Weak stability of non-autonomous discrete dynamical systems

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Abstract

In this paper we introduce a concept of weak stability in non-autonomous dynamical system. We show that the set of weak stable points is residual and investigate the relation between weak stability and shadowing property. We also discuss the relation between weak stability of non-autonomous dynamical system and its induced set-valued system.

Keywords: weak stability, non-autonomous dynamical system, set-valued system, shadowing property

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1. Introduction

Let $f : X \to X$ be a continuous map acting on a compact metric space (X, d). A autonomous discrete dynamical system is a pair (X, f). A non-autonomous discrete system difference equation is the following:

$$x_{n+1} = f_n(x_n), \quad n \geq 0, \tag{1}$$

where $\{f_n\}_{n=0}^{\infty}$ is a sequence of continuous maps and each f_n is a self-map on X. Set $F = \{f_n\}_{n=0}^{\infty}$ for the sake of simplicity. Note that the autonomous dynamical system is a special case of system (1) when $f_n = f$ for all $n \geq 0$.

We refer to Section 2 for other notions and notations mentioned in this section.

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Non-autonomous dynamical systems deal with the situations which dynamics can vary with time. Recently, the study of non-autonomous dynamical systems become active and many elegant results have been obtained [1, 2, 3, 4, 5, 6]. The dynamics in non-autonomous case can be vary complicated. Hence it is natural to study the pseudo-orbits for a better understanding of true orbits. Along this line, the study of shadowing property in autonomous dynamical systems attracts lots of attention [7, 8, 10, 11, 12, 13, 14, and the references therein]. In [9], a concept of weak stability has been introduced, and it is shown that orbital shadowing property is generic in the set of weak stable homeomorphisms. Motivated by this idea, we discuss weak stability in nonautonomous dynamical systems.

On the other hand, a discrete dynamical system uniquely induces its set-valued system which on the space of compact subsets. It is natural to ask the following question: What is the relation between dynamical properties of the original and set-valued systems? The study of the dynamics of the induced system has been extensively studied and many elegant results have been obtained[15, 16, 17, and the references therein].

In this present paper, a concept of weak stability has been introduced and the relation between shadowing and weak stability has been discussed. The relations between some chaotic properties of the nonautonomous discrete dynamical system and its set-valued system have also been investigted.

Below, basic notions are introduced in Section 2. Main results are presented in Section 3.

2. Basic concepts and notations

Let $F = \{f_n\}_{n=0}^{\infty}$ be a sequence of continuous selfmaps defined on a compact metric space X. An *orbit* of a point $x_0 \in X$, denoted by $o(x, F) = \{x_n\}_{n=0}^{\infty}$, is defined as follows:

$$x_n = f_n(x_{n-1}), \quad n = 1, 2, \cdots$$

Denote $F_n : X \to X$ by

$$F_n = f_n \circ f_{n-1} \cdots \circ f_2 \circ f_1.$$

For $\delta > 0$, a δ-*pseudo-orbit* for F is a sequence $\{x_n\}_{n=0}^{\infty}$ in X such that $d(f_{i+1}(x_i), x_{i+1}) < \delta$ for $i \in \mathbb{N}$. A finite δ-*pseudo-orbit* $\{x_i\}_{i=0}^{b}$ is called a δ-*chain* from x_0 to x_b with length $b + 1$.

For $\epsilon > 0$, F has *shadowing property* if, there is a $\delta > 0$ such that every δ-*pseudo-orbit* for F can be ϵ-*shadowed* by some point $y \in X$, that is $d(F_i(y), x_i) < \delta$ for all $i \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all positive integers. F is *chain transitive* if for any $x, y \in X$ there is a δ-*chain* of F from x to y.

Let $\mathcal{K}(X)$ be the collection of all non-empty compact subsets of X. Define the ε-neighborhood of a nonempty subset A in X to be the set

$$\mathrm{N}_\varepsilon(A) = \{x \mid d(x, A) < \varepsilon\},$$

where $d(x, A) = \inf_{a \in A} \rho(x - a)$.

The Hausdorff separation $\rho(A, B)$ of $A, B \in \mathcal{K}(X)$ is defined by

$$\rho(A, B) = \inf\{\varepsilon > 0 |\ A \subseteq \mathrm{N}_\varepsilon(B)\},$$

The Hausdorff metric on $\mathcal{K}(X)$ is defined by letting

$$H_d(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

For a compact space X, the topology generated by H_d coincides with the finite topology. In this case $\mathcal{K}_{\mathcal{F}}(X)$, the set of all finite subsets of X is dense in $\mathcal{K}(X)$. Also, $\mathcal{K}(X)$ is compact if and only if X is compact.

