Factorization of (q,p)-summing polynomials through Lorentz spaces

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Abstract

We present a vector valued duality between factorable (q,p)-summing polynomials and (q,p)-summing linear operators on symmetric tensor products of Banach spaces. Several applications are provided. First, we prove a polynomial characterization of cotype of Banach spaces. We also give a variant of Pisier's factorization through Lorentz spaces of factorable (q,p)-summing polynomials from $C(K)$-spaces. Finally, we show a coincidence result for (q,p)-concave polynomials.

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1. Introduction

Polynomials burst upon the scene of linear operator theory via duality, for the scalar valued case, and linearization, for the vector valued case. These two procedures allow to identify the space of continuous m-homogeneous polynomials defined on a Banach space X (with values in a Banach space Y) with the space of continuous linear operators defined on the symmetric projective tensor product of X (and values in Y). Some subspaces of polynomials have also been studied from the duality point of view. For instance, the space of integral polynomials on a Banach space X can be identified with the dual space of the injective symmetric tensor product of X (see for instance [14]). Our main objective is to study the duality related to the subspace of (q, p)-summing linear operators and prove that the space of all vector valued factorable (q, p)-summing polynomials on X can be identified with the space of all vector valued (q, p)-summing linear operators defined on the symmetric injective tensor product of X. Our main application of this duality theory is to get a polynomial version of Pisier factorization theorem: factorable (q, p)-summing polynomials factorize through a Lorentz space $L_{q,1}$ on a probability Borel measure space. Further characterizations of two important properties related to the Banach space theory, are given in terms of polynomials. The first one characterizes cotype in terms of factorable summing polynomials and the second one gives the equivalence between (q, p)- and $(q, 1)$-concavity for polynomials on Banach lattices.

One of the main problems when dealing with summability of homogeneous polynomials is that linear properties are not inherited, in general, by polynomials. Opposed to what happens in the case of linear maps, domination and factorizations of multilinear maps, and their symmetric version m-homogeneous polynomials, are not equivalent properties, in the sense that a domination property for example, p-summability, does not coincide in general with a canonical factorization through the inclusion map $i\colon C(K) \to L^p(K, \mu)$ for a compact space K and a regular Borel probability measure μ on K. This is the case of dominated polynomials, which is one of the first classes that were considered when trying to state a parallel theory to the theory of linear summing operators. For instance, it is well known that the linearization of a dominated polynomial may not be absolutely summing (see [1]).

The factorization results for operators play an important role in the modern Banach space theory. Domination inequalities can be considered a first step in the factorization process (see [3, 10, 11, 17]). It is of special interest those classes of operators that satisfy a domination, as well as some relevant factorization scheme. This was the primary motivation for the introduction in [22] and [19] of the classes of factorable p-dominated polynomials and strongly factorable p-summing polynomials, subspaces of the corresponding class of p-dominated polynomials satisfying, besides the domination property, factorization schemes similar to the ones appearing in the linear case.

Furthermore, the second class relates summability of linear operators with summability of homogeneous polynomials by means of the linearization: an m-homogeneous polynomial P is strongly factorable p-summing if and only if its π-linearization P_L on projective tensors products is absolutely p-summing. In particular, in this paper, we will show that the first class also relates the summability of a polynomial with the summability of a linearization via injective tensors products. We prove that the space of factorable (q,p)-summing polynomials on a Banach space X is isometrically isomorphic to the space of all (q,p)-summing linear operators defined on the injective symmetric tensor product of X via the ε-linearization of the polynomial. As a direct application we extend the known characterization of cotype of a Banach space in terms of polynomials: a Banach space Y has cotype $q > 2$ if ad only if every homogeneous polynomial is factorable $(q,1)$-summing.

We also prove that this ε-linearization has a crucial role when analyzing variants of the Pisier factorization theorem for homogeneous polynomials.

Recall that, if $1 \leq p < q < \infty$, K is a compact Hausdorff space and Y is a Banach space, the factorization theorem of Pisier [21] establishes that an operator T from $C(K)$ to Y is (q,p)-summing with if and only if there is a probability Borel measure μ on K such that T admits a factorization

$$T\colon C(K) \xrightarrow{j} L_{q,1}(\mu) \xrightarrow{S} Y,$$

where j is the inclusion map and $L_{q,1}(\mu)$ is a Lorentz space.

Using several results on the structure of symmetric tensor products of $C(K)$-spaces, we give a complete description of the m-homogeneous polynomials that satisfy a factorization through the a Lorentz space $L_{q,1}(\mu)$ with the canonical polynomial $\Delta\colon C(K) \to L_{q,1}(\mu)$ given by $\Delta(f) := f^m$ for every $f \in C(K)$.

Following the circle of ideas around (q,p)-summability of linear operators and Pisier's Theorem [21], we introduce the definition of a factorable version of (q,p)-concave operators. It is shown that, as in the linear case, for $1 \leq p < q < \infty$, (q,p)-concave polynomials and $(q,1)$-concave polynomials coincide.

2. Preliminaries and notation

We use standard definitions and notation. $\mathbb{K}$ will denote the field of real or complex numbers.

