. L is a locally convex space and its dual is $L' = \ell_2\left((E, \|\cdot\|_1)', (E, \|\cdot\|_2)', \ldots\right)$. If $\varphi \subset L'$ denotes the subspace consisting of vectors with only a finite number of non-zero "coordinates", φ is $\|\cdot\|_2$-dense in L' (where $\|\cdot\|_2$ on L' denotes also the corresponding $\|\cdot\|_2$-seminorm).
The set C is of course $\|\cdot\|_2$-bounded in L. It follows that on $T(C)$ the two topologies $w(L'', L')$ and $w(L, \varphi)$ coincide. Observe that $w(L'', L')$ induces on $T(C)$ the same topology inherited by $w(E_c, E_c')$, while $w(L, \varphi)$ induces on $T(C)$ the topology $w(E, E')$ (we identify C with $T(C)$). Since C is weakly compact, the Banach space $(E_C, \|\cdot\|_C)$ is reflexive. ∎

In quite a striking contrast with the previous result, Valdivia proved Theorem 125. We shall provide here a different proof based on a result on biorthogonal systems given by Pták. Pták's result is worth to be recalled in full. We need some preliminaries.

We recall here that, if X is a Banach space, a *biorthogonal system* $\{x_\gamma; x'_\gamma\}_{\gamma \in \Gamma}$ in $X \times X'$ is a subset of $X \times X'$ such that $\langle x_\gamma, x'_\beta \rangle = \delta_{\gamma,\beta}$ for all $\gamma, \beta \in \Gamma$, where δ is the Kronecker delta. Such a system is called *M-bounded* if there exists a constant $M > 0$ such that $\|x_\gamma\| \leq M$ and $\|x'_\gamma\| \leq M$ for all $\gamma \in \Gamma$. A biorthogonal system $\{x_\gamma; x'_\gamma\}_{\gamma\in\Gamma}$ in $X \times X'$ is called *fundamental* if $\{x_\gamma;\ \gamma \in \Gamma\}$ is linearly dense in X, and it is called *total* if $\{x'_\gamma\}_{\gamma\in\Gamma}$ is $w(X', X)$-linearly dense in X'. A biorthogonal system that is both fundamental and total is called a *Markushevich basis.*

It is worth mentioning that a deep theorem due to Pełczyński and Plichko says that, for $\varepsilon > 0$, *every separable Banach space has a* $(1 + \varepsilon)$*-bounded Markushevich basis* (see, e.g., [HMVZ08, Thm. 1.27]).

In order to motivate Theorem 122, let us present some easy facts about biorthogonal systems in Banach spaces. Although they are not very deep, we did not see them described in the literature, and be believe that they provide the right motivation to the result mentioned above. First, we isolate a property of sets that plays an important role in the study of the structure of WCG Banach spaces, and that was used by Amir and Lindenstrauss in their seminal paper [AmLi68].

Definition 116 *We say that a subset* Γ *of a Banach space* X countably supports X' *if, for every* $x' \in X'$*, the set* $\{\gamma \in \Gamma;\ \langle\gamma, x'\rangle \neq 0\}$ *is countable. We say that* Γ *has the* Amir-Lindenstrauss property *(the* (AL)-property*, in short), if for every* $x' \in X'$ *and every* $c > 0$*, the set* $\{\gamma \in \Gamma;\ |\langle\gamma, x'\rangle| > c\}$ *is finite.*

Proposition 117 *Let* X *be a Banach space. A set* $\Gamma \subset X$ *with the (AL)-property countably supports* X'*, and moreover, the set* $\Gamma \cup \{0\}$ *is weakly compact.*

Proof. Let $x' \in X'$. The set $\Gamma_n := \{\gamma \in \Gamma; |\langle\gamma, x'\rangle| > 1/n\}$ is finite for every $n \in \mathbb{N}$. Since $\{\gamma \in \Gamma; \langle\gamma, x'\rangle \neq 0\} = \bigcup_{n=1}^\infty \Gamma_n$, the conclusion follows. The uniform boundedness principle yields that the set Γ is bounded. Let γ'' be an element in $\overline{\Gamma \cup \{0\}}^{w(X'',X')} \setminus X$. Find $x' \in X'$ such that $\langle\gamma'', x'\rangle > c > 0$. Then the set $\{\gamma \in \Gamma;\ \langle\gamma, x'\rangle > c\}$ is infinite, a contradiction. ∎

Remark 118 Let $\{x_\lambda; f_\lambda\}_{\lambda\in\Lambda}$ be a *total* biorthogonal system in $X \times X^*$. Then the only possible $w(X, X')$-accumulation point *in* X of the set $\{x_\lambda;\ \lambda \in \Lambda\}$ is 0. This is easy to prove: if there exists a net (of distinct points) $(x_{\lambda_\iota})_{\iota\in I}$ in $\{x_\lambda;\ \lambda \in \Lambda\}$ that $w(X, X')$-converges to some point $x \in X$ then, obviously, $\langle x, f_\lambda\rangle = 0$ for all $\lambda \in \Lambda$, so $x = 0$.

The following proposition is now almost trivial.

Proposition 119 *Let* $\{x_\lambda;\ f_\lambda\}_{\lambda\in\Lambda}$ *be a total biorthogonal system in* $X \times X'$*. Then the following are equivalent:*

(i) *The set* $\{x_\lambda;\ \lambda \in \Lambda\}$ *has the (AL) property.*

(ii) *The set* $\{x_\lambda;\ \lambda \in \Lambda\}$ *is weakly relatively compact (and so* $\{x_\lambda;\ \lambda \in \Lambda\} \cup \{0\}$ *is weakly compact).*

Proof. (i)⇒(ii) follows from Proposition 117 and Remark 118.
(ii)⇒(i). Assume that the set $\{x_\lambda;\ \lambda \in \Lambda\}$ does not have the (AL)-property. Then there exists $x' \in X'$ and $c > 0$ such that the set $\{\lambda \in \Lambda;\ \langle x_\lambda, x'\rangle > c\}$ is infinite. Since this set is weakly relatively compact, it has an accumulation point in X, say x ($\neq 0$). This is impossible in view of Remark 118. ∎

The following simple proposition is a consequence of the orthogonality.

Proposition 120 *Let* X *be a Banach space. Let* $\{x_i; f_i\}_{i\in\mathbb{N}}$ *be a biorthogonal system in* $X \times X'$ *and assume that* $(\sum_{i=1}^{n} f_i)_{n\in\mathbb{N}}$ *has a bounded subsequence. Then* $\{x_i;\ i \in \mathbb{N}\}$ *lies in a hyperplane missing* 0.

Proof. Let (n_p) be an increasing sequence in $\mathbb{N}$ such that the sequence $(\sum_{i=1}^{n_p} f_i)_{p=1}^{\infty}$ is bounded and let x' be a $w(X', X)$-cluster point of it. Then, for every $j \in \mathbb{N}$ we have

$$\left\langle x_j, \sum_{i=1}^{n_p} f_i \right\rangle = 1 \quad \text{for all large} \quad p \in \mathbb{N}.$$

Then $\langle x_j, x'\rangle = 1$ for all $j \in \mathbb{N}$. ∎

Corollary 121 *Let* X *be a Banach space and let* $\{x_\lambda;\ f_\lambda\}_{\lambda\in\Lambda}$ *be a biorthogonal system in* $X \times X'$*. Assume that* $\{x_\lambda;\ \lambda \in \Lambda\}$ *has the (AL)-property. Then, for every one-to-one sequence* (λ_n) *in* Λ *we have* $\|\sum_{i=1}^{n} f_{\lambda_i}\| \to \infty$ *whenever* $n \to \infty$.

Proof. If the conclusion does not hold for some one-to-one sequence (λ_n), there is an increasing sequence (n_p) such that $(\sum_{i=1}^{n_p} f_{\lambda_i})_{p=1}^{\infty}$ is bounded. It follows from Proposition 120 that $\{x_{\lambda_n};\ n \in \mathbb{N}\}$ is in a hyperplane missing 0, and this violates the (AL)-property. ∎

We give here a slightly more precise formulation of Pták's result.

Theorem 122 (Pták, [Pt59]) *Let X be a Banach space. The following statements are equivalent.*

(i) *X is reflexive.*

(ii) *For every biorthogonal system $\{x_n; x'_n\}_{n=1}^{\infty}$ in $X \times X'$ such that $\{x'_n\}_{n=1}^{\infty}$ is bounded, the sequence $(\sum_{k=1}^{n} x_k)_{n \in N}$ is unbounded.*

(iii) *For every biorthogonal system $\{x_n; x'_n\}_{n=1}^{\infty}$ in $X \times X'$ such that $\{x_n\}_{n=1}^{\infty}$ is bounded, the sequence $(\sum_{k=1}^{n} x_k^*)_{n \in \mathbb{N}}$ is unbounded.*

We quote in the next three paragraphs the reviewer (B. R. Gelbaum) of the original paper in MathSciNet (here, notations follow the tradition in Banach spaces; X^* is the topological dual of a Banach space X).

> The original proof is based on the following intermediate results.
>
> *For a Banach space X the following statements are equivalent: (a) X is non-reflexive. (b) There is a bounded biorthogonal system $S_1 = \{e_i; f_i\}_{i \in \mathbb{N}}$ and a $\Delta > 0$ such that if $\alpha_n \uparrow 0$ or if $\alpha_n \downarrow 0$ then $x = \sum_{i=1}^{\infty} \alpha_i e_i$ exists and $\|x\| \leq \Delta |\alpha_1|$. (c) Let $B(S) = w^*$-closure of f_j in E^*. There exists a bounded biorthogonal system S_2 such that, considered as a biorthogonal system in $X/B(S_2)^0$, S_2 enjoys the property: there is a $\Delta > 0$ such that if $0 \leq \lambda_i$, $\sum_{i=1}^{\infty} \lambda_i < \infty$, then $\sum_{i=1}^{\infty} \lambda_i e_i \geq \Delta \sum_{i=1}^{\infty} \lambda_i$.*
>
> The key to Theorem 122 lies in constructing for a given non-reflexive X a system S for which (b) and (c) are true. Once this is achieved, the remaining syllogisms follow readily. The root idea is then the following. Choose $r \in X^{**} \backslash X$. Then for $\delta > 0$, by induction construct sequences $\{b_i\} \subset X$, $\{y_j\} \subset X^*$ such that $\|b_i\| \leq 1 + \delta$, $\|y_j\| = 1$, $(b_i, y_j) = \beta_j > \frac{1}{2}\delta_j > 0$ for $j \leq i$, $(b_i, y_j) = 0$ for $j > i$, where $\delta_j = \sup\{(r, y) | \|y\| \leq 1, y \in X_{j-1} =$ linear space spanned by $b_1, b_2, \cdots, b_{j-1}\}$. Then there is a $\beta > 0$ such that $\delta_j \geq 2\beta$. Set $e_i = b_i - b_{i-1}$, $f_j = (1/\beta_j) y_j$. $\{e_i; f_j\}$ is a system S_1 for (b). Set $g_i = b_i$, $h_j = (1/\beta_j) y_j - (1/\beta_{j+1}) y_{j+1}$. $\{g_i; h_j\}$ is a system S_2 for (c). S_1 is also a system for which the sums $\sum_{i=1}^{n} e_i$, $n \in \mathbb{N}$, are bounded. The remaining arguments are less involved.

We return now to our accepted notation for the topological dual of a locally convex space.
We shall present here a different approach, based on a well-known characterization of reflexivity due to James. James' result depends, essentially, on Helly's Theorem (for a proof of the equivalence between (i) and (iii) below, based only on Riesz' Lemma, we refer to [FHHMPZ, Theorem 3.57]). We recall here this characterization, as it is presented, for example, in [Beau82, Theorem III.6].

Theorem 123 *Let X be a Banach space. The following are equivalent.*

(i) *X is not reflexive.*

(ii) *For every $0 < \theta < 1$, there is a sequence (x_n) in S_X and a sequence (x'_n) in $S_{X'}$ such that*

$$\begin{cases} \langle x_n, x'_m\rangle = \theta & \text{for all } n \geq m, \\ \langle x_n, x'_m\rangle = 0 & \text{for all } n < m. \end{cases}$$

(iii) *For every $0 < \theta < 1$, there is a sequence (x_n) in S_X such that $\inf\{\|u\|;\ u \in \operatorname{conv}\{x_n;\ n \in \mathbb{N}\}\} \geq \theta$ and*

$$\operatorname{dist}\left(\operatorname{conv}\{x_k\}_{k=1}^{n}, \operatorname{conv}\{x_k\}_{k=n+1}^{\infty}\right) \geq \theta \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof of Theorem 122. (i)⇒(ii) and (iii). Assume that the space X is reflexive. Let $\{x_n; x'_n\}_{n\in\mathbb{N}}$ be a biorthogonal system in $X \times X'$ with $\{x_n\}_{n=1}^{\infty}$ bounded. Let $Y := \overline{\operatorname{span}}\{x_n;\ n \in \mathbb{N}\}$; this is a reflexive space. Let $q : X' \to Y'$ be the restriction mapping. Then $\{x_n; q(x'_n)\}$ is a biorthogonal system in $Y \times Y'$. And, as Y is reflexive, the sets $\{x_1, x_2, \ldots\} \cup \{0\}$ and $\{q(x'_1), q(x'_2), \ldots\} \cup \{0\}$ are both weakly compact. Since $\{q(x'_n); x_n\}$ is a total biorthogonal system in $Y' \times Y$, Proposition 119 and Corollary 121 give (ii). Reversing the roles of X and X' we get (iii).

(ii)⇒(i) Assume that X is not reflexive. Theorem 123 says that, given $0 < \theta < 1$, there exist two sequences, (x_n) in S_X and (x'_n) in $S_{X'}$, such that $\langle x_n, x'_m\rangle = \theta$ if $n \geq m$, and $\langle x_n, x'_m\rangle = 0$ if $n < m$. Let $d_1 := x_1$, $d_n := x_n - x_{n-1}$, $n = 2, 3, \ldots$. Then, it is clear that the family $\{(1/\theta)d_n; x'_n\}_{n\in\mathbb{N}}$ is a biorthogonal system in $X \times X'$. Moreover, $\{x'_n;\ n \in \mathbb{N}\}$ is bounded. Observe, too, that $\sum_{k=1}^{n} d_k = x_n$ for all $n \in \mathbb{N}$. We obtain a contradiction with (ii).