3. Main Results

In this section, we investigate the so-called weak stability in (X, F) (recall that $F = \{f_n\}_{n=0}^{\infty}$).

Definition 3.1. *We call x a weak stable point of F, or F is weak stable at x, if for every $\epsilon > 0$ there exist $\delta > 0$ and an integer T such that $o(z, F) \subset \mathrm{N}_\epsilon(\{F_i(z)\ ;\ i = -T, \ldots, T\})$ for any $z \in X$ with $d(z, x) < \delta$.*

Theorem 3.2. *Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of homeomorphisms on a compact space X. Then the set of weak stable points is residual in X.*

Proof. Let $\epsilon > 0$ and $U = \{U_i \mid i = 1, 2, \cdots, k\}$ be a finite open covering of X with $diam(U_i) < \frac{\epsilon}{2}$. Set $\mathrm{K} = \{1, 2, \cdots, k\}$. For every $x \in X$, choose $L_x \subset \mathrm{K}$ satisfying the following:

1. $o(x, F) \subset \cup\{U_i \mid i \in L_x\}$
2. $o(x, F) \cap U_i \neq \emptyset$

Let W_ϵ be the set of all $a \in X$ such that for $\epsilon > 0$, there exist $\delta_a > 0$ and positive integer T_a with $d(a, x) < \delta_a$ implies $o(x, F) \subset \mathrm{N}_\epsilon(\{F_i(x)\})$ for $i = -T_a, \cdots, T_a$. Obviously, W_ϵ is open. To prove that W_ϵ is dense in X, fix any $a \in X$. Choose $\lambda_1 > 0$ such that for every $x \in \mathrm{N}_{\lambda_1}(a)$,

$$d(F_i(a), F_i(x)) < \frac{\epsilon}{2}.$$

where $i = -T_a, \cdots, T_a$.

Assume that $a \notin W_\epsilon$. For $0 < \delta_1 < \lambda_1$ there exists $a_1 \in \mathrm{N}_{\delta_1}(a)$ such that for $i = -T_a, \cdots, T_a$,

$$d(F_{m_1}(a_1), F_i(a_1)) \geq \epsilon,$$

where $|m_1| > T_a$. We also have for $i = -T_a, \cdots, T_a$,

$$d(F_{m_1}(a_1), F_i(a)) \geq \frac{\epsilon}{2},$$

Indeed, if $d(F_{m_1}(a_1), F_i(a)) < \frac{\epsilon}{2}$, then

$$d(F_{m_1}(a_1), F_i(a_1)) \leq d(F_{m_1}(a_1), F_i(a)) + d(F_i(a), F_i(a_1)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is a contradiction. Consequently,

$$F_{m_1}(a_1) \notin \mathrm{N}_{\frac{\epsilon}{2}}(\{F_i(a)\}_{i=-T_a}^{T_a}).$$

Notice that $diam(U_i) < \frac{\epsilon}{2}$, thus $F_{m_1}(a_1) \notin U_i$ for all $i \in L_a$, and then there exists $j \in \mathrm{K} - L_a$ such that $F_{m_1}(a_1) \in U_j$. Thus $L_a \subset L_{a_1}$. Choose a positive integer $m_2 > m_1$ such that for all $j \in L_{a_1}$,

$$\{F_i(a_1)\}_{i=-m_2}^{m_2} \cap U_j \neq \emptyset.$$

Thus

$$o(a_1, F) \subset \mathrm{N}_\epsilon(\{F_i(a_1)\}_{i=-m_2}^{m_2}).$$

Still, one could choose $\lambda_2 > 0$ such that for every $x \in \mathrm{N}_{\lambda_2}(a_1)$,

$$d(F_i(a_1), F_i(x)) < \frac{\epsilon}{2}.$$

where $i = -m_2, \cdots, m_2$.

If $a_1 \in W_\epsilon$ then the proof is done, otherwise there exists $a_2 \in \mathrm{N}_{\delta_2}(a_1) \subset \mathrm{N}_{\delta_1}(a)$ implies for $i = -m_2, \cdots, m_2$,

$$d(F_{m_3}(a_2), F_i(a_2)) \geq \epsilon,$$

where $|m_3| > m_2$.

Using the same technique as above we obtain

$$F_{m_3}(a_2) \notin \mathrm{N}_{\frac{\epsilon}{2}}(\{F_i(a_1)\}_{i=-m_2}^{m_2}),$$

and then $F_{m_3}(a_2) \notin U_i$ for all $i \in L_{a_1}$, hence there exists $j \in \mathrm{K} - L_{a_1}$ such that $F_{m_3}(a_2) \in U_j$. Consequently, $L_{a_1} \subset L_{a_2}$.