Recall that, given m Banach spaces $X_1, \dots, X_m$, we denote by ε the injective norm on the tensor product $X_1 \otimes \cdots \otimes X_m$ given by

$$\varepsilon(u) = \sup\Big\{ \Big| \sum_{j=1}^{n} \langle x_j^1, \varphi_1 \rangle \cdots \langle x_j^m, \varphi_m \rangle \Big|;\ \varphi_1 \in B_{X_1^*}, \dots, \varphi_m \in B_{X_m^*} \Big\},$$

$u \in X_1 \otimes \cdots \otimes X_m$, where $\sum_{j=1}^{n} x_j^1 \otimes \cdots \otimes x_j^m$ is any representation of u. The completion of this space is the injective tensor product $X_1 \widehat{\otimes}_\varepsilon \cdots \widehat{\otimes}_\varepsilon X_n$ of Banach spaces $X_1, \dots, X_n$.

It is well-known that, if K is compact Hausdorff space, then the injective tensor product $C(K) \widehat{\otimes}_\varepsilon \cdots \widehat{\otimes}_\varepsilon C(K)$ can be identified with the space of continuous functions $C(K \times \cdots \times K)$.

Consider now a Banach space X and the symmetric m-fold tensor product $\otimes^{m,s} X$, that is defined by the linear span of the elements of $X \otimes \cdots \otimes X$ of the form $x \otimes \cdots \otimes x$, $x \in X$.

We will require two norms on this space:

- The projective s-tensor norm π_s given by

$$\pi_s(z) = \inf\Big\{ \sum_{j=1}^{k} |\lambda_j| \|x_j\|^n;\ k \in \mathbb{N}, z = \sum_{j=1}^{k} \lambda_j x_j \otimes \cdots \otimes x_j \Big\},$$

for $z \in \otimes_{m,s} X$.

- The injective s-tensor norm ε_s given by

$$\varepsilon_s(u) = \sup\Big\{ \Big| \sum_{j=1}^{n} \lambda_j \langle x_j, x^* \rangle \cdots \langle x_j, x^* \rangle \Big|;\ x^* \in B_{X^*} \Big\},$$

and this expression makes sense for any representation of u as $\sum_{j=1}^{n} \lambda_j x_j$.

By $\widehat{\otimes}_{\pi_s}^{m,s} X$ and $\widehat{\otimes}_{\varepsilon_s}^{m,s} X$ we mean the completion of $\otimes_{\pi_s}^{m,s} X$ and $\otimes_{\varepsilon_s}^{m,s} X$ respectively.

A mapping $P\colon X \to Y$, defined between Banach spaces X and Y, is called an *m-homogeneous polynomial* if there is an m-linear mapping $A\colon X \times \cdots \times X \to Y$ such that $P(x) = A(x, \dots, x)$. The space of all continuous m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}(^m X; Y)$, and becomes a Banach space equipped with the norm $\sup_{x\in B_X}\|Px\|$, where B_X denotes the closed unit ball of X. If $Y=\mathbb{K}$ we write $\mathcal{P}(^mX)$ for short.

Given $P\in\mathcal{P}(^mX;Y)$, the π-linearization of P is the unique linear operator $P_{L,s}\colon \widehat{\otimes}_{\pi_s}^{m,s}X\to Y$ such that $P_{L,s}(x\otimes\cdots\otimes x)=P(x)$ for all $x\in X$. Notice that Ryan [23] proved that the space $\mathcal{P}(^mX)$, endowed with the usual sup norm, and the strong dual of $\widehat{\otimes}_{\pi_s}^{m,s}X$ are isometrically isomorphic via the correspondence $P\leftrightarrow P_{L,s}$.

Next, we recall that an m-homogeneous polynomial $P\colon X\to\mathbb{K}$ is of *integral type* if there is a regular Borel measure μ of finite variation on B_{X^*}, equipped with the weak*-topology topology, such that

$$P(x)=\int_{B_{X^*}}\langle x,x^*\rangle^m\,d\mu(x^*),\qquad x\in X.$$

We let $\|P\|_{\mathcal{P}_I}$ to be the infimum of all $\|\mu\|$ when μ varies over all measures as in the definition. Let $\mathcal{P}_I(^mX)$ denote the Banach space of all integral polynomials on X equipped with the norm $\|\cdot\|_{\mathcal{P}_I}$. Integral polynomials were introduced by Dineen in [13], where it was shown that the dual of $\widehat{\otimes}_{\varepsilon_s}^{m,s}X$ is isometrically isomorphic to $(\mathcal{P}_I(^mX),\|\cdot\|_{\mathcal{P}_I})$.

Following the classical definition by Piestch, a linear operator $T\colon X\to Y$ defined between Banach spaces is (q,p)-*absolutely summing* $(1\le p\le q<\infty)$ if there is a constant $C>0$ such that for every finite sequence $x_1,\ldots,x_n\in X$

$$\Big(\sum_{j=1}^n\|Tx_j\|^q\Big)^{1/q}\le C\sup_{x^*\in B_{X^*}}\Big(\sum_{j=1}^n|\langle x_j,x^*\rangle|^p\Big)^{1/p}.$$

It is well known that the space of all (q,p)-absolutely summing operators, denoted by $\Pi_{q,p}(X,Y)$, is a Banach space equipped with the norm $\pi_{q,p}$, where $\pi_{q,p}(T)$ for $T\in\Pi_{q,p}(X,Y)$ is the infimum of the constants $C>0$ satisfying the above definition. As usual, we write $\Pi_p(X,Y)$ for short instead of $\Pi_{p,p}(X,Y)$. Note that the same definition makes sense if X and Y are normed spaces.