(iii)⇒(i) Starting from the assumption that X is not reflexive, we proceed as in the proof of (ii)⇒(i). Once we have the two sequences (x_n) and (x'_n), put $d'_n = x'_n - x'_{n+1}$ for $n \in \mathbb{N}$. The system $\{x_n; (1/\theta)d'_n\}_{n\in\mathbb{N}}$ is again a biorthogonal system such that $\{x_n; n \in \mathbb{N}\}$ is bounded, and now $\sum_{k=1}^{n} d'_n = x'_1 - x'_{n+1}$ for all $n \in \mathbb{N}$. We obtain again a contradiction, this time with (iii). ∎

Remark 124 From the proof of Theorem 122 we obtain something a little bit more precise: If X is reflexive, the sequences $(\|s_n\|)$ and $(\|s'_n\|)$ defined there indeed tend to $+\infty$.

Theorem 125 (Valdivia, [Val72-2]) *In every infinite-dimensional Fréchet space $(E, \mathcal{T})$ there exists a compact absolutely convex subset A of E such that $(E_A, \|\cdot\|_A)$ is a nonreflexive Banach space.*

Proof A simple argument proves that there exists a biorthogonal system $\{x_n; u_n\}_{n\in\mathbb{N}}$ in $E \times E'$ such that $x_n \to 0$. Let $A := \overline{\Gamma}\{x_n; n \in \mathbb{N}\}$, a compact subset of $(E, \mathcal{T})$ (see, e.g., [Ko69, §20.6.(3)]). By [Ko69, §20.9.(6)] we have $A = T(B_{\ell_1})$, where $T : \ell_1 \to E$ is the continuous linear mapping defined in Proposition 96 taking $S := \{x_n; n \in \mathbb{N}\}$. We shall prove that $(u_m(x))_{m\in\mathbb{N}} \in B_{\ell_1}$ for all $x \in A$. Indeed, given $x \in A$ we can find $(a_n) \in B_{\ell_1}$ such that $T((a_n)) = x$. Then

$$\begin{aligned}(\langle x, u_m\rangle)_m &= (\langle T((a_n)), u_m\rangle)_m \\ &= (\langle (a_n), T'u_m\rangle)_m = \left(\sum_n a_n\langle e_n, T'u_m\rangle\right)_m = \\ \left(\sum_n a_n\langle Te_n, u_m\rangle\right)_m &= \left(\sum_n a_n\langle x_n, u_m\rangle\right)_m = (a_m) \in B_{\ell_1}. \quad (2.2)\end{aligned}$$

This proves that $\sum_{m=1}^{\infty} |u_m(x)| \leq 1$ for all $x \in A$. The system $\{x_n; u_n\}_{n\in\mathbb{N}}$, considered in $E_A \times (E_A)'$, is also biorthogonal. The previous argument proves that it is bounded. From (2.2) it follows that $\sum_{k=1}^{n} u_k$ is a bounded sequence in $(E_A)'$, so, by Theorem 122, the space $(E_A, \|\cdot\|_A)$ is not reflexive. ∎

Valdivia's Theorem 125 is used in [Val72-2] to prove the following result.

Theorem 126 (Valdivia, [Val72-2]) *In every infinite-dimensional Fréchet space* $(E, \mathcal{T})$ *there is a non-closed disk* D *such that* $(E_D, \|\cdot\|_D)$ *is a Banach space.*

It is easy to check that the conclusion in both Theorems 125 and 126 fail if we omit either completeness or metrizability in the locally convex space. This is done, respectively, in Examples 127 and 128.

Example 127 *There exists a normed space where the conclusions of Theorems 125 and 126 fail.*

Proof. Consider $(\varphi(\mathbb{N}), \|\cdot\|_\infty)$, the space of sequences with finite support, endowed with the supremum norm. Observe that $\varphi(\mathbb{N})$ is a vector space of countable algebraic dimension. Assume that, for some disk $D \subset \varphi(\mathbb{N})$, the space $(\varphi(\mathbb{N})_D, \|\cdot\|_D)$ is a Banach space. As a vector space, $\varphi(\mathbb{N})_D$ is a vector subaspace of $\varphi(\mathbb{N})$, so it has countable algebraic dimension. However, from the Baire category theorem, no Banach space can be of countable dimension. Therefore, every Banach disk D of $(\varphi(\mathbb{N}), \|\cdot\|_\infty)$ is of finite dimension, and so $(E_D, \|\cdot\|_D)$ is always reflexive. ∎

Example 128 *There exists a complete locally convex space where the conclusions of Theorems 125 and 126 fail.*

Proof. Consider the (complete) space $\varphi(\mathbb{N})$ endowed with the topology $w(\varphi(\mathbb{N}), \mathbb{R}^{\mathbb{N}})$. As in example 127, every Banach disk is of finite dimension. ∎

2.4 On local convergence

This section is again instrumental in nature. The simple results we prove here and the well-known concepts we mention below will be used in the next chapter.
Let $(E, \mathcal{T})$ be a topological vector space. A sequence (x_n) in E is said to be *locally convergent* or *Mackey convergent* to an element $x \in E$ if there is a disc $B \subset E$ such that the sequence converges to x in E_B. If this is the case we write $x_n \xrightarrow{\text{loc}} x$. The sequence (x_n) is said to be *locally null* if $x_n \xrightarrow{\text{loc}} 0$, and *locally Cauchy* or *Mackey Cauchy* if it is a Cauchy sequence in E_B.

Obviously, a locally null sequence is a null sequence (but not conversely, in general, see, e.g. [BP87, Chapter 5]). The same applies to locally convergent and locally Cauchy sequences.
The following basic result appears, for example, in [BP87, Proposition 5.1.3].

Proposition 129 (i) *Let* (x_n) *be a sequence in a topological vector space* $(E, \mathcal{T})$ *and let* $x \in E$*. Then* $x_n \xrightarrow{loc} x$ *if and only if* $(x_n - x) \xrightarrow{loc} 0$*.*
(ii) *Let* (x_n) *be a sequence in a topological vector space* $(E, \mathcal{T})$*. Then* $x_n \xrightarrow{loc} 0$ *if and only if there is an increasing sequence* (r_n) *of positive real numbers such that* $r_n \to +\infty$ *and* $r_n x_n \xrightarrow{\mathcal{T}} 0$*.*

The following result is easy. We include it here for the sake of completeness.

Proposition 130 *Let* $(E, \mathcal{T})$ *be a locally convex space. Let* $x,\ y \in E$*. Let* (x_n)*,* (y_n) *be two sequences in* E *such that* $x_n \xrightarrow{loc} x$ *and* $y_n \xrightarrow{loc} y$*. Let* (λ_n) *be a sequence of scalars such that* $\lambda_n \to \lambda$ *for some scalar* λ*. Then*
(i) $(x_n + y_n) \xrightarrow{loc} x + y$*.*
(ii) $\lambda_n x_n \xrightarrow{loc} \lambda x$*.*

Proof
(i) Since $x_n \xrightarrow{\text{loc}} x$, there exists a disc B in E such that $x_n \to x$ in E_B. In the same way, there exists another disc C in E such that $y_n \to y$ in E_C. The set $D := \Gamma(B \cup C)$ is a disc in E, since it is absolutely convex and bounded. Obviously, as sets, $E_B \cup E_C \subset E_D$, and convergence in E_B implies convergence in E_D. The same applies to E_C. This ensures that $(x_n + y_n)$ converges to $(x + y)$ in E_D, so $(x_n + y_n) \xrightarrow{\text{loc}} (x + y)$.
(ii) This is trivial, since the statement is true in every normed space. ∎

Let A be a subset of a topological vector space $(E, \mathcal{T})$. A point $x \in E$ is a *local limit point of* A if there is a sequence (a_n) in A locally convergent to x. We say that A is *locally closed* if every local limit point of A belongs to A.

Proposition 131 *Let* $(E, \mathcal{T})$ *be a locally convex space. Let* A, B *be two subsets of* E*. Assume that* A *satisfies that every sequence in* A *has a subsequence that locally converges to some point in* A*. Assume that* B *is locally closed and bounded. Then the set* $\operatorname{conv}(A \cup B)$ *is locally closed.*

Proof Let (a_n) be a sequence in A and (b_n) a sequence in B. Let (λ_n) be a sequence in $[0,1]$ and $x \in E$. Assume that $\lambda_n a_n + (1-\lambda_n)b_n \xrightarrow{\text{loc}} x$. By passing to a subsequence if necessary, we may assume, without loss of generality, that $\lambda_n \to \lambda$ for some $\lambda \in [0,1]$ and $a_n \xrightarrow{\text{loc}} a$ for some $a \in A$.
By (ii) in Proposition 130, $\lambda_n x_n \xrightarrow{\text{loc}} \lambda a$. By (i) in the same proposition, it follows that $(1-\lambda_n)b_n \xrightarrow{\text{loc}} x - \lambda a$.
Assume first that $\lambda \neq 1$. Then, again by (ii) in Proposition 130, $b_n \xrightarrow{\text{loc}} (1-\lambda)^{-1}(x-\lambda a)$. Since B is locally closed, $b := (1-\lambda)^{-1}(x-\lambda a) \in B$. It follows that $x = \lambda a + (1-\lambda)b \in \text{conv}(A \cup B)$.
Assume now that $\lambda = 1$. We have $(1-\lambda_n)b_n \xrightarrow{\text{loc}} x - a$. In particular, $(1-\lambda_n)b_n \xrightarrow{\mathcal{T}} x - a$. Since B is bounded, we have, at the same time, $(1-\lambda_n)b_n \xrightarrow{\mathcal{T}} 0$. It follows that $x = a$ $(\in A)$.
In both cases we got $x \in \text{conv}(A \cup B)$. ∎

Chapter 3
Drop property

3.1 Introduction

Let B a non-empty closed bounded convex subset of a Banach space X. By the *drop* induced by a point $a \in X \setminus B$ we mean the set $D(a, B) := \text{conv}(\{a\} \cup B)$. Daneš showed in [Da72] that *if A is a non-empty closed subset of X at positive distance of the closed unit ball B_X, then there exists $a \in A$ such that $D(a, B_X) \cap A = \{a\}$*. In [Da85] he showed that *the same is true if B is any non-empty closed convex and bounded subset of X*. This is referred to as Daneš Drop Theorem. The following definition describes the kind of objects considered in this theorem.

Definition 132 *A closed convex and bounded subset B of a Banach space X is said to have the* drop property *if for any non-empty closed subset A of X such that $A \cap B = \emptyset$, there exists a point $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

Rolewicz proved in [Ro85] that *if X is reflexive Banach space then B_X has the drop property.* Montesinos proved in [Mo87] that *a Banach space is reflexive if and only if it can be renormed to have the drop property.* In [Ku87] it was proved that *every weakly compact convex subset of a Banach space has the drop property.*

Different versions of the drop property in locally convex spaces were considered by several authors. They use corresponding concepts—that we introduce below—closely related to the original definition of the drop property. In order to formulate their results in their same words we stick to the same names for those concepts (Definitions 133 and 134). Later on, in Definition 140, we propose a different terminology.

Definition 133 (Giles, Sims and Yorke, [GSY90]) *A closed convex and bounded subset B of a locally convex space $(E, \mathcal{T})$ is said to have the* weak drop property *if for every non-empty weakly sequentially closed subset A of E disjoint from B there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

Definition 134 (Qiu, [Q02-b]) *A closed convex bounded subset B of a locally convex space $(E, \mathcal{T})$ is said to have the* quasi-weak drop property *if for every non-empty weakly closed subset A of E disjoint from B there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

Obviously, the weak drop property implies the quasi-weak drop property.

Sufficient conditions for one or the other have been given in the literature. Chen et al. [Ch96, Th. 3], by using a concept of separation that somehow mimics a strictly positive-distance situation in a locally convex setting, provided Theorem 136. To be precise, the separation concept needed is given in the next definition.

Definition 135 *Two nonempty subsets A, B of a locally convex space E are said to be* Minkowski separated *(*strongly Minkowski separated*) if there exists a continuous Minkowski gauge p on E and a point x_0 in E such that either $p(x) > p(y)$ for all $x \in A_{x_0} \equiv A + x_0$ and $y \in B_{x_0} \equiv B + x_0$ or $p(x) < p(y)$ for all $x \in A_{x_0}$ and $y \in B_{x_0}$ (respectively, either $\inf\{p(x) : x \in A_{x_0}\} > \sup\{p(y) : y \in B_{x_0}\}$ or $\sup\{p(x) : x \in A_{x_0}\} < \inf\{p(y) : y \in B_{x_0}\}$).*

Using this concept, they proved the following result.

Theorem 136 ([Ch96], Theorem 3) *Given a sequentially closed bounded convex set B in a sequentially complete locally convex space E, then for every sequentially closed set A which is strongly Minkowski separated from B, there exists a point $a \in A$, such that $D(a, B) \cap A = \{a\}$.*

J. H. Qiu improved this result in [Q02-a, Cor. 3.2].

Theorem 137 ([Q02-a], Corollary 3.2) *Let A be a locally closed subset of a locally complete locally convex space E and B be a locally closed, bounded, convex subset of E. If there exists a locally convex topology $\mathcal{T}$ on E, such that $0 \notin \overline{A - B}^{\mathcal{T}}$, then there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

The same author showed some connections between weak compactness and weak drop property along several papers. For example, he proved the two following results.

Theorem 138 ([Q03a], Theorem 2.1) *Every non-empty closed convex and weakly sequentially compact subset B of a locally convex space $(E, \mathcal{T})$ has the weak drop property.*

Theorem 139 ([Q04], Theorem 2.1) *Every non-empty closed convex and weakly countably compact subset B of a locally convex space $(E, \mathcal{T})$ has the quasi-weak drop property.*

More recent results from the same author about the drop property can been seen in [Q06] and [Q07].

3.2 Results

In this section we introduce the basic definitions and we present short proofs of some of the results mentioned in Section 3.1. In fact, we extend those results, since we formulated them in the context of arbitrary locally convex spaces not necessarily equipped with the the weak topology. We believe that part of the interest of our approach lies in the fact that we are able to reduce many of the arguments and techniques in the current literature to the classical Daneš Drop Theorem, just by embedding the sets we are dealing with in certain Banach spaces. This approach unifies different procedures and is able to produce, as we show below, new results. So, we study some extra features of the quasi-drop property that do not appear in the existing literature.

