Asymptotically Exact Stabilization for Constrained Discrete Takagi-Sugeno Systems via Set-Invariance

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Abstract

Given a Takagi-Sugeno (TS) system, this paper proposes a novel methodology to obtain the state feedback controller guaranteeing, asymptotically as a Polya-related complexity parameter grows, the largest (membership-shape independent) possible domain-of-attraction with contraction-rate performance λ, based on polyhedral λ-contractive sets from constrained linear systems literature. The resulting controller is valid for any realisation of the memberships, as usual in most TS literature. For a finite complexity parameter, an inner estimate of such largest set is obtained; the frontier of of such approximation can be understood as the level set of a polyhedral control-Lyapunov function. Convergence of a proposed iterative algorithm is asymptotically necessary and sufficient for TS system stabilisation: for a high-enough value of the complexity parameter, any conceivable shape-independent Lyapunov controller design procedure will yield a proven domain of attraction smaller or equal to the algorithm's output.

Keywords: Fuzzy control, Invariant sets, Takagi-Sugeno models, Contractive sets, polyhedral Lyapunov functions

1. Introduction

A large class of nonlinear systems can be *exactly* expressed, locally in a *compact* region of interest (denoted as Ω in the sequel), as a fuzzy Takagi-Sugeno (TS) model, using the "sector nonlinearity" methodology [1, 2], embedding the nonlinearity into a convex time-varying combination of "vertex" linear equations, where the convex combination's coefficients, say μ, are usually denoted as membership functions.

Once these *locally exact* fuzzy models are available, model-based stability analysis and control design for such systems can be handled via some conditions on the vertex models; conditions which involve only the vertex models and disregard the actual "shape" of the memberships are called *shape-independent* [3]: they introduce some conservativeness, as shape-independent conditions refer to the "family" of systems sharing the same vertices, instead of the single nonlinear one which originated the TS model.

The most widespread approach to the above shape-independent stability and control design problems for TS systems are the Linear Matrix Inequality (LMI) results in literature [4, 5, 6, 7, 8].

If decay-rate performance is pursued, most of the above LMI results can be understood as finding a Lyapunov function such that $V(\lambda^{-1}x_{k+1}) \leq V(x_k)$, for a given value of the contraction rate λ (or optimising it via, say, bisection). The classes of controllers are called parallel distributed compensator (PDC) [1] if the controller is chosen as a combination of vertex actions sharing the same membership functions as the controlled plant; or non-PDC if other functions of the memberships are used [5]. Past and future memberships may be involved in the Lyapunov function and non-PDC controllers [9, 10, 11].

In most literature, once a feasible Lyapunov function is found, either quadratic $V(x) = x^T Px$ [12] or nonquadratic [5], the control problems are considered solved, and the proven stability domain is the largest level set $\{V(x) < V_c\}$ inside the region Ω. Actually, a slightly larger set is possible [13]; furthermore, the LMI solution $V(x)$ may be non-unique: so, the actual domain of attraction can be much larger than the Lyapunov level set. The developments in this paper will be also compared to the above-cited options considering delayed/future values of membership functions in nonquadratic Lyapunov functions.

Apart from state constraints arising from the local modelling region, control action saturation is also an important issue. LMI analysis of saturated controllers needs additional restrictions forcing non-saturation on a particular level set [1] or, for instance, iterative approaches [14]. Determining the largest stabilisable domain of attraction in a given region Ω via LMI under constraints remains basically unsolved: there are powerful results using polynomial-fuzzy Lyapunov functions and multi-sum controllers, but changes of variable render some steps conservative in controller synthesis and, also, maximum-volume formulae do not exist for non-quadratic level sets.

In robust (polytopic) linear control, the above problem has been successfully addressed based in set-invariance ideas, originating in the 70's [15], with later refinements [16, 17, 18, 19, 20]; extensions to switching/Markov setups appear in [21] and references therein. The relationship between robust and fuzzy approaches lies in the fact that condition $V(\lambda^{-1}x_{k+1}) \leq V(x_k)$ means that level sets of the Lyapunov functions are λ-contractive, in the sense introduced in [17].

The connection to fuzzy control systems hasn't been, however, exploited in literature to the author's knowledge. A first work in such direction appears in [22], and extending such results motivates the research presented in this paper.

The goal of this paper is studying stabilisation of *discrete-time* TS systems based on geometric set invariance considerations under affine state and control constraints, avoiding LMIs. Inspired on that idea, a prior paper [22] proposes using polytope-handling software to find the maximal (i.e., largest) λ-contractive set in Ω, for a given open-loop or closed-loop (being the controller fixed, *a priori*) fuzzy system, using an asymptotically exact algorithm. It is shown in such a paper that, by sheer definition, such a set will be larger than any level set obtained with a *shape-independent* Lyapunov approach. Algorithms from earlier polytopic system literature are adapted in the above-cited work to the multiple summations arising in closed-loop PDC fuzzy systems, by combining those results with the ones using Polya's theorem [6]. The above paper does exploit that information under state and input constraints but, however, considers only stability analysis of a *pre-existing* PDC controller.

The objective of this work is to extend the results in [22] to (possibly non-PDC) fuzzy controller *synthesis*, obtaining an estimate of the largest set inside a polytopic region of interest Ω in which there exists an admissible (i.e., within saturation limits) controller such that the set is made λ-contractive in closed loop. The paper improves on current shape-independent fuzzy LMI-based literature in several key aspects:

- A Lyapunov function is not needed (although a polyhedral one is obtained as a by-product), as the argumentation is purely based on set-invariance results.
- The algorithm is asymptotically exact, so given enough computing resources it would equal or beat any shape-independent Lyapunov result, by sheer definition of the maximal λ-contractive set.
- The controller structure can also be expanded so that it may approach any continuous non-PDC controller (using polynomials in memberships, which are "universal function approximators" in the unit simplex [23]).

Of course, it also improves over earlier robust-linear polytopic controllers using related approaches [16, 17], by the fact that the knowledge of the membership functions is actually exploited in fuzzy control systems.

There are three issues left out of the scope of this work: (a) for brevity, only a disturbance-free case is considered; extensions could be made in systems with additive disturbances adapting [24]; (b) there are other shape-*dependent* results [25, 26] whose output might be less conservative than the ones in this work; (c) although the results in this work would overcome any shape-independent result with enough computational resources, there may exist LMI results which obtain acceptable controllers in practice with less computational resources than those needed to match them via the proposals in this work.

The structure of the paper is as follows: Sections 2 and 3 state the goal of the paper and discuss preliminary definitions and results. Section 4 precisely defines shape-independent sets for fuzzy control systems. Section 5 details an algorithm for the computation of polytopic λ-contractive sets which can be proved to *asymptotically* obtain the *maximal* shape-independent λ-contractive set. Section 6 presents two different procedures to compute the control action: an online optimisation and an explicit offline solution. Further discussion and comparative analysis with prior literature appears in Section 7. Finally, some examples appear in Section 8, and a conclusion section closes the paper.

2. Problem statement

Consider a discrete-time nonlinear system:

$$x_{k+1} = f(x_k, u_k) \tag{1}$$

such that f has continuous partial derivatives and $f(0,0) = 0$, where $x_k \in \mathbb{R}^n$ represents the state vector and $u_k \in \mathbb{R}^m$ stands for the control actions at time instant k.

It is well known that such system can be equivalently expressed (*locally* in a compact region $\mathbb{X}$ of the state-space [1] containing the origin in its interior), as a TS fuzzy system with r rules or local models:

$$x_{k+1} = \tilde{f}(\mu(x_k), x_k, u_k) := \sum_{i=1}^{r} \mu_i(x_k)(A_i x_k + B_i u_k) \tag{2}$$

where A_i, B_i are the so-called consequent model matrices and $\mu_i : \mathbb{X} \mapsto [0, 1]$ represent membership functions, grouped for convenience onto a vector of membership functions, $\mu(x) := (\mu_1(x) \ \ldots \ \mu_r(x))^T$. Membership functions are defined in such a way so that, for any $x \in \mathbb{X}$, $\mu(x)$ belongs to the $(r-1)$-dimensional standard simplex $\Delta \subset \mathbb{R}^r$, defined as:

$$\Delta := \{\mu = (\mu_1, \ldots, \mu_r) \in \mathbb{R}^r \mid \sum_{i=1}^{r} \mu_i = 1, \quad \mu_i \geq 0 \ \ i : 1 \ldots r\} \tag{3}$$

Notation $\tilde{f}(\mu, x, u)$ is a shorthand for future developments; note that that $\tilde{f}$ is linear in μ and, (separately) in (x_k, u_k). Also, when memberships in several instants of time are involved, notation $h_i \in \Delta$, $h_i := \mu(x_{k+i})$ will be used. The TS model of a nonlinear system is not unique, so different TS models yield different performance values in subsequent steps; an initial approach to optimal modelling appears in [2]; also, sector nonlinearity can be extended to a polynomial case [27]. These issues, however, are out of the scope of this work.

The problem this paper aims to solve is the determination of a fuzzy control law which stabilizes the TS system (2) in the largest possible subset of a polytopic region Ω, with $\Omega \subset \mathbb{X}$. Such controller design procedure must be understood as finding a set of valid initial conditions $\mathcal{C}^\lambda$ and a feedback law $u(x, \mu)$ which ensures that $x_k \in \Omega$ for all $k \geq 0$, and $\lim_{k\to\infty} x_k = 0$ if $x_0 \in \mathcal{C}^\lambda$, while fulfilling control constraints $u(x, \mu(x)) \in \mathbb{U}$ for all $x \in \Omega$, for all possible shapes of the membership $\mu(x)$ as long as $\mu(x) \in \Delta$ (shape-independent stabilisation). Coefficient λ will be related to a "contraction-rate" performance measure. The formal meaning of shape-independent stabilisation with contraction rate λ will be made clear later in Section 4.

By assumption, modelling region $\mathbb{X}$ and input constraint set $\mathbb{U}$ will be compact, convex, polytopes, containing the origin. So, they can be defined by affine constraints, expressed as vector inequalities:

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid Rx \leq l\} \tag{4}$$

$$\mathbb{U} = \{u \in \mathbb{R}^m \mid Su \leq s\} \tag{5}$$

being R, S matrices and l, s vectors with compatible dimensions, with vector inequalities to be understood as element-wise; abusing the notation, a scalar at the right-hand side of an inequality should be understood as affecting each of the rows at the left-hand side.

Actually, in most cases of practical interest Ω will be intentionally set to be equal to the modelling region $\mathbb{X}$, but the developments in this work do not necessarily require so from a theoretical point of view.

3. Preliminary definitions and results

Given an arbitrary set Ω, notation $\lambda\Omega$ will denote the linear scaling of the set Ω by $\lambda \geq 0$. If Ω is defined as $\Omega := \{x \in \mathbb{R}^n : M(x) \leq 0\}$, for an arbitrary vector of constraint functions $M(\cdot)$, the scaled set is $\lambda\Omega := \{x : M(\lambda^{-1}x) \leq 0\}$.

