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Analytical Methods for Computing the Density of Differential Equations with Uncertainties. Theory and Applications.

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Chapter 1

Introduction

1.1 Motivation

The key role played by differential equations is mainly justified by their wide range of applications in many scientific fields including Physics, Chemistry, Engineering, Biology, Epidemiology, Economics, etc. In practice, model inputs (coefficients, source terms and initial/boundary conditions) appearing in the formulation of differential equations often need to be set from sampling, experimental measurements or metadata excerpted from the extant literature. This approach entails that model inputs involve uncertainties and then they must be treated as random variables or stochastic processes rather than deterministic constants or functions, respectively. This approach leads to formulate RDEs and SDEs in mathematical modeling (see [1, 2, 3]). Apart from studying relevant theoretical questions, like existence and uniqueness conditions to IVPs, as well as to devise methods for computing their solutions, say $X(t) = X(t)(\omega)$ ($\omega \in \Omega$, being $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probability space), in the setting of both RDEs and SDEs a primary objective is to determine the main statistical functions of the solution stochastic process. The most important information in this latter regard includes determining the mean ($\mathbb{E}[X(t)]$) and the variance ($\mathbb{V}[X(t)]$), since they provide the average and the variability of the solution, and moreover, they allow to construct pointwise and probabilistic predictions. However, a more complete goal consists of computing the first probability density function (1-PDF), say $f(t, x)$, since from this deterministic function one can calculate the one-dimensional moments of arbitrary order

$$\mu_k = \mathbb{E}[(X(t))^k] = \int_{-\infty}^{+\infty} x^k f(t, x)\, \mathrm{d}x, \quad k = 1, 2, \ldots, \tag{1.1}$$

provided they exist. Observe that, in particular, $\mathbb{E}[X(t)] = \mu_1$ and $\mathbb{V}[X(t)] = \mu_2 - \mu_1^2$. Besides, the computation of the 1-PDF enables the calculation that at any specific time instant, say $\hat{t}$, the solution lies within a specific interval of interest, say $[a, b]$, via its direct integration

$$\mathbb{P}[\{\omega \in \Omega : a \leq X(\hat{t})(\omega) \leq b\}] = \int_a^b f(\hat{t}, x)\, \mathrm{d}x.$$

This project is organized as follows. In Chapter 2, we revisit the classical Liouville theorem for dynamical systems focusing on its adaptation to the context of RDEs. In the deterministic context, this result establishes that the density of the solution of a dynamical system is an integral invariant of motion and satisfies a PDE, called Liouville-Gibbs equation (also termed continuity equation in the context of Hydrodynamics) [4]. In the probabilistic setting, we will see that this result can be interpreted as the PDE satisfied for the 1-PDF of the solution stochastic process of a RDE. Specifically, in Section 2.1 we will derive the Liouville-Gibbs PDE in the context of random dynamical systems of type (2.1), and then, in Section 2.2 we will obtain an explicit expression of its solution, which represents the 1-PDF of the solution stochastic process to the random IVP (2.1). Generalizing this same idea, in Chapter 3 we will see how we can build a PDE verified by the 1-PDF of an arbitrary stochastic process. Furthermore, in Section 3.2 we will consider the particular case of Itô SDEs and the Fokker-Planck PDE. Chapter 4 is devoted to study several mathematical models with various type of uncertainties, formulated via differential equations, that appear in different scientific fields. By taking advantage of results exhibited in Chapter 2 and 3, we will obtain explicit expressions of the 1-PDF of their corresponding solution stochastic process. Then, we will carry out some numerical simulations to illustrate the main theoretical results studied in previous chapters. Conclusions are drawn in Chapter 5.

1.2 Preliminaries

For the sake of completeness, hereinafter we introduce some definitions, notations and results that will be required throughout this project. To avoid an excess of information, we will only introduce whatever is necessary and sufficient for the complete understanding of this project. However, we assume certain basic knowledge on probability theory and mathematical analysis. We will divide this section in two parts. The first one will focus on the notions and definitions from mathematical analysis and the second one will focus on those from probability theory.

1.2.1 Mathematical Analysis

The following definitions and properties are taken from [5]. Let us begin by defining the normed function spaces of absolutely integrable functions $(\mathrm{L}^1(\mathbb{R}^n,\mathbb{R}),\|\cdot\|_{\mathrm{L}^1})$, or simply $\mathrm{L}^1(\mathbb{R}^n)$,

$$\mathrm{L}^1(\mathbb{R}^n,\mathbb{R}) = \{f:\mathbb{R}^n\to\mathbb{R} \text{ such that } \|f\|_{\mathrm{L}^1} := \int_{\mathbb{R}^n} |f(\mathbf{x})|\,\mathrm{d}\mathbf{x} < \infty\},$$

and $(\mathrm{L}^2(\mathbb{R}^n,\mathbb{R}),\|\cdot\|_{\mathrm{L}^2}))$, or simply $\mathrm{L}^2(\mathbb{R}^n)$

$$\mathrm{L}^2(\mathbb{R}^n,\mathbb{R}) = \{f:\mathbb{R}^n\to\mathbb{R} \text{ such that } \|f\|^2_{\mathrm{L}^2} := \int_{\mathbb{R}^n} |f(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} < \infty\}.$$

It must be noted that we have made use of the usual notation for the functions in these spaces. However, the elements of these spaces are actually classes of functions, where

two functions are related, or in the same class, if the are equal everywhere but in a zero-measure set. We say, in that case, that two functions are equal almost everywhere. If we take $f \in \mathrm{L}^1(\mathbb{R}^n)$, then we can actually take any function that is equal to f almost everywhere.

As we will see in the following chapters, the notion of Fourier transform is a natural one when dealing with statistical information of stochastic processes or random variables. We define the **Fourier transform** of an absolutely integrable function f as

$$\mathcal{F}[f(\mathbf{x})](\mathbf{v}) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{v}^\mathsf{T} \mathbf{x}} f(\mathbf{x}) \, \mathrm{d}\mathbf{x}, \quad f \in \mathrm{L}^1(\mathbb{R}^n),$$

where $\mathcal{F}$ denotes the Fourier transform operator, $\mathrm{i} = \sqrt{-1}$ denotes the imaginary unit and $\mathbf{v} = (v_1, \ldots, v_n)^\mathsf{T}$ and $\mathbf{x} = (x_1, \ldots, x_n)^\mathsf{T}$ are column vectors in $\mathbb{R}^n$. Some basic and trivial properties of the Fourier transform are its existence (well defined) and linearity. Now we will state, without proof, some more interesting properties we will also need.

Note that we may use the **multi-index** or conventional notation for partial derivatives and products throughout the project; that is, given a multi-index $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, we denote

$$\partial_{\mathbf{x}}^{\mathbf{m}} f(\mathbf{x}) = \frac{\partial^{|\mathbf{m}|} f(\mathbf{x})}{\partial x_1^{m_1} \ldots \partial x_n^{m_n}}, \quad \mathbf{x}^{\mathbf{m}} = x_1^{m_1} \ldots x_n^{m_n},$$

where f is a real valued function defined in $\mathbb{R}^n$, and $|\mathbf{m}| = \sum_{j=1}^n m_j$. Also, we will denote $\|x\|$ as the Euclidean norm in $\mathbb{R}^n$.

A basic and interesting property from the Fourier transform appears when we concatenate the operator:

$$\mathcal{F}[\mathcal{F}[f]](\mathbf{x}) = f(-\mathbf{x}), \quad \text{a.e. } \mathbf{x} \in \mathbb{R}^n,$$

whenever f, $\mathcal{F}[f]$ are absolutely integrable functions (see [5, Prop. 18.2.1]). The following two properties show how the Fourier transform changes differentiation with multiplication.