Let Z be a dense linear subspace of a normed space X. Then a bounded linear map $T\colon X\to Y$ is (q,p)-absolutely summing if and only if its restriction to Z, $T|_Z$, is (q,p)-absolutely summing. Moreover we have $\pi_{q,p}(T)=\pi_{q,p}(T|_Z)$. This follows by a simple observation that the space $\ell_p^w(Z)$ is a dense subspace in the space $\ell_p^w(X)$ of weakly p-summable sequences in X with the weak-p norm given by

$$\|(x_n)\|_p^w = \sup_{x^* \in B_{X^*}} \Big(\sum_{n=1}^{\infty} |\langle x, x^* \rangle|^p \Big)^{1/p}.$$

For $1 \leq q < \infty$, $L_{q,1}(\mu)$ is the Lorentz space on the probability Borel measure space $(K, \mathcal{B}(K), \mu)$ equipped with the norm

$$\|f\| := \int_0^1 f^*(t) t^{1/q-1} \, dt.$$

Here, $\mathcal{B}(K)$ is the σ-algebra of the Borel sets in a compact Hausdorff space K and f^* is the decreasing rearrangement of $|f|$ with respect to the distribution function $s \mapsto \lambda(\{|f| > t\})$, $t \in [0, \infty)$.

The notions that we use of p-convexity and (q,p)-concavity for linear operators on Banach lattices are standard and can be found, for example, in [16] and [18].

3. Duality for spaces of summing polynomials

Let $1 \leq p \leq q < \infty$ and let X be a Banach space. An m-homogeneous polynomial $P\colon X \to Y$ is said to be *factorable* (q,p)*-summing* if for every positive integers M, N and all $M \times N$ matrices $\big(\lambda_{jk}\big)_{jk}$ and $\big(x_{jk}\big)_{jk}$, with $\lambda_{jk} \in \mathbb{K}$ and $x_{jk} \in X$, we have

$$\Big(\sum_{j=1}^{M} \Big\| \sum_{k=1}^{N} \lambda_{jk} P(x_{jk}) \Big\|^q \Big)^{1/q} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{j=1}^{M} \Big| \sum_{k=1}^{N} \lambda_{jk} \langle x_{jk}, x^* \rangle^m \Big|^p \Big)^{1/p}. \quad (1)$$

We denote by $\mathcal{P}_{F(q,p)}({}^m X; Y)$ the space of all factorable (q,p)-summing m-homogeneous polynomials from X to Y ($\mathcal{P}_{F(p)}({}^m X; Y)$ for short in the case $q = p$). The factorable (q,p)-summing norm $\|P\|_{F(q,p)}$ is given by the infimum of all $C > 0$ that satisfy the above inequality.

Given a continuous m-homogeneous polynomial $P\colon X \to Y$, let us define the linear operator $u_P\colon \otimes^{m,s} X \to Y$ by $u_P(\theta) := \sum_{i=1}^{N} \lambda_i P(x_i)$ for $\theta = \sum_{i=1}^{N} \lambda_i \otimes_m x_i \in \otimes^{m,s} X$ (the map u can be view as the restriction of the π-linearization of P to $\otimes^{m,s} X$).

Theorem 1. *The Banach spaces $\mathcal{P}_{F(q,p)}(^mX;Y)$ and $\Pi_{q,p}(\widehat{\otimes}^{m,s}_{\varepsilon_s}X,Y)$ are isometrically isomorphic.*

Proof. Consider a polynomial $P\in\mathcal{P}_{F(q,p)}(^mX;Y)$ and its linearization $u:=u_P\colon \otimes^{m,s}X\to Y$ given by $u(\theta):=\sum_{i=1}^N\lambda_iP(x_i)$ for $\theta=\sum_{i=1}^N\lambda_i\otimes_m x_i\in\otimes^{m,s}X$. Then, using (1),

$$\|u(\theta)\|=\Big\|\sum_{i=1}^N\lambda_iP(x_i)\Big\|\le\|P\|_{F(q,p)}\sup_{x^*\in B_{X^*}}\Big|\sum_{i=1}^N\lambda_i\langle x_i,x^*\rangle^m\Big|=\|P\|_{F(q,p)}\varepsilon_s(\theta),$$

for all $\theta=\sum_{i=1}^N\lambda_i\otimes_m x_i\in\otimes^{m,s}X$. So, $u:\otimes^{m,s}_{\varepsilon_s}X\to Y$ is continuous and we extend it to a continuous linear operator $P_{\varepsilon_s}:\widehat{\otimes}^{m,s}_{\varepsilon_s}X\to Y$. Moreover, given $\theta_j=\sum_{k=1}^N\lambda_{jk}\otimes_m x_{jk}\in\otimes^{m,s}X$, $j=1,\dots,M$,

$$\begin{aligned}\Big(\sum_{j=1}^M\|u(\theta_j)\|^q\Big)^{1/q}&=\Big(\sum_{j=1}^M\Big\|\sum_{k=1}^N\lambda_{jk}P(x_{jk})\Big\|^q\Big)^{1/q}\\&\le\|P\|_{F(q,p)}\sup_{x^*\in B_{X^*}}\Big(\sum_{j=1}^M\Big|\sum_{k=1}^N\lambda_{jk}\langle x_{jk},x^*\rangle^m\Big|^p\Big)^{1/p}\\&\le\|P\|_{F(q,p)}\sup_{Q\in B_{\mathcal{P}_I(^mX)}}\Big(\sum_{j=1}^M\Big|\sum_{k=1}^N\lambda_{jk}Q(x_{jk})\Big|^p\Big)^{1/p}\\&=\|P\|_{F(q,p)}\sup_{\phi\in B_{(\widehat{\otimes}^{m,s}_{\varepsilon_s}X)^*}}\Big(\sum_{j=1}^M\big|\phi(\theta_j)\big|^p\Big)^{1/p}.\end{aligned}\tag{2}$$

The density argument described in the preliminaries shows that $P_{\varepsilon_s}\in\Pi_{q,p}(\widehat{\otimes}^{m,s}_{\varepsilon_s}X,Y)$ and $\pi_{q,p}(P_{\varepsilon_s})\le\|P\|_{F(q,p)}$.