3.2.1 Definitions

Definition 140 *A closed convex subset B of a locally convex space $(E, \mathcal{T})$ is said to have the* drop *(*quasi-drop*)* property *if for every non-empty sequentially closed (respectively, closed) subset A of E disjoint from B there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

A subset A of a locally convex space $(E, \mathcal{T})$ is *countably closed* (*sequentially closed*) if it contains the closure of every of its countable subsets (respectively, the limit of every sequence in A which converges in E). The following implications are clear:

$$\text{closed} \Rightarrow \text{countably closed} \Rightarrow \text{sequentially closed.}$$

However, none of the converse implications hold true. For instance, take $(\ell_\infty[0,1], \mathcal{T}_p)$, the space of bounded functions in $[0,1]$ endowed with the topology of the pointwise convergence. Consider the set $A := \{f \in \ell_\infty : \|f\|_\infty \leq 1\}$ and let B be the subset of A consisting of all countably supported elements. Then B is countably closed, but not closed, since its closure is A. On the other hand, the subset ℓ_1 of $(\ell_\infty^*, w(\ell_\infty^*, \ell_\infty))$ is sequentially closed (by Schur Lemma) but not countably closed, since the countable set $\{e_n; n \in \mathbb{N}\} \cup \{0\}$ is not weakly compact in ℓ_1. A bounded set with the same property is the closed unit ball of ℓ_1 as a subset of $(\ell_\infty^*, w(\ell_\infty^*, \ell_\infty))$.

Qiu studied and separated the concepts of drop property dealing (in the weak topology) with sequentially closed subsets (weak drop property) and with closed subsets (weak quasi-drop property). It is natural to ask about a seemingly intermediate concept, the drop property dealing with countably closed subsets. We show below that this apparently new property is indeed the same as the quasi-drop property.

3.2.2 Drop property for bounded subsets of a locally convex space

The key point in our proofs is the ability to embed a subset of a locally convex space in a Banach space. We shall use the results proved in Chapter 2. For example, using Corollary 102 we can easily reduce Theorem 138 to the classical Daneš Theorem. We can even formulate this result in a more general setting.

Theorem 141 *Every non-empty closed convex and sequentially compact subset B of a locally convex space $(E, \mathcal{T})$ has the drop property.*

Proof. Let A be a sequentially closed subset of E disjoint from B. Let $a_0 \in A$.

It is easy to see that $D(a_0, B)$ is closed and sequentially compact. Indeed, first of all, it is a standard fact that the convex hull of a closed convex set and a closed convex and compact set in a locally convex space is again closed. Assume now that $(\lambda_n a_0 + (1-\lambda_n)b_n)$ is a sequence in $D(a_0, B)$, being (b_n) a sequence in B and (λ_n) a sequence in $[0,1]$. Since B is sequentially compact, we can extract a convergent subsequence (b_{n_i}) of (b_n). Let $b \in B$ be its limit. By passing to a further subsequence of (n_i), denoted again (n_i), we may assume that (λ_{n_i}) converges to some $\lambda_0 \in [0,1]$. Then $(\lambda_{n_i} a_0 + (1-\lambda_{n_i})b_{n_i})$ converges to $\lambda_0 a_0 + (1-\lambda_0)b$, and, obviously, $\lambda_0 a_0 + (1-\lambda_0) \in D(a_0, B)$.
Use now Corollary 102 to ensure that $D(a_0, B)$ is contained in a Banach disk U. Let d_U be the metric induced in E_U by $\|\cdot\|_U$. Suppose that $d_U(A \cap D(a_0, B), B) = 0$. Then we can find two sequences (a_n) in $A \cap D(a_0, B)$ and (b_n) in B such that $\|a_n - b_n\|_U \to 0$ (so $a_n - b_n \xrightarrow{\mathcal{T}} 0$). Since B is $\mathcal{T}$-sequentially compact, we can extract a subsequence (b_{n_i}) such that $b_{n_i} \xrightarrow{\mathcal{T}} b \in B$. But then, $a_{n_i} \xrightarrow{\mathcal{T}} b$, so $b \in A$, since A is $\mathcal{T}$-sequentially closed, a contradiction. Therefore $d_U(A \cap D(a_0, B), B) > 0$. Obviously, $A \cap D(a_0, B)$ is $\|\cdot\|_U$-closed; we can apply Daneš theorem in the Banach space $(E_U, \|\cdot\|_U)$ to conclude the result. ∎

Theorem 139—even an extension of it—can also be proved in a very simple way.

Theorem 142 *Every non-empty closed convex and countably compact subset B of a locally convex space $(E, \mathcal{T})$ has the quasi-drop property.*

Proof. Let A be a closed subset of E disjoint from B and let $a_0 \in A$. $D(a_0, B)$ is closed and countably compact (these two assertions can be proved similarly to what was done in the proof of Theorem 141; now we need to pass to convergent subnets instead of subsequences, both in the case of vectors and scalars). Again by Corollary 102, $D(a_0, B)$ is contained in a Banach disk U. As above, suppose that $d_U(A \cap D(a_0, B), B) = 0$. We can take then two sequences (a_n) in $A \cap D(a_0, B)$ and (b_n) in B such that $\|a_n - b_n\|_U \to 0$ (so $a_n - b_n \xrightarrow{\mathcal{T}} 0$). Since B is $\mathcal{T}$-countably compact, we can extract a subnet (b_{n_i}) such that $b_{n_i} \xrightarrow{\mathcal{T}} b \in B$. But then, $a_{n_i} \xrightarrow{\mathcal{T}} b$, so $b \in A$, since A is $\mathcal{T}$-closed, a contradiction. Again an application of Daneš Theorem in the Banach space $(E_U, \|\cdot\|_U)$ gives the result. ∎

Some other results concerning the drop property include conditions of completeness together with conditions of separateness of sets B and A in their hypothesis; we claim that they are again immediate consequences of Daneš Theorem. This is the case of Theorem 136 and its extension, Theorem 137. We provide a proof, simpler than the original, of a slightly weaker version of this last theorem.

Theorem 143 *Let A be a locally closed subset of a locally complete locally convex space E and B be a locally closed, bounded, convex subset of E. If there exists a locally convex topology $\mathcal{T}$ on E with a basis of closed neighborhoods for the original topology, such that $0 \notin \overline{A-B}^{\mathcal{T}}$, then there exists $a \in A$ such that $D(a,B) \cap A = \{a\}$.*

Proof. By the condition $0 \notin \overline{A-B}^{\mathcal{T}}$, we can ensure that there exists a closed, absorbent and absolutely convex set U in E, such that $U \cap (A-B) = \emptyset$. Take $x_0 \in A$ and let D be the closed absolutely convex hull of $(B \cup \{x_0\})$. The set D is a disc in E. The set $U \cap E_D$ is $\|\cdot\|_D$-closed in E_D, and this last space is complete by the assumptions. The set $U \cap E_D$ is absorbing in E_D. By the Baire Category Theorem, $U \cap E_D$ contains a multiple D' of D. Since B is locally closed and bounded, it is also $\|\cdot\|_D$-closed. Proposition 131 ensures that $D(x_0,B)$ is also locally closed, so $A \cap D(x_0,B)$ is locally closed, hence $\|\cdot\|_D$-closed, too. Finally, $D' \cap (A-B) = \emptyset$ implies that $d(B, A \cap D(x_0,B))_D > 0$, where d_D denotes the distance induced by $\|\cdot\|_D$ in E_D, so we can apply Daneš theorem to the sets B and $A \cap D(x_0,B)$ in the Banach space $(E_D, \|\cdot\|_D)$. ∎

Theorem 142 is based on the fact the every convex countably compact set is contained in a Banach disk. In Corollary 104 we saw that the same is true for sets that are convex and convex-compact. Then we can provide, in Theorem 146, a statement similar to Theorem 142, this time for convex-compact sets. We need first a lemma.

Lemma 144 *Let B be a convex-compact, closed convex subset of a locally convex space $(E,\mathcal{T})$ and $a_0 \notin B$. Then $D(a_0,B)$ is also convex-compact.*

Proof. Without loss of generality we may suppose $a_0 = 0$. Let (K_n) be any decreasing sequence of closed convex sets which intersects $D(a_0,B)$ ($= \bigcup_{0 \le \lambda \le 1} \lambda B$) and let $I_n := \{\lambda \in [0,1] : K_n \cap \lambda B \neq \emptyset\}$. Then, by Lemma 16, (I_n) is a decreasing sequence of (non-empty) closed subsets of $[0,1]$, so they have a non-empty intersection. Take $\lambda \in \bigcap I_n$. It follows that every K_n

intersects λB (a convex-compact set) and so there exists an adherent point of the sequence $(K_n \cap D(a_0, B))_n$ in $\lambda B \subset D(a_0, B)$. ∎

The following is a simple observation about convex sets that will be used in the proof of Theorem 146. Most (if not all) of the statements are almost trivial. We collect them here for future references. Given a subset A of a locally convex space $(E, \mathcal{T})$ we denote $\mathrm{conv}_{\mathbb{Q}}(A) := \{\lambda a_1 + (1-\lambda)a_2;\ a_1, a_2 \in A, \lambda \in \mathbb{Q} \cap [0,1]\}$, the *rational convex hull of* A. We will say that a A is $\mathbb{Q}$-*convex* if $A = \mathrm{conv}_{\mathbb{Q}}(A)$.

Lemma 145 *Let A be a subset of a locally convex space $(E, \mathcal{T})$. Then*

(i) $\mathrm{conv}_{\mathbb{Q}}(A)$ *is $\mathbb{Q}$-convex.*

(ii) *If A is $\mathbb{Q}$-convex and closed, then A is convex.*

(iii) *If A is $\mathbb{Q}$-convex, then $\overline{A}$ is convex.*

(iv) $\overline{\mathrm{conv}}(A) = \overline{\mathrm{conv}_{\mathbb{Q}}}(A)$.

(v) *If A is separable, so it is* $\mathrm{conv}(A)$ *(and hence* $\overline{\mathrm{conv}}(A)$*).*

Proof. (i) is obvious. (ii) Let $a, b \in A$. Let $\lambda \in [0,1]$. Let (q_n) be a sequence in $\mathbb{Q} \cap [0,1]$ such that $q_n \to \lambda$. Then $A \ni q_n a + (1-q_n)b \to \lambda a + (1-\lambda)b$, so $\lambda a + (1-\lambda)b \in A$. (iii) Let $\overline{a}, \overline{b} \in \overline{A}$. We can find nets (a_i), (b_i) in A such that $a_i \to \overline{a}$ and $b_i \to \overline{b}$. Then, given $q \in \mathbb{Q} \cap [0,1]$, we have $qa_i + (1-q)b_i \to q\overline{a} + (1-q)\overline{b}$. It follows that $\overline{A}$ is $\mathbb{Q}$-convex. Now, it is enough to apply (ii). (iv) Obviously, $\overline{\mathrm{conv}}(A) \supset \overline{\mathrm{conv}_{\mathbb{Q}}}(A)$. It follows from (i) that the set $\mathrm{conv}_{\mathbb{Q}}(A)$ is $\mathbb{Q}$-convex. By (iii), $\overline{\mathrm{conv}_{\mathbb{Q}}}(A)$ is convex, and contains A, so $\overline{\mathrm{conv}}(A) \subset \overline{\mathrm{conv}_{\mathbb{Q}}}(A)$ and the conclusion follows. (v) Let D be a dense countable subset of A. We have $\mathrm{conv}_{\mathbb{Q}}(D) \subset \mathrm{conv}_{\mathbb{Q}}(A) \subset \mathrm{conv}(A)$. Because of (iv), $\overline{\mathrm{conv}_{\mathbb{Q}}}(D) = \overline{\mathrm{conv}}(D) (= \overline{\mathrm{conv}}(A))$. To finish, it is enough to observe that $\mathrm{conv}_{\mathbb{Q}}(D)$ is countable. ∎

We are ready to prove our result for convex-compact sets.

Theorem 146 *Let B be a non-empty closed convex and convex-compact subset of a locally convex space $(E, \mathcal{T})$. Then, for every convex countably closed subset A such that $A \cap B = \emptyset$, there exists a point $a \in A$ such that $D(a, B) \bigcap A = \{a\}$.*

Proof. Let $a_0 \in A$. Corollary 104 and Lemma 144 guarantee the existence of a Banach disk $U \in E$ containing $D(a_0, B)$ (it is worth to recall here that $D(a_0, B)$ is also closed and convex, and it is bounded, as B is also bounded [Ko69, §24.3(4)]). Let d_U be the metric in E_U induced by $\| \cdot \|_U$. We shall prove the following
`Claim`: $d_U(A \cap D(a_0, B), B) > 0$.
Assuming that the `Claim` is false, we can find two sequences (a_n) in $A \cap D(a_0, B)$ and (b_n) in B such that $\|a_n - b_n\|_U \to 0$. Let K_n be defined as follows:

$$K_n := \overline{\text{conv}}^{\mathcal{T}} \left(\{a_i : i \in \mathbb{N}\} + (1/n)\, U \right)$$

K_n is a decreasing sequence of closed convex sets in $(E, \mathcal{T})$ which intersects B, so there is a point $b \in B$, such that

$$b \in \bigcap_n \overline{K_n \cap B}^{\mathcal{T}} = \bigcap_n K_n \cap B.$$

Observe that b is also a $\mathcal{T}$-adherent point of the set $\text{conv}\{a_i : i \in \mathbb{N}\}$. This can be seen as follows: let $U(0)$ be a convex (otherwise arbitrary) $\mathcal{T}$-neighborhood of 0. We can find $n \in \mathbb{N}$ such that $(1/n)U \subset U(0)$. As $b \in K_n$ we get

$$(b + U(0)) \cap \left(\text{conv}\{a_n : n \in \mathbb{N}\} + \frac{1}{n} U \right) \neq \emptyset,$$

so

$$(b + U(0)) \cap \left(\text{conv}\{a_n : n \in \mathbb{N}\} + U(0) \right) \neq \emptyset$$

and finally $(b + 2U(0)) \cap \text{conv}\{a_n : n \in \mathbb{N}\} \neq \emptyset$.
Now, A is convex. Then $\text{conv}\{a_n : n \in \mathbb{N}\}$ is a separable subset of A (see (v) in Lemma 145) and A is countably closed. It is elementary to prove that the closure of a separable subset of a countably closed set A is contained in A. Therefore, $b \in A$, and we reach a contradiction. The `Claim` is proved. To finish the proof it is enough to apply, in the Banach space $(E_U, \| \cdot \|_U)$, the classical Daneš Drop Theorem [Da85] to B and $A \cap D(a_0, B)$, both closed sets in $(E_U, \| \cdot \|_U)$. ∎

According to Theorem 146, a non-empty closed convex and convex-compact subsets B of a locally convex space enjoys, apparently, a stronger property than the quasi-drop, precisely the possibility to find “drop points” for every non-empty countably closed subset of $(E, \mathcal{T})$ disjoint from B (we showed above that there are countably closed subsets of a locally convex space which are not closed). We shall prove below that these two seemingly different drop properties, the one just mentioned and the quasi-drop property, coincide. To begin with, we need a simple lemma.