Theorem 1.1. *([5, Prop. 17.2.1 (ii)] and [6, Prop. 0.22]). Let $f \in \mathrm{L}^1(\mathbb{R}^n)$ be continuously differentiable and assume that $\partial_{\mathbf{x}}^{\mathbf{m}} f \in \mathrm{L}^1(\mathbb{R}^n)$ for all multi-index $\mathbf{m}$ and $\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 0$, then*

$$\mathcal{F}\left[\partial_{\mathbf{x}}^{\mathbf{m}} f(\mathbf{x})\right](\mathbf{v}) = (2\pi \mathrm{i} \mathbf{v})^{\mathbf{m}} \mathcal{F}[f(\mathbf{x})](\mathbf{v}). \tag{1.2}$$

Also, assume $\|\mathbf{x}^{\mathbf{m}} f(\mathbf{x})\| \in \mathrm{L}^1(\mathbb{R}^n)$ for all multi-index $\mathbf{m}$. Then $\mathcal{F}[f](\mathbf{v})$ is smooth and

$$\partial_{\mathbf{v}}^{\mathbf{m}} \mathcal{F}[f](\mathbf{v}) = \mathcal{F}[(-2\pi \mathrm{i}\, x)^{\mathbf{m}} f(\mathbf{x})](\mathbf{v}). \tag{1.3}$$

If we consider the change of variable: $\mathbf{u} = -2\pi \mathbf{v}$ in the definition of the Fourier transform, properties (1.2) and (1.3) can be written, respectively, as

$$\mathcal{F}\left[\partial_{\mathbf{x}}^{\mathbf{m}} f(\mathbf{x})\right](\mathbf{u}) = (-\mathrm{i}\mathbf{u})^{\mathbf{m}} \mathcal{F}[f(\mathbf{x})](\mathbf{u}), \tag{1.4}$$

and

$$\partial_{\mathbf{u}}^{\mathbf{m}} \mathcal{F}[f](\mathbf{u}) = \mathcal{F}[(\mathrm{i}\mathbf{x})^{\mathbf{m}} f(\mathbf{x})](\mathbf{v}). \tag{1.5}$$

The following theorem is known as the inversion Fourier identity.

Theorem 1.2. *([5, Th. 18.1.1]). If $f, \mathcal{F}[f] \in \mathrm{L}^1(\mathbb{R}^n)$ and*

$$\overline{\mathcal{F}}[f(\mathbf{v})](\mathbf{x}) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi \mathrm{i} \mathbf{x}^\mathsf{T} \mathbf{v}} f(\mathbf{v}) \mathrm{d}\mathbf{v},$$

then

$$\overline{\mathcal{F}}[\mathcal{F}[f]](\mathbf{x}) = f(\mathbf{x}),$$

almost everywhere; particularly, in every $\mathbf{x} \in \mathbb{R}^n$ where f is continuous.

Now, let $\mathcal{D}(\mathbb{R}^n)$ denote the space of **test functions**, that is

$$\mathcal{D}(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \to \mathbb{R} \text{ such that } \operatorname{supp}(\varphi) \text{ is compact}\},$$

where $\operatorname{supp}(\varphi) = \overline{\{\mathbf{x} \in \mathbb{R}^n \,:\, \varphi(\mathbf{x}) \neq 0\}}$, that is, denotes the closure of the largest set where φ does not vanish.

A **distribution** (see [5, Lessons 27, 28]) is a continuous linear functional with real or complex values defined on the space of test functions. In particular, we will consider distributions T_f given by

$$T_f(\varphi) = \langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(\mathbf{x}) \varphi(\mathbf{x}) \mathrm{d}\mathbf{x}, \quad \varphi \in \mathcal{D}(\mathbb{R}^n). \tag{1.6}$$

Notice that this definition is consistent if f is, at least, integrable in every compact subset of $\mathbb{R}^n$ (locally integrable, denoted by $\mathrm{L}^1_{\mathrm{loc}}$). Distributions defined as (1.6) are called **regular** distributions.

There are also non-regular distributions. The only one we will consider is the **Dirac delta distribution**, given by

$$\langle \delta_{\mathbf{a}}, \varphi \rangle = \varphi(\mathbf{a}).$$

We can extend most properties (such as differentiability and Fourier transform) from integrable functions to distributions, despite the function does not need to be differentiable. Particularly, we define the following operations on distributions:

- Product of a distribution T with a smooth function g (see [5, S. 28.3]):

$$g\, T(\varphi) = T(g\, \varphi).$$

- Derivative of a distribution T (see [5, S. 28.4]):

$$\partial^{\mathbf{m}} T(\varphi) = (-1)^{|\mathbf{m}|} T(\partial^{\mathbf{m}} \varphi),$$

  for every multi-index $\mathbf{m} \in \mathbb{N}^n$.

- Fourier Transform of a distribution T (see [5, S. 31.2]):

$$\mathcal{F}[T](\varphi) = T(\mathcal{F}[\varphi]).$$

Let us see a particular case. Consider the Dirac delta distribution $\delta_{\mathbf{a}}$. Its Fourier transform is

$$\langle \mathcal{F}[\delta_{\mathbf{a}}], \varphi \rangle = \langle \delta_{\mathbf{a}}, \mathcal{F}[\varphi] \rangle = \mathcal{F}[\varphi](\mathbf{a}) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{a}^\top \mathbf{u}}\, \varphi(\mathbf{u})\, \mathrm{d}\mathbf{u},$$

therefore the distributional Fourier transform of the Dirac delta distribution is $\mathcal{F}[\delta_{\mathbf{a}}](\mathbf{u}) = \frac{1}{(2\pi)^n} \mathrm{e}^{\mathrm{i}\mathbf{a}^\top \mathbf{u}}$, in the distribution sense. In the case $\mathbf{a} = \mathbf{0}$, we have $\mathcal{F}[\delta_{\mathbf{0}}](\mathbf{u}) = \frac{1}{(2\pi)^n}$, a constant function.

It can be proven that Theorems 1.1 and 1.2 can be extended to distributions giving the same property as in classical functions (see [5, Prop. 31.2.4]). Particularly, we have for this case

$$\mathcal{F}[\partial_{\mathbf{x}}^{\mathbf{m}} \delta_{\mathbf{0}}](\mathbf{u}) = \frac{1}{(2\pi)^n} (-\mathrm{i}\mathbf{u})^{\mathbf{m}},$$

in the distribution sense. Now, taking the Fourier transform on both sides,

$$\frac{1}{(2\pi)^n} \mathcal{F}[(\mathrm{i}\mathbf{u})^{\mathbf{m}}](\mathbf{x}) = (-1)^n \partial_{\mathbf{x}}^{\mathbf{m}} \delta_{\mathbf{0}}(\mathbf{x}), \tag{1.7}$$

in the distribution sense.

Note that we will make a clear distinction when referring to distributions as defined in this section, and when talking about the distribution function of a random variable or stochastic process. For further details on Fourier Analysis, distributions and its applications, see [7, 5, 6, 8, 9].

1.2.2 Probability theory

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We will work in the Hilbert space of random variables with a finite second order moment, denoted by $(\mathrm{L}_2(\Omega), \langle , \rangle)$. Specifically, its elements are real random variables $X : \Omega \to \mathbb{R}$ and the inner product is defined by $\langle X, Y \rangle = \mathbb{E}[XY]$, $X, Y \in \mathrm{L}_2(\Omega)$, being $\mathbb{E}[\cdot] = \int_\Omega \cdot\, \mathrm{d}\mathbb{P}(\omega)$ the expectation operator. From this inner product one derives the corresponding norm $\|X\|_2 = (\mathbb{E}[X^2])^{1/2}$. Elements in space $\mathrm{L}_2(\Omega)$ are termed second-order random variables and they have finite variance ($\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 < \infty$, since the finiteness of $\mathbb{E}[X^2]$ entails $\mathbb{E}[X] < \infty$ too). As it was mentioned, L^1 is a Hilbert Space and, therefore, its elements verify the **Schwartz inequality**; that is

$$\langle X, Y \rangle^2 = \mathbb{E}[X\, Y]^2 \leq \mathbb{E}[X^2]\mathbb{E}[Y^2] = \|X\|_2^2 \|Y\|_2^2, \quad \forall X,\, Y \in \mathrm{L}^2(\Omega). \tag{1.8}$$