Definition 1 ([17]) *A set $\Omega \subset \mathbb{X}$ is* control λ-contractive *(given $0 \leq \lambda \leq 1$) for the system* (1) *if and only if, for any x in Ω there exists an admissible input such that the successor state lies in $\lambda\Omega$, i.e., if $x \in \Omega \Rightarrow \exists u \in \mathbb{U} : f(x, u) \in \lambda\Omega$.*

Obviously, u above might be non-unique, and, too, the set of feasible u depends on x, denoted as $U_\Omega(x) := \{u \in \mathbb{U} \,|\, f(x,u) \in \Omega\}$. If $\mathbb{U}$ is a polytope, and $f(x,u)$ is *affine* in control, i.e., $f(x,u) = \bar{f}(x) + \bar{g}(x)u$, then $U_\Omega(x)$ is a polytope, too. Trivially, a contracting state-feedback controller $u(x)$ can be implemented by any arbitrary selection from the set-valued map $U_\Omega(x)$; however, additional hypothesis are needed on $\mathbb{U}$, Ω and f so that there exists a *continuous* selection $u(x)$ [28]. The scalar λ will be denoted as geometric contraction rate. *Decay-rate* stability requires contraction at all future time, requiring suitable Lyapunov functions:

Definition 2 *A convex function $V(x)$ such that $V(0) = 0$ is a (local) control Lyapunov function (CLF) ensuring geometric contraction rate λ for system* (1) *if there exists a convex set $\Omega \subset \mathbb{X}$ including the origin in which, for all $x \in \Omega \sim \{0\}$, $V(x) > 0$ and there exists $u \in \mathbb{U}$ such that $V(\lambda^{-1} f(x,u)) \leq V(x)$.*

The above definition is an adaptation to the discrete-time and contraction-rate setting of well-known concepts defined in, for instance, [29] for continuous-time stabilization. The motivation of the definition is the fact that V is a Lyapunov function as, by convexity, $V(f(x,u)) \leq \lambda V(\lambda^{-1} f(x,u)) \leq \lambda V(x)$ and the level sets are control λ-contractive, as a level set $\Omega := \{V(x) \leq \gamma\}$ scaled would be $\lambda\Omega = \{V(\lambda^{-1}x) \leq \gamma\}$, so if $x \in \Omega$, next state $f(x,u)$ will lie in $\lambda\Omega$ for some u. If $f(x,u)$ were linear, condition in Definition 2 could be stated as $V(f(x,u)) \leq V(\lambda x)$, because $V(f(x,u)) = V(\lambda^{-1} f(\lambda x, \lambda u)) \leq V(\lambda x)$; this fact will be latter used in Lemma 4.

In many common cases, $V(x)$ is a homogeneous degree-q polynomial in x, then $V(\lambda^{-1}x) = \lambda^{-q} V(x)$; standard discrete decay-rate formulas $V(\lambda^{-2} x_{k+1}) \leq V(x_k)$ arise with, for instance, $q = 2$. In the homogeneous case, we have

$$V(\lambda^{-k} x_k) = \lambda^{-(k-1)q} V(\lambda^{-1} x_k) \leq \lambda^{-(k-1)q} V(x_{k-1}) = V(\lambda^{-(k-1)} x_{k-1})$$

so, by induction, we can easily prove $V(\lambda^{-k} x_k) \leq V(x_0)$ or, equivalently, by multiplication by λ^{kq} we get $V(x_k) \leq V(\lambda^k x_0)$.

Definition 3 (Maximal control λ-contractive Set) *A set, to be denoted as $\mathcal{C}_\infty^\lambda(\Omega)$, is the* maximal *control λ-contractive set contained in a region Ω for the system $x_{k+1} = f(x_k, u_k)$ if and only if $\mathcal{C}_\infty^\lambda(\Omega)$ is control λ-contractive and contains all the control λ-contractive sets contained in Ω.*

The following result is evident from the sheer definition of maximality and the above discussion.

Corollary 1 *Any level set of a local CLF in $\mathbb{X}$ ensuring contration rate λ is a subset of the maximal control λ-contractive set in $\mathbb{X}$.*

Proof: Evident, because of the above-mentioned fact that the referred level sets are control λ-contractive and all such sets are subsets of the maximal one. ∎

In the particular case of $\lambda = 1$, a control λ-contractive set is also denoted in literature [17] as *control invariant* set, and the maximal control λ-contractive set is denoted as the *maximal control invariant set* $\mathcal{C}_\infty(\Omega)$. Generalisations of these invariance concepts to nonlinear disturbed systems have also been proposed, but they may require conservative BMI manipulations, see [30].

Definition 4 *Given an arbitrary target set* Ω*, the one-step set* $\mathcal{Q}(\Omega)$ *is the set of states* x *in* $\mathbb{X}$ *from which the next state of system* (1) *can be driven to* Ω *with an admissible* $u \in \mathbb{U}$*, i.e.,*

$$\mathcal{Q}(\Omega) := \{x \in \mathbb{X} | \exists u \in \mathbb{U} : f(x,u) \in \Omega\}$$

Note that $x \in \mathcal{Q}(\Omega)$ iff $U_\Omega(x) \neq \emptyset$. Also, Definition 1 could be rewritten saying that Ω is control λ-contractive iff $\Omega \subset Q(\lambda\Omega)$.

Definition 5 ([16]) *The so-called i-step set* $C_i^\lambda(\Omega)$ *is recursively defined, starting with* $\mathcal{C}_0^\lambda(\Omega) := \Omega$ *as* $\mathcal{C}_{i+1}^\lambda(\Omega) := \mathcal{Q}\left(\lambda\mathcal{C}_i^\lambda(\Omega)\right) \cap \Omega$*, for* $i \geq 0$*.*

If there exists a finite i such that $\mathcal{C}_{i+1}^\lambda(\Omega) = \mathcal{C}_i^\lambda(\Omega)$, it can be proved [17] that $\mathcal{C}_i^\lambda(\Omega)$ is the maximal one in Definition 3. Such set will be denoted as $\mathcal{C}_\infty^\lambda(\Omega)$. Also, in case such finite i does not exist, but there exists $\mathcal{C}_\infty^\lambda$, for any $1 \geq \lambda^* > \lambda$, there exist a finite i^* such that $\mathcal{C}_i^\lambda$ is control λ^*-contractive for all $i \geq i^*$, albeit possibly non-maximal [31, Theorem 3.2].

Efficient computational characterisation of the one-step set $\mathcal{Q}$ in Definition 4 can only be easily carried out for special cases of f; for instance, the linear case [17]. Actually, extending the idea to the TS case is the main motivation of this work.

In order to do that, we recall Polya's theorem, which is a key tool for the results presented in the Takagi-Sugeno controller synthesis in later sections.

Theorem 1 (Polya) *[32] If a real homogeneous polynomial* $F(\mu_1, \ldots, \mu_r)$*,* $F : \Delta \mapsto \mathbb{R}$*, is (strictly) positive in the simplex* Δ*, then there exists a sufficiently large* $d \geq 0$ *such that all the coefficients of the polynomial* $(\mu_1 + \cdots + \mu_r)^d F(\mu_1, \ldots, \mu_r)$ *are positive.*

4. Shape-Independent one-step and λ-contractive sets for fuzzy control systems

In order to obtain the i-step sets in Definition 5, iterative computation of the one-step set in Definition 4 is needed. For a TS system, such set is:

$$\mathcal{Q}(\Omega) = \{x \in \mathbb{X} \,|\, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i(x)(A_i x + B_i u) \in \Omega\} \tag{6}$$

The shape of $\mathcal{Q}(\Omega)$ may be very hard to compute, due to the nonlinearities in the membership functions $\mu_i(x)$. Indeed, determining if a particular x belongs to $\mathcal{Q}(\Omega)$, for convex Ω, is computationally simple as $\tilde{f}(\mu(x), x, u)$ is, actually, an affine function[1] of u; however, the difficulty lies in determining an *explicit* expression for the boundary of $\mathcal{Q}(\Omega)$ needed for the iterations in Definition 5.

A reasonable approach, in order to deal with this drawback, is disregarding the information about the actual value of the membership functions, dealing with the Takagi-Sugeno model for *any* possible value of μ_i –assumed known to the controller, as done in most TS literature (i.e., a shape-independent analysis [3])–. Hence, the one-step set in Definition 4 should be *replaced* by the one below:

[1]For instance, for fixed x, $\mu(x)$, if Ω is a polytope, the problem is a linear programming feasibility one.

Definition 6 *The* shape-independent one-step set *of a TS system* (2) *is*

$$Q_{si}(\Omega) := \{x \in \mathbb{X} \,|\, \forall \mu \in \Delta\, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i \,(A_i x + B_i u) \in \Omega\} \tag{7}$$

The definition ensures that for each $(x, \mu) \in \mathcal{Q}_{si}(\Omega) \times \Delta$ there exists a non-empty set of fuzzy (i.e., membership-dependent) control actions defined as:

$$U_\Omega(x, \mu) := \{u \in \mathbb{U} \,|\, \tilde{f}(\mu, x, u) \in \Omega\} \tag{8}$$

If Ω is polyhedral, the set $U_\Omega(x, \mu)$ is itself a polytope, for fixed x and μ; optimisation problems on $U_\Omega(x, \mu)$ will be discussed in Section 6. Unfortunately, exact computation of $\mathcal{Q}_{si}$ is still cumbersome, due to the nonlinearities involving products of μ_i with x and u.

Let us show that $\mathcal{Q}_{si}(\Omega) \subset \mathcal{Q}(\Omega)$. Indeed,

$$\mathcal{Q}(\Omega) = \{x \in \mathbb{X} \,|\, \text{for } \mu \equiv \mu(x)\, \exists u \in \mathbb{U} : \sum_{i=1}^{r} \mu_i \,(A_i x + B_i u) \in \Omega\} \supset \mathcal{Q}_{si}(\Omega) \tag{9}$$

as the conditions in the left-hand side of (9) involve only the single point $\mu(x)$, instead of the whole simplex in (7).

Any function $u(x, \mu)$, $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$, so that $u(x, \mu) \in U_\Omega(x, \mu)$ would be a valid fuzzy state-feedback control law to steer any state in $\mathcal{Q}_{si}(\Omega)$ to Ω in one step applying $u(x, \mu(x))$, valid for any actual shape of $\mu(x)$. Although there might be many options, the referred controller $u(x, \mu)$ can be selected to be *continuous*, which will be important for later developments:

Lemma 1 *Let us assume* Ω *is described by* $\Omega := \{x : g(x) \leq 0\}$ *with* g *being a vector of affine functions (polytopic* Ω*). Then, there exists a* continuous *function* $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$*, such that* $\tilde{f}(\mu, x, u(x, \mu)) \in \Omega$.