Lemma 147 *Let* $(X, \|\cdot\|)$ *be a normed space and* B *a non-empty closed convex and bounded subset of* X*. Let* $a \in X$ *such that* $a \notin B$*. Then, for every* $x \in D(a, B)$ *with* $x \neq a$*, we have* $\operatorname{dist}(x, B) < \operatorname{dist}(a, B)$*, where* dist *denotes the distance in* X *induced by* $\|\cdot\|$*.*

Proof. Let $d := \operatorname{dist}(a, B)$ (> 0). There exists a sequence (b_n) in B such that $\|a - b_n\| < d + 1/n$ for all $n \in \mathbb{N}$. Take $x \in D(a, B)$, $x \neq a$. Then there exists $0 \leq \lambda < 1$ and $b \in B$ such that $x = \lambda a + (1 - \lambda)b$. Observe that $y_n = \lambda b_n + (1 - \lambda)b \in B$, since B is convex. If $\lambda = 0$ then $x = b \in B$, so $\operatorname{dist}(x, B) = 0 < d = \operatorname{dist}(a, B)$ and we are done. Otherwise, for $n \in \mathbb{N}$,

$$\begin{aligned}\operatorname{dist}(x, B) &\leq \|x - y_n\| \\ &= \|\lambda a + (1 - \lambda)b - \lambda b_n - (1 - \lambda)b\| = \lambda \, \|a - b_n\| < \lambda(d + 1/n).\end{aligned}$$

Since this is true for every n, we get $\operatorname{dist}(x, B) \leq \lambda d < d = \operatorname{dist}(a, B)$. ∎

Remark 148 Observe that closedness or boundedness of the set B are not used in the proof of the former lemma.

The following result proves that the quasi-drop property implies (therefore is equivalent) to an apparently stronger property. This will have implications in the separate behavior of the quasi-drop property (see Corollary 150).

Theorem 149 *Let* $(E, \mathcal{T})$ *be a locally convex space and* B *a closed, convex and bounded subset of* E *with the quasi-drop property. Then, for every non-empty countably closed set* A *disjoint from* B*, there exists* $a \in A$ *such that* $D(a, B) \cap A = \{a\}$*.*

Proof. We shall argue by contradiction. Let B be a closed, convex and bounded subset of E with the quasi-drop property and suppose that we can find A, a nonempty countably closed subset of X disjoint from B, such that there is no $x \in A$ satisfying $D(x, B) \cap A = \{x\}$. Fix $a \in A$ and let $D := \overline{\Gamma(D(a, B))}$ be the absolutely convex and closed hull of $D(a, B)$, a disk in E. Let dist be the $\|\cdot\|_D$-distance in the normed space $(E_D, \|\cdot\|_D)$.

We shall construct a sequence (x_n) in A with the following properties.
(i) $x_1 := a$.
(ii) (x_n) is a stream, i.e., $x_{n+1} \in D(x_n, B)$ for all $n \in \mathbb{N}$, and $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
(iii) $\operatorname{dist}(x_{n+1}, B) < \operatorname{dist}(D(x_n, B) \cap A, B) + 1/n$ for all $n \in \mathbb{N}$.
This will be done by induction. Let us start by taking $x_1 := a$. Assume now that, for some $i \in \mathbb{N}$, elements $x_1, x_2, \ldots, x_i$ in A have already been defined such that (i), (ii) and (iii) above hold for $n = 1, 2, \ldots, i-1$. Compute then $\operatorname{dist}(D(x_i, B) \cap A, B)$ and choose $x_{i+1} \in D(x_i, B) \cap A$ such that

$$(\operatorname{dist}(D(x_i, B) \cap A, B) \leq) \operatorname{dist}(x_{i+1}, B) < \operatorname{dist}(D(x_i, B) \cap A, B) + 1/i.$$

The fact that $D(x_i, B) \cap A \neq \{x_i\}$ allow us to choose $x_{i+1} \neq x_i$. This finish the construction.
The set B has the quasi-drop property, hence, if $\{x_i; i \in \mathbb{N}\}$ is $\mathcal{T}$-closed, there is some x_n such that $D(x_n, B) \cap \{x_i; i \in \mathbb{N}\} = \{x_n\}$, a contradiction. Then, $\{x_n; n \in \mathbb{N}\}$ is not $\mathcal{T}$-closed. Let $\tilde{x} \in \overline{\{x_n; n \in \mathbb{N}\}}^{\mathcal{T}} \setminus \{x_n; n \in \mathbb{N}\}$. Since A is countably closed, we have $\tilde{x} \in A$. Moreover, $\tilde{x} \in \overline{\{x_m; m \geq n\}}^{\mathcal{T}}$ for all $n \in \mathbb{N}$. Obviously, the set $D(x_n, B)$ is $\mathcal{T}$-closed, so $\tilde{x} \in \overline{D(x_n, B)}^{\mathcal{T}} \cap A = D(x_n, B) \cap A$ for all $n \in \mathbb{N}$. In particular, $D(\tilde{x}, B) \subset D(x_n, B)$ for all $n \in \mathbb{N}$. Let $x \in D(\tilde{x}, B) \cap A$. Then, since $x \in D(x_{n+1}, B) \subset D(x_n, B)$, we have, using Lemma 147,

$$\begin{aligned}\operatorname{dist}(D(x_n, B) \cap A, B) &\leq \operatorname{dist}(x, B) \leq \operatorname{dist}(x_{n+1}, B) \\ &< \operatorname{dist}(D(x_n, B) \cap A, B) + \frac{1}{n}.\end{aligned}$$

We have, too,

$$\begin{aligned}\operatorname{dist}(D(x_n, B) \cap A, B) &\leq \operatorname{dist}(\tilde{x}, B) \leq \operatorname{dist}(x_{n+1}, B) \\ &< \operatorname{dist}(D(x_n, B) \cap A, B) + \frac{1}{n}.\end{aligned}$$

Then,

$$0 \leq \operatorname{dist}(\tilde{x}, B) - \operatorname{dist}(x, B) < \frac{1}{n}, \text{ for all } n \in \mathbb{N},$$

hence, for all $x \in D(\tilde{x}, B) \cap A$ we have $\operatorname{dist}(x, B) = \operatorname{dist}(\tilde{x}, B) \neq 0$. In view of Lemma 147, it follows that $x = \tilde{x}$ for every $x \in D(\tilde{x}, B) \cap A$, i.e., $D(\tilde{x}, B) \cap A = \{\tilde{x}\}$, and we arrive to a contradiction. ∎

A byproduct of the former theorem is that drop and quasi-drop properties are separably determined, something that is not evident from the very definition, since no metrizability nor, more generally, angelicity are present in this context. This will be formulated in the following two results.

Corollary 150 *Let $(E, \mathcal{T})$ be a locally convex space. Let B be a closed convex and bounded subset of E. Then B has the quasi-drop property if and only if $B \cap S$ has the quasi-drop property for every closed separable subspace $S \subset E$.*

Proof. One direction is quite obvious. Indeed, suppose that B has the quasi-drop property and let S be a closed separable subspace of E. Let A be a closed subset of S disjoint from B. Then A is closed in E and we can find $a \in A$ such that $D(a, B) \cap A = \{a\}$. Therefore $D(a, B \cap S) \cap A = \{a\}$.
Assume now that $B \cap S$ has the quasi-drop property for every closed separable subspace $S \subset E$ and assume that B has not the quasi-drop property in E. Let A be a closed subset of E such that no $x \in A$ has the property that $D(x, B) \cap A = \{x\}$. As in the proof of Theorem 149, we can find a sequence (x_n) in A with the properties listed there. Let $S := \overline{\text{span}(x_n)}$, a closed separable subspace of E. The set $B \cap S$ has the quasi-drop property. Then $\{x_n; n \in \mathbb{N}\}$ is not closed in S, nor in E. Find $\tilde{x} \in A$ as in the proof of Theorem 149 and continue this proof to get $D(\tilde{x}, B) \cap A = \{\tilde{x}\}$, a contradiction. ∎

An analogous proof works for the drop property.

Corollary 151 *Let $(E, \mathcal{T})$ be a locally convex space. Let B be a closed convex and bounded subset of E. Then B has the drop property if and only if $B \cap S$ has the drop property for every closed separable subspace $S \subset E$.*

Proof. Again, a direction is almost obvious. For the other implication, assume that $B \cap S$ has the drop property for every separable subspace S of E. If B has no the drop property, we can find a sequentially closed set $A \subset E$ such that no $x \in A$ satisfies $D(x, B) \cap A = \{x\}$. As in the previous proof we can get a sequence (x_n) in A with the properties listed in Theorem 149. Let $S := \overline{\text{span}(x_n)}$. The set $\{x_n; n \in \mathbb{N}\}$ is not sequentially closed in S, nor in E. Then there exists a subsequence (x_i) of (x_n) which converges to a point $\tilde{x}$ in A (actually $x_n \to \tilde{x}$). Now it is easy to prove that $D(\tilde{x}, B) \cap A = \{\tilde{x}\}$, a contradiction. ∎

3.2.3 Drop property for unbounded subsets of a locally convex space

A basic key to prove Theorem 149 for bounded sets is to be able to use some kind of "distance" in order to find a point in A at "minimum" distance from B. That was done there by considering the disk $D := \overline{\Gamma(D(a,B))}$ and embedding the set $D(a,B)$ in the normed space $(E_D, \|\cdot\|_D)$.
If B is unbounded, such a technique cannot be employed; however, we can still find some kind of "distance" from points in a drop to B. Precisely, let B be a closed convex subset of the locally convex space E, A any subset of E disjoint from B and $a \in A$. For a point $x \in D(a,B)$ we can define $g_a(x)$ from a to B as follows.

$$g_a(x) := \inf\{t \in [0,1] : \exists b \in B, x = ta + (1-t)b\},\ \ x \in D(a,B).$$

Observe that, since B is closed, the infimum is attained, so, in fact,

$$g_a(x) := \min\{t \in [0,1] : \exists b \in B, x = ta + (1-t)b\}.$$

The following simple result, similar to Lemma 147, holds.

Lemma 152 *Let $(E,\mathcal{T})$ be a locally convex space, B a closed convex subset of E and A a subset of E disjoint from B. Let $a \in A$ and $x \in D(a,B) \cap A$. Then, for every $y \in D(x,B) \cap A$ with $y \neq x$, we have $g_a(y) < g_a(x)$.*

Proof. Let $b_1 \in B$ such that $x = g_a(x)a + (1-g_a(x))b_1$. There exists $b_2 \in B$ and $t \in (0,1)$ such that $y = tx + (1-t)b_2$. Therefore

$$y = t\Big(g_a(x)a + (1-g_a(x))b_1\Big) + (1-t)b_2 =$$

$$= tg_a(x)a + \Big(1 - tg_a(x)\Big)\frac{(1-g_a(x))tb_1 + (1-t)b_2}{1-tg_a(x)}.$$

Observe that

$$\frac{(1-g_a(x))tb_1 + (1-t)b_2}{1-tg_a(x)} \in B,$$

since B is convex, so $g_a(y) \le tg_a(x) < g_a(x)$. ∎

We will also use the following simple lemma.

Lemma 153 *Let $(E, \mathcal{T})$ be a locally convex space and B a closed convex subset of E. Let $x \notin B$ and $y \in D(x, B)$ such that $y \neq x$. Then $\overline{D(y, B)} \subset D(x, B)$.*

Proof. Let $z \in \overline{D(y, B)}$. We can find sequences (λ_n) in $[0, 1]$ and (b_n) in B such that $z_n := \lambda_n y + (1 - \lambda_n) b_n \to z$. We may assume that $\lambda_n \to \lambda$ for some $\lambda \in [0, 1]$.
Assume first that $\lambda \neq 1$. Then $b_n = (z - \lambda_n y)/(1 - \lambda_n) \to (z - \lambda y)/(1 - \lambda) \in B$, since B is closed. Let b_0 be the former limit. Then $z = \lambda y + (1 - \lambda) b_0 \in D(y, B) \subset D(x, B)$.
Assume now that $\lambda = 1$. Since $y \in D(x, B)$ and $y \neq x$ there exists $\mu \in [0, 1[$ and $b \in B$ such that $y = \mu x + (1 - \mu) b$. We have

$$\begin{aligned} z_n &= \lambda_n(\mu x + (1 - \mu) b) + (1 - \lambda_n) b_n \\ &= \lambda_n \mu x + (1 - \mu\lambda_n) \frac{\lambda_n (1 - \mu) b + (1 - \lambda_n) b_n}{1 - \mu\lambda_n} = \lambda_n \mu x + (1 - \mu\lambda_n) b\,'_n, \end{aligned}$$

where

$$b\,'_n := \frac{\lambda_n (1 - \mu) b + (1 - \lambda_n) b_n}{1 - \mu\lambda_n} \in B,$$

by convexity. We know that $z_n \to z$. As before, $b\,'_n = (z_n - \lambda_n \mu x)/(1 - \mu\lambda_n) \to b\,' := (z - \mu x)/(1 - \mu) \in B$, since B is closed. We get $z = \mu x + (1 - \mu) b\,' \in D(x, B)$. This finishes the proof of the lemma. ∎

The following result extends to the unbounded setting Theorem 149.

Theorem 154 *Let $(E, \mathcal{T})$ be a locally convex space and B a closed, convex and unbounded subset of E with the quasi-drop property. Then, for every non-empty countably closed set A disjoint from B, there exists $a \in A$ such that $D(a, B) \cap A = \{a\}$.*

Proof. We shall argue by contradiction. Let B be a closed, convex and unbounded subset of E with the quasi-drop property and suppose that we can find A, a nonempty countably closed subset of X disjoint from B, such that there is no $x \in A$ satisfying $D(x, B) \cap A = \{x\}$.
Take any point $a \in A$. We construct a sequence (x_n) in A with the following properties.
(i) $x_1 := a$.
(ii) (x_n) is a stream, i.e., $x_{n+1} \in D(x_n, B)$ for all $n \in \mathbb{N}$, and $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.