The convergence associated to $\|\cdot\|_2$–norm is referred to as mean square (m.s.) convergence and it is defined as follows. Let $\{X_n : n \geq 0\}$ be a sequence of random variables in $\mathrm{L}_2(\Omega)$ and $X \in \mathrm{L}_2(\Omega)$, we say that X_n is m.s. convergent to X if and only if $\|X_n - X\|_2 \to 0$

as $n \to \infty$. This fact is denoted by $X_n \xrightarrow[n\to\infty]{\|\cdot\|_2} X$. This stochastic convergence has the following distinctive property [1, Th. 4.3.1]

$$X_n \xrightarrow[n\to\infty]{\|\cdot\|_2} X \Rightarrow \mathbb{E}[X_n] \xrightarrow[n\to\infty]{} \mathbb{E}[X] \text{ and } \mathbb{V}[X_n] \xrightarrow[n\to\infty]{} \mathbb{V}[X]. \tag{1.9}$$

A random function (or stochastic process), say $X(t) \equiv \{X(t) : t \in T \subset \mathbb{R}\}$, defined in the space $\mathrm{L}_2(\Omega)$ is such that $X(t) \in \mathrm{L}_2(\Omega)$ for each $t \in T$, i.e., $X(t)$ is a 2-r.v., and it is called a second-order stochastic process (2-stochastic process). The concept of m.s. derivative of a 2-stochastic process, $\dot{X}(t)$, is defined in terms of m.s. convergence via the incremental quotient

$$\lim_{h\to 0} \left\| \frac{X(t+h) - X(t)}{h} - \dot{X}(t) \right\|_2 = 0.$$

As a consequence of property (1.9) and the fact that m.s. derivatives are defined in terms of m.s. limits, one gets that the expectation operator and the m.s. derivative commute [1, p. 97], i.e.,

$$\mathbb{E}[\dot{X}(t)] = \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathbb{E}[X(t)] \right). \tag{1.10}$$

The previous scalar concepts can be straightforwardly extended to the multidimensional scenario leading to the Banach space of second-order random vectors, $(\mathrm{L}_2^n(\Omega), \|\cdot\|_n)$, where

$$\begin{aligned} \mathrm{L}_2^n(\Omega) &:= \left\{ \mathbf{X} = (X_1, \ldots, X_n)^\mathsf{T} : \ X_j \in \mathrm{L}_2(\Omega),\, 1 \le j \le n \right\}, \\ \|\mathbf{X}\|_n &:= \max_{1\le j\le n} \|X_j\|_2. \end{aligned}$$

A very interesting and useful property when studying stochastic processes is the relationship between the PDF and the characteristic function of the stochastic process at a fixed time. Let $\{\mathbf{X}(t)\}_{t\in T}$ be a stochastic process and $t \in T$ fixed. We have

$$\Phi(\mathbf{u}, t) = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{u}^\mathsf{T}\mathbf{x}} f(\mathbf{x}, t) \mathrm{d}\mathbf{x} = (2\pi)^n \mathcal{F}[f(\mathbf{x}, t)](\mathbf{u}, t).$$

This relationship will be particularly useful in Chapters 2 and 3.

A stochastic processes can be classified in two classes according to its regularity: **stationary** and **non stationary**. Most stochastic processes modeling physical phenomena are non stationary, such as ground motion due to strong earthquakes, noise processes in devices with starting transient, seasonal temperature variations and even epidemic models. However, stationary stochastic processes also appear when modeling physical phenomena such as the surface of the sea in spatial and time coordinates, noise in time electric circuits under steady state operation, impurities in materials and media as functions of spatial coordinates. Also, there is a large amount of mathematical machinery to deal with stationary stochastic processes, which we will define below.

Definition 1.3. (see [1, Pgs. 42, 43]). A stochastic process $\mathbf{X}(t)$, $t \in T$ is said to be **stationary**, or **strictly stationary**, if its collection of probability distributions stay invariant under an arbitrary translation of the time parameter; that is, for each n and for an arbitrary τ,

$$F_n(\mathbf{x}_1, t_1; \ldots; \mathbf{x}_n, t_n) = F_n(\mathbf{x}_1, t_1 + \tau; \ldots; \mathbf{x}_n, t_n + \tau), \quad \forall (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in \mathbb{R}^{n^2},$$

where $t_j + \tau \in T$ for all j. Also, F_n is the n-th joint distribution function.

Now, we say a stochastic process $\mathbf{X}(t)$, $t \in T$ is **wide sense stationary**, **weakly stationary**, or **covariance stationary**, if it verifies in every $t \in T$

$$|\mathbb{E}[\mathbf{X}(t)]| = \text{constant} < \infty, \quad \|\mathbf{X}(t)\|_n < \infty, \quad \mathbb{E}[\mathbf{X}(t_1)^{\mathsf{T}} \mathbf{X}(t_2)] = \sum_{i=1}^{n} \Gamma_{i\,i}(t_1 - t_2),$$

where t_1, $t_2 \in T$, the second inequality is true for all $t \in T$ and $\Gamma_{i\,i}$ is the correlation, or cross-correlation function of the components X_i, X_j; that is,

$$\Gamma_{i\,j}(t_1, t_2) = \mathbb{E}[X_i(t_1)\, X_j(t_2)], \quad \Gamma_{i\,j}(\tau) := \Gamma_{i\,j}(t, t + \tau).$$

Now, we will consider the very special kind of stochastic processes called **Gaussian** stochastic processes. Gaussian stochastic processes are most often encountered in practice, since the **central limit theorem** guarantees its usefulness when the stochastic process represents a sum of a very large number of small independent random effects at each instant.

Definition 1.4. (see [1, Pg. 66]). A stochastic process $\mathbf{X}(t)$, $t \in T$ is called a **Gaussian stochastic process** if, for any finite set $t_1, \ldots, t_n$, the random vectors $\mathbf{X}(t_1), \ldots, \mathbf{X}(t_n)$ have the joint characteristic function

$$\Phi(\mathbf{u}^1, t_1; \ldots; \mathbf{u}^n, t_n) = \exp\left[\mathrm{i} \sum_{i=1}^{n} \mathbf{m}_i^{\mathsf{T}} \mathbf{u}^i - \frac{1}{2} \sum_{i,\,j=1}^{n} (\mathbf{u}^i)^{\mathsf{T}} \Lambda_{i\,j} \mathbf{u}^j \right],$$

where $(\mathbf{u}^i)^{\mathsf{T}} = [u_1^i, \ldots, u_n^i]$, $\mathbf{m}_i^{\mathsf{T}} = \mathbb{E}[\mathbf{X}(t_i)]^{\mathsf{T}}$, and

$$\Lambda_{i\,j} = \mathbb{E}[(\mathbf{X}(t_i) - \mathbf{m}_i)^{\mathsf{T}} (\mathbf{X}(t_j) - \mathbf{m}_j)]$$

is the covariance matrix.

Gaussian stochastic processes play a fundamental role in the study of SDEs due to the properties they satisfy. To name only a few, Gaussian stochastic processes remain Gaussian under linear transformations, such as differentiation and integration (in the mean square sense). Gaussian stochastic processes are completely determined by its mean and covariance functions. Finally, a Gaussian stochastic process is stationary if and only if it is wide-sense stationary. Now, we will define a very special Gaussian stochastic process, named **Wiener process**, or also **Brownian Motion**. Although the properties that we will use to define the Wiener process actually follow directly from the **Kolmogorov extension theorem** (see [3, Th. 2.1.5]), it is a rather hard task to construct it directly from this theorem.

Definition 1.5. (see [3, Ch. 2]). Let $\mathbf{B}(t,\omega) : [t_0, +\infty) \times \Omega \to \mathbb{R}^m$, be a function such that the following conditions hold:

1. $\mathbb{P}[\{\omega \in \Omega : \mathbf{B}(t_0,\omega) = \mathbf{b}_0\}] = 1$, and we say that the Wiener process starts at $(\mathbf{b}_0, t_0)$.
2. The function $t \mapsto \mathbf{B}(t,\omega)$ is continuous for almost every $\omega \in \Omega$[1].
3. $\mathbf{B}$ is a Gaussian process, and for all t, $s \geq t_0$,

   $$\mathbb{E}[\mathbf{B}(t,\cdot)] = \mathbf{0}, \quad \mathbb{E}[\mathbf{B}(t,\cdot)^{\mathsf{T}}\mathbf{B}(s,\cdot)] = m\, \min\{t,\, s\}, \quad \mathbb{E}[(\mathbf{B}(t,\cdot) - \mathbf{B}(s,\cdot))^2] = m\, |t-s|\,.$$

   From the last equality, we see that $\mathbf{B}(t,\cdot) - \mathbf{B}(s,\cdot)$ is, in distribution, equal to $\mathbf{B}(t-s,\cdot)$. Therefore they have the same expectation, variance, other higher order moments and more important statistical information.
4. $\mathbf{B}(t,\cdot)$ has independent increments. That is,

   $$\mathbf{B}(t,\cdot) - \mathbf{B}(r,\cdot),\, \mathbf{B}(s,\cdot) - \mathbf{B}(t,\cdot),$$

   are independent for all $t_0 \leq r < t < s$.