Proof: The proof follows an argumentation analogous to the linear case in [33, Proposition 3.2]. In this case, $U_\Omega(x, \mu) = \{u \in \mathbb{U} \,|\, g(\tilde{f}(\mu, x, u)) \leq 0\}$ can be understood as a *set-valued* map. Convexity of $\mathbb{U}$, plus $g \circ f$ being affine in u (for fixed μ and x), ensure $U_\Omega(x, \mu)$ is a closed convex set for all $(x, \mu) \in \mathcal{Q}_{si}(\Omega) \times \Delta$. Also, it is a set-valued map which can be proved to be continuous (because, again, $g \circ f$ is continuous). The classical Michael's *convex selection* theorem [28, Theorem 3.2] implies that a continuous selection $u : \mathcal{Q}_{si}(\Omega) \times \Delta \mapsto \mathbb{U}$ exists. ∎

A shape-independent definition of λ-contractiveness for TS systems is now presented:

Definition 7 *Given* $0 \leq \lambda \leq 1$*, a set* $\Omega \subset \mathbb{X}$ *is* shape-independent control λ-contractive *for the system* (2) *if and only if, for any* (x, μ) *in* $\Omega \times \Delta$ *there exists an admissible (possible non-unique) input* $u \in \mathbb{U}$ *such that* $\tilde{f}(\mu, x, u) \in \lambda\Omega$*; equivalently, iff* $\Omega \subset \mathcal{Q}_{si}(\lambda\Omega)$*. Given a region* $\mathbb{X}$*, a shape-independent control* λ*-contractive set* Ω *is* maximal *if any other shape-independent control* λ*-contractive set in* $\mathbb{X}$ *is contained in* Ω*.*

As $Q_{si}(\lambda\Omega) \subset Q(\lambda\Omega)$, any shape-independent λ-contractive sets are also λ-contractive sets for the system (1) from which the TS model came from, as $\Omega \subset Q_{si}(\lambda\Omega) \subset Q(\lambda\Omega)$. So, studying shape-independent control λ-contractive sets is a way to guarantee similar contraction properties for nonlinear systems; of course this is, actually, the leitmotif of most TS fuzzy control developments.

Basically, the generic goal of shape-independent fuzzy controllers (designed with a contraction performance objective in mind) should be approaching the above maximal shape-independent set: indeed, no algorithm can prove a larger set by definition. The results in this paper will present a constructive procedure to generate a family of λ-contractive sets wich approach the maximal one with increasing accuracy.

Proposition 1 *If Ω is shape-independent control λ-contractive for the TS system* (2)*, so it is its convex hull $Co(\Omega)$. Thus, the maximal shape-independent control λ-contractive set is convex.*

Proof: Consider any x_1, x_2 in Ω, such that u_1 and u_2 make $\tilde{f}(\mu, x_1, u_1)$ and $\tilde{f}(\mu, x_2, u_2)$ belong to $\lambda\Omega$ for any $\mu \in \Delta$, respectively. Then $\alpha x_1 + (1 - \alpha)x_2 \in Co(\Omega)$ and, subsequently, $\tilde{f}(\mu, \alpha x_1 + (1 - \alpha)x_2, \alpha u_1 + (1 - \alpha)u_2) = \alpha\tilde{f}(\mu, x_1, u_1) + (1 - \alpha)\tilde{f}(\mu, x_2, u_2) \in \lambda Co(\Omega)$. ∎

Proposition 2 *If Ω is shape-independent control λ-contractive for the TS system* (2)*, then any linear scaling $\gamma\Omega$, with $0 < \gamma \leq 1$, is shape-independent control λ-contractive, too.*

Proof: Considering $x \in \gamma\Omega$, with any arbitrary $\gamma \leq 1$. Then, as $\gamma\Omega \subset \Omega$, for any $(x, \mu) \in \gamma\Omega \times \Delta$ there exists u such that $\tilde{f}(\mu, \gamma^{-1}x, u) \in \lambda\Omega$, because $\gamma^{-1}x \in \Omega$. Linearity of $\tilde{f}$ in 2nd and 3rd arguments allows to state that:

$$\tilde{f}(\mu, \gamma^{-1}x, u) = \gamma^{-1}\tilde{f}(\mu, x, \gamma u) \in \lambda\Omega$$

hence, $\tilde{f}(\mu, x, \gamma u) \in \lambda(\gamma\Omega)$. So, the control $\gamma u \in \mathbb{U}$ drives $x \in \gamma\Omega$ to $\lambda(\gamma\Omega)$. ∎

Hence, as shape-independent control λ-contractive sets are control λ-contractive, the following well-known result and Proposition 2 can be joined to induce a control Lyapunov function, if a shape-independent control λ-contractive set is found:

Proposition 3 ([19]) *Consider $\Omega = \{x \in \mathbb{R}^n | \max_{1 \leq i \leq n_h} H_i x \leq 1\}$. If $\gamma\Omega$ is control λ-contractive, for the TS system* (2)*, for all $0 \leq \gamma$ such that $\gamma\Omega \in \mathbb{X}$ then*

$$V(x) := \max_{1 \leq i \leq n_h} (H_i x) \tag{10}$$

is a control Lyapunov function ensuring contraction rate λ.

The nesting of contractive sets in Proposition 2 allows, too, the following corollary to be stated (proof omitted for brevity):

Corollary 2 *A set Ω is shape-independent control λ-contractive for a TS system, if and only if, for any $x_0 \in \Omega$, for any membership sequence $(h_0, h_1, \ldots, h_{k-1}) \in \Delta^k$, there exists a control law $u(x, \mu)$, with $\mu = h_k$ at time k, such that $x_k = \tilde{f}(h_{k-1}, x_{k-1}, u(x_{k-1}, h_{k-1})) \in \lambda^k\Omega$, i.e., any initial state in it converges to the origin with a geometric contraction rate λ.*

Modifying the iterations in Definition 5, the following result can be stated:

Lemma 2 *For $\lambda < 1$, the maximal shape-independent control λ-contractive set in a region $\Omega \subset \mathbb{X}$ would be obtained if the iteration*

$$\bar{\mathcal{C}}_{i+1}^{\lambda}(\Omega) = \mathcal{Q}_{si}\left(\lambda \bar{\mathcal{C}}_i^{\lambda}(\Omega)\right) \cap \Omega,$$

initialised with $\bar{\mathcal{C}}_0^{\lambda}(\Omega) = \Omega$, converges in a finite number of steps, i.e., $\bar{\mathcal{C}}_{\infty}^{\lambda}(\Omega) := \bar{\mathcal{C}}_{i+1}^{\lambda}(\Omega) = \bar{\mathcal{C}}_i^{\lambda}(\Omega)$ for some finite i. The set $\bar{\mathcal{C}}_i^{\lambda}(\Omega)$ will be denoted as i-step shape-independent set.

Proof: The proof comprises three steps:

1. First, the fact that $\bar{\mathcal{C}}_i^{\lambda}$ is shape-independent control λ-contractive if $\bar{\mathcal{C}}_i^{\lambda} = \bar{\mathcal{C}}_{i+1}^{\lambda}$. Indeed, for any i, if there exists a state $x \in \bar{\mathcal{C}}_i^{\lambda}$ and a membership $\mu \in \Delta$ such that $\tilde{f}(\mu, x, u)$ cannot be steered to $\lambda \bar{\mathcal{C}}_i^{\lambda}$ with an admissible u, then such state will *not* belong to $\bar{\mathcal{C}}_{i+1}^{\lambda}$. Hence, convergence will not occur until such x does not exist.
2. Second, let us prove that no point $x \in \Omega$, $x \notin \bar{\mathcal{C}}_i^{\lambda}$ can be steered to $\lambda \bar{\mathcal{C}}_{\infty}^{\lambda}$ for all μ: if there existed x which could be steered to $\lambda \bar{\mathcal{C}}_i^{\lambda}$ such point would belong to $\bar{\mathcal{C}}_{i+1}^{\lambda}$; again, convergence cannot happen until no such x exists.
3. Finally, as $\lambda \bar{\mathcal{C}}_{\infty}^{\lambda}$ contains the origin, and $\lambda < 1$, any stabilising trajectory should eventually enter $\lambda \bar{\mathcal{C}}_{\infty}^{\lambda}$ (Corollary 2). However, the above second assertion states that for states outside $\bar{\mathcal{C}}_{\infty}$ there exists at least one value of membership for which entering $\lambda \bar{\mathcal{C}}_{\infty}^{\lambda}$ is impossible. Hence, no larger shape-independent control λ-contractive set exists. ∎

Given a nonlinear system, the set $\bar{\mathcal{C}}_{\infty}^{\lambda}$ obtained from a TS model of it is a subset of the "true" $\mathcal{C}_{\infty}^{\lambda}$ discussed in Section 3, due to the inherent conservatism of shape-independent TS analysis [3].

5. Inner approximation of shape-independent control λ-contractive sets for TS systems

The above shape-independent sets need choosing a particular controller parametrisation $u(x, \mu)$ in order to be computable whith available computational geometry software such as Multi-Parametric Toolbox (MPT) [34]. This is the topic of this section.

The simplest approximation is choosing u not depending on memberships. Indeed, let us consider:

$$\mathcal{Q}_{si}^{0}(\Omega) := \{x \in \mathbb{X} \,|\, \exists u \in \mathbb{U} : A_i x + B_i u \in \Omega \;\; \forall\, i = 1 \ldots r\} \tag{11}$$

The above expression comes from plugging a membership-independent $u(x, \mu) := u(x)$ into (7) and considering that $\sum_{i=1}^{r} \mu_i (A_i x + B_i u) \in \Omega$ if and only if $A_i x + B_i u \in \Omega$ for all i. Obviously, $\mathcal{Q}_{si}^{0}(\Omega) \subset \mathcal{Q}_{si}(\Omega)$.

In fact, the set $\mathcal{Q}_{si}^{0}(\Omega)$ is the *robust* one-step set in uncertain polytopic systems literature [35]: its main drawback is its conservativeness coming from the fact that, for a given state, the control action should be the same for any value of the membership functions.

5.1. Fuzzy controllers (single-sum)

Given that $\mu_i(x_k)$ are actually known, a clear improvement is defining a so-called parallel distributed control parametrisation in the form:

$$u(x) = \sum_{j=1}^{r} \mu_j(x) u_j(x) \tag{12}$$

which defines a different "vertex controller" $u_j(x)$ for each model. This well-known formula is, of course, the key idea behind "fuzzy" controllers since the 1990s, building a "complicated" nonlinear controller from "simpler" components $u_j(x)$; in particular, later results in Section 6 will provide design methods for vertex controllers based on convex optimisation, either numerical (once x and μ are measured) or as piecewise affine state-feedback laws.

The closed-loop system with the parametrisation (12) can be written as

$$x_{k+1} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_k)\mu_j(x_k)(A_i x_k + B_i u_j(x_k)) \tag{13}$$

Let us introduce the augmented notation

$$\bar{u}(x) = \begin{pmatrix} u_1(x) \\ \vdots \\ u_r(x) \end{pmatrix}, \quad E_j = [0_{m\times m} 0_{m\times m} \dots I_{m\times m} \dots 0_{m\times m}] \tag{14}$$

being E_j an $m \times (mr)$ matrix with an identity matrix in the j-th block position, for $1 \le j \le r$. In this way, we have $u_j(x_k) = E_j \bar{u}(x_k)$, so the closed-loop system can be written as the augmented-input one:

$$x_{k+1} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_k)\mu_j(x_k)(A_i x_k + B_i E_j \bar{u}(x_k)) \tag{15}$$

where the new input is a vector of length $r \times m$. In this case, disregarding again the fact that memberships depend on state, the shape-independent one-step set of system (15), to be denoted as $\mathcal{Q}^1_{si}(\Omega)$, is readily expressed as:

$$\mathcal{Q}^1_{si}(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r, \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i \mu_j (A_i x + B_i E_j \bar{u}) \in \Omega \quad \forall \mu \in \Delta \right\} \tag{16}$$

where $\bar{u}$ is understood as a length-r vector whose elements belong to $\mathbb{U}$; such elements must be the same for all membership values but might be different for different states; in an analogous way to (8), a set $\bar{U}_\Omega(x)$ could be suitably defined, and a continuous $\bar{u}(x) : \mathcal{Q}^1_{si}(\Omega) \mapsto \mathbb{U}^r$ can be proven to exist[2], thus justifying the chosen parametrisation (12), obtained from $\bar{u}(x)$ by reverting back the vertical stacking to a fuzzy summation.