We claim that the map given by

$$\mathcal{P}_{F(q,p)}(^mX;Y)\ni P\mapsto P_{\varepsilon_s}\in\Pi_{q,p}(\widehat{\otimes}^{m,s}_{\varepsilon_s}X,Y)$$

is an isometric isomorphism onto.

To see this we define for a given $v\in\Pi_{q,p}(\widehat{\otimes}^{m,s}_{\varepsilon_s}X,Y)$ a map $P(x):=v(\otimes_m x)$ for every $x\in X$. Since

$$\|P(x)\|=\|v(\otimes_m x)\|\le\|v\|\varepsilon_s(\otimes_m x)=\|v\|\|x\|^m$$

the m-homogeneous polynomial $P\colon X\to Y$ is continuous. Let us denote Ext the set of extreme points of the unit ball of $(\widehat{\otimes}^{m,s}_{\varepsilon_s}X)^*$. By [5, Proposition

1], Ext is contained in the set $\{\pm(x^*)^m : x^* \in B_{X^*}\}$ (see also [8], where the equality of both sets is proved for the real case). Therefore, for every positive integers M, N and all $M \times N$ matrices $\big(\lambda_{jk}\big)_{jk}$ and $\big(x_{jk}\big)_{jk}$, with $\lambda_{jk} \in \mathbb{K}$ and $x_{jk} \in X$, we have

$$\begin{aligned}\Big(\sum_{j=1}^{M}\Big\|\sum_{k=1}^{N}\lambda_{jk}P(x_{jk})\Big\|^q\Big)^{1/q} &= \Big(\sum_{j=1}^{M}\Big\|v\Big(\sum_{k=1}^{N}\lambda_{jk}\otimes_m x_{jk}\Big)\Big\|^q\Big)^{1/q}\\ &\le \pi_{q,p}(v)\sup_{\phi\in B_{(\widehat{\otimes}^{m,s}_{\varepsilon_s}X)^*}}\Big(\sum_{j=1}^{M}\Big|\phi\Big(\sum_{k=1}^{N}\lambda_{jk}\otimes_m x_{jk}\Big)\Big|^p\Big)^{1/p}\\ &= \pi_{q,p}(v)\sup_{\phi\in Ext}\Big(\sum_{j=1}^{M}\Big|\sum_{k=1}^{N}\lambda_{jk}\phi(\otimes_m x_{jk})\Big|^p\Big)^{1/p}\\ &\le \pi_{q,p}(v)\sup_{x^*\in B_{X^*}}\Big(\sum_{j=1}^{M}\Big|\sum_{k=1}^{N}\lambda_{jk}\langle x_{jk},x^*\rangle^m\Big|^p\Big)^{1/p}.\end{aligned}$$

Then $P \in \mathcal{P}_{F(q,p)}(^mX;Y)$ and $\|P\|_{F(q,p)} \le \pi_{q,p}(v)$. Clearly, $P_{\varepsilon_s} = v$. □

Before stating the following corollary we recall that a Banach space X is of *cotype* q $(2 \le q < \infty)$ if there is a constant $C > 0$ such that

$$\Big(\sum_{j=1}^{n}\|x_j\|_X^q\Big)^{1/q} \le C\Big(\int_0^1\Big\|\sum_{j=1}^{n}r_j(t)x_j\Big\|_X^2\,dt\Big)^{1/2}$$

for every choice of elements $x_1, \dots, x_n \in X$. Here as usual r_j denotes the j-th Rademacher function on $[0,1]$.

Corollary 2. *Let X and Y be Banach spaces.*

(i) *If Y has cotype 2 then $\mathcal{P}_{F(2,1)}(^mX;Y) = \mathcal{L}(\widehat{\otimes}^{m,s}_{\epsilon_s}X, Y)$ with equivalent norms, for each $m \in \mathbb{N}$.*

(ii) *Let $2 < q < \infty$. Then Y has cotype q if and only if $\mathcal{P}_{F(q,1)}(^mX;Y) = \mathcal{L}(\widehat{\otimes}^{m,s}_{\epsilon_s}X, Y)$. In this case both norms are equivalent.*

Proof. It is enough to take into account that the identity map in a space of cotype q is always $(q,1)$-summing (see for example Corollary 11.17 in [12]; see also Section 32 in [9]). Then, using the ideal property of the $(q,1)$-summing operators we obtain that every operator in $\mathcal{L}(\widehat{\otimes}^{m,s}_{\varepsilon_s} X, Y)$ is in $\Pi_{q,1}(\widehat{\otimes}^{m,s}_{\varepsilon_s} X, Y)$ and so the result holds. For the converse of (ii) just take $m = 1$ and the result follows from [12, Theorem 14.5]. □