We define the canonical Wiener process as any stochastic process verifying these four conditions.

Let us reconsider something about Wiener processes. As we can see from the third condition, $\mathbf{B}(t-s,\cdot) \sim \mathrm{N}(\mathbf{0}, \mathbf{C})$, where $\mathbf{C} \in \mathbb{R}^{m\times m}$ is the covariance matrix, whose diagonal is the vector whose components are all $|t-s|$. However, we can also consider the non-canonical Wiener process. By rewriting the matrix $\mathbf{C}$ with components written as $2\,D_{i\,j}$, for i, $j \in \{1,\ldots,m\}$ (that is, this value depends on the vector component, not the time at which we consider the process), we can consider more general cases in which the variance vector of the process is a multiple of the absolute value of the time distance. Specifically, we will have in this case, $\mathbb{E}[B_i(t,\cdot)B_j(s,\cdot)] = 2D_{i\,j}\min\{t,s\}$ and, therefore

$$\mathbb{E}[\mathbf{B}(t,\cdot)^{\mathsf{T}}\mathbf{B}(s,\cdot)] = 2\min\{t,\, s\}\sum_{i=1}^{m} D_{i\,i}, \quad \mathbb{E}[(\mathbf{B}(t,\cdot) - \mathbf{B}(s,\cdot))^2] = 2\,|t-s|\sum_{i=1}^{m} D_{i\,i}.$$

The Wiener process plays a fundamental role when describing random noise in a system. The desirable properties of white noise cannot be satisfied by any "reasonable" stochastic process. Let $\mathbf{W}(t)$ be the assumed stochastic process that describes noise. The properties to be verified are

1. If $t_1 \neq t_2$, then $\mathbf{W}(t_1)$ and $\mathbf{W}(t_2)$ are independent component-wise.
2. $\{\mathbf{W}(t)\}_{t\geq t_0}$ is strictly stationary.

[1]Following the most rigorous construction of the Wiener process, $\mathbf{B}(\cdot,)$ is not necessarily continuous. However, it can be proven (by the Kolmogorov continuity theorem, see [3, Th. 2.2.3]) that a continuous version of the Wiener process exists, having the same finite dimensional distributions.

3. $\mathbb{E}[\mathbf{W}(t)] = 0$ for all $t \geq t_0$.

Although the white noise process can be represented as a generalized stochastic process (the stochastic counterpart for distributions, or generalized functions) on the space of tempered distributions in $[t_0, \infty)$ (see [1, Ch. 3] and [9, Ch. 2]), we can also use the "derivative" of the Wiener process, which verifies the previous conditions. That is, **formally** we have

$$\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}(t,\omega) := \mathbf{W}(t,\omega) = \lim_{h\to 0} \frac{1}{h}(\mathbf{B}(t+h,\omega) - \mathbf{B}(t,\omega)), \quad t \geq t_0, \quad \omega \in \Omega,$$

and define **white noise** as the process $\mathbf{W}(t,\omega)$. We can easily see that white noise, as defined above verifies the desired properties. However, the Wiener process does not have differentiable trajectories (it has $\frac{1}{2}$-Hölder continuous trajectories (see [3, Ch. 2])), it is not m.s. differentiable and it does not have finite variation. Therefore, we cannot define its derivative nor its integral in the classical and usual m.s. sense. However, we will consider the derivative of the Wiener process in a formal sense and will make use of all of its statistical properties.

Also, a different type of integral must be defined: the Itô integral (for a rigorous construction of the Itô integral, see [3, Ch. 3]). The multidimensional Itô integral is defined as follows

$$\int_{t_0}^{t} \mathbf{X}(t)^\mathsf{T}\, \mathrm{d}\mathbf{B}(t) := \lim_{n\to\infty}^{m.s.} \sum_{k=0}^{n-1} \mathbf{X}(t_k)^\mathsf{T}[\mathbf{B}(t_{k+1}) - \mathbf{B}(t_k)],$$

where $\lim\limits_{n\to\infty}^{m.s.}$ denotes the limit in the mean square sense and $\{t_0 < t_1 < \ldots < t_n = t\}$ is a partition of the interval $[t_0, t]$.

For further details on stochastic processes and probability theory, see [1, Ch. 4], [10] and/or [11].

Chapter 2

Random Differential Equations

RDEs deal with differential equations where uncertainty enters the equation via random variables in the coefficients, source terms and initial/boundary conditions. Equations of this type form the basis of Liouville-Gibbs theory in statistical mechanics. Also, in pre-flight analysis of space missions, it is necessary to use the minimum amount of fuel to reach a certain target. However, due to small errors in fuel injection systems, orbit determination processes, atmospheric pressure variations during launch and other many factors, it is extremely unlikely that our spacecraft will achieve its desired target without any orbital corrective maneuvers. To take all these factors into consideration, one may use a random description of initial conditions, coefficients and/or source terms in the differential equations that describe the flight dynamics.

In this context, the main approach to determine the 1-PDF of the solution of dynamical systems with randomness is the random variable transformation technique. This method has been successfully applied in dealing with random difference equations [12, 13], random ODEs [14, 15] and random partial differential equations [16, 17]. As it can be checked in these papers, the successful application of random variable transformation method relies heavily on defining an appropriate invertible mapping based on the knowledge of an explicit expression of the solution stochastic process and then computing its jacobian. In this chapter, we study an alternative approach, based on the Liouville-Gibbs partial differential equation (PDE) for dynamical systems, in order to determine the 1-PDF of RDEs. The main advantage of this approach, with respect to the random variable transformation method, is that it provides an explicit expression of the 1-PDF in terms of the data avoiding the search of any *ad-hoc* invertible mapping as well as the computation of its jacobian.

We are going to consider our model problem to be formulated via random initial value problems (IVPs) of the form

$$\left\{ \begin{array}{rcl} \dot{\mathbf{X}}(t) & = & \mathbf{g}(t, \mathbf{X}(t)), \quad t \geq t_0, \\ \mathbf{X}(t_0) & = & \mathbf{X}_0, \end{array} \right. \tag{2.1}$$

where $\mathbf{g} : [t_0, \infty[\times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a continuously differentiable function, $\mathbf{X}_0$ is an absolutely

continuous random vector in $\mathrm{L}_2^n(\Omega)$, $\mathbf{X}(t)$ is a 2-stochastic process defined in $\mathrm{L}_2^n(\Omega)$, and $\dot{\mathbf{X}}(t)$ denotes the m.s. derivative with respect to time.

In the first section we will see the relationship between the 1-PDF of the solution stochastic process of a random IVP and the Liouville-Gibbs, or continuity, equation. Afterwards, in the following section, we will see when we can guarantee a solution to IVP (2.1) and how to obtain it.

2.1 The Liouville-Gibbs Partial Differential Equation

In this section, we shall show that a PDF, $f = f(t, \mathbf{x})$, associated to the solution stochastic process of the random IVP (2.1) satisfies the following PDE

$$\frac{\partial f}{\partial t} + \sum_{j=1}^{n} \frac{\partial (f g_j)}{\partial x_j} = 0, \tag{2.2}$$

where $g_j = g_j(x_1, \ldots, x_n)$, $1 \leq j \leq n$, denotes the components of mapping $\mathbf{g}$ defining the right-hand side of the RDE in (2.1). The equation (2.2) is called the Liouville-Gibbs PDE.

