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# Subspace-Based Takagi-Sugeno Modeling for Improved LMI Performance

Ruben Robles, Antonio Sala, Senior Member, IEEE, Miguel Bernal, and Temoatzin González

Abstract—Given a nonlinear system, the sector-nonlinearity methodology provides a systematic way of transforming it in an equivalent Takagi-Sugeno model. However, such transformation is not unique: conservatism of shape-independent performance conditions in the form of linear matrix inequalities results in some models yielding better results than others. This paper provides some guidelines on choosing a sector-nonlinearity Takagi-Sugeno model, with provable optimality (in a particular sense) in the case of quadratic nonlinearities. The approach is based on Hessian and restrictions of a function onto a subspace.

#### I. Introduction

Analysis and design of nonlinear control systems via Takagi-Sugeno (TS) models is well developed, evolving from model-free heuristics [1], [2] to model-based exact representations, combined with the direct Lyapunov method in order to obtain linear matrix inequalities (LMIs) [3], [4], [5]. The latter case is based on the sector nonlinearity approach, obtaining an exact TS model via maximum and minimum bounds of a nonlinearity in a compact modelling region. Approximate TS models can, too, be obtained, via linearisation at several points [3], or based on approximate fitting via  $\mathcal{H}_2$  or SVD argumentations (linear or polynomial in the scheduling parameters [6], tensor-product summation [7]); SVD-based techniques for rule reduction of complex TS systems appear in [8]. However, these "approximate" TS models are intentionally left out of the scope of this paper, concentrating on presenting improvements to the exact sector-nonlinearity technique. Polynomial-fuzzy models [9] will also not be considered in the present work.

Although the models are exact, a first drawback comes from the conservatism of considering only the vertices and not the combination coefficients in the stability conditions. Thus, stability is actually proven for a family of linear timevarying (LTV) systems in which the plant is embedded; these results are therefore called *shape-independent* [10]. Shape independency is the easiest way to get conditions in the form of LMIs, which in turn are advantageous because they belong to the class of convex optimization problems, which are efficiently solved<sup>1</sup> [11]. A few shape-dependent options are available [12], not considered here.

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<sup>1</sup>In some cases, performance optimisations are in generalised-eigenvalue problem form (GEVP); bisection plus LMI is a well-known option: this paper will understand "LMI" as the convex or quasi-convex problems involving matrix inequalities, including GEVP, discussed in [11].

Another drawback of the TS/LMI methodology is the well-known fact that TS representations may not be unique [13]. Hence, different performance levels can be proven with shape-independent LMIs for the same nonlinear system, depending on the chosen TS model. To handle this issue, apart from naive trial-and-error, no systematic procedure of choosing a "good" TS model from the many options (infinitely many, actually) is available in literature, to the authors' knowledge.

From the above discussion, the objective of this work is choosing an appropriate TS model in order to maximise a performance objective in regions close to the origin. The approach is based in first and second partial derivatives (Jacobian and Hessian). As shape-independent conditions consider all the convex hull of vertices  $A_i$  as polytopic uncertainty, different TS models will, hence, have different shape and orientation of such uncertainty polytope. The key idea to be presented is making the intersection of such polytope with some vector subspaces (appearing in performance-related LMIs) as small as possible. A preliminary approach appears in [14].

This work is organized as follows: section II introduces preliminaries and motivates the problem; section III presents shape-independent uncertainty measures; section IV shows how a Hessian based transformation can optimise such measures; restrictions onto a subspace are discussed in section V; consequences in an LMI context appear in section VI. Discussion, examples and conclusion, are sections VII, VIII and IX, respectively. An appendix is provided reviewing basic ideas and notation in sector-nonlinearity fuzzy modelling.

## II. PRELIMINARIES AND PROBLEM STATEMENT

Consider a nonlinear dynamic system in the form

$$\dot{x}(t) = f(x(t)) \tag{1}$$

with  $f: \mathbb{R}^n \to \mathbb{R}^n$ , having continuous second derivatives and f(0) = 0. Consider the linearised model of (1) to be:

$$\dot{x} = Ax, \quad A := \frac{\partial f(x)}{\partial x} \Big|_{x=0}$$
 (2)

1) Takagi-Sugeno modelling: The well-known sector nonlinearity methodology [15] allows algebraically rewriting (1) as an *equivalent* convex sum of linear models

$$\dot{x} = \sum_{i=1}^{r} h_i(x) A_i x,\tag{3}$$

where the membership functions (MFs), grouped in a vector  $h \in \mathbb{R}^r$ , belong to the r-1-dimensional standard simplex:

$$\Delta := \{ h \in \mathbb{R}^r : \sum_{i=1}^r h_i = 1, h_i \ge 0 \ \forall i \}$$
 (4)

1

Basically, each  $A_i \in \mathbb{R}^{n \times n}$  is the matrix corresponding to a particular combination of maxima/minima of previously defined nonlinearities in a compact set of the state space  $\Omega$ . Although the methodology is well known, a brief outline, introducing some notation needed later on, appears in Appendix, justifying that the number of rules in (3) is a power of two, see (72). For later developments, let us denote as  $\mathcal{A}$  the ordered list of consequents matrices  $\mathcal{A} := \{A_1, \dots, A_r\}$ . The sector-nonlinearity technique may not produce a unique TS model, resulting in possible conservatism [10]. It is also well known that the linearised matrix A can be cast as a convex combination of the vertices  $\mathcal{A}$  of any TS model of (1).

From the developments in the Appendix, the actual structure of  $h_i$  coming from sector-nonlinearity TS model of  $f: \mathbb{R}^n \mapsto \mathbb{R}$  is either in the form (71), as the sum of s two-rule models:

$$f(x) = \sum_{i=1}^{s} \sum_{j=0}^{1} \mu_{ij}(x) A_{ij} x$$
 (5)

where the MFs belong to:

$$\Delta_s = \{ \mu_{ij} : \mu_{i0} = 1 - \mu_{i1}, \mu_{ij} \ge 0, i = 1, \dots, s, j = 0, 1 \}$$

or as a tensor-product (72), where  $h_i$  in (3) has the structure:

$$h_i(x) = \prod_{l=1}^{s} \mu_{l \, bit_l(i)}(x)$$
 (6)

The reader is referred to the Appendix for detailed definition and obtention of the above expressions.

2) Performance measures: Once a TS model is obtained, analysis and design can be done taking advantage of its convex structure combining them with Lyapunov functions such as  $V=x^TPx$ ,  $P=P^T>0$ , which naturally leads to conditions in the form of LMIs in P. A generic assumption on the problem structure will be made:

**Assumption 1.** The pursued control objective is the minimisation of a performance measure  $\gamma$  subject to some model-independent matrix-definiteness constraints

$$\Psi(D) \succ 0 \tag{7}$$

and model-dependent constraints:

$$x^T \Upsilon(A_i, D, \gamma) x > 0 \quad \forall x \neq 0, \forall i = 1, \dots, r$$
 (8)

where D denotes the decision variables (Lyapunov function, controller gains, etc.) and all  $A_i \in \mathcal{A}$  are given by the TS model under consideration. Matrix expression  $\Upsilon(\cdot,\cdot,\cdot)$  will be assumed symmetric, convex in its first argument, and linear in the third argument. We will assume, too, that  $\Psi$  and  $\Upsilon$  can be transformed to tractable problems such as, for instance, LMI, so that suitable convex optimisation software will find the optimal  $\gamma$  and D.

Many contributions, referred to in the introduction, set up problems which can be expressed as the above assumption (for instance, decay-rate or  $\mathcal{H}_{\infty}$  norm computations for continuous- and discrete-time TS systems, see example section). Note that problem (8) is *shape-independent* as memberships h do *not* appear there: in (conservative) *shape-independent* analysis non-uniqueness of sector-nonlinearity models ends up in different performance levels being proven for different (supposedly equivalent) TS models [10].

Under the above assumption, linearity in the third argument will force that the optimal solution of (8), to be denoted as  $\gamma^{opt,TS}$ , hits the boundary of the constraint set: there will exist  $i^*$  such that  $\Upsilon(A_{i^*},D,\gamma^{opt,TS})$  will be positive *semi*-definite (non-empty nullspace) for all feasible D.

3) Relation with performance of linearised model: When problem (8) is solved with single matrix A, the optimal performance of the linearised model is obtained.

**Proposition 1.** The optimal performance measure for (2), say  $\gamma^{opt}$ , is obtained when there exists  $D^{opt}$  such that the above conditions (8), particularised to a single matrix A, are

$$x^T \Upsilon(A, D^{opt}, \gamma^{opt}) x = 0 \quad \forall x \neq 0, x \in \mathcal{C}$$
 (9)

$$x^T \Upsilon(A, D^{opt}, \gamma^{opt}) x > 0 \quad \forall x \neq 0, x \in \mathcal{C}^{\perp}$$
 (10)

for some vector subspace  $C \subset \mathbb{R}^n$ , being  $C^{\perp}$  its orthogonal complement.

*Proof.* As  $\Upsilon(\cdot,\cdot,\cdot)$  is a symmetric matrix, it has an orthonormal basis of eigenvectors, so when conditions cease to be strictly feasible they will be equal to zero in a subspace (associated to the null eigenvalues of  $\Upsilon(A,D^{opt},\gamma^{opt})$ , denoted as  $\mathcal{C}$ ) and strictly positive in vectors associated to the nonzero eigenvalues of  $\Upsilon(A,D^{opt},\gamma^{opt})$  (which will belong to the orthogonal complement of  $\mathcal{C}$ , i.e.,  $\mathcal{C}^{\perp}$ ).

With a suitable change of variable in the original linearised dynamics, there is no loss of generality in assuming that the linearised dynamics yields some constraints which fail when x lies in the canonical q-dimensional subspace<sup>2</sup>

$$C = \{ x \in \mathbb{R}^n : x = \begin{pmatrix} 0 \\ I \end{pmatrix} \eta, \quad \eta \in \mathbb{R}^q \}$$
 (11)

In these coordinates, the constraints for the linearised system (9) and (10) can be equivalently written as a single one in the form below, for some matrix  $\Upsilon_{11}$ :

$$x^{T} \Upsilon(A, D^{opt}, \gamma^{opt}) x = x_{c}^{T} \begin{pmatrix} \Upsilon_{11} & 0 \\ 0 & 0 \end{pmatrix} x_{c}$$
 (12)

**Proposition 2.** The (shape-independent) optimal performance for (3) proven with (8),  $\gamma^{opt,TS}$ , is equal or worse than  $\gamma^{opt}$  for the linearised system (2) proven by (9)–(10):  $\gamma^{opt,TS} \geq \gamma^{opt}$ .