Example 3. *In particular, this result allows us to show an example of a pair of (infinite dimensional) spaces X and Y for which*

$$\mathcal{P}(^2X; Y) = \mathcal{P}_{F(q,1)}(^2X; Y).$$

To see this we take X to be the Pisier space for which it is known that $\widehat{\otimes}^2_\epsilon X = \widehat{\otimes}^2_\pi X$ (the equality given by the identity map, see [20, Th.3.2]), and let Y be a Banach space of cotype q, $2 \leq q < \infty$. Since the symmetric tensor products $\otimes^{2,s}_{\varepsilon_s} X$ and $\otimes^{2,s}_{\pi_s} X$ are complemented subspaces of $\widehat{\otimes}^{2,s}_{\varepsilon_s} X$ and $\widehat{\otimes}^{2,s}_{\pi_s} X$, respectively (see 2.3. Prop. and 3.1. Prop. in [14]), we obtain the (topological and algebraic) equality $\otimes^{2,s}_{\epsilon_s} X = \otimes^{2,s}_{\pi_s} X$.

Recall now that the space of m-homogeneous polynomials on Y can be identified with all the linear operators from the symmetric projective s-tensor product on Y. Then, using Corollary 2 we obtain

$$\mathcal{P}_{F(q,1)}(^2X; Y) = \mathcal{L}(\widehat{\otimes}^{2,s}_{\varepsilon_s} X, Y) = \mathcal{L}(\widehat{\otimes}^{2,s}_{\pi_s} X, Y) = \mathcal{P}(^2X; Y).$$

It is interesting to note that the equality between the symmetric ε_s and π_s tensor product is never true for any symmetric tensor product of more than 2 spaces being infinite dimensional (see Corollary 2.6 in [6] and the references therein). Thus, our example cannot be extended to the case of homogeneous polynomials of degree bigger than 2.

Corollary 4. *The spaces $\mathcal{P}_{F(q,p)}(^mX)$ and $\mathcal{P}_I(^mX)$ are isometrically isomorphic.*

Proof. By [13] (see also [14]) the space $\mathcal{P}_I(^mX)$ is isometrically isomorphic to the dual of $\widehat{\otimes}^{m,s}_{\varepsilon_s} X$. Theorem 1 applied for $Y = \mathbb{K}$ gives the result. □

The same idea of the scalar polynomials of integral type can be translated to the case of vector valued polynomials. An m-homogeneous polynomial $P\colon X \to Y$ is *Pietsch integral* if there is a regular (vector valued) Borel measure $\lambda \in \mathcal{B}(K, Y)$ of finite variation on B_{X^*}, endowed with the weak star topology, such that

$$P(x) = \int_{B_{X^*}} \langle x, x^* \rangle^m \, d\lambda(x^*), \quad x \in X.$$

Again we write $\|P\|_{\mathcal{P}_I}$ for the infimum of all $\|\mu\|$ when μ varies over all measures as in the definition. Let $\mathcal{P}_I(^m X; Y)$ denote the Banach space of all integral polynomials on X and taking values on Y with the norm $\|\cdot\|_{\mathcal{P}_I}$. We put $\mathcal{L}_I(X, Y) := \mathcal{P}_I(^1 X; Y)$. It is well-known that the space of (Y-valued) Pietsch integral polynomials is isometric to $\mathcal{L}_I(\widehat{\otimes}^{m,s}_{\epsilon_s} X, Y)$ (see [7, 8, 25], taking into account that two different notions of integral polynomial are used there). We write $\mathcal{P}_I(^m X; Y)$ for the space of all such polynomials. Stegall and Retherford [24] characterized $\mathcal{L}_\infty$ spaces as those on which every absolutely summing operator is integral. Cilia, D'Anna and Gutiérrez [8] extended such a result for spaces of polynomials by proving that a Banach space X is an $\mathcal{L}_\infty$ space if and only if 1-dominated m-homogeneous polynomials on X is integral. Our following corollary is a related result.

Corollary 5. *Let $2 \le q < r < \infty$ and $1 \le p < q$. For every Banach space Y of cotype q we have $\mathcal{P}_{F(r)}(^m C(K); Y) = \mathcal{P}_{F(q,p)}(^m C(K); Y) = \mathcal{P}_I(^m C(K); Y)$.*

Proof. This is a direct consequence of Corollary 2, and Theorem 11.14 in [12]. Indeed, under the assumption that Y has cotype q for $2 \le q$ we have that

$$\mathcal{L}(C(K^m), Y) = \Pi_{q,1}(C(K^m), Y) = \Pi_r(C(K^m), Y).$$

Thus, we can identify polynomials factoring through $\widehat{\otimes}^{m,s}_{\epsilon_s} C(K)$ by means of a linear map (i.e., integral polynomials) with the elements of $\mathcal{P}_{F(q,p)}(^m C(K); Y)$ and $\mathcal{P}_{F(r)}(^m C(K); Y)$. $\square$

By [14, Proposition 3.1] the space $\otimes^{m,s}_{\epsilon_s} X$ is a topologically complemented subspace of $\otimes^m_\varepsilon X$. Moreover, if we denote by $\iota\colon \otimes^{m,s}_{\varepsilon_s} X \to \otimes^m_\epsilon X$ and $\sigma : \otimes^m_\epsilon X \to \otimes^{m,s}_{\epsilon_s} X$ the inclusion and the projection respectively, then $\|\sigma\| = 1$ and $\|\iota\| \le c(m, X^*)$, where $c(m, X^*)$ is the polarization constant of X^*. Given $P \in \mathcal{P}_{F(q,p)}(^m X; Y)$, let P_ϵ denote the extension of $u_P \circ \sigma$ to the completion $\widehat{\otimes}^m_\epsilon X$.