By continuing this process there is $a^* \in \mathrm{N}_{\delta_1}(a)$ such that $L_{a^*} = \mathrm{K}$, since K is finite. Thus $a^* \in W_\epsilon$, which completes the proof of density of the set W_ϵ. Set $W = \cap_{n=1}^{\infty} W_{\frac{1}{n}}$, then W is residual in X. □

Lemma 3.3. *If F has the shadowing property, then so does F_k for $k \in \mathbb{N}$.*

Lemma 3.4. *Let F_k be chain transitive for $k \in \mathbb{N}$. If F has the shadowing property, then F_k is topological transitive.*

Proof. By Lemma 3.3, F_k has the shadowing property. Let $\mathrm{B}(x, r_1)$ and $\mathrm{B}(y, r_2)$ be balls of $x, y \in X$, respectively. For $0 < \epsilon < \min\{r_1, r_2\}$, there exists $\delta > 0$ such that every δ-*pseudo-orbit* of F_k can be ϵ-shadowed by some point of X. Since F_k is chain transitive, there exists a δ-*chain* $\{x = x_0, \cdots, x_n = y\}$ from x to y. Thus there is $z \in X$ such that $d(z, x) < \epsilon$ and $d(F_{kn}(z), y) < \epsilon$. Consequently, $F_{kn}(\mathrm{B}(x, r_1)) \cap \mathrm{B}(y, r_2) \neq \emptyset$. It follows that F_k is topological transitive. □

Theorem 3.5. *Let (X, d) be a compact metric space. Let F_n be chain transitive for $n \in \mathbb{N}$. If F has a weak stable point, then F does not have the shadowing property.*

Proof. Let $\epsilon > 0$ and $x \in X$ be a weak stable point of F. Let $U = \cup_{i=1}^{s} U_i$ be a finite open covering of X with $diam(U_i) < \frac{\epsilon}{6}$. Then there exist $0 < \eta < \frac{\epsilon}{6}$ and $n_1, n_2, \cdots, n_s \in \mathbb{N}$ such that for $y \in \mathrm{B}(x, \eta)$, $F_{n_i}(y) \in U_i$ for $i = 1, 2, \cdots, s$. Take $T = \max\{|n_i| : 1 \leq i \leq s\}$. Then

$$d(F_n(y), F_{n_i}(y)) < \frac{\epsilon}{6}$$

for $n \in \mathbb{N}, -T \leq i \leq T$. If F has the shadowing property, then there exists $0 < \delta < \eta$ such that each δ-*pseudo-orbit* of F can be η-shadowed by some point $t \in X$. By Lemma 3.4, F is topological transitive, there exists $k \in \mathbb{N}$ such that $F_{-k}(\mathrm{B}(x, \frac{\delta}{2})) \cap \mathrm{B}(x, \frac{\delta}{2}) \neq \emptyset$. Take $z \in F_{-k}(\mathrm{B}(x, \frac{\delta}{2})) \cap \mathrm{B}(x, \frac{\delta}{2})$. Since F_k is chain transitive, there exists a δ-chain $\{y = y_0, y_1, \cdots, y_m = z\}$ from y to

z. Thus $\{y, f_1(y), \cdots, F_{k-1}(y), y_1, f_1(y_1), \cdots, F_{k-1}(y_1), y_2, \cdots, y_{m-1}, f_1(y_{m-1}), \cdots, F_{k-1}(y_1), z\}$ is a δ-chain of F, which can be η-shadowed by some point $t \in X$. It follows that

$$d(t, y) < \eta, \quad d(F_{(m+l)k}(t), z) < \eta, \quad l = 0, 1, \cdots.$$

Note that $F_{(m+l)k}(t) \in X = \cup_{i=1}^{s} U_i$, then $F_{(m+l)k}(t) \in U_i$ for some $i = 1, \cdots, s$. However, $F_{n_i}(y) \in U_i$. Therefore,

$$\begin{aligned} d(F_{(m+l)k}(y), z) \quad & \leq d(F_{(m+l)k}(y), F_{n_i}(y)) + d(F_{n_i}(y), F_{(m+l)k}(t)) + d(F_{(m+l)k}(t), z) \\ & < \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}. \end{aligned}$$

Then

$$d(F_{(m+l)k}(y), x) \leq d(F_{(m+l)k}(y), z) + d(z, x) < \frac{\epsilon}{2} + \frac{\epsilon}{12} = \frac{7\epsilon}{12}.$$

Consequently, $\overline{o(y, F_k)} - B(x, \epsilon) \subset \{ y, F_k(y), \cdots, F_{(m-1)k}(y)\}$ is a finite set. Thus $\overline{o(y, F_k)} \neq X$, there exist $y^* \in X$ and $\lambda > 0$ such that $B(y^*, \lambda) \subset X - \overline{o(y, F_k)}$.