(iii) $\inf\{g_a(y) : y \in D(x_n, B) \cap A\} \leq g_a(x_{n+1}) < \inf\{g_a(y) : y \in D(x_n, B) \cap A\} + 1/n$ for all $n \in \mathbb{N}$.
This will be done by induction. Let us start by taking $x_1 := a$. Assume now that, for some $i \in \mathbb{N}$, elements $x_1, x_2, \ldots, x_i$ in A have already been defined such that (i), (ii) and (iii) above hold for $n = 1, 2, \ldots, i-1$. Choose $x_{i+1} \in D(x_i, B) \cap A$ such that

$$\inf\{g_a(y) : y \in D(x_i, B) \cap A\} \leq g_a(x_{i+1}) < \inf\{g_a(y) : y \in D(x_i, B) \cap A\} + 1/i.$$

The fact that $D(x_i, B) \cap A \neq \{x_i\}$ and Lemma 152 allow us to choose $x_{i+1} \neq x_i$. This finishes the construction.
Observe that, by Lemma 152, if $y \in D(x_{n+1}, B)$, then

$$\inf\{g_a(y) : y \in D(x_n, B) \cap A\} \leq g_a(y) < \inf\{g_a(y) : y \in D(x_n, B) \cap A\} + 1/n.$$

On ther other hand the sequence $\{x_n\}$ is not closed, since otherwise B would not have the quasi-drop property, so $\{x_n\}$ has an adherent point $\tilde{x} \in A$. Since $\{x_n\} \subset \overline{D(x_i, B)}$ for every $i \in \mathbb{N}$, we have that $\tilde{x} \in \overline{D(x_n, B)}$ for every $n \in \mathbb{N}$. By Lemma 153, we conclude that $\tilde{x} \in \cap_1^\infty D(x_n, B) \cap A$.
Finally, take $y \in D(\tilde{x}, B)$. We get that $g_a(\tilde{x}) - g_a(y) < 1/n$ for every $n \in \mathbb{N}$ and so $g_a(\tilde{x}) = g_a(y)$. Therefore, by Lemma 152, we conclude that $y = \tilde{x}$ and so $D(\tilde{x}, B) \cap A = \{\tilde{x}\}$, a contradiction. This finishes the proof. ∎

In [Mo91], slices of subsets with drop property are analyzed in Banach spaces. Precisely, the following result holds.

Proposition 155 ([Mo91], Prop. 2.3) *Let B be a non-void closed convex subset of a Banach space X. Let $f \in X'$ be bounded above on B, $f \neq 0$ and $M := \sup\{f(x) : x \in B\}$. Then, if B has the drop property, every section $S(f, B, \delta) := \{x : x \in B, M - \delta \leq f(x) \leq M\}$, δ an arbitrary positive number, is bounded.*

A slight modification of the proof of that result permits us to prove a similar statement for the quasi-drop property in locally convex spaces.

Proposition 156 *Let B be a closed convex subset of a locally convex space E and let $f \in E'$ be bounded above on B, $f \neq 0$ and $M := \sup\{f(x) : x \in B\}$. Then, if B has the quasi-drop property, every section $S(f, B, \delta) := \{x : x \in B, M - \delta \leq f(x) \leq M\}$, δ an arbitrary positive number, is bounded.*

Proof. As in the proof of [Mo91, Prop. 2.3], suppose that for some $\delta > 0$, $\{x : x \in B, M - \delta \leq f(x) \leq M\}$ is not bounded. Then it is not bounded in $(E, \|\cdot\|_U)$, for some seminorm U. Starting by any point $x_1 \notin B$, such that $f(x_1) > M$, we construct a sequence (x_n) with the following properties:
(i) (x_n) is a stream, i.e., $x_{n+1} \in D(x_n, B)$ for all $n \in \mathbb{N}$, and $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$.
(ii) $f(x_n) > M$.
(iii) $\|x_i - x_j\|_U \geq 1$, $i, j \in \mathbb{N}$, $i \neq j$.
Suppose we have already constructed $(x_i)_1^n$.
Choose $0 < \lambda < 1$ such that $(1-\lambda)f(x_n) - \lambda M > M$ and take $z \in S(f, B, \delta)$ such that $\|\lambda z + (1-\lambda)x_n - x_i\|_U \geq 1$, $i = 1, 2, \ldots, n$. Define

$$x_{n+1} := \lambda z + (1-\lambda)x_n.$$

Then x_{n+1} satisfies (ii) and (iii).
We have constructed a stream closed sequence, a contradiction. ∎

This result will be used in the proof of Theorem 162.

3.3 Property (α)

Given a non-void closed convex (not necessarily bounded) subset B of a normed space X, let

$$F(B) := \{f \in X', f \text{ is bounded above on } B\}. \tag{3.1}$$

For $f \in F(B)$, put $\sup f[B] := \sup\{f(x) : x \in B\}$. Given $\delta > 0$ and $f \in F(B)$, the *δ-slice* determined by f on B is defined as

$$S(f, B, \delta) := \{x \in B : f(x) \geq \sup f[B] - \delta\}. \tag{3.2}$$

Definition 157 *Let X be a normed space. The* Kuratowski index of non-compactness *$\alpha(M)$ of a subset M of X is the infimum of all $\varepsilon > 0$ such that M can be covered by a finite number of subsets of X with diameter less than ε.*

The set B is said to have *property (α)* if, for every $f \in F(B)$,

$$\lim_{\delta \to 0^+} \alpha[S(f, B, \delta)] = 0.$$

Several authors have studied property (α) in Banach spaces, for bounded subsets or, more generally, for unbounded sets (see [Ro85], [Ku87], [KR91], [Mo87] and [Mo93]). In [KR91], Rolewicz and Kutzarova proved that *in a Banach space X, every convex closed subset with drop property has property (α).* Montesinos proved in [Mo87] that *the closed unit ball of a Banach space X has property (α) iff it has drop property, iff X is reflexive and has property (KK).*
In this section we study property (α) in locally convex space. We shall treat simultaneously the bounded and unbounded case. Given a closed convex subset B of a locally convex space E, the sets $F(B)$ and $S(f, B, \delta)$ are defined as in (3.1) and (3.2) for $\delta > 0$ and $f \in F(B) \subset E'$. The set $S(B, f, \delta)$ is called again a *section of B defined by f* (or, more precisely, the *δ-section of B defined by f*). Given $0 \neq f \in E'$ and some $r \in \mathbb{R}$, the set $H_r := \{x \in E; f(x) = r\}$ is an *hyperplane defined by f*, and the set $H_r \cap B$ is a *section of B defined by f*.

Definition 158 *Let B be a closed convex subset of a locally convex space $(E, \mathcal{T})$. B is said to have* property (α) *is for every $f \in F(B)$ and for every neighborhood U of the origin, there exists $\delta > 0$ such that $S(f, B, \delta)$ can be covered by a finite number of translates of U.*

In Proposition 184 we shall give an equivalent formulation of property α in terms of the Kuratowski index of non-compactness.

Let us start by some simple remarks concerning translates of neighborhoods of zero. In this section, $\mathcal{U}(0)$ will denote the family of all closed absolutely convex neighborhoods of 0 in a locally convex space $(E, \mathcal{T})$. Given $U \in \mathcal{U}(0)$, a subset S of E is called *U-small* if $s_1 - s_2 \in U$ for every $s_k \in S$, $k = 1, 2$.

Proposition 159 *Let S be a non-empty subset of a locally convex space $(E, \mathcal{T})$. Let U be an absolutely convex neighborhood of* 0 *in E. The following hold true.*
(i) *If S can be covered by a finite number of translates of U, then S is a finite union of $2U$-small sets.*
(ii) *If S is a finite union of U-small sets, then S can be covered by a finite number of translates of U.*
In particular, the two following conditions are equivalent: (a) *For every absolutely convex neighborhood U of* 0 *in E, the set S can be covered by a finite number of translates of U.* (b) *For every absolutely convex neighborhood U of* 0 *in E, the set S is a finite union of U-small sets.*

Proof. (i) Since U is absolutely convex, a translate of U is, obviously, a $2U$-small set. This proves (i).
(ii) Assume that $S = \bigcup_{i=1}^{n} S_i$, where $\emptyset \neq S_i$ is U-small for every $i = 1, 2, \ldots, n$. Fix $s_i \in S_i$, $i = 1, 2, \ldots, n$. Given $s \in S$, find $j \in \{1, 2, \ldots, n\}$ such that $s \in S_j$. Then $s - s_j \in U$. It follows that $s \in s_j + U \subset \bigcup_{i=1}^{n}(s_i + U)$. This happens for every $s \in S$, so $S \subset \bigcup_{i=1}^{n}(s_i + U)$.
The rest of the proposition follows from it: given an absolutely convex $U \in \mathcal{U}(0)$, we can find an absolutely convex $V \in \mathcal{U}(0)$ such that $2V \subset U$. Apply now simultaneously (i) and (ii) to prove the equivalence between (a) and (b).
∎

Remark 160 In fact, we proved in (ii) in Proposition 159 that, under the stated condition, S can be covered by a finite number of translates of U *to elements in* S. Hence, this last property can be added to (a) and (b) above still keeping the equivalence.

Remark 161 Property (α) depends on the topology on E and not only on the dual pair $\langle E, E' \rangle$. Actually, every closed convex and bounded subset B of a locally convex space has property (α) for the weak topology. That this is so is a consequence of the fact that the concepts of boundedness and total boundedness (i.e., precompactness) coincide in the weak topology of a locally convex space (see, e.g., [Ko69, §20.1(3)]). So, for every absolutely convex $U \in \mathcal{U}_w(0)$ (i.e., the family of all weak neighborhoods of 0 in E), *every* section of the set B by an element in $F(B)$ can be covered by a finite number of U-small sets, and this, in view of Proposition 159, is equivalent to the fact that *every* such a section B can be covered by a finite number of translates of U. (By the way, it is enough that B is a closed convex subset of a locally convex space with the property that every $f \in F(B)$ defines *at least* one bounded section of B. Then, B has property α in the weak topology. In this case, in view of Proposition 181, *all* sections defined in B by elements of $F(B)$ are bounded.) However, we cannot expect to have property (α) for an arbitrary closed convex and bounded subset of a locally convex space (see Theorem 169).

There is a close connection between property (α) and the drop property. In fact, property (α) is more general than the quasi-drop property. This is the content of the following result.

Theorem 162 *Let B be a closed convex subset of a locally convex space $(E, \mathcal{T})$. Suppose that B has the quasi-drop property. Then, B has property (α).*

Proof. Suppose B does not have property (α). Then, there exists $f \in F(B)$ and an absolutely convex closed neighborhood U of the origin, such that no $S(f, B, \delta)$ can be covered by a finite number of translates of U. If $S(f, B, \delta)$ is not bounded, by Proposition 156, B cannot have the quasi-drop property and the results is proved. We can consider then, that $S(f, B, \delta)$ is bounded. Let $\|\cdot\|$ be the seminorm defined by the Minkowski gauge of U. We shall prove the following

$$\texttt{Claim:} \qquad \sup_{x \in S(f,B,\delta)} \inf_{y \in L} \|x - y\| \geq 1, \tag{3.3}$$

for every $\delta > 0$ and every finite-dimensional subspace $L \subset E$.
In order to prove the `Claim`, proceed by contraposition; assume that for some finite-dimensional subspace L and some $\delta > 0$ we have

$$\sup_{x \in S(f,B,\delta)} \inf_{y \in L} \|x - y\| := \gamma < 1.$$

Take $1 > \beta > \gamma$. For every $x \in S(f, B, \delta)$ the set $(x + U) \cap L$ is non-empty. The set

$$K := \{l \in L : \exists x \in S(f, B, \delta) \text{ such that } \|x - l\| < \beta\}$$

is a bounded (hence relatively compact) subset of $(L, \|\cdot\|)$; therefore, for $0 < \varepsilon < (1 - \beta)$, we can find an ε-net $\{l_1, l_2, \ldots, l_n\}$ in $(K, \|\cdot\|)$. It follows that the set $\bigcup_{i=1}^{n} l_i + U$ covers $S(f, B, \delta)$, a contradiction, and the `Claim` is proved.
The rest of the proof is analogous to the proof provided in [Ro85]: let $M := \sup f[B]$. We shall construct by induction a sequence (x_n) such that $x_n \in D(x_{n-1}, B)$, $f(x_n) > M$, and $\|x_n - z\| > 1/3$ for every n and every $z \in \operatorname{span}(x_i)_1^{n-1}$. As the starting point, choose any $x_1 \in E$ such that $f(x_1) > M$. Suppose we have already obtained $x_1, \ldots, x_n$ and take $0 < \delta < f(x_n) - M$. Consider the finite dimensional subspace $L := \operatorname{span}\{x_i\}_1^n$. Then, by the `Claim`, there is an element $\overline{x}_{n+1} \in S(f, B, \delta)$, such that $\|\overline{x}_{n+1} - z\| > 2/3$ for every $z \in L$. It is easy to see that

$$x_{n+1} := \frac{\overline{x}_{n+1} + x_n}{2}$$

satisfies the requirements. The set $\{x_n; n \in \mathbb{N}\}$ is closed, disjoint from B, and $x_{n+1} \in D(x_n, B)$ for all n, so B does not have quasi-drop property. ∎

In Theorem 142 we proved that every closed convex and countably compact subset of a locally convex space has the quasi-drop property. Thus, we get:

Corollary 163 *Every closed convex and countably compact subset of a locally convex space has property (α).*

Property (α) is strictly more general than the quasi-drop property. Below we provide examples, in different settings, that separate both concepts.

Example 164 *There exists a normed space X whose closed unit ball has property (α) although it has not the quasi-drop property.*

Proof. Let $(X, \|\cdot\|)$ be a Banach space such that $\|\cdot\|$ in X^* is Fréchet differentiable (the space X is then reflexive) and let $E \neq X$ be a dense subspace of X. We claim that the closed unit ball B_E of the normed space $(E, \|\cdot\|)$ has property (α) although it has not drop property. That it has property (α)—in fact, a stronger property—follows from the Šmulyan criterium of Fréchet differentiability (see, for example, [DGZ93, Chapter I, 1]). In order to see that it has not drop property, let $x_0 \in S_X$ such that $x_0 \notin E$. There exists a continuous linear functional f which attains its maximum in B_X on x_0. Let $H := \{x \in E; f(x) > f(x_0)\}$ be an open half-space of E defined by f. Given two vectors $a \neq b$ in X we denote by $]a, b[$ the "open" interval defined by a and b, i.e., $]a, b[:= \{\lambda a + (1-\lambda)b : \ \lambda \in]0, 1[\}$. Take $x_1 \in H$ such that $\|x_1 - x_0\| < 1$. It is easy to see that we can find $b_1 \in B_E$ such that there exists $x_2 \in]b_1, x_1[\cap H$ with $\|x_2 - x_0\| < 1/2$. Next, choose $b_2 \in B_E$ such that there exists $x_3 \in]b_2, x_2[\cap H$ with $\|x_3 - x_0\| < 1/4$. Proceed in this way to define the sequence (x_n). No subsequence of (x_n) converges in $(E, \|\cdot\|)$, so $\{x_n; n \in \mathbb{N}\}$ is a closed subset of E disjoint from B_E, and there is no $x \in \{x_n; n \in \mathbb{N}\}$ such that $D(x, B_E) \cap \{x_n; n \in \mathbb{N}\} = \{x\}$. ∎

Example 165 A similar example for the weak topology is even easier, modulus a result of Qiu (Theorem 166). Recall that a subset S of a locally convex space E is called *Mackey complete* if every locally Cauchy sequence in S locally converges to some point in S (for a thorough study of this property see, e.g., [BP87, 5.1.5 and fw.]).