Now, $\mathcal{Q}^0_{si}(\Omega) \subseteq \mathcal{Q}^1_{si}(\Omega) \subset \mathcal{Q}_{si}(\Omega)$ because forcing all u_j to be equal converts (16) into the particular case (11) and, on the other hand, the parametrisation

[2]The proof in this case would be analogous to Lemma 1, adding the fact that the infinite intersection of closed convex sets is itself also closed and convex; details omitted for brevity.

of the underlying $u(x,\mu)$ in (7) is generic, not restricted to being linear as (12) postulates. Notation $\mathcal{Q}^1_{si}$ is used to emphasise that the candidate controller is a polynomial of degree 1 in the memberships. More general controller parametrisations will be discussed in Section 5.3.

5.2. Asymptotically exact polytopic inner approximation of $\mathcal{Q}^1_{si}(\Omega)$

On the sequel, we will assume that Ω is a polytope defined as

$$\Omega := \{x \,|\, R_\Omega x \leq l_\Omega\} \tag{17}$$

for some matrices R_Ω and vector l_Ω. As $\sum_{i=1}^r \mu_i = 1$, $\mathcal{Q}^1_{si}(\Omega)$ can be expressed as

$$\mathcal{Q}^1_{si}(\Omega) = \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r,\ \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \left(R_\Omega (A_i x + B_i E_j \bar{u}) - l_\Omega \right) \leq 0 \quad \forall \mu \in \Delta \right\} \tag{18}$$

The main issue regarding $\mathcal{Q}^1_{si}(\Omega)$ in (16) is the fact that a double-fuzzy summation [36] appears in its expressions, so necessary and sufficient conditions for computing (18) cannot be stated in a convex form. So, relaxations of the double sum are needed. This kind of problems have been widely studied in the field of copositive programming and LMI control for TS systems [36, 37].

The goal of this section is adapting the procedures in the referred works, based on Polya's theorem (here recalled as Theorem 1), to the problem of computing approximations to $\mathcal{Q}^1_{si}(\Omega)$. In order to do that, the notation for d-dimensional indices in [6, 36, 22] will be used:

$$\mathbf{i} = (i_1, i_2, \ldots i_d), \quad \mathbb{I}_d = \{1, \ldots, r\}^d, \quad \mathbb{I}^+_d = \{\mathbf{i} \in \mathbb{I}_d | i_s \leq i_{s+1},\ s = 1, \ldots, d-1\}$$

so $\mathbb{I}^+_d$ indexes all the different monomials $\mu_\mathbf{i}$ of an homogeneous degree-d polynomial (taking into account commutativity). For instance, for $d = 3$ and $r = 2$ we can define $\mathbb{I}^+_3 = \{111, 112, 122, 222\}$ – with some abuse of notation shorthanding $(1,1,1)$ as 111, etc.

Notation $n_\mathbf{i}$ will denote the number of elements of $perm(\mathbf{i})$, being $perm(\mathbf{i})$ the set of permutations of an element of $\mathbb{I}^+_d$ in $\mathbb{I}_d$. In the above case $n_{111} = n_{222} = 1$, $n_{112} = n_{221} = 3$ (because $perm(111) = \{111\}$, $perm(112) = \{112, 121, 211\}$, $\ldots$).

Denoting as $\mu_\mathbf{i} := \mu_{i_1}\mu_{i_2}\ldots\mu_{i_d}$, the following identities are straightforward:

$$\sum_{i_1=1}^r \sum_{i_2=1}^r \cdots \sum_{i_d=1}^r \mu_{i_1}\mu_{i_2}\ldots\mu_{i_d} = \left(\sum_{i=1}^r \mu_i\right)^d = \sum_{\mathbf{i}\in\mathbb{I}_d} \mu_\mathbf{i} = \sum_{\mathbf{i}\in\mathbb{I}^+_d} n_\mathbf{i}\mu_\mathbf{i} = 1 \tag{19}$$

because $\mu_\mathbf{i} = \mu_\mathbf{j}$ if $\mathbf{j} \in perm(\mathbf{i})$. The reader is referred to [6, 36, 22] for further details on the multiindex notation and relevant properties.

Continuing with the example with $d = 3$ and $r = 2$, we have

$$1 = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \sum_{i_3=1}^2 \mu_{i_1}\mu_{i_2}\mu_{i_3} = \underbrace{\mu_1^3}_{\mu_{111}} + \underbrace{3}_{n_{112}}\underbrace{\mu_1^2\mu_2}_{\mu_{112}} + \underbrace{3}_{n_{122}}\underbrace{\mu_1\mu_2^2}_{\mu_{122}} + \underbrace{\mu_2^3}_{\mu_{222}} = \sum_{\mathbf{i}\in\mathbb{I}^+_3} n_\mathbf{i}\mu_\mathbf{i}$$

.

In order to apply Theorem 1, as $\left(\sum_{i=1}^r \mu_i\right)^{d-2} = 1$ for any d, we rewrite equation (15), denoting $\bar{u}(x_k)$ with shorthand $\bar{u}_k$, as

$$x_{k+1} = \left(\sum_{i=1}^{r} \mu_i\right)^{d-2} \sum_{i=1}^{r}\sum_{j=1}^{r} \mu_i\mu_j(A_i x_k + B_i E_j \bar{u}_k) =$$
$$= \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} \dots \sum_{i_d=1}^{r} \mu_{i_1}\mu_{i_2}\dots\mu_{i_d}(A_{i_1}x_k + B_{i_1}E_{i_2}\bar{u}_k) \quad (20)$$

Denoting $G_{i_1 i_2} = [A_{i_1} \quad B_{i_1}E_{i_2}]$, and reordering the terms of the summation (20) we get

$$x_{k+1} = \sum_{i_1=1}^{r}\sum_{i_2=1}^{r} \dots \sum_{i_d=1}^{r} \mu_{i_1}\mu_{i_2}\dots\mu_{i_d} G_{i_1 i_2} \begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{I}_d} \mu_{\mathbf{i}} G_{i_1 i_2} \begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} =$$
$$= \sum_{\mathbf{i}\in\mathbb{I}_d^+} \mu_{\mathbf{i}} \left(\sum_{\mathbf{j}\in perm(\mathbf{i})} G_{j_1 j_2} \right) \begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} = \sum_{\mathbf{i}\in\mathbb{I}_d^+} \mu_{\mathbf{i}} n_{\mathbf{i}} \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x_k \\ \bar{u}_k \end{bmatrix} \quad (21)$$

being $\widetilde{G}_{\mathbf{i}}$ the average values of $G_{i_1 i_2}$ over all permutations of a particular ordered multidimensional index, i.e.;

$$\widetilde{G}_{\mathbf{i}} := \frac{1}{n_{\mathbf{i}}} \cdot \sum_{\mathbf{j}\in perm(\mathbf{i})} G_{j_1 j_2} \quad (22)$$

For instance, in the above case $d = 3$, $r = 2$, we would get:

$$\widetilde{G}_{111} = G_{11}, \quad \widetilde{G}_{112} = \frac{1}{3}(G_{11} + G_{12} + G_{21}),$$
$$\widetilde{G}_{122} = \frac{1}{3}(G_{12} + G_{21} + G_{22}), \quad \widetilde{G}_{222} = G_{22} \quad (23)$$

With the new equivalent expression (21) of the system dynamics, the one step set (of course, identical to that in (16), as (21) is a mere rewriting of (15) in the new notation for any $d \geq 2$), can be written as

$$\mathcal{Q}_{si}^1(\Omega) = \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r : \sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}\widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \ \forall \mu \in \Delta \right\} \quad (24)$$

Now, if Ω were a polytope (17), from (19) we can cast the evident equivalence:

$$R_\Omega \left(\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}\widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \right) \leq l_\Omega \ \Leftrightarrow \ R_\Omega \left(\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}}\widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \right) \leq \left(\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}} \right) l_\Omega \quad (25)$$

Now, the right-hand side inequality is actually equivalent to:

$$\sum_{\mathbf{i}\in\mathbb{I}_d^+} n_{\mathbf{i}}\mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \leq 0 \quad (26)$$

so, as $n_{\mathbf{i}}\mu_{\mathbf{i}}$ are all non-negative, we can assert that existence of $\bar{u}$ such that

$$R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \leq l_\Omega \quad (27)$$

is a sufficient condition for $x \in \mathcal{Q}^1_{si}(\Omega)$. So, it becomes clear that a sufficient condition for a given point to belong to $\mathcal{Q}^1_{si}(\Omega)$ is that it belongs to the polytopic complexity-d subset arising from the inequality in (27), denoted as:

$$\widetilde{\mathcal{Q}}^1_d(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^r : \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \quad \forall \mathbf{i} \in \mathbb{I}^+_d \right\} \tag{28}$$

because, indeed, the above argumentation ensures $\widetilde{\mathcal{Q}}^1_d(\Omega) \subset \mathcal{Q}^1_{si}(\Omega)$.

It can be proved, following the asymptotic exactness results derived from Polya argumentations [32], that the polytopic set $\widetilde{\mathcal{Q}}^1_d(\Omega)$ will tend to $\mathcal{Q}^1_{si}(\Omega)$ as the complexity parameter d tends to infinity. This is done in the lemma below:

Lemma 3 *If x belongs to the interior of $\mathcal{Q}^1_{si}(\Omega)$, for a polytopic Ω expressed as* (17)*, there exists a finite d such that $x \in \widetilde{\mathcal{Q}}^1_d(\Omega)$.*

Proof: Indeed, for x in the interior of $\mathcal{Q}^1_{si}(\Omega)$, using the original definition (15), i.e., $d = 2$, there exists $\gamma < 0$ such that

$$\sum_{\mathbf{i} \in \mathbb{I}^+_2} n_{\mathbf{i}} \mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{i_1 i_2} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \leq \gamma < 0 \tag{29}$$

So standard Polya-argumentations [32, 36] show that there is a finite d such that, expanding (29) in the same way as done in (21), i.e.,

$$\sum_{\mathbf{i} \in \mathbb{I}^+_d} n_{\mathbf{i}} \mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) = \sum_{\mathbf{i} \in \mathbb{I}^+_2} n_{\mathbf{i}} \mu_{\mathbf{i}} \left(R_\Omega \widetilde{G}_{i_1 i_2} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} - l_\Omega \right) \tag{30}$$

results in a d-th degree homogeneous polynomial in $\mu_{\mathbf{i}}$ at the left-hand side of the equation such that the polynomial coefficients will be all non-positive. Requiring non-positiveness of all such coefficients is, actually, what (28) states once matrices R_Ω, l_Ω defining the shape of Ω are plugged in. ■

5.3. Extension to multiple-parametrization controllers.

A more flexible controller parametrisation can be set up as a c-dimensional fuzzy summation, being $c > 1$ an arbitrarily chosen integer, as suggested in, for instance, [6]:

$$u(x) = u^c(x, \mu) := \sum_{\mathbf{i} \in \mathbb{I}^+_c} n_{\mathbf{i}} \mu_{\mathbf{i}}(x) u_{i_1 i_2 \ldots i_c}(x) \tag{31}$$

For each state, this fuzzy control parametrisation is a degree-c homogeneous polynomial in the memberships.