*Proof.* Note that the linearised A is in the convex hull of the matrices in the TS consequents  $\mathcal{A}$  in exact sector-nonlinearity models. Convexity in the first argument of  $\Upsilon(\cdot,\cdot,\cdot)$  entails that for any D such that  $\Upsilon(A_i,D,\gamma^{opt,TS})\geq 0$  we would have  $\Upsilon(A,D,\gamma^{opt,TS})\geq 0$ . Evidently, then, the best performance provable with (8) will be larger or equal than  $\gamma^{opt}$  from the linearised model (9)–(10).

#### A. Other preliminary results

Through this paper, some other results/notation will be used.

 $<sup>^2</sup>$ Indeed, if  $\Upsilon$  fail in a subspace  $\mathcal C$  (in original coordinates), canonical expression (11) is obtained by conforming a transformation matrix  $x=\overline{T}x_c,$  with  $x_c=(\xi^T,\eta^T)^T$  as  $\overline{T}=\begin{pmatrix} T_{\mathcal C^\perp} & T_{\mathcal C} \end{pmatrix},$  where columns of  $T_{\mathcal C}$  are a basis of  $\mathcal C,$  and those of  $T_{\mathcal C^\perp}$  are a basis of  $\mathcal C^\perp$ .

**Proposition 3.** For any two vectors  $\psi$ , x in  $\mathbb{R}^n$ ,

$$\max_{\|x\| \le 1} \psi^T x = \|\psi\|, \quad \min_{\|x\| \le 1} \psi^T x = -\|\psi\|.$$

Proof is trivial from scalar product properties.

Given  $M \in \mathbb{R}^{m \times n}$ , the *Frobenius norm* of M is defined as  $\|M\|_F := \sqrt{\sum_{i=1}^m \sum_{j=1}^n m_{ij}^2}$ , where  $m_{ij}$  denotes the element of M at row i and column j; it verifies  $\|M\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} \lambda_i^2}$ , where  $\lambda_i$  are the *singular* values of M [16]. If M is square and symmetric,  $\lambda_i$  are, actually, its eigenvalues.

Classical interval arithmetic: An interval  $\eta = [a,b]$  is a convex subset of the real line, with minimum a and maximum  $b, a \leq b$ . The sum [a,b]+[c,d] will be defined as [a+c,b+d]. The product  $x \times [a,b]$  will be defined as [ax,bx] if  $x \geq 0$ , and as [bx,ax] otherwise. The width of an interval [a,b], with  $b \geq a$ , will be denoted as w([a,b]) := b-a. The absolute value will be defined as  $|[a,b]| = \max(|a|,|b|)$ . Let us denote sym([a,b]) = [-|[a,b]|,|[a,b]|]. Obviously,  $w([a,b]) \leq 2|[a,b]|$ , and  $[a,b] \subset sym([a,b])$ .

**Proposition 4.** Let  $\eta = \sum_{i=1}^{n} x_i \times [a_i, b_i]$  be an interval. Then,

$$\max_{\|x\| \le 1} w(\eta) \le 2\sqrt{\sum_{i=1}^{n} |[a_i, b_i]|^2}$$

*Proof.* Obviously,  $\eta \subset \sum_{i=1}^n x_i \times sym([a_i, b_i])$ . Elementary manupulations and Proposition 3 yield the required result.  $\square$ 

**Second-order approximation.** Smooth functions f(x) around the origin based on Taylor series can be seen as:

$$f(x) \approx Jx + \frac{1}{2}x^T Hx,$$

where J stands for the Jacobian while H stands for the Hessian matrix, evaluated at the origin:

$$H := \frac{\partial^2 \rho(x)}{\partial x_i \partial x_j} \bigg|_{x=0}, \quad i, j \in \{1, \dots, n\}.$$

Since the Hessian is a symmetric matrix, there exists an orthonormal basis which diagonalises it; this implies that each nonlinearity close to the origin can be represented as a sum of squares of single independent variables. If  $H = V^T \Lambda V$ , the transformation  $\eta = Vx$  can express

$$x^T H x = g(\eta) = \lambda_1 \eta_1^2 + \dots + \lambda_n \eta_n^2, \tag{13}$$

where  $\lambda_1, \ldots, \lambda_n$  are the Hessian eigenvalues.

#### B. Motivation and problem statement

The basic idea arising from the above propositions is that, as TS vertex matrices  $A_i$  drift away from the linearisation A, then, the "closer" the matrices  $A_i$  could be made to the such linearisation, the better the obtained proven performance  $\gamma^{opt,TS}$  might be. As the worst-case directions are those in subspace  $\mathcal C$ , the goal of the TS modelling will be fitting "as closely as possible" the model in the subspace  $\mathcal C$ . This idea motivates this manuscript.

Given the non-uniqueness of the TS modelling, the first objective of this paper is providing a systematic methodology to build an infinite family of possible TS models based on some coordinate transformations, from a set of functions  $\eta_i(x)$ , see (69) in Appendix.

Then, the next objective will be defining what the abovementioned fit means in formal terms, proposing a choice of the aforementioned  $\eta_i(x)$  derived from the equations which define subspace  $\mathcal{C}$ , proving optimality for quadratic nonlinearities. Given that all smooth functions are locally quadratic by the Taylor series up to degree 2 when close enough to the origin, the proposal in this paper allow to find the optimal TS model in the above settings in small enough modelling regions.

Examples will show that the proposed TS models preserve performance of the linearised model (the best one, from Proposition 2) in a better way than other arbitrary choices as the modelling region increases.

## III. SYSTEMATIC TS MODELLING

Let us consider a TS model in box form (5) with a suitably ordered consequent list A.

**Definition 1.** The shape-independent TS model  $f^{si}(x, A)$  is defined as a set-valued map  $f^{si}: \mathbb{R}^n \to \mathbb{C}(\mathbb{R}^m)$ , where  $\mathbb{C}(\mathbb{R}^m)$  denotes the convex subsets of  $\mathbb{R}^m$ , given by:

$$f^{si}(x, \mathcal{A}) := \{ y \in \mathbb{R} : \exists \mu_{ij} \in \Delta_s \ s.t. \ y = \sum_{i=1}^s \sum_{j=1}^2 \mu_{ij} A_{ij} x \}$$
 (14)

where  $\mathcal{A}$  in the left-hand side has been implicitly used to denote the whole list of consequent models.

With the above definition, the following is evident:

**Proposition 5.** For any of the possible TS models of a given f(x), evidently,  $f(x) \in f^{si}(x, A)$ .

The core step in the sector-nonlinearity methodology deals with single-output nonlinear functions  $\rho_i:\mathbb{R}^n\mapsto\mathbb{R}$ , conforming a vector  $\rho$  such that  $f(x)=Ax+M\rho(x)$ , see (68). Each  $\rho_i$  is a nonlinearity with *one* output, to be bounded between *two* linear functions, see (70); therefore, this paper focuses first on analising these mappings  $\mathbb{R}^n\mapsto\mathbb{R}$  in order to tackle the problem of choosing a TS model for improved performance. Later on, the case of multiple nonlinearities is discussed.

Let us, then, consider a function  $f: \mathbb{R}^n \to \mathbb{R}$ , with f(0) = 0. Due to convexity,  $f^{si}(x, A)$  is an interval for the chosen class of one-output functions. The width of such interval will be related to the conservatism of shape-independent developments with the TS model.

As TS consequents are linear, considering properties of the TS model over  $\{\|x\|=1\}$  will be informative enough. This motivates the definition below:

**Definition 2.** The normalised worst-case width (WCW) of a one-output TS model with consequent list  $\mathcal{A}$  is:

$$\bar{\sigma}(\mathcal{A}) := \max_{x \in \Omega, x \neq 0} \frac{w(f^{si}(x, \mathcal{A}))}{\|x\|}$$
 (15)

**Definition 3.** A TS model with consequent list A is WCW-optimal if there is no other choice of consequent matrices with better  $\bar{\sigma}(A)$ .

Actually, this paper will prove that a Hessian-based methodology obtains such optimal TS model if f(x) is quadratic. Later, the optimality criteria will be recast to finding the model having the lowest uncertainty width in the intersection of modelling region  $\Omega$  with a particular subspace<sup>3</sup> C.

**Definition 4.** Given a vector subspace C, the subspace-constrained WCW of a one-output TS model is defined as:

$$\bar{\sigma}_{\mathcal{C}}(\mathcal{A}) := \max_{x \in \Omega \cap \mathcal{C}, x \neq 0} \frac{w(f^{si}(x, \mathcal{A}))}{\|x\|}$$

So, the optimal TS model will be redefined to be the one minimising  $\bar{\sigma}_{\mathcal{C}}$ . Again, in a quadratic case, the solution to the minimum  $\bar{\sigma}_{\mathcal{C}}$  will be provided in this paper.

#### A. Optimal shape-independent TS model for SISO nonlinearity

Consider a single-variable nonlinearity with  $f : \mathbb{R} \to \mathbb{R}$ , f(0) = 0 and its classical sector-nonlinearity TS model, (67) in Appendix, here repeated for convenience:

$$f(x) = h(x)\tilde{f}_0 x + (1 - h(x))\tilde{f}_1 x \tag{16}$$

being  $\tilde{f}_0 = \max_{x \in \Omega} \tilde{f}(x)$  and  $\tilde{f}_1 = \min_{x \in \Omega} \tilde{f}(x)$ , with  $\tilde{f}(x) = f(x)/x$ . Consider now any other possible consequent models  $\hat{q}_0$  and  $\hat{q}_1$  such that there exists  $\hat{h}(x)$  allowing writing

$$f(x) = \hat{h}(x)\hat{q}_0x + (1 - \hat{h}(x))\hat{q}_1x \tag{17}$$

Using (14), with s = m = 1, so  $f^{si}$  is an interval, we have:

$$f(x) \in f^{si}(x, \{\hat{q}_0, \hat{q}_1\}) = \begin{cases} [\hat{q}_0 x, \hat{q}_1 x], & x \le 0\\ [\hat{q}_1 x, \hat{q}_0 x], & x \ge 0. \end{cases}$$

In order for the above  $\hat{h}$  to exist, the consequents must verify (proof is straightforward, omitted for brevity):

$$\hat{q}_0 \ge \tilde{f}_0, \quad \hat{q}_1 \le \tilde{f}_1 \tag{18}$$

Then (18) translates to:

**Proposition 6.** The shape-independent TS model (16) fulfills

$$f^{si}(x, \{\tilde{f}_0, \tilde{f}_1\}) \subset f^{si}(x, \{\hat{q}_0, \hat{q}_1\})$$

for any  $\hat{q}_0$ ,  $\hat{q}_1$  such that  $f(x) \in f^{si}(x, \{\hat{q}_0, \hat{q}_1\})$  for all  $x \in \Omega$ .