In order to derive this equation, let us fix $t \in [t_0, \infty[$ and consider the definition of the characteristic function of the random vector $\mathbf{X}(t) = (X_1(t), \ldots, X_n(t))^{\mathsf{T}}$ (which corresponds to the evaluation of the solution stochastic process to IVP (2.1) at the time instant $t \in [t_0, \infty[$),

$$\Phi(t, \mathbf{u}) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{X}(t)}\right] = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{x}} f(t, \mathbf{x})\, \mathrm{d}\mathbf{x} = (2\pi)^n \mathcal{F}[f(t, \mathbf{x})](\mathbf{u}), \tag{2.3}$$

where $\mathbf{u} = (u_1, \ldots, u_n)^{\mathsf{T}} \in \mathbb{R}^n$ and $\mathcal{F}[\cdot]$ is the Fourier Transform operator.

Now, we differentiate expression (2.3) with respect to t and apply the commutation between the m.s. derivative and the expectation operator (see (1.10)). This yields

$$\begin{aligned}
\frac{\partial \Phi(t, \mathbf{u})}{\partial t} &= \frac{\partial}{\partial t}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^n u_k X_k(t)}\right] = \mathbb{E}\left[\frac{\partial}{\partial t}\left(\mathrm{e}^{\mathrm{i}\sum_{k=1}^n u_k X_k(t)}\right)\right] \\
&= \mathbb{E}\left[\mathrm{i}\sum_{k=1}^n u_k \dot{X}_k(t) \mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{X}(t)}\right] = \mathrm{i}\sum_{k=1}^n u_k \mathbb{E}[\dot{X}_k(t)\mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{X}(t)}] \\
&= \mathrm{i}\sum_{k=1}^n u_k \mathbb{E}[g_k(t, \mathbf{X}(t))\mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{X}(t)}] = \sum_{k=1}^n \mathrm{i}u_k \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{x}} g_k(t, \mathbf{x}) f(t, \mathbf{x})\, \mathrm{d}\mathbf{x}.
\end{aligned} \tag{2.4}$$

On the one hand, according to (1.4) and (2.3), each addend in the last sum can be expressed in terms of the Fourier transform,

$$\mathrm{i}u_k \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{x}} g_k(t, \mathbf{x}) f(t, \mathbf{x})\, \mathrm{d}\mathbf{x} = (2\pi)^n \mathcal{F}\left[-\frac{\partial}{\partial x_k}(g_k(t, \mathbf{x}) f(t, \mathbf{x}))\right](\mathbf{u}).$$

Then using this latter representation together with the linearity of the Fourier transform operator, expression (2.4) can be written as

$$\frac{\partial \Phi(t,\mathbf{u})}{\partial t} = (2\pi)^n \mathcal{F}\left[-\sum_{k=1}^{n} \frac{\partial}{\partial x_k}(g_k(t,\mathbf{x}) f(t,\mathbf{x}))\right](\mathbf{u}). \tag{2.5}$$

On the other hand, if we directly differentiate (2.3) under the integral sign with respect to t, one gets

$$\frac{\partial \Phi(t,\mathbf{u})}{\partial t} = \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}\mathbf{u}^{\mathsf{T}}\mathbf{x}} \frac{\partial f(t,\mathbf{x})}{\partial t} \mathrm{d}\mathbf{x} = (2\pi)^n \mathcal{F}\left[\frac{\partial f(t,\mathbf{x})}{\partial t}\right](\mathbf{u}). \tag{2.6}$$

Finally, subtracting (2.5) and (2.6) and using Th. 1.2, i.e., the inversion Fourier transformation, one obtains the Liouville-Gibbs equation

$$\frac{\partial f(t,\mathbf{x})}{\partial t} + \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(g_k(t,\mathbf{x}) f(t,\mathbf{x})) = 0. \tag{2.7}$$

Remark. Notice that if we differentiate the product in (2.7), the Liouville-Gibbs equation can be expressed equivalently in terms of the divergence operator of mapping $\mathbf{g}$, i.e., $\nabla_{\mathbf{x}} \cdot \mathbf{g} = \sum_{k=1}^{n} \frac{\partial g_k}{\partial x_k}$, leading to

$$\frac{\partial f(t,\mathbf{x})}{\partial t} + \sum_{k=1}^{n} g_k(t,\mathbf{x}) \frac{\partial f(t,\mathbf{x})}{\partial x_k} + f(t,\mathbf{x}) \nabla_{\mathbf{x}} \cdot \mathbf{g}(t,\mathbf{x}) = 0.$$

2.2 Solving the Liouville-Gibbs equation

In the context of applications, it is assumed that the PDF, $f_0(\mathbf{x})$, of the absolutely continuous random initial condition, $\mathbf{X}_0$, of random IVP, (2.1), is known after sampling or experimental measurements. So, our goal is to determine the solution of the following IVP for the Liouville-Gibbs PDE,

$$\begin{cases} \dfrac{\partial f(t,\mathbf{x})}{\partial t} + \displaystyle\sum_{k=1}^{n} g_k(t,\mathbf{x}) \frac{\partial f(t,\mathbf{x})}{\partial x_k} = -f(t,\mathbf{x}) \nabla_{\mathbf{x}} \cdot \mathbf{g}(t,\mathbf{x}), \quad t > t_0,\ \mathbf{x} \in \mathbb{R}^n, \\ f(t_0,\mathbf{x}) = f_0(\mathbf{x}), \quad \mathbf{x} \in S_0, \end{cases} \tag{2.8}$$

where S_0 denotes the interior of the support of f_0, which is assumed to be a C^1 hypersurface.

We can prove that there exists a unique solution to the IVP (2.8), we will use the following theorem.

Theorem 2.1. *([6, Th. 1.10]). Let S be a C^1 hypersurface in $\mathbb{R}^n$. Consider the following IVP*

$$\begin{cases} \displaystyle\sum_{k=1}^{n} a_k(\mathbf{x},u) \frac{\partial u}{\partial x_k}(\mathbf{x}) = b(\mathbf{x},u), \quad \mathbf{x} \in \mathbb{R}^n, \\ u(\mathbf{x}) = \psi(\mathbf{x}), \quad \mathbf{x} \in S, \end{cases} \tag{2.9}$$

where a_k, b and ψ are C^1 real-valued functions. Additionally, suppose that the vector

$$(a_1(\mathbf{x}, \psi(\mathbf{x})), \ldots, a_n(\mathbf{x}, \psi(\mathbf{x})))$$

is not tangent to S at any point. Then, there exists a neighborhood Ω of S in $\mathbb{R}^n$ such that there exists a unique solution $u \in C^1(\Omega)$ of the IVP (2.9).

Remark. It is well known that hypotheses can be weakened in certain cases to prove the existence and uniqueness of weak solutions for the continuity equation (see [8, S. 3.4]). However, we will use this theorem due to the fact that the examples considered in this project verify the hypotheses and gives the existence of a strong solution.

Let us check that IVP (2.8) verifies the hypotheses of Th. 2.1. We define $S := \{(t_0, \mathbf{x}) : \mathbf{x} \in S_0\}$, which is a smooth hypersurface parametrically given by

$$\sigma : S_0 \longrightarrow \mathbb{R}^{n+1}, \quad \sigma(\mathbf{s}) := (\sigma_1(\mathbf{s}), \ldots, \sigma_{n+1}(\mathbf{s})) = (t_0, \mathbf{s}),$$

where $\mathbf{s} = (s_1, \ldots, s_n) \in S_0 \subset \mathbb{R}^n$. Let us see that $(1, g_1(t_0, f_0(\mathbf{x})), \ldots, g_n(t_0, f_0(\mathbf{x})))$ is not tangent to S at any $(t_0, \mathbf{x}) \in S$. Using the previous parametrization we obtain,

$$\det \begin{bmatrix} \frac{\partial \sigma_1(\mathbf{s})}{\partial s_1} & \cdots & \frac{\partial \sigma_1(\mathbf{s})}{\partial s_n} & 1 \\ \frac{\partial \sigma_2(\mathbf{s})}{\partial s_1} & \cdots & \frac{\partial \sigma_2(\mathbf{s})}{\partial s_n} & g_1(t_0, f_0(\mathbf{s})) \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial \sigma_{n+1}(\mathbf{s})}{\partial s_1} & \cdots & \frac{\partial \sigma_{n+1}(\mathbf{s})}{\partial s_n} & g_n(t_0, f_0(\mathbf{s})) \end{bmatrix}$$

$$= \det \left[\begin{array}{ccc|c} 0 & \cdots & 0 & 1 \\ \hline & & & g_1(t_0, f_0(\mathbf{s})) \\ & \mathbf{I}_n & & \vdots \\ & & & g_n(t_0, f_0(\mathbf{s})) \end{array}\right] = (-1)^{n+1} \neq 0, \ \forall \mathbf{s} \in S_0,$$

where $\mathbf{I}_n$ denotes the identity matrix of size n. Therefore, IVP (2.8) verifies the hypothesis of Th. 2.1 and we can guarantee the local existence and uniqueness of solution for the Liouville-Gibbs IVP (2.8).