Theorem 6. *If $P \in \mathcal{P}_{F(q,p)}(^m X; Y)$ then P_ϵ belongs to $\Pi_{q,p}(\widehat{\otimes}^m_\varepsilon X, Y)$ and*

$$\pi_{q,p}(P_\epsilon) \le \|P\|_{\mathcal{P}_{F(q,p)}} \le c(m, X^*) \pi_{q,p}(P_\varepsilon).$$

Proof. If we denote by $\jmath\colon \otimes_\varepsilon^m X \to \widehat{\otimes}_\varepsilon^m X$ the natural inclusion, then $P_\varepsilon \circ \jmath \circ \iota = u_P \circ \sigma \circ \iota = u_P$. By (2),

$$\begin{aligned}\Big(\sum_{j=1}^M \|P_\varepsilon \circ \jmath(\theta_j)\|^q\Big)^{1/q} &= \Big(\sum_{j=1}^M \|u_P \circ \sigma(\theta_j)\|^q\Big)^{1/q} \\ &\leq \|P\|_{\mathcal{P}_{F(q,p)}} \sup_{\phi \in B_{(\otimes_{\varepsilon_s}^{m,s} X)^*}} \Big(\sum_{j=1}^M |\phi(\sigma(\theta_j))|^p\Big)^{1/p} \\ &\leq \|P\|_{\mathcal{P}_{F(q,p)}} \sup_{\phi \in B_{(\otimes_\varepsilon^m X)^*}} \Big(\sum_{j=1}^M |\phi(\theta_j)|^p\Big)^{1/p}\end{aligned}$$

for all $\theta_1, \dots, \theta_M \in \otimes_\varepsilon^m X$. From this and in a similar way to the proof of Theorem 1, we get that P_ε belongs to $\Pi_{q,p}(\widehat{\otimes}_\varepsilon^m X, Y)$ and $\pi_{q,p}(P_\varepsilon) \leq \|P\|_{\mathcal{P}_{F(q,p)}}$. Furthermore,

$$\begin{aligned}\|P\|_{\mathcal{P}_{F(q,p)}} &= \pi_{q,p}(P_{\varepsilon_s}) = \pi_{q,p}(u_P) = \pi_{q,p}(P_\varepsilon \circ \jmath \circ \iota) \\ &\leq \pi_{q,p}(P_\varepsilon)\|\jmath\|\|\iota\| \leq \pi_{q,p}(P_\varepsilon) c(m, X^*),\end{aligned}$$

and this completes the proof. □

Note that there could be operators $S \in \Pi_{q,p}(\widehat{\otimes}_\varepsilon^m X, Y)$ other than $P_{\epsilon_s} \circ q$ such that $S \circ \jmath \circ \iota = u_p$.

4. Applications: Pisier factorization theorem and (q,p)-concave polynomials

4.1. A variant of Pisier theorem for polynomials

In what follows we prove the polynomial version of Pisier's Theorem on factorization of (q,p)-summing operators from $C(K)$-spaces for $1 \leq p < q < \infty$. Let $m \in \mathbb{N}$, Y a Banach space and K a compact topological Hausdorff space. As usual, $C(K)$ denotes the space of all real or complex continuous functions on K endowed with the sup norm. Given a probability Borel measure μ on K^m, consider the canonical m-homogeneous $\odot_m\colon C(K) \to C(K^m)$ given by

$$\odot_m f(t_1, \dots, t_m) := f(t_1) \cdots f(t_m)$$

for all $f \in C(K)$ and $t_1, \dots, t_m \in K$.

Let $j_{q,1}\colon C(K^m) \to L_{q,1}(\mu)$ be the canonical map and $\kappa\colon \widehat{\otimes}_{\varepsilon}^{m} C(K) \to C(K^m)$ is the isometric isomorphism onto fulfilling

$$\kappa(f_1 \otimes \cdots \otimes f_m)(t_1, \ldots, t_m) = f_1(t_1) \cdots f_m(t_m)$$

for all $f_j \in C(K)$ and all $t_j \in K$, $1 \leq j \leq m$.

We can state and prove the polynomial version of Pisier's factorization theorem.

Theorem 7. *Let $1 \leq p < q < \infty$ and $m \in \mathbb{N}$. The following assertions are equivalent for a Banach space valued m-homogeneous polynomial $P\colon C(K) \to Y$.*

(i) *P is factorable (q,p)-summing.*

(ii) *For each positive integers M, N and all $M \times N$ matrices (λ_{jk}) and (f_{jk}) in $\mathbb{K}$ and $C(K)$, respectively, the following inequality holds*

$$\Big(\sum_{j=1}^{M} \Big\| \sum_{k=1}^{N} \lambda_{jk} P(f_{jk}) \Big\|^q\Big)^{1/q} \leq C \Big\| \Big(\sum_{j=1}^{M} \Big| \sum_{k=1}^{N} \lambda_{jk} (f_{jk})^m \Big|^p\Big)^{1/p} \Big\|_{C(K)}.$$

(iii) *There is a probability Borel measure μ on K^m such that P admits a factorization:*

$$\begin{array}{ccc} C(K) & \xrightarrow{\;P\;} & Y, \\ {\scriptstyle \odot_m}\big\downarrow & & \big\uparrow{\scriptstyle v} \\ C(K^m) & \xrightarrow{\;j_{q,1}\;} & L_{q,1}(\mu) \end{array}$$

where v is a continuous linear operator.