Note: on the sequel,  $f^{si}(x,\mathcal{A})$  will be shorthanded to  $f^{si}(x)$  when no confusion on the consequent parameters arises. In functions of one variable, the only reasonable choice of consequents is that in (67), because of the above proposition. Such model  $f^{si}(x,\{\tilde{f}_0,\tilde{f}_1\})$  will be, also, shorthanded to  $f^{si,opt}(x)$ .

The objective of this work is generalising the easily provable Proposition 6 above to functions of several variables. In order to do that, a reformulation of the sector nonlinearity methodology, altogether with a coordinate transformation will be presented in next section.

#### IV. COORDINATE TRANSFORMATIONS

In ordinary TS modelling, as discussed in the previous section and the Appendix, the selection of a particular element of  $\rho(x)$  in model (68), say  $\rho_i(x)$ , assumes the existence of a specific linear function of the state  $\eta_i(x)$ ; these selections are usually chosen by "manual inspection", such that, defining

$$\tilde{\rho}_i(x) := \frac{\rho_i(x)}{\eta_i(x)}, \quad \rho_i(x) = \tilde{\rho}_i(x)\eta_i(x)$$

suitable limits of  $\tilde{\rho}_i(x)$  exist, so a 2-rule model of  $\rho_i$  can be crafted (see Appendix).

**Example 1** (Ad-hoc modelling). Consider  $f(x) := -4x_1 + 4x_1x_2$ , expressed as  $f(x) = -4x_1 + 4\rho(x)$ ,  $\rho(x) := x_1x_2$ , to be modelled in the unit circle. Either  $\{\tilde{\rho}(x) := x_1, \eta(x) := x_2\}$  or  $\{\tilde{\rho}(x) := x_2, \eta(x) := x_1\}$  could be reasonable choices to craft a TS model. These two possible choices for  $\eta$  can be visually found in the aforementioned inspection, leading to:

- $f(x) = -4x + h_1(x_1)x_2 + h_2(x_1) \cdot (-x_2)$ ,  $h_1 = 0.5(x_1 + 1)$ ,  $h_2 = 1 h_1$ , with the associated shape-independent model  $f^{si}(x) = [-x_2, x_2]$ , or
- $f(x) = -4x + h_1(x_2)x_1 + h_2(x_2) \cdot (-x_1), h_1 = 0.5(x_2 + 1), h_2 = 1 h_1, being f^{si}(x) = [-x_1, x_1].$

Introducing generic coordinate transformations, the above TS models can be expressed as a particular case of an infinite family of choices, as the discussed below.

**Example 2** (i.e., Example 1, continued). The motivation of this section is that, actually, there are infinitely many other choices for the TS models of f(x) in example 1. If we express

$$f(x) = -4x_1 + \frac{1}{\alpha\beta}\rho_1(x) - \frac{1}{\alpha\beta}\rho_2(x)$$

with  $\rho_1(x)=(\alpha x_1+\beta x_2)^2$  and  $\rho_2(x)=(\alpha x_1-\beta x_2)^2$ , we could also think of  $\eta_1=(\alpha x_1+\beta x_2)$  for the TS model of  $\rho_1(x)=\eta_1^2(x)$ , and, on the other hand, choose  $\eta_2=(\alpha x_1-\beta x_2)$  for the TS model of  $\rho_2(x)=\eta_2^2(x)$ . For notational convenience, let us define  $v_1:=(\alpha \beta)$ ,  $v_2:=(-\alpha \beta)$ , so we have  $\eta_1=v_1x$ ,  $\eta_2=v_2x$ .

The resulting TS model, in box form (5), would be a four vertex representation:

$$f(x) = ((\mu_{10}A_{10} + \mu_{11}A_{11}) + (\mu_{20}A_{20} + \mu_{21}A_{21}))x$$

where  $\mu_{10} + \mu_{11} = 1$ ,  $\mu_{20} + \mu_{21} = 1$ , and

$$A_{i0} = (-2\ 0) + \frac{1}{\alpha\beta}\xi_{i0}v_i, \ A_{i1} = (-2\ 0) + \frac{1}{\alpha\beta}\xi_{i1}v_i$$

 $\xi_{i0} = \max_{x \in \Omega} \eta_i(x)$ ,  $\xi_{i1} = \min_{x \in \Omega} \eta_i(x)$ , details omitted for brevity. Note that the two prior "manually" obtained TS models in Example 1 correspond to  $(\alpha = 1, \beta = 0)$  or  $(\alpha = 0, \beta = 1)$ , respectively.

As the number of possible models is infinite, the question of which is the "best" one arises. Using the WCW-optimality criteria in Definition 3, in the quadratic case, such best model can be found via eigenvalue decomposition, leading to the main result in this section below.

<sup>&</sup>lt;sup>3</sup>Such subspace will actually come from constraints (9).

**Theorem 1.** Consider a quadratic nonlinearity  $f: \mathbb{R}^n \mapsto \mathbb{R}$ ,  $f(x) := x^T M x$ , with M symmetric, with an eigenvalue-eigenvector decomposition  $M = V \Lambda V^T$  with  $\Lambda$  diagonal and V orthonormal matrices. Consider, too, a modelling region  $\Omega := \{x: ||x|| \leq 1\}$ . Then, the WCW-optimal TS model is given by expressing

$$f(x) = \sum_{i=1}^{n} \lambda_i \rho_i(x)$$
 (19)

being  $\lambda_i$  the eigenvalues of M and  $\rho_i(x) = \eta_i^2(x)$ , for  $\eta_i(x) = V_i^T x$ , i.e.  $\eta_i$  being the projection of x over the unit-norm eigenvector  $V_i^T$  associated to  $\lambda_i$ . Then, the optimal TS model has the form:

$$f(x) = \sum_{i=1}^{n} \sum_{j=0}^{1} \mu_{ij}(x) \psi_{ij} V_i^T x$$
 (20)

where  $\psi_{i0} := \max_{x \in \Omega} \lambda_i \eta_i(x)$ ,  $\psi_{i1} := \min_{x \in \Omega} \lambda_i \eta_i(x)$  and membership functions are:

$$\mu_{i0}(x) := \frac{V_i^T x - \psi_{i1}}{\psi_{i0} - \psi_{i1}}, \ \mu_{i1}(x) := 1 - \mu_{i0}(x), \ i = \{1, \dots, n\}.$$

*Proof.* For symmetric M, f(x) can be expressed as:

$$f(x) = \sum_{i=1}^{n} \left( M_{ii} x_i^2 + \sum_{j>i}^{n} 2M_{ij} x_i x_j \right)$$
  
=  $\sum_{i=1}^{n} \left( M_{ii} x_i + \sum_{j>i}^{n} 2M_{ij} x_j \right) x_i.$  (21)

The last expression can be equivalently written as

$$f(x) = \sum_{i=1}^{n} \tilde{\rho}_i(x) x_i, \qquad (22)$$

with  $\tilde{\rho}_i$  defined as:

$$\tilde{\rho}_i(x) := M_{ii} x_i + \sum_{j>i}^n 2M_{ij} x_j.$$
 (23)

Note that each  $\tilde{\rho}_i(x)$  is linear in x; its maximum and minimum over the unit ball, defined as  $\underline{\rho}_i$ ,  $\overline{\rho}_i$  respectively, so  $\underline{\rho}_i \leq \tilde{\rho}_i(x) \leq \overline{\rho}_i$ , are from Proposition  $\overline{3}$ :

$$\overline{\rho}_i := \sqrt{M_{ii}^2 + 4\sum_{j>i}^n M_{ij}^2}, \quad \underline{\rho}_i := -\overline{\rho}_i, \tag{24}$$

hence the TS model arising from (22) will be:

$$f(x) = \sum_{i=1}^{n} \left( \mu_{i0}(x) \overline{\rho}_i + \mu_{i1}(x) \underline{\rho}_i \right) x_i$$

and the map in (14) will be the interval:

$$f^{si}(x,\mathcal{A}) = \left[ -\sum_{i=1}^{n} \overline{\rho}_{i} |x_{i}|, \sum_{i=1}^{n} \overline{\rho}_{i} |x_{i}| \right]$$
 (25)

So, from Definition 2 we have

$$\bar{\sigma} = \max_{\|x\|=1} \sum_{i=1}^{n} 2\overline{\rho}_i \cdot |x_i|$$

and such maximum on the unit sphere (Proposition 3 again) is given by:

$$\bar{\sigma} = \sqrt{\sum_{i=1}^{n} 4\overline{\rho}_i^2}$$

which, substituting (24), results in:

$$\bar{\sigma}^2 = \sum_{i=1}^n 4 \left( M_{ii}^2 + 4 \sum_{j>i}^n M_{ij}^2 \right) = 4 \sum_{i=1}^n \left( M_{ii}^2 + 2 \sum_{j\neq i}^n M_{ij}^2 \right)$$

which can be bounded as

$$\bar{\sigma} \ge 2\sqrt{\sum_{i=1}^{n} \left(M_{ii}^2 + \sum_{j \ne i}^{n} M_{ij}^2\right)} \tag{26}$$

where the term at the right-hand side of the inequality is, actually, twice the Frobenius-norm, i.e.,  $\bar{\sigma} \geq 2\|M\|_F$ .

The key idea for the theorem is the fact that the above bound is tight if M is diagonal, i.e.,

$$\bar{\sigma} = 2||M||_F = 2\sqrt{\sum_{i=1}^n \lambda_i^2}.$$
 (27)

Hence, if the representation of  $f(x) = x^T M x$  had been chosen in diagonalised coordinates  $x^T V \Lambda V^T x$ ,  $V^T x = \eta$ ,  $f(\eta) = \eta^T \Lambda \eta$ , the resulting  $\bar{\sigma}$  would have been the lowest possible one. Note that, as M is symmetric, there exists an orthonormal basis of eigenvectors so the transformation  $\eta = V^T x$  preserves the norm and, hence, exploration over  $\|x\| = 1$  is the same as exploring over  $\|\eta\| = 1$ . This proves that the diagonalised representation is WCW-optimal.

**Remark 1.** If the modelling region is not the unit ball, a scaling/change of variable should be carried out before obtaining the optimal TS model for a quadratic nonlinearity so that the new modelling region coincides with the one required in the above theorem.

**Remark 2.** If the function to be modelled is non-quadratic, the diagonalisation-based approach no longer applies. However, close to x = 0, the function may be approximated to:

$$f(x) \approx Jx + \frac{1}{2}x^T Hx$$

where J is the Jacobian and H is the Hessian at x=0. Hence, the coordinate changes arising from diagonalisation of the Hessian would obtain a TS model guaranteed to be optimal in a small enough sphere around the origin (so that higher-order terms can be neglected).