To calculate its solution, first we apply the Lagrange-Charpit technique [18], that leads to the following set of differential equations

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}f}{-f\nabla_{\mathbf{x}} \cdot \mathbf{g}} = \frac{\mathrm{d}x_1}{g_1} = \ldots = \frac{\mathrm{d}x_n}{g_n}. \tag{2.10}$$

Secondly, we apply the method of characteristics [19] for the above equations involving the variables $t, x_1, \ldots, x_n$. This allow us to establish the following set of differential

equations formulated in terms of the auxiliary variables s and $\mathbf{r} = (r_1, \ldots, r_n)$,

$$\begin{aligned}
\frac{\mathrm{d}t}{\mathrm{d}s}(s,\mathbf{r}) &= 1, & t(0,\mathbf{r}) &= 0,\\
\frac{\mathrm{d}x_1}{\mathrm{d}s}(s,\mathbf{r}) &= g_1(t,\mathbf{x}), & x_1(0,\mathbf{r}) &= r_1,\\
&\vdots & &\vdots\\
\frac{\mathrm{d}x_n}{\mathrm{d}s}(s,\mathbf{r}) &= g_n(t,\mathbf{x}), & x_n(0,\mathbf{r}) &= r_n.
\end{aligned}$$

Solving these ODEs, we obtain a parametrization of the variables $t = t(s,\mathbf{r})$ and $\mathbf{x} = \mathbf{x}(s,\mathbf{r})$. Now, using the equation that relates variables t and f in the chain of equations given in (2.10), one obtains the following ODE,

$$\frac{\partial f}{\partial s}(s,\mathbf{r}) = -f(t,\mathbf{x})\nabla_{\mathbf{x}} \cdot \mathbf{g}(t,\mathbf{x}), \quad f(0,\mathbf{r}) = f_0(\mathbf{r}),$$

which has a well-known solution,

$$f(s,\mathbf{r}) = f_0(\mathbf{r}) \exp\left\{ -\int_0^s \nabla_{\mathbf{x}} \cdot \mathbf{g}(\sigma,\mathbf{r})\, \mathrm{d}\sigma \right\}. \tag{2.11}$$

To express the solution in terms of the original variables, we use the fact that our initial conditions verify the hypotheses of Th. 2.1. This result guarantees that we can invert the parametrization, hence obtaining the functions $s(t,\mathbf{x})$ and $\mathbf{r}(t,\mathbf{x})$ that we can substitute, respectively, into expression (2.11). Therefore, the solution can be written as

$$f(t,\mathbf{x}) = f_0(\mathbf{r}(t,\mathbf{x})) \exp\left\{ -\int_{t_0}^t \nabla_{\mathbf{x}} \cdot \mathbf{g}(s(\tau,\mathbf{x}),\mathbf{r}(\tau,\mathbf{x}))\, \mathrm{d}\tau \right\}. \tag{2.12}$$

It is interesting to note that the function $\mathbf{r}(t,\mathbf{x})$ is actually the function we would obtain when solving the IVP (2.1), say $\mathbf{X}(t) = \mathbf{h}(t,\mathbf{X}_0)$, and then solving for $\mathbf{X}_0$. With the corresponding notation this writes as $\mathbf{r}(t,\mathbf{x}) = \mathbf{h}^{-1}(t,\mathbf{x}) = \mathbf{x}_0$. As a consequence, the 1-PDF given in (2.12) can be finally expressed in terms of the data

$$f(t,\mathbf{x}) = \left(f_0(\mathbf{x}_0) \exp\left\{ -\int_{t_0}^t \nabla_{\mathbf{x}} \cdot \mathbf{g}(\tau, \mathbf{x} = h(\tau,\mathbf{x}_0))\, \mathrm{d}\tau \right\} \right) \Big|_{\mathbf{x}_0 = h^{-1}(t,\mathbf{x})}. \tag{2.13}$$

So far we have dealt with the case where randomness appears through initial conditions only. Observe that the expression (2.13) is given in terms of the PDF, f_0, of the random initial condition $\mathbf{X}_0$. Now, we extend this result to the general case where the RDE also depends on a finite number of random variables, represented by the absolutely continuous random vector $\mathbf{A} = (A_1, \ldots, A_m)$, i.e.,

$$\begin{cases} \dot{\mathbf{X}}(t) = \mathbf{g}(t,\mathbf{X}(t),\mathbf{A}), & t > t_0, \\ \mathbf{X}(t_0) = \mathbf{X}_0, \end{cases} \tag{2.14}$$

where $\mathbf{g} : [t_0, \infty[\times \mathrm{L}_2^n([t_0, +\infty) \times \Omega) \times \mathrm{L}_2^m(\Omega) \mapsto \mathrm{L}_2^n([t_0, +\infty) \times \Omega)$. These kind of random IVPs involving a finite number of random variables (in this case $n+m$ random variables) are usually called IVPs with a finite degree of randomness [1, Ch. 3]. Henceforth, the PDF, $f_0(\mathbf{x}_0, \mathbf{a})$, of the random vector made up of all random inputs $(\mathbf{X}_0, \mathbf{A})$ is assumed to be known. Below, we shall show how to obtain a similar expression to (2.13) to the random IVP (2.14), i.e., when uncertainties appear in initial condition and in the differential equation. To achieve this goal, the strategy will consist of transforming the random IVP (2.14) into another random IVP (having higher dimension) where randomness only appears through the initial condition, and then we will apply (2.13). To this end, let us consider the extended random IVP

$$\begin{cases} \dot{\mathbf{Y}}(t) = \mathbf{G}(t, \mathbf{Y}(t)), \\ \mathbf{Y}(t_0) = \mathbf{Y}_0, \end{cases} \tag{2.15}$$

where $\mathbf{Y}(t) = (\mathbf{X}(t), \mathbf{A}) = (X_1(t), \ldots, X_n(t), A_1, \ldots, A_m)$, $\mathbf{Y}_0 = (\mathbf{X}_0, \mathbf{A})$ and $\mathbf{G}(t, \mathbf{Y}(t)) = (\mathbf{g}(t, \mathbf{X}(t)), \mathbf{0}) \in \mathrm{L}_2^{n+m}(\Omega)$ is continuously differentiable. With this new reformulation of IVP (2.14), randomness only appears via the initial condition $\mathbf{Y}_0$, and then the results shown in the first part of this section are applicable. Therefore, according to (2.7), the Liouville-Gibbs PDE to the random IVP (2.15) is given by

$$\frac{\partial f(t, \mathbf{y})}{\partial t} + \sum_{k=1}^{n+m} \frac{\partial}{\partial y_k} \left(G_k(t, \mathbf{y}) f(t, \mathbf{y}) \right) = 0. \tag{2.16}$$

Notice that $G_k(t, \mathbf{y}) = g_k(t, \mathbf{y})$, $1 \leq k \leq n$, and $G_k(t, \mathbf{y}) = 0$, $n+1 \leq k \leq n+m$. Therefore, taking into account that $\mathbf{y} = (\mathbf{x}, \mathbf{a}) \in \mathbb{R}^{n+m}$, PDE (2.16) can be written as

$$\begin{cases} \dfrac{\partial f(t, \mathbf{x}; \mathbf{a})}{\partial t} + \displaystyle\sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(g_k(t, \mathbf{x}, \mathbf{a}) f(t, \mathbf{x}; \mathbf{a}) \right) = 0, \quad t > t_0, \\ f(t_0, \mathbf{x}; \mathbf{a}) = f_0(\mathbf{x}_0, \mathbf{a}), \end{cases}$$

where we have included the initial condition, $f_0(\mathbf{x}_0, \mathbf{a})$, corresponding to the PDF of the random initial vector $(\mathbf{X}_0, \mathbf{A})$. Then, the solution of this IVP turns straightforwardly out

$$f(t, \mathbf{x}; \mathbf{a}) = f_0(\mathbf{x}_0, \mathbf{a}) \exp \left\{ - \int_{t_0}^{t} \nabla_{\mathbf{x}} \cdot \mathbf{g}(\tau, \mathbf{x} = \mathbf{h}(\tau, \mathbf{x}_0, \mathbf{a}), \mathbf{a}) \, \mathrm{d}\tau \right\} \Big|_{\mathbf{x}_0 = h^{-1}(t, \mathbf{x}, \mathbf{a})}, \tag{2.17}$$

where $\mathbf{h}(t, \mathbf{X}_0, \mathbf{A}) = \mathbf{X}(t)$. Finally, the 1-PDF of the solution stochastic process to random IVP (2.14) is derived by marginalizing $f(t, \mathbf{x}; \mathbf{a})$ with respect to random vector $\mathbf{A}$,

$$f(t, \mathbf{x}) = \int_{\mathbb{R}^m} f(t, \mathbf{x}; \mathbf{a}) \, \mathrm{d}\mathbf{a}. \tag{2.18}$$