Proof. (i) $\Rightarrow$ (iii) Assume that $P \in \mathcal{P}_{F(q,p)}(^m C(K); Y)$. Consider the continuous linear operator $u := P_{\varepsilon} \circ \kappa^{-1}\colon C(K^m) \to Y$. By Theorem 6 and the ideal property, $u \in \Pi_{q,p}(C(K^m), Y)$. The classical Pisier factorization theorem asserts that there exists a probability Borel measure μ on K^m and a continuous linear operator $v\colon L_{q,1}(\mu) \to Y$ such that $u = v \circ j_{q,1}$. Note that $\odot_m = \kappa \circ \jmath \circ \iota \circ \otimes$. Then

$$P = u_P \circ \otimes = P_{\varepsilon} \circ \jmath \circ \iota \circ \otimes = P_{\varepsilon} \circ \kappa^{-1} \circ \odot_m = u \circ \odot_m = v \circ j_{q,1} \circ \odot_m.$$

That proves (i) implies (iii).

Let us now prove that (iii) implies (i). Take $M \times N$ matrices (λ_{jk}) and (f_{jk}) in $\mathbb{K}$ and $C(K)$, respectively. Since $j_{q,1} \in \Pi_{q,p}(C(K^m), Y)$, using the ideal property and the fact that the set of the extreme points of $B_{(\otimes_{\varepsilon_s}^{m,s} C(K))^*}$ is a norming set contained in $\{\pm\phi^m; \phi \in B_{C(K)^*}\}$, we get

$$\Big(\sum_{j=1}^{M} \Big\| \sum_{k=1}^{N} \lambda_{jk} j_{q,1} \circ \odot_m f_{jk} \Big\|^q_{L_{q,1}(\mu)} \Big)^{1/q}$$

$$\leq \pi_{q,p}(j_{q,1}) \|\iota\| \sup_{\phi \in B_{(\otimes_{\varepsilon_s}^{m,s} C(K))^*}} \Big(\sum_{j=1}^{M} \Big| \phi\Big(\sum_{k=1}^{N} \lambda_{jk} \otimes_m f_{jk} \Big) \Big|^p \Big)^{1/p}$$

$$\leq \pi_{q,p}(j_{q,1}) c(m, C(K)^*) \sup_{\phi \in B_{C(K)^*}} \Big(\sum_{j=1}^{M} \Big| \sum_{k=1}^{N} \lambda_{jk} \langle f_{jk}, \phi \rangle^m \Big|^p \Big)^{1/p}.$$

We easily conclude that $P = v \circ j_{q,1} \circ \odot_m \in \mathcal{P}_{F(q,p)}(^m C(K); Y)$ and that $\|P\|_{F(q,p)} \leq \pi_{q,p}(j_{q,1}) c(m, C(K)^*) \|v\|$. This finishes the proof of (iii) $\Rightarrow$ (i).

To show the equivalence of (i) and (ii), recall that the dual of $\otimes_{\varepsilon_s}^{m,s} X$ is isometrically isomorphic to the space of integral m-homogeneous polynomials endowed with its norm (see [13]). Now it suffices to use the fact that the set of extreme points of the unit ball of $C(K)^*$ coincides with $\Lambda := \{\lambda\delta_x; x \in K, |\lambda| = 1\}$, where $\delta_x(f) = f(x)$ is the evaluation at $x \in K$, and this set is a norming set. So, since the function of ϕ involved in the expression below is convex, the suprema

$$\sup_{\phi \in B_{(\otimes_{\varepsilon_s}^{m,s} C(K))^*}} \Big(\sum_{j=1}^{M} \Big| \phi\Big(\sum_{k=1}^{N} \lambda_{jk} \otimes_m f_{jk} \Big) \Big|^p \Big)^{1/p} = \sup_{\phi \in B_{C(K)^*}} \Big(\sum_{j=1}^{M} \Big| \phi\Big(\sum_{k=1}^{N} \lambda_{jk} \otimes_m f_{jk} \Big) \Big|^p \Big)^{1/p}$$

coincide with

$$\sup_{\eta \in \Lambda} \Big(\sum_{j=1}^{M} \Big| \eta\Big(\sum_{k=1}^{N} \lambda_{jk} \otimes_m f_{jk} \Big) \Big|^p \Big)^{1/p}$$

that is equal to

$$\sup_{x \in K} \Big(\sum_{j=1}^{M} \Big| \sum_{k=1}^{N} \lambda_{jk} f_{jk}(x)^m \Big|^p \Big)^{1/p} = \Big\| \Big(\sum_{j=1}^{M} \Big| \sum_{k=1}^{N} \lambda_{jk} (f_{jk})^m \Big|^p \Big)^{1/p} \Big\|_{C(K)}.$$

□

4.2. (q,p)-Concavity for polynomials

In this subsection, we are interested in showing the lattice version of the factorization through the symmetric ε tensor product that has been presented in the previous section. The main difference with respect to the previous results is that, in general, we cannot assume that the symmetric ε tensor product of Banach lattices has a Banach lattice structure. Also, the canonical polynomial that plays the role of the polynomial $x \mapsto \otimes^m x$ is now the pointwise power, i.e., $f \mapsto f^m$ that in general does not belong to X but to the m-power of a Banach lattice X on a given measure space.