In later developments, the optimal shape-independent TS model from (20) will be denoted as  $f^{si,opt}(x)$ .

# V. TS MODELS WITH OPTIMAL PERFORMANCE IN A SUBSPACE

As discussed in the problem statement, an accurate fit of the shape-independent TS model in the performance-critical subspace  $\mathcal{C}$  is often needed. This issue will be now adddressed.

#### A. Restrictions

**Definition 5** (Restriction of a function or set-valued map). The restriction of a function  $f: \mathbb{R}^n \to \mathbb{R}$  to a linear q-dimensional (q < n) vector subspace C, will be denoted as  $f|_C: C \to \mathbb{R}$ , trivially defined as  $f|_C(x) := f(x) \ \forall x \in C$ . An analogous definition will be assumed for the restriction of a set valued map, such as  $f^{si}$ , i.e.,  $f^{si}|_C(x) := f^{si}(x) \ \forall x \in C$ .

Consider now the subspace C being defined as:

$$C = \{ x \in \mathbb{R}^n : \exists \eta \in \mathbb{R}^q \text{ s.t. } x = H\eta \}$$
 (28)

thus, being H a  $n \times q$  matrix mapping from canonical coordinates in  $\mathbb{R}^q$  to  $\mathcal{C}$ .

Then, abusing the notation, the restriction can be also defined in terms of q-dimensional subspace coordinates  $\eta$ , as

$$f|_{\mathcal{C}} : \mathbb{R}^q \mapsto \mathbb{R}$$
  
$$f|_{\mathcal{C}}(\eta) := f(H\eta), \quad \eta \in \mathbb{R}^q$$
 (29)

Given a function and its restriction, a complementary function can be defined fulfilling the following proposition.

**Proposition 7** (Complementary function). For any subspace C, a function  $f: \mathbb{R}^n \to \mathbb{R}$  can be decomposed as:

$$f(x) = f_{\neg \mathcal{C}}(x) + f|_{\mathcal{C}}(x)$$

where  $f_{\neg C}(x)$ , denoted as complementary function, fulfills  $f_{\neg C}(x) = 0$  for  $x \in C$ .

*Proof.* It is evident, setting 
$$f_{\neg C}(x) := (f(x) - f|_{C}(x))$$
.

Consider an invertible  $n \times n$  matrix  $\overline{T} := (H_{\xi} H)$ , formed by completing H with suitable n-q linearly independent columns, so the following linear transformation  $T: \mathbb{R}^n \mapsto \mathbb{R}^n$  is set up:

$$x = T(\xi, \eta) := \overline{T} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{30}$$

with  $\eta \in \mathbb{R}^q$  the subspace coordinates, and  $\xi \in \mathbb{R}^{n-q}$  being dummy complementary coordinates. Then, using the above transformation T between x and  $(\xi, \eta)$  and notation (29), we can express:

$$f(x) = f(T(\xi, \eta)) = f^{[1]}(\xi, \eta) + f^{[2]}(\eta)$$
(31)

being  $f^{[1]}$  and  $f^{[2]}$  defined as:

$$f^{[1]}(\xi,\eta) := f(T(\xi,\eta)) - f(T(0,\eta)) = f_{\neg \mathcal{C}}(x)$$
$$f^{[2]}(\eta) := f(T(0,\eta)) = f|_{\mathcal{C}}(\eta)$$

If the original function f verifies f(0)=0 then these functions verify  $f^{[1]}(0,\eta)=0, \ f^{[1]}(0,0)=0, \ f^{[2]}(0)=0.$ 

**Definition 6** (Restriction of a TS model). Consider a box-TS model of f(x) given by (5). The restriction of such model to a linear subspace C given in (28) results in:

$$f|_{\mathcal{C}}(\eta) = \sum_{i=1}^{s} \sum_{j=0}^{1} \mu_{ij}(H\eta) A_{ij} H\eta$$
 (32)

The shape-independent TS model  $f^{si}(x, A)$  from (14) can be also restricted to C, allowing to prove:

**Proposition 8.** The restriction of f to C is contained in the restriction of the shape-independent TS model, i.e.,

$$f|_{\mathcal{C}}(\eta) \in f^{si}|_{\mathcal{C}}(H\eta, \mathcal{A})$$

Proof is obvious from Definition 6 and Proposition 5,

**Proposition 9.** If the subspace C is one-dimensional, denoting  $g(\eta) := f|_{C}(\eta)$ , the optimal TS model of the univariate function  $g(\eta) : \mathbb{R} \mapsto \mathbb{R}$  fulfills

$$g^{si,opt}(\eta) \subseteq f^{si}|_{\mathcal{C}}(H\eta,\mathcal{A})$$
 (33)

for any choice of consequents A in the original model.

*Proof.* Representation (32) is a TS model of  $f|_{\mathcal{C}}(\eta)$ , i.e., of  $g(\eta)$ . Then, Proposition 6 yields the required result.

In subspaces whose dimension might not be one, we can assert the following result for quadratic functions, which extends Theorem 1 in order to consider restrictions:

**Theorem 2.** Consider f(x) being a quadratic function. Denote as  $g(\eta) := f|_{\mathcal{C}}(\eta)$  the restriction of f to  $\mathcal{C}$ , being  $g^{si,opt}(\eta)$  the Hessian-based optimal shape-independent TS model of g. Then:

$$\bar{\sigma}(g^{si,opt}) \le \bar{\sigma}(f^{si}|_{\mathcal{C}}(\eta,\mathcal{A}))$$
 (34)

*Proof.* Proof is analogous to Proposition 9: as  $g(\eta) \subset f^{si}|_{\mathcal{C}}(\eta, \mathcal{A})$ , the Hessian-based representation of  $g(\eta)$  has the lowest maximum uncertainty width, by Theorem 1.

Basically, as intuitively expected, Proposition 9 and Theorem 2 say that it is better (or at least equal) to get a direct TS model on the restriction of a function –left-hand side of (33) and (34)– than, first, modelling on a larger space and, later, restricting the resulting TS model –right-hand side of the referred inequalities—. This motivates using the decomposition in Proposition 7 –equivalently, (31)– to obtain WCW-optimal TS models of the restrictions, as detailed in next section.

#### B. Global models with optimal performance in C

Up to now, results in previous section have discussed optimality of certain TS models defined *only* on a subspace  $\mathcal{C}$  (indeed, restrictions from Definition 5 are meaningless outside  $\mathcal{C}$ ). However, applications usually require TS modelling in all of  $\mathbb{R}^n$  and not only in  $\mathcal{C}$ . Expression (31) comes handy now.

Consider a quadratic function  $f(x) = x^T M x$  and the q-dimensional subspace defined in (28). Consider, too, any of the possible linear transformations T and its associated matrix  $\overline{T}$  in (30), and express f in the new coordinates as:

$$f(x) = \begin{pmatrix} \xi^T & \eta^T \end{pmatrix} F \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{35}$$

for  $F = \overline{T}^T M \overline{T}$ . Using the new coordinates  $(\xi, \eta)$ , the subspace  $\mathcal{C}$  becomes the canonical subspace  $(0, \eta), \eta \in \mathbb{R}^q$ .

Express, then f(x) decomposed in the form (31). Given the fact that  $f^{[1]}(\eta,0)=0$ , as  $f^{[1]}$  is quadratic we can express, with trivial manipulations of matrix F above:

$$f^{[1]}(\xi,\eta) = \begin{pmatrix} \xi^T & \eta^T \end{pmatrix} \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$
(36)

$$f^{[2]}(\eta) = \eta^T F_{22} \eta \tag{37}$$

With suitable choices of H and  $H_{\xi}$  when conforming  $\overline{T}$ , are assumed  $F_{11}$  and  $F_{22}$  diagonal, without loss of generality<sup>4</sup>.

<sup>4</sup>Indeed, if it were not the case, consider the diagonalisations  $F_{22}=V_\eta^T\Lambda_\eta V_\eta,\ F_{11}=V_\xi^T\Lambda_\xi V_\xi.$  Then, the change of variable  $\eta^*=V_\eta\eta,\ \xi^*=V_\xi\xi$  would render a representation:

$$f^{[1]}(\xi^*, \eta^*) = \begin{pmatrix} \xi^{*T} & \eta^{*T} \end{pmatrix} \begin{pmatrix} \Lambda_{\xi} & 0 \\ V_{\eta} F_{21} V_{\xi}^T & 0 \end{pmatrix} \begin{pmatrix} \xi^* \\ \eta^* \end{pmatrix}$$

$$f^{[2]}(\eta^*) = \eta^{*T} \Lambda_{\xi} \eta^*$$

So, replacing a supposed (initial guess)  $\hat{H}_{\eta}$  by a corrected  $H_{\eta}=V_{\eta}\hat{H}_{\eta}$  and  $\hat{H}_{\xi}$  by  $H_{\xi}=V_{\xi}\hat{H}_{\xi}$ , the requested diagonal form would be obtained.

Denote as  $\lambda_i^{[2]}$ ,  $i=1,\ldots,q$  the diagonal elements (i.e., eigenvalues) of  $F_{22}$ ; denote as  $\lambda_i^{[1]}$ ,  $i=1,\ldots,n-q$  the diagonal elements of  $F_{11}$ . Then, in the  $\eta$ ,  $\xi$  coordinates, we

$$f^{[1]}(\xi,\eta) = \begin{pmatrix} \xi^T & \eta^T \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix} \xi \tag{38}$$

Let us denote  $f^{[1]}(\xi,\eta) = \mathcal{F}(\xi,\eta)\xi$ , being  $\mathcal{F}$  the linear function  $\mathcal{F}: \mathbb{R}^n \mapsto \mathbb{R}^{n-q}$  multiplying  $\xi$  in (38), i.e.,

$$\mathcal{F}(\xi, \eta) := \begin{pmatrix} \xi^T & \eta^T \end{pmatrix} \begin{pmatrix} F_{11} \\ F_{21} \end{pmatrix}$$

actually expressed as a  $1 \times (n-q)$  row vector, and denote as  $\mathcal{F}_i$  its elements, for  $1 \leq i \leq n-q$ . Then, (38) can be expressed as:

$$f^{[1]}(\xi, \eta) = \sum_{i=1}^{n-q} \mathcal{F}_i(\xi, \eta) \xi_i$$

Denoting  $\overline{\mathcal{F}}_{i0} := \max_{x \in \Omega} \mathcal{F}_i(T^{-1}x)$ , and  $\overline{\mathcal{F}}_{i1} := \min_{x \in \Omega} \mathcal{F}_i(T^{-1}x)$ , we can express  $f^{[1]}$  as the TS model:

$$f^{[1]}(\xi,\eta) = \sum_{i=1}^{n-q} \sum_{j=0}^{1} \mu_{ij}(x) \overline{\mathcal{F}}_{ij} \xi_i$$
 (39)

Manipulating  $f^{[2]}(\eta) = \sum_{i=1}^{q} \lambda_i^{[2]} \eta_i^2$ , we can express it as:

$$f^{[2]}(\eta) = \sum_{i=1}^{q} \sum_{j=0}^{1} \mu_{ij}(x) \overline{\mathcal{G}}_{ij} \eta_i$$
 (40)

with  $\overline{\mathcal{G}}_{i0} := \max_{x \in \Omega} \lambda_i^{[2]} \eta_i$  and  $\overline{\mathcal{G}}_{i1} := \min_{x \in \Omega} \lambda_i^{[2]} \eta_i$ . From Theorem 2, because of the diagonal representation of  $f^{[2]}(\eta)$ , the above TS model (40) is the WCW-optimal one, in the sense of Definition 3, for the restriction  $f|_{\mathcal{C}}$  in  $\eta$  coordinates.