Lemma 1 ensures that, for a fixed x, there exists a control function which is continuous in the memberships (in fact, so it will be in the state, too, but this will not be needed for the moment) fulfilling the required constraints on successor states. Any arbitrary continuous controller parametrization, in the compact region Δ can be approximated to any desired accuracy by a polynomial in μ, as (31) proposes (polynomials are universal function approximators [23]); the idea will be later used to prove asymptotic exactness of some algorithms, via increasing the degree c.

With the above general parametrization (31), consider conforming $\bar{u}$ vertically stacking all $u_{i_1 i_2 \ldots i_c}(x)$ and suitably defining matrices $E_{i_1 i_2 \ldots i_c}$ so that $u_{i_1 i_2 \ldots i_c}(x) = E_{i_1 i_2 \ldots i_c}\bar{u}(x)$ in the same way as it was done in (14). For instance, for $c = 2$ in a system with 2 rules, $\bar{u}$ would be defined as $\bar{u} = (u_{11}^T \ u_{12}^T \ u_{22}^T)^T$, as well as suitable $E_{11} = (I\ 0\ 0)$, $E_{12} = E_{21} = (0\ I\ 0)$ and $E_{22} = (0\ 0\ I)$. Then, for $d = 3$, the closed loop would be

$$x_{k+1} = \sum_{\mathbf{i} \in \mathbb{I}_3} \mu_{\mathbf{i}}(x_k) \left(A_{i_1} x_k + B_{i_1} E_{i_2 i_3} \bar{u}(x_k)\right) \tag{32}$$

With the extra decision variables in $\bar{u}$, a larger polytopic approximation $\widetilde{\mathcal{Q}}_d^c$, $d > c$ of the "ideal" shape-independent one-step set $\mathcal{Q}_{si}$ will be defined below, where superscript c denotes the degree of the controller parametrisation, and subscript d denotes the total Polya complexity parameter. The definition of such $\widetilde{\mathcal{Q}}_d^c$ will be analogous to (28) but with different sizes of $\bar{u}$ and $\mathbf{i}$:

$$\widetilde{\mathcal{Q}}_d^c(\Omega) := \left\{ x \in \mathbb{X} \mid \exists \bar{u} \in \mathbb{U}^\rho, \rho = card(\mathbb{I}_c^+) : \widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix} \in \Omega \quad \forall \mathbf{i} \in \mathbb{I}_d^+ \right\} \tag{33}$$

where, actually, the expression of $\widetilde{G}_{\mathbf{i}}$ in (21) should be reworked in order to fit the higher dimensionality. For illustration, in the above example (32), in order to define $\widetilde{\mathcal{Q}}_4^2(\Omega)$ we would need $G_{ijk} = (A_i \ \ B_i E_{jk})$ and, subsequently:

$$\widetilde{G}_{1111} = G_{111}, \quad \widetilde{G}_{1112} = \frac{1}{4}(G_{111} + G_{112} + G_{121} + G_{211}),$$
$$\widetilde{G}_{1222} = \frac{1}{4}(G_{122} + G_{212} + G_{221} + G_{222}),$$
$$\widetilde{G}_{1122} = \frac{1}{6}(G_{112} + G_{121} + G_{211} + G_{221} + G_{212} + G_{122}), \quad \widetilde{G}_{2222} = G_{222}$$

For brevity, details on the construction of $\bar{u}$ and $\tilde{G}_{\mathbf{i}}$ in other cases are left to the reader. The above definition (33) generalises the cases of controller degrees $c = 0$, implicitly assumed in (11), and $c = 1$, explicitly defined in (12) and used in (28). It can be proved that $\widetilde{\mathcal{Q}}_d^c \subset \widetilde{\mathcal{Q}}_{d'}^{c'}$ when $c' \geq c$ and $d' \geq d$ (details omitted for brevity).

5.4. Polytopic inner approximation of the maximal shape-independent control λ-contractive set

As the actual $\mathcal{Q}_{si}$ used in Lemma 2 is out of reach with finite computational resources, we will modify it by substituting $\mathcal{Q}_{si}$ by the polytopic shape-independent *approximation* $\widetilde{\mathcal{Q}}_d^c$. The result is Algorithm 1. Once restricted to polytopic sets, the computational geometry tools in the MPT toolbox [34] allow implementing the above algorithm to find $\hat{\mathcal{C}}_i^\lambda$ in a few lines of MATLAB® code.

Proposition 4 *For any positive c, d, i, we have: $\widehat{\mathcal{C}}_i^\lambda \subset \bar{\mathcal{C}}_i^\lambda(\Omega)$; hence, if the corresponding iterations converge $\widehat{\mathcal{C}}_\infty^\lambda \subset \bar{\mathcal{C}}_\infty^\lambda(\Omega)$ (inner approximation). Also, the converged $\widehat{\mathcal{C}}_\infty^\lambda$ is shape-independent λ-contractive for the TS model* (2).

Proof: The first statement arises from the fact that $\widetilde{\mathcal{Q}}_d^c \subset \mathcal{Q}_{si}$ so each iteration yields a progressively smaller set. The second statement is proved from

Algorithm 1 Computation of the λ-contractive set $\widehat{\mathcal{C}}_\infty^\lambda$

Inputs: c, d, Ω, λ.

1. Make $i = 0$, $\widehat{\mathcal{C}}_0^\lambda = \Omega$
2. Repeat :
   (a) i=i+1
   (b) $\widehat{\mathcal{C}}_i^\lambda = \widetilde{\mathcal{Q}}_d^c \left(\lambda \widehat{\mathcal{C}}_{i-1}^\lambda\right) \cap \Omega$

   Until $\widehat{\mathcal{C}}_i^\lambda = \widehat{\mathcal{C}}_{i-1}^\lambda$;
3. Set $\widehat{\mathcal{C}}_\infty^\lambda = \widehat{\mathcal{C}}_i^\lambda$; END.

the fact that $\widehat{\mathcal{C}}_\infty^\lambda = (\widetilde{\mathcal{Q}}_d^c(\lambda\widehat{\mathcal{C}}_\infty^\lambda) \cap \Omega) \subset \mathcal{Q}_{si}(\lambda\widehat{\mathcal{C}}_\infty^\lambda)$. ■

The above proposition states that the algorithm may have obtained a *non-maximal* λ-contractive set. However, the asymptotic exactness of the Polya result allows to state the following result extending Lemma 3, using $int(S)$ to denote the interior of a set S:

Theorem 2 *Given any integer $i > 0$, for every $x \in int(\bar{\mathcal{C}}_i^\lambda(\Omega))$, there exists a pair of finite c, d such that, when Algorithm 1 is run with such complexity parameters, then $x \in int(\widehat{\mathcal{C}}_i^\lambda)$.*

Proof: Considering any arbitrary i, let us assume the polytopic S is expressed as $S = \{R_i x \le l_i\}$ for some R_i, l_i. Then, if x belongs to the interior of $\mathcal{Q}_{si}(S)$ there exists $\gamma > 0$ and there exists a continuous $u(x,\mu)$ such that $R_i\tilde{f}(\mu, x, u(x,\mu)) - l_i \le \gamma < 0$ for all $\mu \in \Delta$, by Lemma 1 and the fact that being x an interior point, inequalities defining the set must be strictly fulfilled.

Now, universal approximation of polynomials enables us to ensure that there exists a degree-c polynomial in μ in the form (31), say $u^c(x,\mu)$ which, for fixed x, approximates the continuous function $u(x,\mu)$ in the compact set Δ up to a precision $\|u(x,\mu) - u^c(x,\mu)\| \le \varepsilon$ with ε as small as needed so that $R_i\tilde{f}(\mu, x, u(x,\mu) - u^c(x,\mu)) \le \gamma/2$. This allows us to assert that there exists a finite c such that:

$$R_i\tilde{f}(\mu, x, u^c(x,\mu)) - l_i \le \gamma/2 < 0$$

Now, the left-hand side of the above expression can be trivially converted to an homogeneous polynomial of degree $c+1$ on the simplex Δ. Hence, asymptotic exactness of Polya theorem (Theorem 1) ensures that there exists a finite complexity parameter d such that all coefficients of the degree d expansion of $R_i\tilde{f} - l_i$ are strictly negative. Hence, $x \in int(\widetilde{\mathcal{Q}}_d^c(\lambda S))$, by definition of $\widetilde{\mathcal{Q}}_d^c$.

Now, an induction argumentation is needed. Starting from $\bar{\mathcal{C}}_0^\lambda = \widehat{\mathcal{C}}_0^\lambda = \Omega$, if $x_1 \in int(\bar{\mathcal{C}}_1^\lambda)$ then there exist c_1,d_1 such that $x_1 \in int(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega)$. if $x_2 \in int(\bar{\mathcal{C}}_2^\lambda)$, then there exists u, depending on x_2 and μ, in $\mathbb{U}$ such that $x_1 := \tilde{f}(\mu, x_2, u) \in int(\bar{\mathcal{C}}_1^\lambda)$ so the above c_1, d_1 ensure $\tilde{f}(\mu, x_2, u) \in int(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega)$: hence, $x_2 \in \mathcal{Q}_{si}(\widetilde{Q}_{d_1}^{c_1}(\Omega) \cap \Omega)$. Now, letting $S = \widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega$ we can assert that there exist c_2, d_2 such that $x \in int(\widehat{\mathcal{Q}}_{d_2}^{c_2}(\widetilde{\mathcal{Q}}_{d_1}^{c_1}(\Omega) \cap \Omega) \cap \Omega)$. The argumentation can follow on for any i: if $x \in int(\bar{\mathcal{C}}_i^\lambda)$ there exist a sequence $d_1, \ldots, d_i$, $c_1, \ldots, c_i$ such that x belongs to $\widehat{\mathcal{C}}_i^\lambda$. The required c and d in the theorem statement will be the maximum c and d of the respective sequences. ■

6. Controller computation from λ-contractive sets

The obtained polytopic λ-contractive sets, after Algorithm 1 convergence in a finite number of iterations, induce a control Lyapunov function and associated controllers, to be discussed in this section.

Let us assume that the converged set $\widehat{\mathcal{C}}_\infty^\lambda$ is defined as a polytope $\widehat{\mathcal{C}}_\infty^\lambda = \{x \mid \max_{1\le i\le n_h} H_i x \le 1\}$ for some row vectors H_i. Then, Proposition 3 immediately allows defining a control Lyapunov function (10). However, the only problem addressed up to this point is the *existence* of a continuous control law (Lemma 1), but not any *constructive* procedure to find it; notwithstanding, it is well known that, once a *control* Lyapunov function is available, computation of a controller is possible [38].

As the set of valid control actions $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x,\mu)$ defined in (8), is polyhedral for known x and μ (actually, μ would be the measured $\mu(x)$), optimisation of a convex cost index over $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x,\mu(x))$ can be efficiently solved via convex programming. Such optimisation is a widely used choice to constructively compute the above-referred control action in the polyhedral-robust control literature referred to in the introduction; details and adaptation to the fuzzy case will be presented next. Let us discuss two possible options: *on-line* and *off-line* optimisation.