On the other hand, since x is a weak stable point of F, it is a weak stable point of F_k. Thus there exists $\xi > 0$ such that if $d(x, y) < \xi$ then $d(F_{kn_i}(x), F_{kn_i}(y)) < \frac{\lambda}{6}$ for $-T \leq i \leq T$. Due the topological transitivity of F_k, there is a point $\omega \in X$ such that $\overline{o(\omega, F_k)} = X$. Hence there exist $m, j \in \mathbb{N}$ with $-T \leq m - j \leq T$ such that

$$d(F_{kj}(\omega), x) < \xi, \quad d(F_{km}(\omega), y^*) < \frac{\lambda}{6}.$$

Therefore,

$$\begin{aligned} & d(F_{(m-j)k}(y), y^*) \leq d(F_{(m-j)k}(y), F_{(m-j)k}(x)) + d(F_{(m-j)k}(x), F_{(mk}(\omega)) + d(F_{(mk}(\omega), y^*) \\ & < \frac{\lambda}{6} + \frac{\lambda}{6} + \frac{\lambda}{6} < \lambda, \end{aligned}$$

which contradicts with $B(y^*, \lambda) \subset X - \overline{o(y, F_k)}$. This completes the proof. □

Theorem 3.6. *Let (X, F) be a non-autonomous dynamical system and A be a dense invariant subset of X. Then F is weak stable if and only if $F \mid_A$ is weak stable.*

Proof. It is obvious that the weak stability of F implies the same one of $F\mid_A$.

Conversely, assume that $F\mid_A$ is weak stable. Fix any $x^* \in X$. Due density of A and uniform continuity of F, for $\epsilon > 0$, there exists $\delta > 0$ such that if $z \in A \cap \mathrm{N}_\delta(x^*)$ then $d(F_n(x^*), F_n(z)) < \frac{\epsilon}{3}$. Since $F\mid_A$ is weak stable, there is $T \in \mathbb{N}$ such that $d(F_n(x^*), F_i(x^*)) < \frac{\epsilon}{3}$ for $i = -T, \cdots, T$.

Take any $y \in X$ with $d(x^*, y) < \delta$, thus

$$\begin{aligned} d(F_n(y), F_i(y)) &< d(F_n(y), F_n(x^*)) + d(F_n(x^*), F_i(x^*)) + d(F_i(x^*), F_i(y)) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

for $i = -T, \cdots, T$. Therefore F is weak chain continuous. This completes the proof. □

Theorem 3.7. *Let (X, F) be a non-autonomous dynamical system and $(\mathcal{K}(X), \overline{F})$ be its induced set-valued system. Then F is weak stable if and only if $\overline{F}$ is weak stable.*

Proof. Assume that F is weak stable. To prove that $\overline{F}$ is weak stable, by Theorem 3.6, it suffices to show that the weak stability of $\overline{F}$ on $\mathcal{K}_{\mathcal{F}}(X)$, as $\mathcal{K}_{\mathcal{F}}(X)$ is dense in $\mathcal{K}(X)$. Take $A = \{x_1, x_2, \cdots, x_k\} \in \mathcal{K}_{\mathcal{F}}(X)$. Since F is weak stable, for $\epsilon > 0$, there exist $\delta_j > 0$ and $T_j \in \mathbb{N}$ such that

$$d(F_n(y_j), F_i(y_j)) < \epsilon$$

for every $y_j \in X$ with $d(x_j, y_j) < \delta_j$, where $i = -T_j, \cdots, T_j$ and $j = 1, \cdots, k$.

Set $\delta = \max\{\delta_j\}$ and $T = \max\{T_j\}$. Let $B = \{y_1, y_2, \cdots, y_k\}$. Then $B \in \mathcal{K}_{\mathcal{F}}(X)$ satisfies the following

$$H_d(A, B) < \delta$$

and

$$H_d(\overline{F}_n(A), \overline{F}_i(B)) < \epsilon,$$

for $i = -T, \cdots, T$. It follows that $\overline{F}$ is weak stable.

Conversely, fix any $x \in X$. Then $\{x\} \in \mathcal{K}_{\mathcal{F}}(X)$. To prove F is weak stable, it is sufficient to observe that

$$d(x, y) = H_d(\{x\}, \{y\})$$

and

$$H_d(\overline{F}_n(\{y\}), \overline{F}_i(\{y\})) = d(F_n(y), F_i(y))$$

for every $y \in X$ with $d(x, y) < \delta$. This completes the proof. □

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