Combination of the optimal  $f^{[2]}$  with its complementary function  $f^{[1]}$  results in the main theorem of this section:

**Theorem 3.** Given a quadratic function  $f(x) = x^T M x$ , subspace C defined in (28), and the change of variable (30), the TS model

$$f(x) = f^{[1]}(\xi, \eta) + f^{[2]}(\eta)$$

$$= \sum_{i=1}^{n-q} \sum_{j=0}^{1} \mu_{ij}(\xi, \eta) \overline{\mathcal{F}}_{ij} \xi_i + \sum_{i=1}^{q} \sum_{j=0}^{1} \mu_{ij}(\eta) \overline{\mathcal{G}}_{ij} \eta_i$$
(41)

is WCW-optimal in subspace C in the sense of Definition 4.

*Proof.* As the restriction of  $f^{[1]}$  onto  $\mathcal{C}$  is zero, and so it is the restriction to C of the shape-independent TS model arising from (39), i.e.,  $f^{[1],si}(0,\eta) = \{0\}$ , it is easily seen that:

$$f^{si}|_{\mathcal{C}}(\xi,\eta)=f^{[2],si}|_{\mathcal{C}}(\eta)$$

so optimality in C is not lost when adding (39) and (40) as the complementary function (and its TS model) is zero on C.  $\square$ 

Note that the diagonal form in  $F_{11}$  is, actually, not needed in the proof. However, there is no loss of generality in assuming it: it has been stated as such in the above discussion because, being both  $F_{11}$  and  $F_{22}$  diagonal, the same matrices  $F_{11}$ ,  $F_{21}$ 

and  $F_{22}$  in (36) and (37) would be usable for obtaining the optimal model in either the subspace  $\xi = 0$  or  $\eta = 0$ .

Note also that the change of variable should be reverted in  $\xi_i$ ,  $\eta_i$ , in order to have the TS model (41) depending on the original x coordinates instead of the transformed ones (details omitted for brevity).

So, basically the procedure to obtain the optimal model for a subspace C of a quadratic function would be:

- 1) Get a basis H of C, complete it and obtain  $\overline{T}$ , F, and subsequently,  $f^{[1]}$  and  $f^{[2]}$ .
- 2) Obtain the Hessian eigenvectors of  $f^{[1]}$  (yielding diagonal  $F_{11}$ ) and  $f^{[2]}$  (yielding diagonal  $F_{22}$ ).
- 3) Combine both steps in a single change of variable.
- 4) [optional] Express the box-TS model (41) in original coordinates.

The outline of the procedure is illustrated in example below.

**Example 3.** Let us model the function  $f: \mathbb{R}^5 \to \mathbb{R}$  given by  $f(x) = x_1^2$  in the 2-dimensional subspace<sup>5</sup> spanned by  $C = \{x \in \mathbb{R}^5 : x = H\eta, \ \eta \in \mathbb{R}^2\}$  with

$$H = \begin{pmatrix} 1 & 1 & -1 & 0.5 & 1 \\ 2 & 0 & 2 & -4 & -2 \end{pmatrix}^T.$$

In order to avoid scalings so that the unit circle in original coordinates keeps being the unit circle in transformed ones, let us obtain an orthonormal basis of C (overwriting Hwith Matlab command H=orth(H), for instance), as well as an orthonormal basis of its complementary space (Matlab H\_xi=null(H')). With these two basis, we can form the change of variable  $\overline{T}$  in (30), with  $\overline{T}$  actually being an orthonormal matrix. So, variables  $x_1$  to  $x_5$  will be mapped to  $(\xi_1, \xi_2, \xi_3, \eta_1, \eta_2)$  and the above subspace will be, in the new coordinates:

$$\mathcal{C} = \{ (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2) \in \mathbb{R}^5 : \xi_1 = \xi_2 = \xi_3 = 0 \}.$$

Let us check the accuracy on C of several possible TS models: a) Inspection-based coordinates: The first TS model to be considered is the "inspection" one given by:

$$f(x) = h_1(x_1)A_0x + h_2(x_1)A_1x \tag{42}$$

with  $h_1(x_1) = 0.5(x_1 + 1)$ ,  $h_2(x_1) = 1 - h_1(x_1)$ ,  $A_1 =$  $-A_0$ , and  $A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$ . The model yields

$$f^{si}(x) = [-x_1, x_1]. (43)$$

Of course, this would end the classical way of TS modelling in prior literature. The issue under discussion is, however, how accurate is such model in subspace C.

In order to assess the accuracy of (42) on C, let us trivially carry out the change of coordinates T by rewriting  $f^{si}$  in (43) replacing  $x_1$  by its expression on the new coordinates  $(\eta, \xi)$  arising from matrix  $\overline{T}$ 

$$x_1 = -0.1379\xi_1 + 0.6081\xi_2 + 0.1379\xi_3 + 0.3385\eta_1 + 0.6911\eta_2$$

<sup>5</sup>The actual subspace is generated via performance optimisation (9) of a 5th-order linearised model. The actual model matrices and LMIs are omitted because they are not relevant for the time being. Full examples will appear on Section VIII.

As  $\eta$  are subspace coordinates, the restriction  $f^{si}|_{\mathcal{C}}$  is

$$f^{si}|_{\mathcal{C}} = [-(0.3385\eta_1 + 0.6911\eta_2), 0.3385\eta_1 + 0.6911\eta_2]$$

for  $\eta$  ranging in the unit ball ( $\overline{T}$  is orthonormal), the worst-case width (Definition 4) on the subspace  $\mathcal{C}$  is given by twice the norm of  $\kappa := (0.3385, 0.6911)^T$ , by Proposition 3 and the fact that all intervals are symmetric. The result for the TS model (42) is:

$$\bar{\sigma}_{\mathcal{C}} = 2\|\kappa\| = 2 \times 0.7695 = 1.5391$$
 (44)

b) Non-optimised change of variable: Now, let us test the ideas in Section V-A, i.e., that carrying out TS modelling after the change of variable will lead to a better model than the one above, where TS modelling was carried before the change  $x = T(\eta, \xi)$ .

Writing now  $f(x) = x^T M x$  being M the matrix with all its entries equal to zero except the (1,1) term (equal to 1), we can express it in the new coordinates as  $F = \overline{T}^T M \overline{T}$  (not displayed, to save space). The restriction to C, in such  $\eta$  coordinates is  $f|_{C}(x) = \eta^T F_{22}\eta$ , with:

$$F_{22} = \begin{pmatrix} 0.1146 & 0.2339 \\ 0.2339 & 0.4777 \end{pmatrix} \tag{45}$$

where, as expected,  $F_{22} = \kappa \cdot \kappa^T$ . So, we can express  $g(\eta) = f|_{\mathcal{C}}(H\eta) = (0.1146\eta_1 + 0.4679\eta_2)\eta_1 + (0.4777\eta_2)\eta_2$ . Computing the norms of  $(0.1146, 0.4679)^T$  and  $(0, 0.4777)^T$ , by Proposition 3, the resulting TS model yields on the unit circle a shape-independent interval:

$$g^{si} = [-0.4817, 0.4817] \times \eta_1 + [-0.4777, 0.4777] \times \eta_2$$
 (46)

It can be shown that, as  $\eta$  range on the unit ball, the worst-case width in this case will be:

$$\bar{\sigma}_{\mathcal{C}} = 2\sqrt{0.4817^2 + 0.4777^2} = 1.357$$
 (47)

which is lower than that from (44), as expected.

c) Optimised coordinates: Obtaining three orthonormal eigenvectors of the top-left  $3 \times 3$  block of F, arranged in a  $3 \times 3$  matrix  $V_1$ , as well as two orthonormal eigenvectors of the bottom-right  $2 \times 2$  block of said F —the  $F_{22}$  matrix in (45) above—, in a  $2 \times 2$  matrix  $V_2$ , the matrix  $V = blockdiag(V_1, V_2)$  is the Hessian-based coordinate transform of each of the subspaces so that the resulting TS model is optimal in the sense of Theorem 1. The overall coordinate change is  $x = \overline{T} \cdot V \cdot (\xi^T \eta^T)^T$ , yielding the transformed model of  $x_1^2$  as:

$$f(x) = 0.4078 \xi_3^2 + 0.9828 \xi_3 \eta_2 + 0.592 \eta_2^2$$
  
=  $(0.4078 \xi_3 + 0.9828 \eta_2) \xi_3 + 0.592 \eta_2^2$ 

which, as  $\sqrt{0.4078^2 + 0.9828^2} = 1.064$  would lead to a 4-rule TS model on the unit circle given by:

$$f^{si}(x) = f^{si} \left( \overline{T} \cdot V \cdot (\xi^T \eta^T)^T \right)$$

$$= [-1.064, 1.064] \times \xi_3 + [-0.592, 0.592] \times \eta_2$$
(48)

The worst-case width on the subspace  $\xi_1 = \xi_2 = \xi_3 = 0$  results, in this case:

$$\bar{\sigma}_{\mathcal{C}} = 2 \times 0.592 = 1.184$$
 (49)

which is, as expected, lower than that from (44) and (47). In fact, as f(x) was quadratic, Theorem 3 states that there is no other linear coordinate change which gives a better figure for  $\bar{\sigma}_{\mathcal{C}}$  than that in (49).