6.1. On-line optimisation

In on-line operation, state and membership values are known at the time of computing the control action, so the model $x_{k+1} = A(\mu(x_k))x_k + B(\mu(x_k))u_k$, affine in the control action u_k, renders:

$$x_{k+1} = M_k + N_k u_k, \qquad M_k := A(\mu(x_k))x_k, \quad N_k := B(\mu(x_k))$$

and M_k and N_k are matrices known at time k once x_k has been measured. A reasonable course of action would be proposing a cost index depending only on the current control action u_k, choosing a suitable one in the convex set $U_{\lambda\widehat{\mathcal{C}}_\infty^\lambda}(x_k,\mu(x_k)) = \{u \in \mathbb{U} \mid \max_i H_i(M_k+N_k u) \le \lambda\}$. In this way, there would be no need to actually build up a "fuzzy" controller (12), as u_k can be directly optimised.

Of course, decrescence of the piecewise-linear Lyapunov function (10) and admissibility of u_k (i.e, $u_k \in \mathbb{U}$) need to be introduced as optimisation constraints, irrespective of the chosen cost index.

Several optimisation criteria may be chosen, for instance:

1. Achieving the fastest decay, by minimising the predicted next value of the polyhedral Lyapunov function (10), i.e., given x_k, selecting u_k equal to the optimal solution below

$$u_k := \arg\min_{u\in\mathbb{U}} V(x_{k+1}) = \arg\min_{u\in\mathbb{U}} \max_{i\in\{1,\dots,n_h\}} H_i(M_k + N_k u) \tag{34}$$

   which is a standard linear minimax problem, which can be cast as a linear programming (LP) one,

$$u_k = \arg\min_u \delta$$
$$\text{subject to:} \quad u \in \mathbb{U},\ H_i(M_k + N_k u) \le \delta \quad \forall i \tag{35}$$

Note that, for $x_k \in \widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$, the minimal δ will be lower than λ because, by construction, $\widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$ is a set in which constraints (35) are feasible for $\delta = \lambda$. If the optimal LP solution were not unique, any arbitrary choice in the set of minimisers would be acceptable.

2. Minimising the "control effort" subject to the contraction condition $V(x_{k+1}) = \max_i H_i(M_k + N_k u_k) \le V(\lambda x_k)$ which forces the Lyapunov function to be decreasing. If the control effort is measured in 1-norm (sum of absolute value of elements) or ∞-norm (elementwise maximum) then the problem is also an LP one; it it is measured in 2-norm, then it is a QP one. Again, feasibility is guaranteed for $x_k \in \widehat{\mathcal{C}}^{\lambda}_{\infty}(\Omega)$.

Note that, even if the controller to be found on-line does not appear to be a "fuzzy" controller, it does indeed depend on the membership values, as $M_k = A(\mu(x_k))$ and $N_k = B(\mu(x_k))$. Note, too, that the above on-line LP/QP problems may be feasible even outside the guaranteed (but conservative, shape-independent) set $\widehat{\mathcal{C}}^{\lambda}_{\infty}$ computed by Algorithm 1; however, such feasibility cannot be guaranteed by the shape-independent analysis in earlier sections.

6.2. Off-line optimisation

Although the above on-line optimisation solution is, actually a one-step optimisation (hence with low computational complexity as the number of decision variables is the number of inputs), an off-line computation of a controller solution can be obtained if so wished.

Indeed, analogously to [39], an explicit piecewise fuzzy controller can be designed under the setup in this work, as an off-line version of (35). Indeed, consider the augmented input $\bar{u}(x)$ defined in (14), either from (12) or, with higher-dimensional controllers (31), composed of vertically stacking $u_{j_1\ldots j_c}(x)$, $\mathbf{j} \in \mathbb{I}_c^+$ in a vector of length $\rho = card(\mathbb{I}_c^+)$. Replacing in (35) the closed-loop fuzzy model (21) –or the higher-complexity versions implicitly considered in (33)–, we have an optimisation problem:

$$\bar{u}^*(x) := \arg\min_{\bar{u}} \delta$$
$$\text{subject to:} \quad u_{j_1\ldots j_c} \in \mathbb{U},\ H_i\left(\textstyle\sum_{\mathbf{j}\in\mathbb{I}_d^+} n_{\mathbf{j}}\mu_{\mathbf{j}}(x)\widetilde{G}_{\mathbf{j}}\begin{bmatrix} x \\ \bar{u}\end{bmatrix}\right) \le (\textstyle\sum_{\mathbf{j}\in\mathbb{I}_d^+} n_{\mathbf{j}}\mu_{\mathbf{j}})\delta \quad \forall i \tag{36}$$

which, as written, cannot yet be solved off-line because memberships are unkown at design time. To overcome such issue, for the controller (31), the proposal is choosing the optimal decision variables given by the solution of

$$\bar{u}^*(x) = \arg\min_{\bar{u}} \delta$$
$$\text{subject to:} \quad u_{j_1\ldots j_c} \in \mathbb{U},\ H_i\widetilde{G}_{\mathbf{j}}\begin{bmatrix} x \\ \bar{u}\end{bmatrix} \le \delta \quad \forall i\ \forall \mathbf{j} \in \mathbb{I}_d^+ \tag{37}$$

because, following analogous argumentations to those leading from (24) to (28), all feasible solutions of problem (37) are feasible, too, in (36). Actually, the developments in previous section prove that problem (37) is feasible in the set $\widehat{\mathcal{C}}^{\lambda}_{\infty}$ resulting from Algorithm 1 (details omitted for brevity).

Explicit solution. As (37) is a linear programming problem once x is fixed (when actually measured), the optimal state-dependent solution $\bar{u}^*(x)$ has an explicit expression, piecewise-affine in x, which can be obtained via *multi-parametric* linear programming, considering x as a parameter, in the form $\bar{u}^*(x) = \overline{F}(x)x + \bar{\sigma}(x)$ with $\overline{F}(x)$ and $\bar{\sigma}(x)$ being a piecewise constant $(m\rho) \times n$ matrix and a $(m\rho) \times 1$ vector, respectively; for details on how such solutions can be obtained with suitable software, the reader is referred to [34].

Now, reverting the vertical stacking in $\bar{u}^*(x)$ to the originating multidimensional fuzzy summation, i.e., writing the controller as in (12) or (31), the optimal controller arising from (37), can be expressed as

$$u^*(x) = \sum_{\mathbf{i} \in \mathbb{I}_c^+} n_{\mathbf{i}} \mu_{\mathbf{i}}(x) \left(F_{\mathbf{i}}(x)x + \sigma_{\mathbf{i}}(x) \right) \tag{38}$$

where $F_{\mathbf{i}}(x)$ and $\sigma_{\mathbf{i}}(x)$ are piecewise constant $m \times n$ matrices and $m \times 1$ vectors, respectively, suitably extracted from $\overline{F}(x)$ and $\bar{\sigma}(x)$.

This formula (*piecewise-affine multi-sum PDC* controller) gives interesting theoretical insights and, as above discussed, does not require on-line optimization. The other proposed optimisation setups (control effort in 1, 2 or ∞ norm) would also give rise to piecewise fuzzy-controllers (details, almost identical, are omitted for brevity). Anyway, the drawback is that, even if likely faster in runtime execution, performance with off-line optimisation will be inferior to that with on-line one (34), due to the explicit use of the measured value of the membership in (34), instead of the setting in (37) where memberships do not appear. Nevertheless, proven *worst-case* performance bounds are identical in both alternatives.

7. Discussion and comparison with existing approaches

This section will compare the result with other approaches in set-invariance and Lyapunov/LMI literature, including the fuzzy control and fuzzy Lyapunov functions.

Set-invariance prior literature. Let us first remind how this work generalises existing set-invariance control approaches: Contractiveness concepts are used in [17] to obtain *necessary and sufficient* constrained robust stability and stabilisation conditions for systems with polytopic uncertainty. The work [22] generalises the idea to asymptotically shape-independent necessary and sufficient stability analysis conditions for TS systems, and the proposal here presented covers the *asymptotically shape-independent necessary and sufficient stabilisation* conditions for constrained TS systems.

Other Lyapunov/LMI approaches. Omitting detail, results from Section 6 in [22], dealing with generic Lyapunov functions, can be adapted to the stabilisation case here, with minor modifications (changing to control Lyapunov functions), so the following can be stated:

Lemma 4 *If a convex function $V(x)$ and a controller $\bar{u}(x)$, conformed as in* (33) *for some controller complexity c, have been proved to exist (with whatever method) such that $V(0) = 0$, $V(x) > 0$ for $x \neq 0$, and*

$$V(\widetilde{G}_{\mathbf{i}} \begin{bmatrix} x \\ \bar{u} \end{bmatrix}) \le V(\gamma x) \qquad \forall \mathbf{i} \in \mathbb{I}_{d+} \tag{39}$$

then, Algorithm 1 with complexity parameters c, d converges in a finite number of iterations for $\gamma < \lambda$ and the resulting λ-contractive set is larger than the Lyapunov level sets.

Proof: Omitted, as it is analogous to Propositions 1, 2 and Corollary 2 (computing an explicit bound on the number of needed iterations) in [22], adapting argumentations dating to [31, 17]. ∎

Theorem 2, combined with the above Lemma, discussing the relationship with any conceivable methods to find convex Lyapunov functions are the key ones in this work: the ideal shape-independent i-step sets cannot be computed, but Polya relaxations allow running the iterations in Algorithm 1 with an approximation to $\bar{\mathcal{C}}_i^\lambda$ which can be made as precise as wished. Then, the obtained sets with Algorithm 1 beat Lyapunov level sets in the sense of Lemma 4. In fact, by maximality and convexity argumentations, they beat the *union* of all feasible solutions of any Lyapunov inequality (39), see Figure 2 in a later example.

Hence, the result *closes* (in theory) the shape-independent control design for *constrained* TS systems with non-fuzzy Lyapunov functions: first, there is no loss in generality when only considering convex Lyapunov functions (Proposition 1); second, say, if any method can find a Lyapunov function proving geometric contraction $\gamma = 0.8$ –in the sense of the above lemma–, our algorithm will, suceed, too, for any $\lambda > 0.8$. Note, however, that as the number of decision variables in $\bar{u}$ and the summation dimension d increases, the computational complexity of the resulting problem grows heavily so only reasonably small values of c and d in $\widetilde{\mathcal{Q}}_d^c$ can actually be sought for in practice. Comparison with fuzzy Lyapunov functions $V(\mu, x)$ will be addressed in Section 7.1 below, with analogous results outperforming a wide class of literature proposals.

Generic controller parametrizations. Note that the proposed piecewise multiple-sum fuzzy affine controller structure in (38) is more general than many *non-piecewise* fuzzy controller choices in literature and, importantly, it is a *result* (asymptotically *exact*) of the proposed *optimal* control, whereas most literature proposes a particular control structure (or a fixed piecewise partition [40, 41]) *a priori*. Importantly, the proposal here allows seamless incorporation of non-symmetric constraint sets, whereas other LMI-based approaches might have difficulties in doing it.