2.3 Example

It is known that dynamics governing several physical systems, such as linear oscillators, can be characterized by a homogeneous second-order linear differential equation. For the

sake of generality, we will derive the 1-PDF of the solution stochastic process of n-th order homogeneous differential equations of the form

$$\begin{cases} \sum_{j=0}^{n} a_j(t) X^{(j)}(t) = 0, \\ X(0) = X_0, \ \ldots, \ X^{(n-1)}(0) = X_{n-1}, \end{cases} \tag{2.19}$$

where $X^{(j)}(t) = \dfrac{\mathrm{d}^j X}{\mathrm{d}t^j}(t)$. We assume $a_n(t) \neq 0$, $\forall t \geq 0$, and $X_0, \ldots, X_{n-1}$ are random variables. Also, we assume that the joint PDF of the initial random vector $\mathbf{X}_0 = (X_0, X_1, \ldots, X_{n-1})^{\mathsf{T}}$, denoted by f_0, is known. In order to take advantage of the results obtained in this chapter, we write (2.19) as n-dimensional first-order matrix ODE To this end, we first divide by $a_n(t)$ in (2.19)

$$X^{(n)}(t) = -\sum_{j=0}^{n-1} \frac{a_j(t)}{a_n(t)} X^{(j)}(t),$$

secondly, we put $\mathbf{X}(t) = (X(t), X'(t), \ldots, X^{(n-1)}(t))^{\mathsf{T}}$, and then we obtain

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t), \ \mathbf{A}(t) := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_0(t)}{a_n(t)} & -\frac{a_1(t)}{a_n(t)} & -\frac{a_2(t)}{a_n(t)} & \ldots & -\frac{a_{n-1}(t)}{a_n(t)} \end{bmatrix}, \ \mathbf{X}(0) = \mathbf{X}_0. \tag{2.20}$$

It corresponds to the random IVP (2.1) by taking $t_0 = 0$, $\mathbf{g}(t, \mathbf{X}(t)) = \mathbf{A}(t)\mathbf{X}(t)$. For $t > 0$ arbitrary but fixed, write $\mathbf{X}(t) = \mathbf{x} = (x_1, \ldots, x_n)$, then $X^{(j)}(t) = x_{j+1}$, $j \in \{0, \ldots, n-1\}$ and

$$g_j(t, \mathbf{x}) = x_{j+1}, \quad g_n(t, \mathbf{x}) = -\sum_{j=0}^{n-1} \frac{a_j(t)}{a_n(t)} x_{j+1}.$$

Therefore,

$$\frac{\partial g_j(t, \mathbf{x})}{\partial x_j} = 0, \quad \forall j \in \{0, \ldots, n-1\}, \qquad \frac{\partial g_n(t, \mathbf{x})}{\partial x_n} = -\frac{a_{n-1}(t)}{a_n(t)},$$

and, as a consequence,

$$\nabla_{\mathbf{x}} \cdot \mathbf{g}(t, \mathbf{x}) = -\frac{a_{n-1}(t)}{a_n(t)}.$$

Now we apply (2.13) to give an explicit expression of the 1-PDF of the solution stochastic process to the random IVP (2.19)

$$f(t, \mathbf{x}) = f_0(\mathbf{x_0} = \mathbf{h}^{-1}(t, \mathbf{x})) \exp\left\{ \int_0^t \frac{a_{n-1}(s)}{a_n(s)} \mathrm{d}s \right\}, \tag{2.21}$$

where $\mathbf{h}^{-1}(t,\mathbf{x}) = \mathrm{e}^{-\int_0^t \mathbf{A}(s)\,\mathrm{d}s}\mathbf{x}$, where $\mathbf{A}(s)$ is defined in (2.20).

To conclude this chapter we want to emphasize that the Liouville-Gibbs method is exact in the sense of providing a closed-form expression for the PDF of the solution stochastic process in terms of an integral (see expressions (2.13) and (2.17)–(2.18)) that can be exactly computable in some cases (otherwise accurate quadrature rules can be applied). An alternative exact method that can also be used is the so-called random variable transformation technique [1, Ch. 3], however its application requires defining an *ad-hoc* injective mapping having non-zero Jacobian that could become difficult to find in some situations. This fact may limit the use of random variable transformation against the Liouville-Gibbs approach in some models. On the other hand, there exist alternate approaches like Monte Carlo simulations, but they only provide approximations in spite of eventually an exact solution is available. Furthermore, using the Monte Carlo method usually requires carrying out many simulations to obtain good approximations since its rate of convergence is slow [20] and this may become prohibitively demanding. All these comments will be pointed out in the forthcoming examples to inform the reader about the advantages of Liouville-Gibbs method against alternate approaches in the models that will be presented later.

Chapter 3

Stochastic Differential Equations

In this chapter, based in [1, Ch. 7, 8] and [3, Ch. 7], we are going to consider the more general case of differential equations where randomness appears, not only as random variables/vectors, but as stochastic processes. These equations are useful when modeling systems where uncertainty evolves over space and time. For example, when taking observations of an experiment, or receiving an electromagnetic signal, we usually obtain a "noisy pattern" of values; that is, we get a disturbed version of the expected values. This noise offsets our signal or observation in a different way at every time instant. How can we clean up the signal, or observations, in an optimal way? This is known as the **filtering problem**, and it is the very basis of modern telecommunications, where hundreds of books on filters and optimal filtering have been written. Also, it is widely used in aeronautics and aerospace engineering where feedback controllers are of such great importance for human pilots, as well as unmanned ships. It is obvious that a clean feedback is absolutely necessary for clear communications and optimal flights and missions.

SDEs have a **drift** term, which indicates the expected evolution of the system, and the **noise** term, which will indicate the intensity and structure of the uncertainty in the system. The most important, or influential, kind of SDEs are those in which uncertainty appears as **white noise**. Contrarily to RDEs, SDEs are much harder to solve. Dealing with SDEs requires a special kind of calculus, the Itô calculus, which is based on the Itô integral and the properties arising from this integral. It is not a natural extension of the deterministic theory as the m.s. calculus used in RDEs. As in the case of RDEs, we can obtain ODEs verified by the mean, covariance or cross-covariance (the covariance of the components of a multidimensional stochastic process at the same time instant) of the solution stochastic process. Most research papers dealing with SDEs in mathematical modeling study these two important functions of the solution stochastic process. However, we can obtain a PDE that will make available, not only the mean, covariance and cross-covariance of the solution, but any moment of the solution and confidence intervals, to say some.

In general, SDEs can be written as

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(t) = \mathbf{g}(\mathbf{X}(t), t) + \mathbf{Y}(t), \quad t > t_0, \\ \mathbf{X}(0) = \mathbf{X}_0, \end{cases} \tag{3.1}$$

where $\mathbf{g} = (g_1 \ldots, g_n)$ is a $C^1(\mathbb{R}^n \times [t_0, +\infty), \mathbb{R}^n)$ deterministic function and $\mathbf{X}(t, \cdot)$, $\mathbf{Y}(t, \cdot) \in \mathrm{L}_2^n(\Omega, \mathbb{R}^n)$ for all $t \in [t_0, +\infty)$. The initial condition $\mathbf{X}_0$ can be taken as a second order random variable/vector or a deterministic constant. Questions and results about existence and uniqueness of solutions of Eq. (3.1) can be found in [1, Ch. 7] and [3, Ch. 5].