For illustration, reverting the change of variable by suitable inversion of the transformation matrices, we can write:

$$\xi_3 = \begin{pmatrix} 0.6386 & -0.5473 & 0.1672 & 0.4865 & -0.1672 \end{pmatrix} x$$
  
 $\eta_2 = \begin{pmatrix} 0.7696 & 0.4542 & -0.1388 & -0.4037 & 0.1388 \end{pmatrix} x$ 

so, replacing these expressions in (48) and multiplying by 1.064 and 0.592, respectively, we can define consequent matrices:

$$A_{10} = \begin{pmatrix} -0.8591 & 0.2088 & 0.1779 & -0.5449 & -0.1477 \end{pmatrix}$$
  
 $A_{20} = \begin{pmatrix} -0.1104 & 0.4794 & -0.0990 & 0.3032 & 0.0822 \end{pmatrix}$   
 $A_{11} = -A_{10}, \quad A_{21} = -A_{20}$ 

such that a final (WCW-optimal in the requested subspace) box model (5) with s=2 can be written in the original x coordinates, as an alternative to the naive (42).

The above example has shown how a rewriting of  $x_1^2$  has reduced the uncertainty due to shape-independence from 1.54 to 1.18 in a particular subspace  $\mathcal{C}$ . These manipulations will be able to improve associated LMI results in fuzzy control, as discussed in next section.

# VI. USE OF OPTIMAL TS MODELS IN LMIS

Let us consider a nonlinear system (1), its linearisation A in (2) and the nonlinearities  $\rho(x)$  in (68). The objective of this section is using the previous developments to suggest a TS model which preserves performance of the linearised system proven with some LMIs (arising by suitable transformations of (7) and (8), if needed) by avoiding larger than necessary uncertainty of  $f^{si}$  in key subspaces, given by Proposition 1.

#### A. Effect of the nonlinearity in the Lyapunov equations

When the optimal solution of a LMI for the linearised system  $\dot{x}=Ax$  has been obtained, we are in the situation in (9)–(10). However, the actual performance proved for a nonlinear system would require replacing the linearised state derivatives  $\dot{x}=Ax$  by the nonlinear ones  $\dot{x}=Ax+M\rho(x)$ . Of course, that would destroy the LMI form as  $\rho$  is nonlinear, so the objective is generating a sector-nonlinearity TS model of  $M\rho$  with low conservatism which still allows proving good enough performance with LMIs.

As the perfomance limit in (9) and (10) is hit for  $x \in \mathcal{C}$ , that means that the restriction of  $\rho(x)$  onto subspace  $\mathcal{C}$  must be modelled with precision in order to lose the least possible performance (ideally). So, at first glance, applying Theorem 3 to each element of  $\rho$  might seem a viable solution and so it is, indeed. However, further improvements may be crafted by considering the structure of matrix M and the obtained Lyapunov function. Such ideas will be detailed next.

First, note that, actually, it is not each component of  $\rho(x)$  the magnitude to be precisely modelled. Indeed, let us assume there exists a Lyapunov function whose time-derivative will

require, in turn, the use of the state derivatives. Say, such Lyapunov function having the form  $V(x) := x^T P x$  will give rise to  $\dot{V} = 2x^T P \dot{x}$ , so the difference between (a) the linearised system's behaviour  $\dot{V}_L := 2x^T P A x$  and (b) that from the nonlinear system  $\dot{V}_{NL} := 2x^T P (Ax + M \rho(x))$  is

$$\Xi(x) := \dot{V}_{NL} - \dot{V}_L = 2x^T P M \rho(x) \tag{50}$$

So, the actual term whose uncertainty must be small, when x lies in subspace C, is  $\Xi(x)$ .

Of course, qualitatively speaking, if each element (nonlinearity) in vector  $\rho(x)$  is modelled with a "precise enough" TS system, the overall  $\Xi$  will be precise. However, the different intervals of uncertainty in  $\rho$  will result in a cumulative uncertainty in  $PM\rho$  given by the rules of classical interval arithmetic [17]. Such uncertainty is larger than that arising from the joint evaluation of each element of the vector  $\tilde{g}(x) := PM\rho$ , because interval arithmetic assumes all intervals may vary independently, which is not usually the case. Suitable canonical structure choices for  $\tilde{g}$  will be discussed below to try to avoid such a source of conservatism.

**Remark 3.** In a discrete-time case, Lyapunov equations would have considered  $\Delta V = (Ax + M\rho)^T P(Ax + M\rho)$  so the difference between linear and non-linear would be:  $2x^T A^T P M \rho + \rho^T M^T P M \rho$ . As  $\rho$  is  $O(x^2)$ , then the first term is  $O(x^3)$  and the second one is  $O(x^4)$ . So, in order to minimise the discrepancy close the origin, concentrating on the terms of the order of  $x^3$  will suggest setting, in this case,

$$\Xi(x) = 2x^T A^T P M \rho(x) \tag{51}$$

B. Diagonalisation of  $\Xi(x)$ 

With PM=I (or  $A^TPM=I$  in the discrete case), we have:

$$\Xi(x) = 2x^T \rho(x) \tag{52}$$

Consider, too, that a suitable TS model for each element of  $\rho(x)$ , yielding a shape-independent interval of uncertainty  $\rho_i^{si}(x)$ , is available. Denote the interval of uncertainty of  $\Xi$  as  $\Xi^{si}$ , given by:

$$\Xi^{si}(x) = 2\sum_{i=1}^{n} x_i \rho_i^{si}(x)$$
 (53)

obtained with standard interval arithmetic from intervals  $\rho_i^{si}$ .

**Proposition 10.** In the above case, the interval of uncertainty  $\Xi^{si}$  fulfills:

$$w(\Xi^{si}(x)) \le 2||x|| \cdot \sqrt{\sum_{i=1}^{n} |\rho_i^{si}(x)|^2}$$
 (54)

*Proof.* Proof is a consequence of Proposition 4 and linearity in x of  $\Xi$ .

Otherwise, with  $PM \neq I$ , such bound would need to include terms regarding the norm (singular values) of PM, and not all elements of  $\rho$  would have the same relevance in  $\Xi$  (depending on alignment with the worst-case singular vectors). As handling  $PM \neq I$  results quite cumbersome, the objective of the next developments is showing that some changes of variable can lead to a TS model in which  $\Xi(x)$  has the expression (52) above (in the relevant q-dimensional subspace given by the failing LMIs).

In that way, all directions would have the same influence (so measuring uncertainty on the unit circle is meaningful) and sums in (54) will be carried only for i ranging from 1 to q.

Restrictions: As discussed in Section II, constraints (8) actually fail in a q-dimensional subspace  $\mathcal{C}$ . Considering the change of variable  $\overline{T}$  leading to canonical form (12), the expression of  $\Xi(x)$  in (50) can be written in the new coordinates  $x_c = (\xi^T, \eta^T)^T$  being  $\eta$  the subspace coordinates as:

$$\Xi(x) = 2(\xi^T \ \eta^T) \overline{T}^T P M \rho(\overline{T} x_c) = \xi^T \Gamma_1 \rho(\overline{T} x_c) + \eta^T \Gamma_2 \rho(\overline{T} x_c) := x_c^T \Gamma \rho(\overline{T} x_c)$$
 (55)

where matrix  $\Gamma_1$  is formed by selecting the first n-q rows of  $\Gamma:=2\overline{T}^TPM$  and  $\Gamma_2$  is built with the last q rows of  $\Gamma$ . Abusing the notation, we will define  $\rho(\xi,\eta):=\rho(\overline{T}x_c)$ .

As subspace  $\mathcal{C}$  is the one with  $\xi=0$ , the proposed goal of the optimal TS modelling is producing a WCW-optimal model of  $\tilde{\rho}(\eta) := \Gamma_2 \rho(\xi, \eta)$  in  $\mathcal{C}$  in order to avoid losing performance with respect to (12), because the restriction of  $\Xi$  to  $\mathcal{C}$  is

$$\Xi|_{\mathcal{C}}(\eta) = \eta^T \tilde{\rho}(0, \eta) \tag{56}$$

Such optimal model of  $\tilde{\rho}$  can be obtained by the techniques in previous sections.

Note that (56) is a reduced-dimensionality version of (52). Hence, with straightforward modifications Proposition 10 applies to bound uncertainty in  $\Xi|_{\mathcal{C}}(\eta)$ , simply changing  $\rho(x)$  to  $\tilde{\rho}(0,\eta)$  and summing over q dimensions (instead of n) in (54).

Last, regarding the remaining nonlinearities in  $\Gamma_1\rho(\xi,\eta)$ , a TS model of them can be crafted using any available technique and choice of coordinates as, from (55), it will not influence modelling accuracy in the requested subspace.

#### VII. DISCUSSION

To conclude the theoretical part of the paper, let us discuss potential limitations and future lines of research.

The first issue is the resulting number of rules. For instance, the inspection-based model of  $x_1^2$  –case (a) in Example 3– has two rules, whereas the ones arising from our recommended changes of variables have four rules. In a quadratic case, it can be shown that, for a nonlinearity  $x^T M x$ , with M being of rank m, the overall number of rules associated to the qdimensional subspace  $\mathcal C$  where it should be optimal will be equal to  $2^{\vartheta}$ , being  $\vartheta = min(q, m)$ . Now, regarding the model on the orthogonal subspace  $\mathcal{C}^{\perp}$  , which must forcedly be carried out to build a global model, the nonlinearities should be modelled in the simplest possible way, but, of course, it would require at least two more rules, so the total number of rules will be  $2^{\vartheta+1}$  or higher, for each of the involved nonlinearities. Hence, our approach may result in a larger number of rules than inspection-based ones looking for "simple" TS representations. In complex cases, combining our approach with the approximate complexity reduction techniques mentioned in the introduction might be needed.

Now, the discussion on what is more conservative, few *inexact* rules (classical approach) versus our new proposal of more rules with *optimal fit in a subspace* should be addressed. If the modelling region is small enough, it has been proved that our model will fit inside the projections onto  $\mathcal{C}$  of the

vertex models of any other TS model. Of course, this fit might not be true on  $\mathcal{C}^{\perp}$  but such possible worse accuracy in the complementary subspace is irrelevant due to the excess performance margin implicit in the *strict* inequality (10). So, for small-enough modelling regions our approach will yield performance equal to or better than alternative options.

In larger modelling regions, for a heavily nonlinear system, the geometry of the state trajectories might change substantially from that around the origin; hence, our proposal cannot claim optimality in such a situation, because the directions critical for performance cease to be those in subspace  $\mathcal C$  from the linearised LMIs (9). Further research is needed in order to generalise the idea in Proposition 1 to, for instance, LMIs arising from preexisting TS models, incorporating a set of  $A_i$  instead of the linearisation A; however, some technical difficulties arise. Detailed analysis of the issues arising with non-quadratic nonlinearities far from the origin is also needed in further research.