7.1. Relation with Fuzzy Lyapunov functions

Some approaches in literature propose fuzzy Lyapunov functions [42, 5, 43] so its level sets are $V_\gamma := \{x : V(\mu(x), x) \le \gamma\}$. Of course, such sets are shape-dependent, as they are defined in terms of $\mu(x)$. Clearly, the largest set that can be certified to belong to V_γ without knowing the specific shape of $\mu(x)$ is:

$$V_{si,\gamma} := \bigcap_{\mu \in \Delta} \{x : V(\mu, x) \le \gamma\} = \{x : \max_{\mu \in \Delta} V(\mu, x) \le \gamma\} \tag{40}$$

Of course, the proposal in this paper can only be compared to level sets in the above form $V_{si,\gamma}$.

Future/delayed fuzzy Lyapunov approaches. More powerful Lyapunov function and controller parametrisations with "future" memberships values have been proposed in the α-samples approach [10], and, also, with "past" memberships ones [9]. Combinations including both past and future memberships appear in [44, 11]. The conditions in the cited references are shape-independent, in the sense that they consider neither the relationship between the memberships in different times nor the one between memberships and states.

The remainder of the section discusses specific details about the relationship between these proposals and the set-invariance one here proposed.

In order to encompass the different fuzzy (delayed/future) Lyapunov function approaches in other literature with an unified notation, generic fuzzy Lyapunov functions will be considered in the form $V(\Upsilon, x)$, where Υ is a delay-line set of membership vectors

$$\Upsilon := \{\mu(x_{k+s}), \dots, \mu(x_{k+1}), \mu(x_k), \mu(x_{k-1}), \dots, \mu(x_{k-l})\} \tag{41}$$

for some chosen values of look-ahead horizon s and delay l parameters.

In order to add causality constraints (control cannot depend on future membership values), the operator $F(\cdot)$ will extract the *future* (non-causal) elements, i.e., $F(\Upsilon) := \{\mu(x_{k+s}), \dots, \mu(x_{k+1})\}$, and $P(\cdot)$ will contain *past* ones,i.e., $P(\Upsilon) := \{\mu(x_{k-1}), \dots, \mu(x_{k-l})\}$. So, under this setting causal fuzzy controllers must be in the form $u(\mu(x), P(\Upsilon), x)$. As $\tilde{f}$ is linear, geometric λ-contractive conditions amount to

$$V(\Upsilon, \lambda x) - V(\Upsilon_+, \tilde{f}(\mu(x), x, u(\mu(x), P(\Upsilon), x))) > 0 \quad \forall x \in \Omega \sim \{0\} \tag{42}$$

where Υ_+ denotes the vector of memberships evaluated one step *forward* in time, i.e., from look-ahead $s+1$ until delay $l-1$.

The above expression is shape-dependent, but we can assert the following general shape-independent stabilization conditions replacing the elements of Υ and Υ_+ by arbitrary vectors (respectively denoted as Υ_{si} and $\Upsilon_{+,si}$) lying in the unit simplex:

Lemma 5 *The closed-loop fuzzy system $x_{k+1} = \tilde{f}(\mu(x_k), x_k, u(\mu(x_k), P(\Upsilon), x_k))$ is locally stable with contraction rate λ if there exist a controller $u(\mu, P(\Upsilon_{si}), x)$ and a Lyapunov function $V(\Upsilon_{si}, x)$ such that*

$$V(\Upsilon_{si}, \lambda x) - V(\Upsilon_{[+,si]}, \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) > 0 \tag{43}$$

being $\Upsilon_{si} = \{h_s, \dots, h_1, \mu, h_{-1}, \cdots h_{-l}\}$, $\Upsilon_{[+,si]} = \{h_{(s+1)}, \dots, h_1, \mu, h_{-1}, \dots, h_{-l+1}\}$, holds for all $x \in \Omega$, $x \neq 0$, for all $h_{s+1}, \dots, h_{-l}, \mu$ in Δ.

In the above assertion, with a slight abuse of notation, $P(\Upsilon_{si})$ should be understood as the operator extracting "past" elements $\{h_{-1}, \dots, h_{-l}\}$. On the following, shorthand notation $\Upsilon_{si} \in \Delta$ should, too, be understood as each element of Υ_{si} belonging to Δ.

Proof: Direct because (43) implies (42) (it is a particular choice of memberships). ■

Conditions (43) are a contraction-rate version analogous to the ones proved in LMI settings such as, for instance, [9, 11]. The relationship of such conditions with the geometric setting in this work is proven in the theorem below:

Theorem 3 *If condition (43) holds for all Υ_{si}, $\Upsilon_{+,si} \in \Delta$, then the level sets in Ω of:*

$$V_{si}(x) = \min_{P(\Upsilon_{si})\in\Delta} \max_{F(\Upsilon_{si})\in\Delta,\mu\in\Delta} V(\Upsilon_{si}, x) \tag{44}$$

are shape-independent control λ-contractive.

Proof: As (43) hold for any possible value of Υ_+, they do for the particular values of $\{h_{s+1}, \ldots, h_1\} = \{\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1\}$ given by

$$\{\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1\} = \arg \max_{F(\Upsilon_{+,si})\in\Delta} V(\Upsilon_{+,si}, \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x)) \tag{45}$$

So, we can assert

$$\begin{aligned} 0 < V(&\bar{h}^+_s, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{si}), \lambda x) \\ &- V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{[+,si]}), \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) \end{aligned} \tag{46}$$

Then, denoting

$$\{\bar{h}_s, \ldots, \bar{h}_1, \bar{h}\} = \arg \max_{h,h_1\ldots,h_s\in\Delta} V(h_s, \ldots, h_1, h, P(\Upsilon_{si}), \lambda x) \tag{47}$$

for any $P(\Upsilon_{[si]})$ in Δ, we have:

$$\begin{aligned} 0 < V(&\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, P(\Upsilon_{si}), \lambda x) \\ &- V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \mu, P(\Upsilon_{[+,si]}), \tilde{f}(\mu, x, u(\mu, P(\Upsilon_{si}), x))) \end{aligned} \tag{48}$$

Denote now:

$$\{\underline{h}_{-1}, \ldots, \underline{h}_{-l}\} = \arg \min_{P(\Upsilon_{si})\in\Delta} \big(V(\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, P(\Upsilon_{si}), \lambda x)\big) \tag{49}$$

so, as the above (48) holds for any $P(\Upsilon_{[si]})$ in Δ, it does for $\{h_{-1}, \ldots h_{-l}\} = \{\underline{h}_{-1}, \ldots, \underline{h}_{-l}\}$, i.e.,

$$\begin{aligned} 0 < V(&\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, \lambda x) - \\ &V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \mu, \underline{h}_{-1}, \ldots, \underline{h}_{-l+1}, \tilde{f}(\mu, x, u(\mu, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, x))) \end{aligned} \tag{50}$$

and, at last, denoting

$$\begin{aligned} &\{\underline{h}^+, \underline{h}^+_1, \ldots, \underline{h}^+_{-l+1}\} = \\ &\arg \min_{h,h_1,\ldots,h_{-l+1}\in\Delta} V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, h, \ldots, h_{-l+1}, \tilde{f}(\mu, x, u(\mu, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, x))) \end{aligned} \tag{51}$$

we have, for all $\mu \in \Delta$:

$$\begin{aligned} 0 < V(&\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, \lambda x) - \\ &V(\bar{h}^+_{s+1}, \ldots, \bar{h}^+_1, \underline{h}^+, \underline{h}^+_{-1}, \ldots, \underline{h}^+_{-l+1}, \tilde{f}(\mu, x, u(\mu, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, x))) \end{aligned} \tag{52}$$

In the last inequality, the only still "free" variable ranging in the unit simplex is the membership at the current instant. All other past of future ones have been replaced by suitable maximisers or minimisers. In particular,

$$V_{si}(\lambda x) = \min_{P(\Upsilon_{si})\in\Delta} \max_{F(\Upsilon_{si})\in\Delta,\mu\in\Delta} V(\Upsilon_{si}, \lambda x) = V(\bar{h}_s, \ldots, \bar{h}_1, \bar{h}, \underline{h}_{-1}, \ldots, \underline{h}_{-l}, \lambda x) \tag{53}$$

and, using the resulting controller $u(\mu, \underline{h}_{-1}, \dots, \underline{h}_{-l}, x)$ in (52), such that $x_{k+1} = \tilde{f}(\mu, x, u(\mu, \underline{h}_{-1}, \dots, \underline{h}_{-l}, x))$, we have:

$$V_{si}(x_{k+1}) = V(\bar{h}^+_{s+1}, \dots, \bar{h}^+_1, \underline{h}^+, \underline{h}^+_{-1}, \dots, \underline{h}^+_{-l+1}, \tilde{f}(\mu, x, u(\mu, \underline{h}_{-1}, \dots, \underline{h}_{-l}, x))) \quad (54)$$

Hence, expression (52), proves that there exists a control action such that $V_{si}(\lambda x) - V_{si}(x_{k+1}) > 0$. ∎

Theorem 3 extends the concept of shape-independent level sets (40) to the case of past and future memberships (indeed, (40) is a particular case of level sets of (44)). The importance of the theorem is twofold:

1. by asymptotical exactness, if any proposal in literature proves a sufficient condition for (43), then, any point in the interior of the level sets[3] of $V_{si}(x)$ in $\mathbb{X}$ will be found by the proposed algorithm with a high-enough value of the complexity parameters.
2. V_{si} is a "standard" Lyapunov function: intuition is "reconciled" with the results, in the sense that Lyapunov functions involving past and future memberships are transformed to standard ones depending only on the current state (at least in the shape-independent case). Also, even if past memberships appear in the controller in (42), the controller actually used to prove contractiveness in the above proof is independent of the past "measured" memberships (as required by the λ-contractiveness definition): the arguments $\underline{h}_{-1}, \dots, \underline{h}_{-l}$ in the controller $u(\mu, \underline{h}_{-1}, \dots, \underline{h}_{-l}, x)$ are actually a function of x, as (49) shows.

Comparison with shape-dependent options. Note that supposedly "future" values of μ and x are, in fact, predictions based on current x, $\mu(x)$. As stability conditions hold for any future memberships, the following shape-dependent Lyapunov function is, too, proven if (43) holds from any LMI literature result[4]:

$$V_{sd}(x_k) := \min_{h_{-1},\dots,h_{-l} \in \Delta} V(h_{-l}, \dots, h_{-1}, \mu(x_k), \mu(x_{k+1}), \dots, \mu(x_{k+s}), x_k) \quad (55)$$

Obviously, given some scalar γ, the level sets $V_{sd,\gamma} = \{V_{sd}(x) \le \gamma\}$ will be larger than those $V_{si,\gamma} = \{V_{si}(x) \le \gamma\}$ with V_{si} from (44), as $V_{sd}(x) \le V_{si}(x)$. Note, however, if comparing the largest level sets, say V_{sd,γ_1^*} and V_{si,γ_2^*}, in a given modelling region Ω, that forcefully $\gamma_2^* \ge \gamma_1^*$, as $V_{si,\gamma_1^*} \subset V_{sd,\gamma_1^*}$. So, the proven domain of attraction with Lyapunov functions (55) or (44) will be "different": there might be some states proven to contract to the origin by V_{sd} but not V_{si} and vice-versa (with no clear inclusion in either sense).