Theorem 166 ([Q03a], Theorem 3.4) *Let $(X, \mathcal{T})$ be a locally convex space and B a Mackey complete closed bounded convex subset of X. Then B has the quasi-weak drop property if and only if B is weakly compact.*

In order to provide the announced example, it is enough to recall that every closed convex and bounded subset B of a locally convex space $(E, \sigma(E, E'))$ has property (α) (see Remark 161). However, according to Theorem 166, if B is Mackey complete and not weakly compact (any non weakly compact closed convex and bounded subset of a Banach space will do the job), it cannot have quasi-weak drop property.

We can provide an example separating drop and (α) properties in the case of unbounded sets of Banach spaces.

Example 167 *There exists an unbounded closed convex subset of a Banach space that has property (α) and has not drop property.*

Proof. Let X be a separable non-reflexive Banach space and let (x_n) be a linearly dense and $\|\cdot\|$-null sequence in X (it is always possible, thanks to the separability of the Banach space, to find such a sequence). By the Krein-Šmulyan Theorem, $B := \overline{\Gamma(x_n)}$ is a $\|\cdot\|$-compact, absolutely convex and closed subset of X, and $0 \in B$. Let us define in $Z := X \oplus \mathbb{R}$ the following set C.

$$C := \bigcup_{r \geq 0} rB \times h(r),$$

where $h : \mathbb{R}^+ \to \mathbb{R}^+$ is a (continuous) convex function such that $h(0) = 0$, $h(r) > 0$ for every $r > 0$ and

$$\lim_{r \to +\infty,\, \forall r \geq 0} \frac{r}{h(r)} = 0.$$

Claim. *C is convex.*
Let $(r_1 b_1, h(r_1))$, $(r_2 b_2, h(r_2))$ in C, and $0 < \lambda < 1$, $r_1 \geq 0$, $r_2 \geq 0$, $b_1 \in B$, $b_2 \in B$. Assume $r_1 + r_2 > 0$. Then $\lambda r_1 + (1-\lambda) r_2 > 0$ and

$$\lambda(r_1 b_1, h(r_1)) + (1-\lambda)(r_2 b_2, h(r_2)) = (\lambda r_1 b_1 + (1-\lambda) r_2 b_2, \lambda h(r_1) + (1-\lambda) h(r_2)).$$

The continuity of h allows us to choose $s \geq \lambda r_1 + (1-\lambda) r_2 > 0$ such that $h(s) = \lambda h(r_1) + (1-\lambda) h(r_2) \geq h(\lambda r_1 + (1-\lambda) r_2)$ (by the fact that h is convex). Then

$$\frac{\lambda r_1 b_1 + (1-\lambda) r_2 b_2}{s} = \frac{\lambda r_1 + (1-\lambda) r_2}{s} \left(\frac{\lambda r_1 b_1 + (1-\lambda) r_2 b_2}{\lambda r_1 + (1-\lambda) r_2} \right).$$

We have that

$$\frac{\lambda r_1 b_1 + (1-\lambda) r_2 b_2}{\lambda r_1 + (1-\lambda) r_2} \in B,$$

since B is convex. Moreover,

$$0 \leq \frac{\lambda r_1 + (1-\lambda) r_2}{s} \leq 1,$$

hence

$$\frac{\lambda r_1 b_1 + (1-\lambda) r_2 b_2}{s} = b \in B$$

and we have

$$\lambda r_1 b_1 + (1-\lambda) r_2 b_2 = sb \in B,$$

since B is balanced. So we obtain that

$$\lambda(r_1 b_1, h(r_1)) + (1-\lambda)(r_2 b_2, h(r_2)) = (sb, h(s)) \in C.$$

On the other hand, obviously, if $r_1 + r_2 = 0$ and $r_1 \geq 0$, $r_2 \geq 0$, we have $r_1 = r_2 = 0$, $h(r_1) = h(r_2) = 0$, and of course

$$\lambda(r_1 b_1, h(r_1)) + (1-\lambda)(r_2 b_2, h(r_2)) \in C.$$

Claim. *C is closed.*
Let $(r_n b_n, h(r_n)) \in C$, such that $(r_n b_n, h(r_n)) \to (x, s) \in Z$. Then $r_n b_n \to x$, $h(r_n) \to x$. Assume $\{r_n\}$ is unbounded. Then we may assume $r_n \to +\infty$. Then $h(r_n) \to +\infty$, a contradiction. So $\{r_n\}$ is bounded and we may assume that $r_n \to r$. If $r = 0$, $h(r_n) \to h(0) = 0 = s$, and $r_n b_n \to 0$ since (b_n) is bounded. Then $x = 0$, $s = 0$ and $(0, 0) \in C$. If $r \neq 0$,

$$b_n = \frac{r_n b_n}{r_n} \to \frac{x}{r} \in B.$$

So $x \in rB$, and $s = h(r)$, so $(x, s) \in C$.

Claim. *C has property* (α).
We shall prove that every non-empty section of C by an element of $f \in F(C)$ is $\|\cdot\|$-compact. Let $f \in Z^*$, $f \in F(C)$. Then $f = (g, \alpha) \in X^* \oplus \mathbb{R}$, $f(x, s) = g(x) + \alpha s$ for every $(x, s) \in Z$ and $f(c) \leq M$ for every $c \in C$ and for some $M > 0$.

We shall prove first that $\alpha \leq 0$. Obviously $(0,s) \in C$ for every $s \geq 0$. Then $f(0,s) = \alpha s \leq M$ for every $s \geq 0$. Therefore $\alpha \leq 0$.
Now, we shall prove that for every $\delta > 0$, $S(C,f,\delta)$ has the following property: there exists $H > 0$ such that for every $(x,s) \in S(C,f,\delta)$, then $s \leq H$.

$$S(C,f,\delta) := \{(x,s) \in C : f(x,s) \geq S - \delta\},$$

where $S := \sup\{f(x,s) : (x,s) \in C\}$. Assume $(x,s) \in C$, $f(x,s) \geq S - \delta$. Then $g(x) + \alpha s \geq S - \delta$. If $\alpha = 0$, $f = (g,0)$. We have $f(x,s) = g(x) \leq M$ for every $x \in \bigcup_{r\geq 0} rB$. Note that g is continuous and $\bigcup_{r\geq 0} rB$ is dense in X. Then g is bounded in X and so $g = 0$. Then $f := 0$ and this case is excluded. Assume then that $\alpha < 0$. Let $(rb, h(r)) \in S(C,f,\delta)$. Then $f(rb,h(r)) \geq S - \delta$, so $g(rb) + \alpha h(r)) \geq S - \delta$. Therefore

$$\frac{r}{h(r)} g(b) \geq -\alpha + \frac{S-\delta}{h(r)}.$$

Now, if it will be possible to choose $r \to +\infty$, we shall arrive to a contradiction, since $r/h(r) \to 0$, $g(b)$ is bounded and $h(r) \to +\infty$. It follows that every section of C by every $f \in F(C)$ is in $HB \times h(H)$, a $\|\cdot\|$-compact set, and so, C has property (α).

Claim C *does not have the drop property.*
Indeed, C is an unbounded non-compact set with no non-empty interior (see [KR91, Theorem 3]). ∎

Remark 168 This example clarifies the main result in [Mo93]. It was proved there that if a Banach space X contains an unbounded closed convex subset B with property (α) *and such that* $\mathrm{int}(B) \neq \emptyset$, then X is reflexive. No example was given to ensure that this last property was, indeed, necessary. One can think that, as soon as such an unbounded closed convex set B with property (α) should lie inside a closed hyperplane, it will obviously had an empty interior and this will provide the desired example. However, it was proved in [Mo93][Corollary 3.3] that if B is such a set, then $\overline{\Gamma(B)} = X$, so this procedure turns out to be inadequate. The previous example gives an unbounded closed convex set B having property (α); moreover, it is obviously not compact nor it has a non-empty interior. By [KR91, Theorem 3], it does not have drop property. The space X is not reflexive, and this proves that the condition about the existence of a non-empty interior in [Mo93, Proposition 3.4] is unavoidable.

By using James theorem, it is easy to see that every closed convex bounded set B with property (α) in a Banach space is weakly compact (see [Ku87]). We prove next that the same is true, more generally, in quasi-complete locally convex spaces (a locally convex space E is *quasi-complete* if every bounded closed subset is complete). It is worth to stress that the completeness and the property (α) required are *in the same topology* (see Remark 175).

Theorem 169 *Let B be a closed convex bounded subset of a locally convex space $(E, \mathcal{T})$. Suppose that B has property (α) and $(E, \mathcal{T})$ is quasi-complete. Then, B is weakly compact.*

Proof. Let $\mathcal{S}$ be the family of all continuous seminorms defined in E. Take $f \in F(B)$, $p \in \mathcal{S}$ and $\varepsilon > 0$, and define $U = \{x \in E : p(x) < \varepsilon\}$. Let (x_μ) be a net in B such that $f(x_\mu) \to \sup f[B]$. We can find $\delta > 0$ such that $S(f, B, \delta)$ can be covered by a finite number of translates of U, i.e., there exist $y_1, \ldots, y_k$ in E such that $S(f, B, \delta) \subset \bigcup_{i=1}^k y_i + U$. There exists μ_0 such that for all $\mu \geq \mu_0$, $x_\mu \in S(f, B, \delta)$. This defines a subnet $\{x_\mu : \mu \geq \mu_0\}$ in $S(f, B, \delta)$. Assume that for all $\mu_1 \geq \mu_0$, there exists $\mu \geq \mu_1$ such that $x_\mu \in y_1 + U$. Then there exists a subnet in $y_1 + U$. If not, there exists $\mu_1 \geq \mu_0$ such that $x_\mu \notin y_1 + U$ for all $\mu \geq \mu_1$. In such a case, assume that for all $\mu_2 \geq \mu_1$ there exists $\mu \geq \mu_2$ such that $x_\mu \in y_2 + U$. Therefore there exists a subnet in $y_2 + U$. Otherwise, there exists $\mu_2 \geq \mu_1$ such that $x_\mu \notin y_2 + U$ for every $\mu \geq \mu_2$. Following this argument we conclude that, for some $i \in \{1, 2, \ldots, n\}$, there is a subnet in $y_i + U$.
Define a mapping

$$E \xrightarrow{\varphi} \mathbb{R}^{E \times \mathcal{S}},$$

by $\varphi(x)(y, p) = p(x - y)$ for $y \in E$ and $p \in \mathcal{S}$. It is easy to see that φ is a homeomorphic embedding when E is endowed with the $\mathcal{T}$-topology and $\mathbb{R}^{E \times \mathcal{S}}$ with the product topology. Moreover, $\varphi(A)$ is bounded for every bounded set A in E. Let (x_μ) be a net in B such that $f(x_\mu) \to \sup f[B]$. By Tychonoff's theorem there exists a subnet (denoted again by (x_μ)) such that $\varphi(x_\mu)(y, p)$ $(= p(x_\mu - y))$ converges for every $y \in E$ and $p \in \mathcal{S}$. Fix $p \in \mathcal{S}$ and $\varepsilon > 0$. By the previous argument we can find y_j and a subnet (x_α) of (x_μ) such that $p(x_\alpha - y_j) < \varepsilon$ for all α. Therefore, $(p(x_\alpha - y_j))_\alpha$ converges to some $l \leq \varepsilon$. But $(p(x_\alpha - y_j))_\alpha$ and $(p(x_\mu - y_j))_\mu$ converge to the same limit l, so $\lim_\mu p(x_\mu - y_j) \leq \varepsilon$. By the triangle inequality, and having in mind that $p \in \mathcal{S}$ and $\varepsilon > 0$ are arbitrary, we obtain that (x_μ) is Cauchy, so it converges to some $x \in B$. We have that $f(x) = \sup f[B]$. It is enough to apply now James' theorem to conclude that B is weakly compact. ∎

A second proof of this theorem relies on the fact that every subset B of a locally convex space $(E, \mathcal{T})$ has the property that "supremum attaining sequences" are precompact if the set B has property (α). More precisely, the following result holds.

Proposition 170 *Let B be a closed convex subset of a locally convex space $(E, \mathcal{T})$. Let $0 \neq f \in F(B)$. Then, if B has property (α), given a sequence (x_n) in B such that $f(x_n) \to M := \sup f(B)$, the set $\{x_n; n \in \mathbb{N}\}$ is precompact.*

Proof. Fix U, a neighborhood of 0 in $(E, \mathcal{T})$. Then we can find $\delta > 0$ such that the slice $S(B, f, \delta)$ can be covered by a finite number of translates of U. There exists $n_0 \in \mathbb{N}$ such that $x_n \in S(B, f, \delta)$ for every $n \geq n_0$. It follows that the set $\{x_n; n \in \mathbb{N}\}$ can be covered by a finite number of translates of U. Since U is arbitrary, the set $\{x_n; n \in \mathbb{N}\}$ is precompact (in particular, bounded). ∎

Now, we can provide a second proof to Theorem 169.

Proof. Let $0 \neq F \in E'$. Let (x_n) be a sequence in B such that $f(x_n) \to M$, where $M := \sup f(B)$. The space $(E, \mathcal{T})$ is quasi-complete, so, from Proposition 170, the set $\{x_n; n \in \mathbb{N}\}$ is relatively compact. Let x_0 be a cluster point of $\{x_n; n \in \mathbb{N}\}$. It belongs to B, and, obviously, $f(x_0) = M$, so f attains it supremum on B. It is enough to apply now James' Theorem. ∎

Therefore, in a quasi-complete locally convex space, every non-weakly compact subset fails to have property (α). From Example 164 we can conclude that quasi-completeness in Theorem 169 cannot be dropped from the statement.

Remark 171 It is worth to isolate the following behavior of a closed convex set B with property (α) in a quasi-complete locally convex space. *Every $f \in F(B)$ attains its supremum on B.* This is a consequence of the precompactness of any "supremum attaining sequence" in B (see Proposition 170).