Before discussing the main result of the paper, we recall that a quasi-Banach lattice $X = (X, \|\cdot\|)$ is said to be *p-convex* $(0 < p < \infty)$, if there exists a constant $C > 0$ such that

$$\Big\|\Big(\sum_{k=1}^{n} |x_k|^p\Big)^{1/p}\Big\| \le C\,\Big(\sum_{k=1}^{n} \|x_k\|^p\Big)^{1/p},$$

for every choice of elements $x_1, \dots, x_n \in X$. The optimal constant C in this inequality is called the p-convexity constant of X, and is denoted, by $M^{(p)}(X)$.

Suppose we are given a quasi-Banach lattice on X on (Ω, Σ, μ). For any $0 < r < \infty$, we define X^r to be the quasi-Banach lattice of all $f \in L^0(\mu)$ such that $|f|^{1/r} \in X$ equipped with the quasi-norm

$$\|f\|_{X^r} = \big\||f|^{1/r}\big\|_X^r.$$

We note that in the case when X is a Banach lattice, then a quasi-norm $\|\cdot\|_{X^r}$ is equivalent to a lattice norm (resp., a norm) on X^r if and only if X is r-convex (resp., the r-convexity constant of X is 1.

We will write δ_m for the canonical m-homogeneous polynomial $\delta_m\colon X \to X^m$ given by $\delta_m(f) := f^m$ for every $f \in X$.

Let $1 \le q, p < \infty$, X be a Banach lattice on a measure space and Y a Banach space. We define the notion of (q,p)-concavity for polynomials in the spirit of [11]. Adapting the inequality given in [11, Cor.3.3(i)] to the factorable case, we say that an m-homogeneous polynomial $P\colon X \to Y$ is (q,p)-concave if for every positive integers M, N and all $M \times N$ matrices $\big(\lambda_{jk}\big)_{jk}$ and $\big(x_{jk}\big)_{jk}$, with $\lambda_{jk} \in \mathbb{K}$ and $x_{jk} \in X(\mu)$, we have

$$\Big(\sum_{j=1}^{M} \Big\|\sum_{k=1}^{N} \lambda_{jk} P(x_{jk})\Big\|^q\Big)^{1/q} \le C \Big\|\Big(\sum_{j=1}^{M} \Big|\sum_{k=1}^{N} \lambda_{jk} x_{jk}^m\Big|^p\Big)^{1/p}\Big\|_{X^m}. \qquad (3)$$

We call $K_{(q,p)}(P)$ the (q,p)-concavity constant of P, that is defined to be the least constant C in the inequality above.

Theorem 8. *Let $1 \le p < q < \infty$ and let X be a Banach lattice on a measure space with the m-convexity constant 1, and let $P\colon X \to Y$ be an m-homogeneous polynomial. Then P is (q,p)-concave if and only if it is $(q,1)$-concave.*

Proof. We claim that a polynomial $P\colon X \to Y$ is (q,p)-concave if and only if it can be written as $P = P_L \circ \delta_m$, where $P_L\colon X^m \to Y$ is a linear (q,p)-concave map.

For the direct implication, it is enough to show that the linear operator P_L can be defined as a consequence of the (q,p)-concavity of P. Suppose that there are two linear combinations $\sum_{i=1}^k \lambda_i x_i^m$ and $\sum_{i=1}^n \mu_i y_i^m$ that are equal as elements of $X(\mu)_{[m]}$. Then the inequality of the (q,p)-concavity implies that

$$\Big\| \sum_{i=1}^k \lambda_i P(x_i) - \sum_{i=1}^n \mu_i P(y_i) \Big\| \le C \Big\| \sum_{i=1}^k \lambda_i x_i^m - \sum_{i=1}^n \mu_i y_i^m \Big\|_{X^m} = 0.$$

This means that the definition of P_L given by $P_L(\sum_{i=1}^k \lambda_i x_i^m) := \sum_{i=1}^k \lambda_i P(x_i)$ is well done and linear in the linear span of the elements of $\{x^m;\, x \in X\}$ that coincides with X^m. The factorization diagram is obviously commutative, and the inequality of the (q,p)-concavity of P shows that P_L is also (q,p)-concave. A direct computation gives the converse and finishes the proof of the claim.

Suppose that P is $(q,1)$-concave. By the claim, there is a factorization as $P = P_L \circ \delta_m$ through a linear map $P_L\colon X^m \to Y$ that is $(q,1)$-concave. Since X^m is a Banach lattice, P_L is also (q,p)-concave, as a consequence of the characterization of (q,p)-concave linear operators given in [12, Th.16.5] and Pisier's Theorem (see [12, Cor.16.6]). The claim above gives that P_L is (q,p)-concave. The converse implication comes directly from the inequality in the definition of (q,p)-concave polynomial. □

We conclude with the following remark that the idea of defining polynomials from Banach lattices over measure spaces to its m-th power is the natural one, when working within spaces of integrable functions, and in general, from Banach lattices. The reason is that the canonical polynomial $X \ni f \mapsto f^m$ is well defined in X^m. Notice that, for the better known case of $C(K)$-spaces, this polynomial is defined from $C(K)$ in the same space $C(K)$ due to the Banach algebra structure of this space.

Actually, the identification that we establish implicitly in the proof of Theorem 8 between polynomials from X to Y and linear maps from X^m to Y has an abstract counterpart in the representation theorem for positive orthogonally additive polynomials on Riesz spaces given in [15, Th.3.4]; see also the references therein for more results regarding polynomials on Banach lattices.

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