8. Examples

Consider a TS system $x_{k+1} = \sum_{i=1}^2 \mu_i (A_i x_k + B_i u_k)$ with model matrices:

$$A_1 = \begin{pmatrix} 0.95 & 0.3 \\ 0.7 & 1.1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0.1 & 0.7 \\ 0.2 & 0.4 \end{pmatrix} \quad (56)$$

[3]Actually, in the convex hull of such level sets, if V were non-convex, by Proposition 1, so a convex Lyapunov function can be built.

[4]As "past" is irrelevant for stability, minimisation on past memberships can be carried out for larger level sets, details omitted for brevity. Such minimisation appears, then, in (55).

$$B_1 = \begin{pmatrix} 0.4 \\ 0.5 \end{pmatrix} \qquad B_2 = \begin{pmatrix} 0.1 \\ 2 \end{pmatrix} \tag{57}$$

Subject to the constraints in inputs and states:

$$-10 \le u_k \le 10, \qquad -10 \le x_k \le 10 \tag{58}$$

For the sake of illustration, even if results are valid for any membership shape, some system trajectories will be later simulated using as membership functions:

$$\mu_1(x) = (10 - (1\ 0)x)/20, \quad \mu_2(x) = 1 - \mu_1(x) \tag{59}$$

8.1. Comparison with fuzzy-delayed Lyapunov function

With the above plant, a comparative study with LMIs in [44, Corollary 1] will be made first.

LMI settings. The cited proposal uses a delayed Lyapunov function and a non-PDC controller, respectively given by:

$$V(x_k, x_{k-1}) = x_k^T \left(\sum_{i=1}^{r} \mu_i(x_{k-1}) P_i \right)^{-1} x_k \tag{60}$$

$$u_k = \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_{k-1}) \mu_j(x_k) F_{ij} \right) \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x_{k-1}) \mu_j(x_k) H_{ij} \right)^{-1} x_k \tag{61}$$

As discussed in Section 7.1, even if the above Lyapunov function is a "fuzzy" one, the (non-fuzzy) Lyapunov function (44) particularised for (60), i.e.,

$$V_{si}(x_k) = \min_{\mu \in \Delta} x^T \left(\sum_{i=1}^{r} \mu_i P_i \right)^{-1} x \tag{62}$$

is also proved and, evidently, the level sets $V_{\gamma, x_{k-1}} = \{x_k : V(x_k, x_{k-1}) \le \gamma\}$ for whichever value of $\mu_i(x_{k-1})$ will be smaller than those in the form $V_{si,\gamma} = \{x : V_{si}(x) \le \gamma\}$ for the same level. In fact, the latter level set is the convex hull of the union of the ellipsoids $\mathcal{E}_i := \{x : x^T P_i^{-1} x \le \gamma\}$, see [42].

In order for the comparison to be fair, apart from the LMIs in [44, Cor. 1], also appearing in [11], extra LMIs have been added to to force that the level set for $\gamma = 1$ of each of the ellipsoid lies inside the constraint region $\mathbb{X}$, and to account for the control saturation. Details are omitted for brevity as they follow standard S-procedure argumentations (and use of [44, Property 2]). In order to obtain a "large" domain of attraction exploiting the non-quadratic and convex-hull ideas, ellipsoids $\mathcal{E}_i$ were forced to contain the point $\gamma \cdot (\cos \phi_i, \sin \phi_i)^T$ for $\phi_1 = \pi/4$ and $\phi_2 = 3\pi/4$, respectively, and the scalar $\gamma \ge 0$ was maximised. In this way, ellipsoids were expanded in orthogonal directions.

Set-invariance settings. In the λ-contractiveness approach presented here, we set $\lambda = 0.9999$ (mere stabilisation), and we test a non-fuzzy controller ($c = 0$, $d = 1$), and a PDC control parametrisation ($c = 1$) with a Polya complexity parameter $d = 6$.

Results. As a result, Figure 1 is obtained. The figure depicts the convex hull of the two ellipsoids arising from the LMI solution(curved line), the maximal set with the non-fuzzy controller (blue region) and the inner approximation to the maximal controllable set with single-sum controllers (union of blue and red region). Increasing d didn't visually appear to generate more stabilisable points. The blue line represents the simulation of the proposed optimal maximum-decay controller, which steers the state to the origin in two samples. The set proposed by our algorithm is larger than the proposed solution in the compared work.

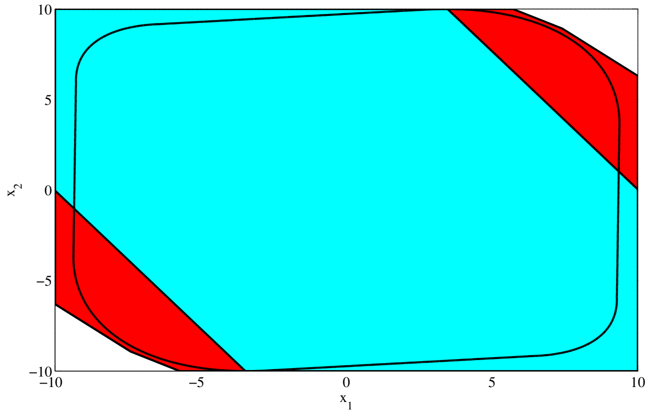

Figure 1: Comparative analysis with delayed-membership non-PDC control (single solution).

Computation time. The LMI solution from [44] in Figure 1, using YALMIP 3.2010.0611 and SEDUMI 1.3, took 1.56 seconds; with MPT Toolbox 2.6.3 [34], obtaining the blue robust-polytopic region with the algorithms in [17] took 16 iterations and 2.03 seconds; the Polya-6 red region in the above figure took 4 iterations and 0.222 seconds; the used computer was an Intel I5 2.56GHz computer with 6 Gb of RAM with Matlab 2010. The more general controller parametrisation allows to prove stability with less iterations (optimal controllers are faster): surprisingly, the more complex setting took less time to compute.

Union of all possible LMI solutions. As the solution of [44, Cor. 1] might be not unique[5], several solutions were crafted by forcing one of the ellipsoids $\mathcal{E}_i$ (i randomly chosen) to contain the largest possible ellipsoid in the form $\mathcal{E}^*_\gamma = \{x : 100x_1^2 + x_2^2 \le \gamma\}$ rotating $\mathcal{E}^*_\gamma$ repeated times, in order to explore whether there exists a solution of the LMIs "stretching out" as much as possible in every direction.

Figure 2 presents the multitude of solutions for different runs of the LMIs[6], with the union emphasised in blue color. All of the solutions lie inside the converged invariant set produced by our algorithm. So, there exists a controller with

[5] In our approach, on-line controllers might be non-unique, but the maximal set is indeed unique (such fact can be proved by convexity argumentations).

[6] Importantly, note that the LMIs in the compared work provide only *one* solution, as in Figure 1: computing the *union* of all feasible LMI solutions requires a theoretically infinite number of LMIs with the settings in prior literature, whereas our proposal takes 0.2 seconds to compute a set which is larger than such union.

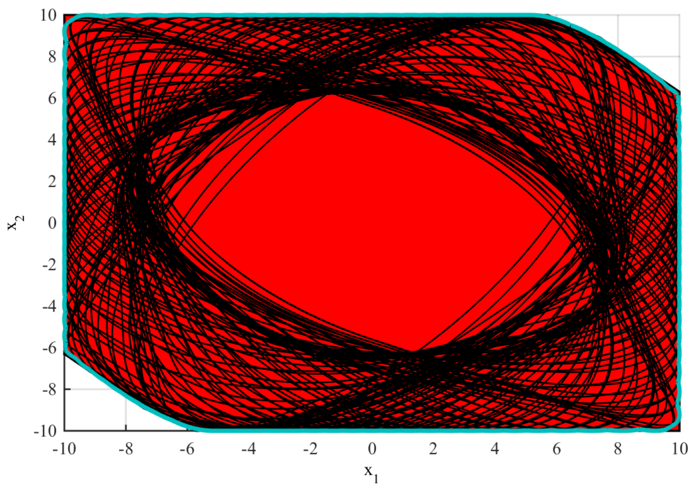

Figure 2: Comparative analysis with delayed-membership non-PDC control (all feasible solutions); the cyan line depicts the union of all such solutions.

piecewise-PDC structure which outperforms (larger domain of attraction) those in [44, Cor. 1], not only individually but also outperforming the *union* of all feasible solutions (which might involve a *different* controller each, so a controller for such union set is *not* found in the cited work) with a *single* controller.

8.2. Off-line piecewise controller

In this subsection, a contraction rate $\lambda = 0.98$ has been chosen as the specification for speed of convergence, so the obtained sets are slightly smaller. The piecewise-PDC fuzzy controller (linear in memberships and affine in the state) in (38) has been computed for complexity parameters $c = 1$, $d = 6$, searching for the fastest decay. As previously discussed, this aims to achieve a faster on-line execution in exchange for a larger computation time in the design phase.

The optimal control problem (37) in the single polytope given by the maximal contractive set has a piecewise solution with a tessellation of 90 polytopic regions[7], depicted in Figure 3. In this example, the computation of the explicit piecewise-PDC-affine optimal feedback law took 2.35 seconds, instead of the 0.222 that took computing "only the *set* in which a controller exists".

Two trajectories are simulated with the piecewise controller. As guaranteed by the algorithms, feasible sequences of control and states can be obtained without violating any constraints.

9. Conclusions

This paper presents an extension of the control λ-contractive set computations in robust control literature to fuzzy Takagi-Sugeno models under state and input constraints (possibly non-symmetric). Based on Polya's asymptotically exact theorems, the obtained closed-loop controllable sets will approach

[7]Note that the resulting regions were not known *a priori*, contrarily to other piecewise results, say [40], in which regions are fixed at start.

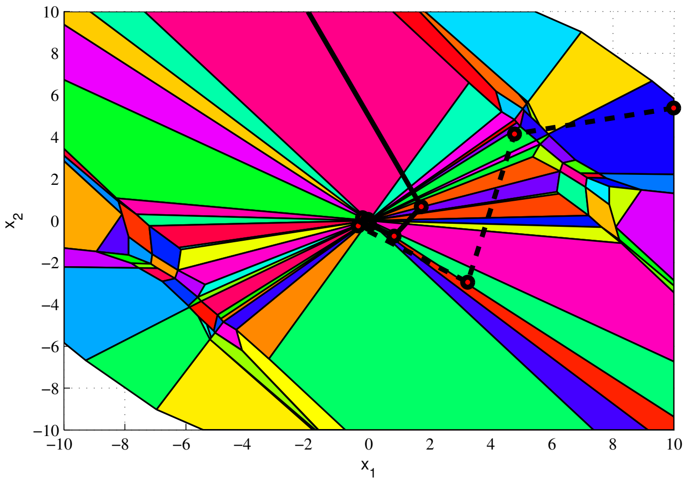

Figure 3: Piecewise state-space tesellation and trajectories for different simulations of the piecewise-affine controller (38).

the maximal shape-independent control λ-contractive set: if some complexity parameters are high enough, the obtained sets and controllers improve over any conceivable (shape-independent) Lyapunov-based controller design technique for TS systems. An implementation requiring on-line one-step optimisation is proposed; as an alternative, by using explicit multi-parametric software tools, a shape-independent piecewise-affine-multidimensional-PDC controller exists whose explicit expression can be obtained off-line, achieving the same worst-case performance. Comparative analysis with delayed-fuzzy Lyapunov functions show that all their shape-independent solutions lie inside the sets produced by the new algorithm for the same complexity parameter values.

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