A last observation is the fact that, if LMIs were shape-dependent (such as [12]), in an ideal case, as "all" possible TS models are equivalent rewritings of the nonlinearities, a "perfect" shape-dependent approach should give the same results whichever the TS model used; however such perfect algorithm has not yet been discovered.

#### VIII. ADDITIONAL EXAMPLES

**Example 4.** This first example illustrates the advantages of the proposed approach in order to obtain an optimal model in the sense of finding the maximum decay rate of the following continuous nonlinear system:

$$\dot{x} = \begin{pmatrix} -3x_1 - 2x_2 - x_3 \\ -4x_2 - x_3 \\ -8(x_1^2 + x_1) + x_2 - 2x_3 \end{pmatrix}$$
 (57)

where  $x \in \Omega$ ,  $\Omega \subset \mathbb{R}^3$  being a spherical modelling region (several values for its radius being tested later on). Taking into account the single nonlinearity  $x_1^2$  in (57), a conventional inspection-based 2-rule TS model can be obtained:

$$\dot{x} = \sum_{i=1}^{2} \mu_i(x) \begin{pmatrix} -3 & -2 & -1 \\ 0 & -4 & -1 \\ -8(\alpha_i + 1) & 1 & -2 \end{pmatrix} x \qquad (58)$$

with  $\mu_1(x) = \frac{x_1 - \alpha_2}{\alpha_1 - \alpha_2}$ ,  $\mu_2(x) = 1 - \mu_1(x)$ ,  $\alpha_1 = \max_{x \in \Omega} x_1$ ,  $\alpha_2 = \min_{x \in \Omega} x_1$ . Maximum decay rate  $\gamma^{opt}$  for  $V = x^T Px$  under TS model (58) is obtained maximising  $\gamma > 0$  subject to:

$$P > 0$$
,  $A_i^T P + P A_i < -2\gamma P$ ,  $i = \{1, 2\}$  (59)

These conditions fulfill Assumption 1 and are in GEVP form. The above model's performance will be compared with the maximum decay rate obtained from the proposed modeling technique. To that end, (57) is rewritten as (68), i.e.,

$$\dot{x} = Ax + M\rho(x) \tag{60}$$

where A is the linearization of (57),  $M = \begin{pmatrix} 0 & 0 & -8 \end{pmatrix}^T$  and,  $\rho(x) = x_1^2$ . Once  $\rho(x)$  is defined, the GEVP test of decay rate (59) was applied for the linearised system, i.e. (60) with  $\rho(x) = 0$ , and achieved a  $\gamma^{opt} = 1$ . No TS model will be able, of course, of getting a faster decay (Proposition 2).

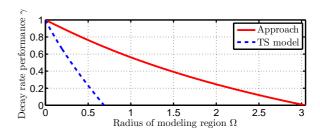


Fig. 1. Comparison of performance of the proposed approach in a solid line against the TS model (58) in dotted lines.

The one-dimensional subspace  $C = \{x \in \mathbb{R}^3 : x = (-0.1764 \ 0.3014 \ 0.9371)^T \eta, \eta \in \mathbb{R}\}$  is the one which prevents the linearised decay-rate problem to progress any further: its basis correspond to the eigenvector of the matrix  $A^TP + PA + 2\gamma^{opt}P$  associated to the minimal eigenvalue  $(\lambda_{min} = 0 \text{ due to } \gamma^{opt})$ .

Following identical modelling procedures to those in Example 3 with the above subspace, the resulting TS model (4 rules), when used in decay-rate optimisation, gives the results in Figure 1 (solid red line), which clearly improve over the first proposed TS model (58) (dashed-blue line). For instance, the improved modelling can prove marginal stability ( $\gamma=0$ ) up to a radius of the modelling region of 3, whereas the conventional non-optimised model only proves stability up to radius 0.7. For any of the radius values in the figure, the proposed TS model gets a faster decay (larger  $\gamma$ ).

**Example 5.** This example illustrates the advantages of the proposed approach for  $\mathcal{H}_{\infty}$  control synthesis. Consider the following discrete-time nonlinear system

$$x[k+1] = \begin{pmatrix} 0.5x_1 + 0.8x_2 + x_1 \sin x_2 + x_2^2 + w_1 - 0.2u \\ 0.25x_1 - 0.45x_2 + 1.5x_1 \sin x_2 + w_2 + 0.1u \end{pmatrix}$$

$$y[k] = \begin{pmatrix} 0.5x_1 & 0.5u \end{pmatrix}^T$$
(61)

where  $x \in \Omega$ ,  $\Omega \subset \mathbb{R}^2$  is the state,  $w \in \mathbb{R}^2$  is the perturbation,  $u \in \mathbb{R}$  is the control input. For comparison a set of 4-rule TS models are proposed, based on extracting the state as a common factor in three different ways:

$$x[k+1] = A_{[i]}^{TS}x + Bu + Ew, \quad i = \{1, 2, 3\}$$
$$y[k] = Cx + Du$$
 (62)

where  $A_{[i]}^{TS}$  denotes the arrangement of nonlinearities of the i-th TS model as follows (see Remark 4 in Appendix):

$$\begin{split} A_{[1]}^{TS} &= \begin{pmatrix} 0.5 & 0.8 + x_1 \operatorname{sinc} x_2 + x_2 \\ 0.25 + 1.5 \operatorname{sin} x_2 & -0.45 \end{pmatrix} \\ A_{[2]}^{TS} &= \begin{pmatrix} 0.5 + \operatorname{sin} x_2 & 0.8 + x_2 \\ 0.25 + 1.5 \operatorname{sin} x_2 & -0.45 \end{pmatrix} \\ A_{[3]}^{TS} &= \begin{pmatrix} 0.5 & 0.8 + x_1 \operatorname{sinc} x_2 + x_2 \\ 0.25 & -0.45 + 1.5 x_1 \operatorname{sinc} x_2 \end{pmatrix} \end{split}$$

being  $\operatorname{sinc}(\alpha) := \frac{\sin(\alpha)}{\alpha}$ , and the respective constant matrices:

$$B\!=\begin{pmatrix} -0.2\\0.1\end{pmatrix},\ C\!=\begin{pmatrix} 0.5&0\\0&0\end{pmatrix},\ D\!=\begin{pmatrix} 0\\0.5\end{pmatrix},\ E\!=\begin{pmatrix} 1&0\\0&1\end{pmatrix}.$$

Of course, all representations are equivalent. Subsequently, corresponding  $TS_{[i]}$  models are obtained using the maximum and minimum in  $\Omega$  of each element of its respective  $2\times 2$  matrix  $A_{[i]}^{TS}$  (details omitted).

Now, in order to apply the methodology in this paper, let us first linearise and then rewrite (61) in the form (68), i.e.,

$$x[k+1] = Ax + Bu + Ew + M\rho(x)$$
$$y[k] = Cx + Du$$
 (63)

with

$$A \!=\! \begin{pmatrix} 0.5 & 0.8 \\ 0.25 & -0.45 \end{pmatrix}, \ M \!=\! \begin{pmatrix} 1 & 1 \\ 0 & 1.5 \end{pmatrix}, \ \rho(x) \!=\! \begin{pmatrix} x_2^2 \\ x_1 \sin x_2 \end{pmatrix}$$

where  $\rho(x)$  is the vector of nonlinearities present in the system. Then, sufficient conditions for the well-know  $\mathcal{H}_{\infty}$  Lyapunov inequality  $\Delta V + y^T y - \gamma_{opt}^2 w^T w \leq 0$  are posed minimising  $\gamma$  subject to:

$$\begin{pmatrix} -X & (*) & (*) \\ A_i X + BF_i & -X + E\gamma^{-2} E^T & (*) \\ CX + DF_i & 0 & -I \end{pmatrix} \le 0$$
 (64)

where X and  $F_i$  are decision variables given by the Lyapunov function  $V = x^T X^{-1} x$  and the control gains, respectively [3]. (\*) refers to completion to obtain a symmetric matrix.

Let us now consider the linearised case, i.e., (63) with  $\rho(x)=0$ , in order to obtain the relevant subspaces where modelling must be precise. In that case, from straightforward Schur complement manipulations, it can be proved that the above problem is feasible if and only if it is so for the worst-case disturbance  $w^*=(\gamma^2I-E^TPE)^{-1}E^TP(A+BK)x$ . Replacing such disturbance in the  $\mathcal{H}_{\infty}$  inequality, it can be proved that if (64) holds, equivalently, P>0 and

$$x^{T} \left( (A+BK)^{T} (P^{-1} - E\gamma^{-2}E^{T})^{-1} (A+BK) - P + (C+DK)^{T} (C+DK) \right) x \le 0 \quad \forall x.$$
 (65)

also hold, with  $P := X^{-1}$  and  $K := FX^{-1}$ . These conditions, too, fulfill Assumption 1 even if (65) is not directly in LMI form; note that convexity in  $A_i$  is ensured by the presence of  $-X + E\gamma^{-2}E^T$  in the constraints in (64).

As (64) is an LMI problem in  $(X, F_i, \gamma^{-2})$ , solving it for the linearized system results in  $\gamma^{opt} = 1.4644$ . Replacing the obtained optimal decision variables in (65), the one-dimensional subspace where (65) marginally holds, see expression (9), is  $C = \{x \in \mathbb{R}^2 : x = \begin{pmatrix} 0.9518 & 0.3068 \end{pmatrix}^T \eta, \eta \in \mathbb{R} \}$ , this allows to obtain a transformation matrix  $x = \overline{T}x_c$  that rewrites (65) in the form (12), i.e.,

$$\overline{T} = \begin{pmatrix} -0.3068 & 0.9518 \\ 0.9518 & 0.3068 \end{pmatrix}.$$

Now, in order to generate the optimal TS model, following analogous reasoning to (51) and (55) in Section VI, nonlinearity enters the Lyapunov equation (65), disregarding quadratic terms in  $\rho$ , as:

$$\Xi = 2x_c^T \overline{T} (A + BK)^T (P^{-1} - E\gamma^{-2} E^T)^{-1} M \rho (\overline{T} x_c)$$

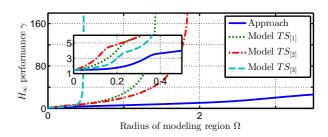


Fig. 2. Comparison of proven LMI performance between the proposed approach in solid line and the TS models in (62) in dotted lines labelled as  $TS_{[i]}$ . A close-up zoom of the radius range [0,0.5] is also provided.

yielding

$$\Xi = x_c^T \Gamma \rho(\overline{T}x_c) = x_c^T \begin{pmatrix} 0.7051 & -0.4719 \\ 1.3140 & 2.4020 \end{pmatrix} \rho(\overline{T}x_c) \quad (66)$$

Defining now  $\hat{\rho}(\xi,\eta)$  as  $\hat{\rho}(\xi,\eta) := \Gamma \rho(\overline{T}x_c)$ , we can express  $\Xi = x_c^T \hat{\rho}(\xi,\eta) = (\xi^T \eta^T) \hat{\rho}(\xi,\eta)$ . In this way, (66) has the overall form (52) and the restriction can be written as (56), after the proposed coordinate changes. By decomposing  $\hat{\rho}$  as in (31), standard TS modelling in the coordinates  $\xi$  and  $\eta$  of  $\hat{\rho}^{[1]}$  and  $\hat{\rho}^{[2]}$ , respectively, is used to find an expression analogous to (41), which concludes the modelling step (details omitted).