In [Q03a, Theorem 3.5], the following is proved.
A locally convex space E quasi-complete in its Mackey topology is semireflexive if every closed convex bounded set has the quasi-drop property.
As a consequence of Theorem 169 we can obtain a similar statement for property (α). Due to Theorem 162, it is formally an improved test for semireflexivity.

Corollary 172 *Let $(E, \mathcal{T})$ be a quasi-complete locally convex space. Then E is semireflexive if and only if every closed convex bounded set has property (α).*

It is natural to ask whether the quasi-completeness condition in Theorem 169 can be weakened. The following example proves that local completeness is not enough.

Example 173 *There exists a locally complete locally convex space $(E, \mathcal{S})$ and an absolutely convex, closed and bounded subset B of E such that B has property (α) and still B is not weakly compact.*

Proof. Every Fréchet space $(E, \mathcal{T})$, endowed with its weak topology $w(E, E')$, is locally complete. This follows from the fact that a locally convex space $(E, \mathcal{T})$ is locally complete if and only if every closed disk in E is a Banach disk (see, e.g., [BP87, Proposition 5.1.6 and Corollary 5.1.7]). In particular, $(E, \mathcal{S}) := (c_0, w(c_0, \ell_1))$ is locally complete. We already mentioned that every closed convex and bounded subset of a locally convex space has property (α) (in the weak topology). In particular, B_{c_0} has property (α) in the space $(c_0, w(c_0, \ell_1))$ and it is not weakly compact, since the space c_0 is not reflexive. ∎

The same idea can be used to provide a more precise example; sequential completeness instead of quasi-completeness in Theorem 169 is still not enough.

Example 174 *There exists a sequentially complete locally convex space $(E, \mathcal{S})$ and an absolutely convex, closed and bounded subset B of E such that B has property (α) and still B is not weakly compact.*

Proof. The space $(E, \mathcal{S}) := (\ell_1, w(\ell_1, \ell_\infty))$ is, thanks to Schur Lemma, sequentially complete. As before, B_{ℓ_1} has property (α) in $(E, \mathcal{S})$. However, B_{ℓ_1} is not weakly compact, since ℓ_1 is not reflexive. ∎

Remark 175 Standard results in weak compactness use as hypothesis that some subset of a locally convex has a certain property and some kind of completeness holds *for the Mackey topology of the space* (the weakest requirement in this direction). Then the weak compactness (or relative weak compactness) of the set can be concluded. However, this is no the case with property (α). Indeed, assume that $(E, \mathcal{T})$ is, say, a Fréchet locally convex space and consider the locally convex space $(E, \mathcal{S}) := (E, w(E, E'))$. Take B a closed convex and bounded subset of E. Then B has property (α) (see again Remark 161) as a subset of $(E, \mathcal{S})$, a space complete in its Mackey topology, although B can be taken non-weakly compact from the beginning.

A modification of the method for proving Theorem 162 implies a separable reduction argument that we explicit now.

Theorem 176 *Let $(E, \mathcal{T})$ a locally convex space. Let B be a convex closed and bounded subset of E. Then B has property (α) if and only if $B \cap F$ has property (α) for every closed separable subspace F of E.*

Proof. A direction is almost trivial. Assume that B has property (α). Let F be a closed separable subspace of E, and assume that $F \cap B \neq \emptyset$. Let $\hat{f} \in F'$ $(= E/F^{\perp})$. Then $\hat{f}$ can be extended to an element $f \in E'$. Obviously, $S(B \cap F, \hat{f}, \delta) = S(B, \hat{f}, \delta) \cap F$ for every $\delta > 0$. If $S(B, \hat{f}, \delta)$ can be covered, for some $U \in \mathcal{U}(0)$, for a finite number of U-small sets, this is true also for the set $S(B \cap F, \hat{f}, \delta)$, and this last set can be also covered by a finite number of $U \cap F$-small sets. It is enough now to use Proposition 159.
To prove the reverse implication, assume that $B \cap F$ has property (α) for every closed separable subspace F of E. If B has no property (α), we can follow the proof of Theorem 162 making the following changes. We find f and U with the properties stated there. Equation (3.3) holds. The sequence (x_n) is defined by induction almost as it was done there. This time, to go from step n to $n+1$, choose $0 < \delta < \min\{1/n, f(x_n) - M\}$, and consider the finite-dimensional subspace $L := \text{span}\{\{x_i; i = 1, 2, \ldots, n\} \cup \{\overline{x}_i; i = 2, 3 \ldots n\}\}$. Then we can find $\overline{x}_{n+1} \in S(f, B, \delta)$ such that $\|\overline{x}_{n+1} - z\| > 2/3$ for every $z \in L$. Define x_{n+1} as in that proof. It is plain that $\|\overline{x}_n - \overline{x}_m\| > 2/3$ for all $n \neq m$ in $\mathbb{N} \setminus \{1\}$. Now, put $F := \overline{\text{span}}\{\overline{x}_{n+1}; n \in \mathbb{N}\}$. This is a separable subspace of E. Given $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that $\overline{x}_n \in S(f, B \cap F, \delta)$ for every $n \geq n_0$. This shows that $S(f, B \cap F, \delta)$ cannot be covered by a finite number of $2/3$-U-small sets, and this is a contradiction, since $B \cap F$ has property (α). ∎

A more restrictive condition on a closed convex subset B of a locally convex space $(E, \mathcal{T})$ is to ask that every slice from B defined by an element $f \in F(B)$ should be precompact. This sounds too demanding. However, in order to have this property, it is enough that *some* slice should be precompact. In the literature, for a certain convex and closed subset of a Banach space, the behavior of its slices versus sections or even the entire set is considered. For example, consider the two following results.

Proposition 177 ([Mo93], Prop. 3.1) *Let B be a closed convex subset of a Banach space, and assume that B has property (α). Then $S(B, f, \delta)$ is bounded, whenever $\delta > 0$ and $f \in F(B)$.*

Lemma 178 ([KR91], Lemma 2) *Let B be a closed convex subset in a Banach space X such that, for some $f \in X'$, $|f|$ is bounded on B. Let $m := \inf f(B) < M := \sup f(B)$, and assume, too, that for some $r \in]m, M[$, the section $H(f, r) \cap B$ is bounded, where $H(f, r) := \{x \in X; f(x) = r\}$. Then B is bounded.*

We shall prove now that a simple geometrical argument is behind these two results. It allows to unify the treatment of statements of this sort, even in the more general context of locally convex spaces. The starting point is the following result, a simple consequence of Tychonoff's theorem.

Proposition 179 (see [Ko69] §20.6(5)) *Let $K_1, K_2, \ldots, K_n$ be a finite sequence of convex (absolutely convex) compact subsets of a locally convex space $(E, \mathcal{T})$. Then $\mathrm{conv}(\bigcup_{i=1}^n K_i)$ is again compact.*

The former proposition works also if $K_1, K_2, \ldots, K_n$ ar supposed only to be precompact (bounded), the conclusion being, of course, that the corresponding convex hull is again precompact (respectively, bounded). The argument to prove the precompact statement is easy: we consider $\tilde{K}_i$, the closure of K_i in the completion $(\tilde{E}, \tilde{\mathcal{T}})$, $i = 1, 2, \ldots, n$. Those closures are compact sets, and the previous proposition applies, so $\mathrm{conv}(\bigcup_{i=1}^n \tilde{K}_i)$ is compact in $(\tilde{E}, \tilde{\mathcal{T}})$. It follows that $\mathrm{conv}(\bigcup_{i=1}^n K_i)$ is precompact. The argument for proving the boundedness statement follows from this: it is enough to recall that, in the weak topology, boundedness and precompactness are equivalent concepts. It is enough now to consider the locally convex space $(E, w(E, E'))$ and use the precompact statement.
In particular, the following corollary holds.

Corollary 180 *Let* $(E, \mathcal{T})$ *be a locally convex space. Let* $f \in E'$, $f \neq 0$. *Fix* $x_0 \in E$ *and* $r \in \mathbb{R}$, $r \neq f(x_0)$. *Assume that a certain set* $S \subset H_r := \{x \in E; f(x) = r\}$ *is convex and compact (precompact, bounded). Then* $T(x_0, S) := \mathrm{conv}(\{x_0\}, S)$ *is compact (respectively, precompact, bounded).*

Now we can prove the following useful fact (here, "non-trivial" means "not a face"). It contains results Prop. 3.1 in [Mo93] and Lemma 2 in [KR91].

Proposition 181 *Let* $(E, \mathcal{T})$ *be a locally convex space. Let* B *be a closed convex subset of* E, *and let* $f \in F(B)$. *Then the following statements are equivalent.*

(i) *There exists a compact (precompact, bounded) slice of* B *defined by* f.

(ii) *There exists a compact (respectively, precompact, bounded) non-trivial section of* B *defined by* f.

(iii) *Every slice of* B *defined by* f *is compact (respectively, precompact, bounded).*

(iv) *Every non-trivial section of* B *defined by* f *is compact (respectively, precompact, bounded).*

If $|f|$ *is bounded on* B *and some non-trivial section of* B *defined by* f *is compact (precompact, bounded), then* B *itself is compact (respectively, precompact, bounded). The same applies to slices.*

Proof. (i)⇒(ii) is trivial.
(ii)⇒(iv). Let $M := \sup f[B]$. Assume that for some $\delta > 0$, the section $H_{M-\delta} \cap B$ is compact, where $H_r := \{x \in E, f(x) = r\}$. To prove that, for $M - \delta < r < M$, the section $H_r \cap B$ is also compact, take $x_0 \in B$, $f(x_0) < M - \delta$, and consider $T(x_0, S)$, where $S := K(x_0, H_{M-\delta} \cap B) \cap H_M$, and $K(x_0, H_{M-\delta} \cap B)$ is the cone with vertex x_0 generated by $H_{M-\delta} \cap B$. It is trivial that S is compact, so $T(x_0, S)$ is also compact, and so it is $H_r \cap B$. If $r < M - \delta$ and $H_r \cap B \neq \emptyset$, a similar argument works now choosing $x_0 \in B$, $f(x_0) \in]M - \delta, M[$ and $S := K(x_0, H_{M-\delta} \cap B) \cap H_r$. The precompact or bounded cases follow the same lines.
(iii)⇒(i) and (iv)⇒(ii) are trivial.
It should be now clear how to prove that (ii)⇒(i) and (iv)⇒(iii).
In order to prove the last statement, it is enough to observe that, in case $|f|$ is bounded on B, then B is a slice of itself determined by f. ∎

It is simple to check that if a convex closed subset B of a locally convex space $(E, \mathcal{T})$ has property (α), then the face defined by every $f \in F(B)$ is $\mathcal{T}$-precompact. It is possible to provide an example of a set B with property (α) such that no section defined by elements in $F(B)$ is precompact. The example is simple modification of the on given in Example 167.

Example 182 *There exists an unbounded closed convex subset B of a Banach space that has property (α) such that no section defined by elements in $F(B)$ is precompact .*

Proof. Let X be a reflexive infinite-dimensional Banach space. We consider the Banach space $Y := X \oplus \mathbb{R}$ endowed with the supremum norm. Let

$$C := \bigcup_{r \geq 0} rB_X \times \{h(r)\},$$

where h is defined as in example 167. Following the same procedures as in example 167 we conclude that C is a closed convex (unbounded) subset of Y.

Let (x_0, r_0) be a support point of C. Let (x_n, r_n) be a sequence in C, and assume that $(x_n, r_n) \stackrel{w}{\to} (x_0, r_0)$. This implies that $x_n \stackrel{w}{\to} x_0$ and $r_n \to r_0$. If $r = 0$, then, since $x_n \in r_n B_X$, the sequence (x_n) is $\|\cdot\|$-null, and $(x_n, r_n) \stackrel{\|\cdot\|}{\to} (0, 0)$. If $r_0 > 0$, we may assume that $r_n > 0$ for every $n \in \mathbb{N}$. Pick $y_n \in r_0 B_X$ such that $\|x_n - y_n\| \leq |r_0 - r_n|$, for every $n \in \mathbb{N}$. If follows that $y_n \stackrel{w}{\to} x_0$, and $y_n \in r_0 B_X$, $x_0 \in r_0 B_X$. It is simple to prove that x_0 is a support point of $r_0 B_X$. Assume from the beginning that $\|\cdot\|$ in X has the Kadec-Klee property. Then $y_n \stackrel{\|\cdot\|}{\to} x_0$. If follows that $x_n \stackrel{\|\cdot\|}{\to} x_0$. Then (x_0, r_0) is a point of continuity. Exactly as in example 167, we may prove that given $f \in Y^*$, $f \in F(C)$, f cuts from C a w-compact section. According to [Mo93, Proposition 3.6.2], the set C has property (α). No section of C can be $\|\cdot\|$-precompact (i.e., $\|\cdot\|$-compact), since such a section contains a section of a translate of an homothetic of B_X. It is clear that no section of B_X can be $\|\cdot\|$-compact, since it contains an open ball. ∎

3.4 Condition (β)

In [Ro87], Rolewicz introduced the following notion for a non-void closed convex subset B of a Banach space X. Let $x \in X$ such that $x \notin B$. Denote

$$R(x) := D(x, B) \backslash B.$$

B is said to satisfy *condition (β)* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $d(x, B) < \delta$ implies $\alpha(R(x)) < \varepsilon$, where $\alpha(S)$, for a set S in a normed space, denotes the *Kuratowski index of non-compactness*, presented in Definition 157. Rolewicz proved in the same paper that if the closed unit ball of X satisfies condition (β), then it is Δ-uniformly convex (see below) and so it has property (α) and drop property [Ro87].
Recall here that a Banach space X is Δ-uniformly convex if $\varepsilon_\alpha(X) = 0$, where

$$\varepsilon_\alpha(X) = \sup\{\varepsilon > 0;\ \Delta_\alpha(\varepsilon) = 0\},$$

and

$$\begin{aligned}\Delta_\alpha(\varepsilon) := \sup\{\eta > 0;\ &(1-\eta)B_X \cap A \neq \emptyset \\ &\text{for every convex subset } A,\ A \subset B_X,\ \text{and } \alpha(A) \geq \varepsilon\}.\end{aligned}$$

A series of results concerning condition (β) have been obtained by Kutzarova and others (for example, see [Ku89-1], [Ku89-2], [Ku90] and references therein). In this section we study condition (β) in locally convex space and establish some relations with property (α) and drop property for closed convex sets.

Let E be a locally convex space. Given a neighborhood $U \subset E$ of the origin, let $\|\cdot\|_U$ be the seminorm defined by the Minkowski gauge of U. Let d_U be the pseudometric in E defined by the seminorm $\|\cdot\|_U$. Precisely,

$$d_U(x, y) := \|x - y\|_U,\ \text{for all } x, y \in E. \tag{3.4}$$

By using this pseudometric we can consider the Kuratowski index of non-compactness associated to U. This is done in the next definition.