This allows finding a 16-rule TS model fulfilling the bound in Proposition 10 for subspace C, with the sum ranging in a single dimension. Such model has been used for locally searching for controllers guaranteeing an  $\mathcal{H}_{\infty}$  norm for the nonlinear system in circular regions.

For illustrative purposes, we present Fig. 2, which describes the performance bound  $\gamma^{opt}$  using the TS models in LMIs (64), as the radius of modeling region  $\Omega$  increases. Results corresponding to the proposed modeling technique are presented in a solid line, whereas those three TS models in (62) are presented with dotted lines. As expected, all TS models match the performance of the linearization at the origin, but as they move away from it, the proposed subspace-based approach outperforms the alternative ones (which disregard the linearised geometry), yielding a lower disturbance-rejection bound (theoretical optimality only for small modelling regions). For completeness, a time simulation presenting ||y[k]|| = ||Cx + Du|| (whose squared sum should be minimised, according to the requested performance criterion) appears in figure 3. The original nonlinear dynamics is simulated with four different controllers arising from each of the TS modelling alternatives for a step disturbance. Results show that the achieved figures for ||y[k]|| with our proposed methodology are the lowest ones, and time response is better damped. Of course, we cannot prove that the chosen disturbance is the worst-case one for the nonlinear system (it is an unsolved problem) but, at least with the chosen w[k], the observed performance of our proposal is better than the classical ones.

Note that optimality in the above examples is only claimed "close enough" to the origin: in fact, changing parameters in system matrices, alternative setups can be built in which the proven performance of the proposed approach is only optimal up to a certain radius.

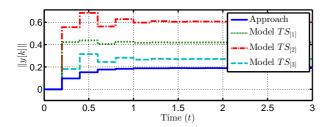


Fig. 3. Time response of ||y[k]|| for a step  $w=(-0.17,-0.09)^T$  of the nonlinear system, with the four controllers arising from each of the considered TS modelling options, using a modelling region radius of 0.3.

#### IX. CONCLUSIONS

This paper has presented a Hessian and subspace based methodology to minimise the conservatism of TS models to be later used in shape-independent LMI conditions. The presented procedure is optimal (in minimax worst-case uncertainty width) for TS models of quadratic functions, hence approximately optimal for any smooth nonlinearity close enough to the origin. Different examples prove that, indeed, the uncertainty width measures, as well as decay-rate and  $\mathcal{H}_{\infty}$  performance figures are better with the proposed TS modelling technique than those obtained with frequently used "inspection" and "extraction of factors" ideas. The LMIs discuss only the preservation of linearised performance. Other LMI setups and reduction/simplification of the number of rules, as well as optimality for large modelling regions, are matter of future research.

#### APPENDIX

Review of sector nonlinearity approach

Consider a single-input nonlinear function  $f : \mathbb{R} \to \mathbb{R}$ . If f(0) = 0 and its derivative is continuous, then the function:

$$\tilde{f}(x) = \begin{cases} f(x)x^{-1} & x \neq 0\\ \lim_{x \to 0} f(x)x^{-1} & x = 0 \end{cases}$$

can be defined, because the required limit exists; furthermore,  $\tilde{f}(x)$  is continuous. As  $f(x)=\tilde{f}(x)x$ , we can trivially express, in any compact region  $\Omega\subset\mathbb{R}$ :

$$f(x) = h(x)\tilde{f}_0x + (1 - h(x))\tilde{f}_1x \tag{67}$$

being  $\tilde{f}_0 = \max_{x \in \Omega} \tilde{f}(x)$  and  $\tilde{f}_1 = \min_{x \in \Omega} \tilde{f}(x)$ .

In a multi-input case, a nonlinear model (1), as f is linearisable at the origin, can be expressed as:

$$\dot{x}(t) = \hat{f}(x, \rho(x))$$

being  $\hat{f}(\cdot,\cdot)$  a linear function and  $\rho$  a vector of nonlinearities, i.e., there exists  $\rho(x):\mathbb{R}^n\mapsto\mathbb{R}^s$  such that (1) is equivalent to

$$\dot{x} = Ax + M\rho(x) \tag{68}$$

being A the Jacobian of f at x=0, from (2). Note that representation (68) may be not unique (there might be several options in choosing M and  $\rho$ ). Each element of  $\rho$ , denoted as  $\rho_i(x)$  will be a function  $\rho_i(x):\mathbb{R}^n\mapsto\mathbb{R}$ ; subindex will be omitted in notation if  $\rho$  has a single element.

Consider a modelling region  $\Omega$ . The objective of sector-nonlinearity TS modelling is bounding in  $\Omega$  each element  $\rho_i(x)$ , by two linear functions: actually , finding linear  $\eta_i(x)$  such that  $\alpha\eta_i(x) \leq \rho_i(x) \leq \beta\eta_i(x)$  for some  $\alpha$ ,  $\beta$  in  $\mathbb{R}$ .

Considering, then, a particular  $\rho_i(x)$ . Assume there exists a function of the state  $\eta_i(x) : \mathbb{R}^n \to \mathbb{R}$ , such that  $\rho_i(x) = 0$  for all x in  $B = \{x \in \Omega : \eta_i(x) = 0\}$ . Defining  $\tilde{\rho}_i(x)$  as:

$$\tilde{\rho}_i(x) = \frac{\rho_i(x)}{\eta_i(x)}, \quad \text{for } x \notin B$$
 (69)

Then, if the following limit exists for all  $x \in B$ :

$$\gamma(x) = \lim_{\xi \to x} \tilde{\rho}_i(\xi)$$

then, the definition of  $\tilde{\rho}_i(x)$  can be extended, i.e., defined everywhere in  $\Omega$  (including the set B), by defining  $\tilde{\rho}_i(x) = \gamma(x)$  for  $x \in B$ , and (69) elsewhere. It is well known that the limit  $\gamma(x)$  exists, and the resulting extended  $\tilde{\rho}_i(x)$  is continuous in  $\Omega$  if  $\rho_i(x)$  has continuous first derivatives (which it does, by assumption). Hence, the relationship

$$\rho_i(x) = \tilde{\rho}_i(x)\eta_i(x)$$

holds in all  $\Omega$ . By compactness of  $\Omega$ , the bounding:

$$\left(\min_{y\in\Omega}\tilde{\rho}_i(y)\right)\eta_i(x) \le \rho_i(x) \le \left(\max_{y\in\Omega}\tilde{\rho}_i(y)\right)\eta_i(x) \quad (70)$$

entails that each  $\rho_i(x)$  can be expressed as an interpolation  $\rho_i(x) = w_i(x) \times \tilde{\rho}_{i0} + (1 - w_i(x)) \times \tilde{\rho}_{i1}$  where:

$$\tilde{\rho}_{i0} = \max_{y \in \Omega} \tilde{\rho}_i(y), \ \tilde{\rho}_{i1} = \min_{y \in \Omega} \tilde{\rho}_i(y), \ w_i(x) = \frac{\tilde{\rho}_i(x) - \tilde{\rho}_{i1}}{\tilde{\rho}_{i0} - \tilde{\rho}_{i1}}$$

When bounding each  $\rho_i$ ,  $i = \{1, ..., s\}$  as above discussed,  $\eta_i(x)$  are linear, say  $\eta_i(x) = N_i^T x$ , denoting by  $M_{[i]}$  the *i*-th column of M in (68), and  $\mu_{i0}(x) = w_i(x)$ ,  $\mu_{i1}(x) = 1 - w_i(x)$ , the result will be an expression in the form:

$$\dot{x} = \sum_{i=1}^{s} \sum_{j=0}^{1} \mu_{ij} \left( A + M_{[i]} \tilde{\rho}_{ij} N_i^T \right) x := \sum_{i=1}^{s} \sum_{j=0}^{1} \mu_{ij} A_{ij} x \quad (71)$$

which is a box-like parameter uncertainty description [18, Eq. (2)]. Converting box representations to tensor-product ones [3], [19], [7] can be done in a straightforward way. Indeed, as  $\sum_{j=0}^{1} \mu_{ij}(x) = 1$  we can express (71) as:

$$\dot{x} = \sum_{i=1}^{s} \sum_{j=0}^{1} \left( \mu_{ij} \left( \prod_{k \neq i} \sum_{j=1}^{2} \mu_{kj} \right) A_{ij} \right) x 
= \sum_{k=1}^{2^{s}} \left( \prod_{i=1}^{s} \mu_{i \, bit_{i}(k)} A_{\text{bit}_{i}(k)i} \right) x = \sum_{k=1}^{2^{s}} h_{k} A_{k} x$$
(72)

where  $\mathrm{bit}_i(k) \in \{0,1\}$  is the *i*-th bit  $(1 \le i \le s)$  of the binary representation of integer k-1, and  $h_k$  is thus formed as the product of a specific combination of  $w_i$  (or  $1-w_i$ , depending on the corresponding binary digit).

In summary, once M,  $\rho(x)$  and the linear functions  $\eta_i(x)$  are chosen, the above well-known steps end up in a TS system with power-of-two vertex models (box or tensor-product form).

Ideally, the TS model of a scalar expression  $\rho(x) = \lambda x^2$  is  $\lambda \left( \mu_1 \underline{x} + (1 - \mu_1) \overline{x} \right) x$ , being  $\underline{x}$  and  $\overline{x}$  the minimum and maximum values of x in a modelling region. For quadratic multivariable functions, the idea is generalised to the Hessian coordinates applying the above to each of the squares in (13).

**Remark 4.** In some references, TS models are generated from a representation

$$\dot{x} = A(x)x\tag{73}$$

instead of (68), such that  $\lim_{x\to 0} A(x)$  exists; then, maximum and minimum in the modelling region of each element of matrix A(x) are sought. Evidently, this is a particular case of the above procedure, considering  $\eta_i = x_i$ .

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