Definition 183 *Given a neighbourhood U of the origin in the locally convex space E, the* U-Kuratowski index of non-compactness $\alpha_U(M)$ *of a subset M of E is the infimum of all $\varepsilon > 0$ such that M can be covered by a finite number of subsets of E with d_U-diameter less than ε.*

Property (α) was introduced in Definition 158. We can rephrase that definition in terms of the U-Kuratowski index of non-compactness. That it is equivalent to the former definition is the content of the next proposition. Recall that $F(B) := \{f \in E';\ f \text{ is bounded above on } B\}$.

Proposition 184 *Let B be a closed convex subset of a locally convex space $(E, \mathcal{T})$. Then B has* property (α) *if for every $f \in F(B)$ and for every neighborhood U of the origin,*

$$\lim_{\delta \to 0^+} \alpha_U[S(f, B, \delta)] = 0. \tag{3.5}$$

Proof. Assume first that B has property α. Take $f \in F(B)$ and U in $\mathcal{U}_0$ (the family of all neighborhoods of 0 in $(E, \mathcal{T})$). Given $\varepsilon > 0$, choose $V \in \mathcal{U}_0$ such that $V \subset (1/4)\varepsilon U$. By property α we can find $\delta > 0$ such that $S(f, B, \delta)$ (see (3.2)) can be covered by a finite family of translates of V. In view of Proposition 159 the set $S(B, f, \delta)$ can be covered by a finite number of $2V$-small sets, and the d_U-diameter of those sets is $< \varepsilon$. This proves that $\lim_{\delta \to 0^+} \alpha_U[S(f, B, \delta)] \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the conclusion follows.
On the other direction, assume that (3.5) holds. Then, given $\varepsilon > 0$ we can find a section $S(f, B, \delta)$ such that $\alpha_U[S(f, B, \delta)] < \varepsilon$. This means that $S(f, B, \delta)$ can be covered by a finite number of sets with d_U-diameter less that ε. Proposition 159 gives the conclusion. ∎

We can extend the definition of condition (β) from Banach spaces to the locally convex space setting by using the same approach.

Definition 185 *Let B be a closed convex subset of a locally convex space $(E, \mathcal{T})$. B is said to satisfy* condition (β) *if, for every neighbourhood $U \subset E$ of the origin and for every $\varepsilon > 0$, there is $\delta > 0$ such that $d_U(x, B) < \delta$ implies $\alpha_U(R(x)) < \varepsilon$.*

Observe that, similarly to the case of property (α), condition (β) depends on the topology on E and not only on the dual pair $\langle E, E' \rangle$. Actually, in the weak topology, every bounded set satisfies condition (β) (the argument follows the same lines as in Remark 161: every bounded set is precompact in the weak topology; obviously, if B is bounded, so it is $R(x)$ for every $x \notin B$). It is also immediate that, in any topology, every compact set satisfies condition (β), but the converse is not true even in normed spaces.

The following proposition extends the result proved by Rolewicz in [Ro87] using a technique similar to the one used there.

Theorem 186 *Let B be a closed convex subset of the locally convex space $(E, \mathcal{T})$. Suppose that B satisfies condition (β). Then, B has property (α).*

Proof. Let U be any neighborhood of 0 and let $f \in F(B)$. Let (δ_n) be a sequence of positive real numbers such that $\delta_n \to 0$. Put $M = \sup f[B]$. We can construct a sequence (x_n) of elements of E such that $f(x_n) = M + 2\delta_n$ and $d_U(x_n, B) \to 0$. Observe that $f\left(\frac{x_n+y}{2}\right) > M$, for every $y \in S(f, B, \delta_n)$. Then $\frac{x_n+y}{2} \in R(x_n)$ and so, $\frac{1}{2}(x_n + S(f, B, \delta_n)) \subset R(x_n)$. Therefore, since $\alpha_U(R(x_n)) \to 0$, we have that $\alpha_U(S(f, B, \delta_n)) \to 0$ and B has property (α).∎

Observe that the result is true even for unbounded sets. We can use Example 167 to give an instance of an unbounded closed convex set satisfying condition (β). This is done below.

Example 187 *There exists an unbounded closed convex subset of a Banach space that satisfies condition (β).*

Proof. Consider the set C in Example 167. We follow notations there. Take a point $(x_0, s_0) \notin C$. As in the proof of the mentioned example, it is easy to check that there exists $H > 0$ such that, if $(x, s) \in R((x_0, s_0), C)$, then $s \leq H$. Therefore $R((x_0, s_0), C)$ is $\|\cdot\|$-compact, so C satisfies condition (β).
∎

Montesinos proved in [Mo93, Prop. 3.1] that, in Banach spaces, every closed convex subsets with property (α) has bounded slices. The extension to the case of condition (β) holds.

Proposition 188 *Let E be a locally convex space and let $U \subset E$ be any neighborhood of the origin. Suppose that the closed convex subset B satisfies condition (β). Then $S(f, B, \delta)$ is bounded (in $(E, \|\cdot\|_U)$), whenever $\varepsilon > 0$ and $f \in F(B)$.*

Proof. Consider (δ_n) and (x_n) as in Theorem 186. For every $n \in \mathbb{N}$, there exists x_n such that $R(x_n)$ is bounded. Then $S(f, B, \delta_n)$ is bounded. Now, as in the proof of [Mo93, Prop. 3.1], the boundedness of $S(f, B, \delta_n)$ implies the boundedness of $S(f, B, \delta)$ for any $\delta > 0$. ∎

Rolewicz showed in [Ro87] a Δ-uniform convex set (and so, with drop property) which does not satisfy condition (β). Therefore, drop property does not imply condition (β). Moreover, the converse implication is also not true

in general: consider a reflexive Banach space $(X, \|\cdot\|)$ and suppose that the closed unit ball B_X is uniformly convex. It is shown in [Ro87] that it satisfies condition (β). Let ($E \neq X$) be a dense subspace. Then B_E still satifies condition (β), but it does not have drop property (see Example 164).
In the weak topology, a similar example is even easier, since every bounded set satisfies condition (β), but not necessary has quasi-drop property (see for example [Q03a, Th. 3.4]).

However, with the assumption of quasi-completeness, condition (β) does imply the quasi-drop property. This is the content of the next result.

Theorem 189 *Let $(E, \mathcal{T})$ be a quasi-complete locally convex space and let B be a closed convex bounded subset of E. Suppose that B satisfies condition (β). Then B has the quasi-drop property.*

Proof. We use some ideas already in the proof of Theorem 169 and Proposition 170. Let A be a non-empty closed subset of E such that $A \cap B = \emptyset$. Fix some $a \in A$. Let $D := \overline{\Gamma}(D(a, B))$, the closed absolutely convex hull of $D(a, B)$, a Banach disc.
Assume first that there exists $x \in D(a, B) \cap A$ such that $d_D(D(x, B) \cap A, B) > 0$. Since E_D is a Banach space, we can apply the Daneš' Drop Theorem in E_D to conclude that there exists $x_0 \in D(x, B) \cap A$ such that $D(x_0, B) \cap A = \{x_0\}$. This should finish the proof.
If this is not the case, for every $x \in D(a, B) \cap A$ we have $d_D(D(x, B) \cap A, B) = 0$. Inductively we may define a stream (x_n) in $D(a, B) \cap A$ such that $d_D(x_n, B) \to 0$ (recall that a stream (x_n) satisfies $x_{n+1} \in D(x_n, B)$ and $x_{n+1} \neq x_n$, for all $n \in \mathbb{N}$). Fix $U \in \mathcal{U}_0$, and $\varepsilon > 0$. By condition (β), there exists $\delta > 0$ satisfying that $d_U(x, B) < \delta$ implies $\alpha_U(R(x)) < \varepsilon$. Therefore, we can find $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $\alpha_U(R(x_n)) < \varepsilon$. The set $\{x_n;\ n \geq n_0\}$ is a subset of $R(x_{n_0})$. This last set can be covered by a finite number of translates of U, hence the same is true for the set $\{x_n;\ n \in \mathbb{N}\}$. Since this happens for every neighborhood U of 0, we get that $\{x_n;\ n \in \mathbb{N}\}$ is a precompact set. The space $(E, \mathcal{T})$ is quasi-complete and $\{x_n;\ n \in \mathbb{N}\}$ is bounded, so this last set a relatively compact subset of E. Let x_0 be a cluster point of the sequence (x_n). Obviously, $d_U(x_0, B)) = 0$, and this happens for all $U \in \mathcal{U}_0$. It follows that $x \in B$. Since A is closed, $x_0 \in A$, and we reach a contradiction. ∎

Remark 190 Theorem 189 cannot be obtained just by combining previous results. It is true that condition (β) for a closed convex subset B of a locally convex space implies property (α) (see Theorem 186), and that, in presence of quasi-completeness of the space, this forces the set B to be weakly compact (see Theorem 169). This shows that B has the quasi-weak drop property (Theorem 142 or just Theorem 139). We cannot obtain from it, in general, that B has the quasi-drop property.

As it happened with drop property and property (α), condition (β) is also separably determined.

Theorem 191 *Let $(E, \mathcal{T})$ a locally convex space. Let B be a convex closed and bounded subset of E. Then B satisfies condition (β) if and only if $B \cap F$ satisfies condition (β) for every closed separable subspace F of E.*

Proof. One direction is immediate. Suppose that B satisfies condition (β) and let F be a closed separable subspace of E. Obviously $B \cap F$ satisfies condition (β).
On the other hand, suppose that B does not satisfy condition (β). This implies that there exists a closed absolutely convex neighbourhood U of the origin and $\varepsilon > 0$, such that for every $\delta > 0$, there exists a point $x \notin B$ satisfying $d_U(x, B) < \delta$ and $\alpha_U(R(x)) \geq \varepsilon$.
For each $\delta_n = 1/n$ $(n = 1, 2, \ldots)$, take x_n such that $d_U(x_n, B) < \delta_n$ and $\alpha_U(R(x_n)) \geq \varepsilon$. Since $\alpha_U(R(x_n)) \geq \varepsilon$, we can construct a sequence $(y_i^n)_{i=1}^{\infty}$, such that $d_U(y_i^n, y_j^n) \geq 1$ for every $i \neq j$. Also, construct a sequence $(b_i^n)_{i=1}^{\infty}$ in B such that $d_U(b_i^n, x_n) \to d_U(x_n, B)$ when $i \to +\infty$. Now define

$$F := \overline{\bigcup_{n=1}^{\infty} (\{x_n\} \cup \{y_i^n\}_{i=1}^{\infty} \cup \{b_i^n\}_{i=1}^{\infty})}.$$

Then it is easy to check that $B \cap F$ does not satisfy condition (β) in F. That finishes the proof. ∎

Chapter 4
Appendix

4.1 Some open problems

1. Give an example of a locally convex space $(E, \mathcal{T})$ and a w-(R)CK subset of E which is not w-(R)NK (see Example 43 and the following item). Certainly, the class of w-CK sets can be separated from the class of w-SK sets (it is enough to find a w-K subset of a locally convex space which is not w-SK. An example (even in a separable locally convex space) can be found in [Fl80, §1.2 (6)]. Moreover, the class of w-CK sets can also be separated from the class of w-K sets. The set A in Example 39 shows this.

2. Several counterexamples to certain questions about stability of the CK concept are provided in topologies different of the weak topology of a locally convex space (closed subsets is an instance of this, see Example 44). We look for examples where this happens in the weak topology.

3. We do not know whether it is possible or not to push further Theorem 59 in the sense of relaxing the class of subsets that have a $w(E', E)$-dense union in the dual of a locally convex space E (see the difficulties shown by Example 60). Observe, however, that the extension cannot go to the class of w-(R)∂K, as it is mentioned in Remark 64, which refers to Example 42.

4. We do not know whether it is possible to extend Theorem 81, this time for a class of bounded subsets of E more general than the class of w-(R)CK, say to one of the intermediate classes that have been considered in this Memoir.

5. We proved an extension of a result of Howard in Theorem 85. We do not know whether it is possible to make a further extension to include more relaxed compactness conditions in the dual Mackey topology.

6. We provided a proof of a slightly weaker form of Theorem 137 by using our techniques based in generation of Banach disks and the classical Daneš Drop Theorem. This was done in Theorem 143. We did not succeed in proving by our methods the result in full generality.

4.2 Further developments

1. We think interesting to study the Ekeland's variational principle in locally convex spaces and its connection with the drop property, the property (α) and the condition (β). Qiu and Rolewicz recently published an article about this topic [QR08]. Other contributions to the same subject are, for example, [DeF89] and [Q03b].

2. The notion of a *direccional derivative* of a real-valued convex function defined on a convex subset of a topological vector space is defined as a natural extension of the $\mathbb{R}^n$-case. Assume that D is a certain closed convex subset of a topological vector space E. If $0 \in D$, we define the set $C(D) := \{x \in E;\ \text{there exists } \lambda > 0 \text{ such that } x \in \lambda D\}$. This is a convex subset of E (indeed, a convex cone). The Minkowski functional p_D of D is well defined in $C(D)$, and it is a real convex function. Differentiability properties of p_D can be then considered. Assume that D is a drop defined by a certain vertex v and a convex subset B of E. Then, according to whether p_D has directional derivatives we speak of the *smooth drop property*. This has been done, for example, in [Maad95b], [GKM96], and [Maad02] in the context of Banach spaces and for the case of *smooth* drops. A through study of the case of a topological vector space should be done.

3. Recently, an improvement of Klee's theorem on convex sets in non-reflexive Banach space (see [Ko69, §24.4]) appeared in [HKVZ07]. The statement reads (Proposition 5): *Let Y be a nonreflexive Banach space and $0 \neq y^* \in Y^*$. Then there exists an equivalent norm $|||\cdot|||$ on Y such that*

   (a) *y^* does not attain its norm on $B_{(Y,|||\cdot|||)}$, and*

(b) *for each* $\varepsilon > 0$, *there is* $M_\varepsilon \subset B_{(Y,|||\cdot|||)}$ *such that* $\operatorname{diam} M_\varepsilon < \varepsilon$ *and* $\sup y^*(M_\varepsilon) = |||y^*|||_*$.

(The authors denote $|||\cdot|||_*$ the corresponding dual norm in X^*.) We think that this result can be further extended in two directions.

(i) To see what it can be said not only for the closed unit ball of a Banach space, but for any closed convex and bounded subset of a Banach space.

(ii) To see what is its right translation in the language of topological vector spaces.

We are working in these directions.

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