This chapter is organized as follows. Section 3.1 shows a very useful property when studying the PDF of an arbitrary stochastic process. In Section 3.2, we will deal with the particular and well known case of Gaussian white noise in the form of an Itô diffusion.

3.1 The Kinetic Equation

We are going to do the following reasoning in the scalar case. Its extension to the vector case is trivial, but much longer. Let $\{X(t)\}_{t \geq t_0}$ be a stochastic process, and $f(x, t)$ its first PDF at a time t. Let us assume f is analytic and the characteristic function Φ is analytic in a neighborhood of $u = 0$. The integral form of Bayes' theorem reads

$$f(x, t + \Delta t) = \int_{\mathbb{R}} f(x, t + \Delta t \,|\, x', t)\, f(x', t)\, \mathrm{d}x', \tag{3.2}$$

where $f(x, t + \Delta t \,|\, x', t)$ denotes the conditional density function of the r.v. $X(t + \Delta t)$ given $X(t) = x'$. If we denote by $\Phi(u, t + \Delta t \,|\, x', t)$ the characteristic function of the r.v. $\Delta X(t) = X(t + \Delta t) - X(t)$ given $X(t) = x'$, that is

$$\begin{aligned} \Phi(u, t + \Delta t \,|\, x', t) &= \mathbb{E}[\mathrm{e}^{\mathrm{i}u\Delta X(t)} \,|\, x', t] \\ &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}u(x - x')}\, f(x, t + \Delta t \,|\, x', t)\, \mathrm{d}x. \end{aligned}$$

Therefore, taking the inverse Fourier transform, which is guaranteed because of the integrability of both f and Φ, we obtain

$$f(x, t + \Delta t \,|\, x', t) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}u(x - x')}\, \Phi(u, t + \Delta t \,|\, x', t)\, \mathrm{d}u. \tag{3.3}$$

Now, let us assume Φ is analytic in $\mathbb{R}$. Taking the Taylor expansion about $u = 0$ we obtain

$$\Phi(u, t + \Delta t \,|\, x', t) = \sum_{k=0}^{\infty} \frac{\partial_u^k \Phi(0, t + \Delta t \,|\, x', t)}{k!} u^k.$$

Now, using the definition of Φ, and by direct calculation, each component of the sum is

$$\partial_u^k \Phi(0, t + \Delta t \,|\, x', t) = \mathrm{i}^k \mathbb{E}[\Delta X^k(t) \,|\, X(t) = x'].$$

Now, using this equality and the Taylor expansion expression back in (3.3) and assuming uniform convergence of the series in $\mathbb{R}$, we obtain

$$f(x, t+\Delta t \,|\, x', t) = \sum_{k=0}^{\infty} \frac{1}{k!} \overbrace{\mathbb{E}[\Delta X^k(t) \,|\, X(t) = x']}^{a_k(x',t)} \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}u(x-x')} \, (\mathrm{i}\, u)^k \, \mathrm{d}u. \tag{3.4}$$

It is clear that the last integral expression in Equation (3.4) does not exist in the classical sense of a function. But it does exist as a distribution (see Chapter 1.2). Particularly Equation (1.7) gives

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}u(x-x')} \, (\mathrm{i}\, u)^k \, \mathrm{d}u = (-1)^k \delta^{(k)}(x - x'),$$

in the sense of distributions, where δ denotes the Dirac delta function. Now, putting everything together with Equation (3.2) we get

$$\begin{aligned} f(x, t+\Delta t) &= \int_{\mathbb{R}} f(x, t+\Delta t \,|\, x', t) \, f(x', t) \, \mathrm{d}x', \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}} a_k(x', t) \delta^{(k)}(x - x') \, f(x', t) \, \mathrm{d}x' \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \partial_x^k \{a_k(x, t) \, f(x, t)\} = f(x, t) + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \partial_x^k \{a_k(x, t) \, f(x, t)\}. \end{aligned} \tag{3.5}$$

If we substract $f(x, t)$ in both sides and divide by Δt in both sides, Equation (3.5) reads:

$$\frac{f(x, t+\Delta t) - f(x, t)}{\Delta t} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \partial_x^k \left\{ \frac{a_k(x, t)}{\Delta t} \, f(x, t) \right\}.$$

All that remains is to take limits when $\Delta t \to 0$ and we get

$$\partial_t f(x, t) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \partial_x^k \left\{ \alpha_k(x, t) \, f(x, t) \right\} = 0, \tag{3.6}$$

where $\alpha_k(x, t) = \lim_{\Delta t \to 0} \frac{a_k(x,t)}{\Delta t}$, if it exists. Equation (3.6) is known as the **Kinetic Equation** (see [1, SS. 7.2.3]), **Kramers-Moyal expansion** (see [21]), or **generalized Fokker-Planck Equation** (see [22]) of the stochastic process $X(t)$.

We have, for the vector case, the following expression

$$\partial_t f(\mathbf{x}, t) + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \sum_{|\mathbf{a}|=k} \partial_{\mathbf{x}}^{\mathbf{a}} \left\{ \alpha_{\mathbf{a}}(\mathbf{x}, t) \, f(\mathbf{x}, t) \right\} = 0, \tag{3.7}$$

where $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is a multi-index and

$$\alpha_{\mathbf{a}}(\mathbf{x}, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\Delta \mathbf{X}^{\mathbf{a}}(t) \,|\, \mathbf{X}(t) = \mathbf{x}\right].$$

Equation (3.7) can be obtained easily using the multidimensional Taylor expansion of the characteristic function and applying the same steps as in the previous reasoning for the scalar case.

Remark. Although we have imposed (very) strong assumptions on the differentiability of the density function and the characteristic function, the proof is also valid on any region where conditions are satisfied. Hence, if there are discontinuities in the PDF, Equation (3.7) will be verified in the weak sense[1] over $\mathbb{R}^n$ (assuming it has weak derivatives of any order), and will be verified in the strong sense in the regions of continuity.

Equation (3.7) gives a **possible** solution of f; in other words, it is a necessary condition, not a sufficient condition. We have obtained a tool in the form of a PDE to study and obtain the expression of the first PDF of an arbitrary stochastic process $\mathbf{X}(t)$, in certain cases. Furthermore, as it can be seen in [1, SS 7.2.3], an analogous reasoning can be applied to obtain any density function (second, third, etc.) of the stochastic process $X(t)$.

Now, despite we have found a way to obtain information about the first PDF of the stochastic process $\mathbf{X}(t)$, having an infinite order PDE is not very useful. However, the following theorem can be extremely helpful, at least in the scalar case.

Theorem 3.1. *(See [1, Th. 7.2.1]) If $\alpha_k(x,t)$, defined as in Equation (3.6), exists $\forall k$ and there is $k_0 \in \mathbb{N}$ such that $\alpha_{k_0}(x,t) = 0$, then $\alpha_k(x,t) = 0$, $\forall k \geq 3$.*

Proof. Let $k \geq 3$ and odd. We consider

$$\begin{aligned}
\alpha_k(x,t)^2 &= \left(\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\Delta X(t)^{\frac{k-1}{2}} \Delta X(t)^{\frac{k+1}{2}} \,|\, X(t) = x\right]\right)^2 \\
&\leq \lim_{\Delta t \to 0} \frac{1}{\Delta t^2} \mathbb{E}\left[\Delta X(t)^{k-1} \,|\, X(t) = x\right] \mathbb{E}\left[\Delta X(t)^{k+1} \,|\, X(t) = x\right] \\
&= \alpha_{k-1}(x,t)\alpha_{k+1}(x,t),
\end{aligned}$$

where the second inequality was obtained due to the Schwartz's inequality (Equation (1.8)).

Now, using the same idea for α_k for $k \geq 4$ and even,we obtain

$$\alpha_k(x,t)^2 \leq \alpha_{k-2}(x,t)\, \alpha_{k+2}(x,t).$$

[1] We say that a PDE is verified in the weak sense if it is verified in the sense of distributions. Contrarily, a PDE is verified in the strong sense if the solution has the sufficient degree of continuous differentiability and it verified in every point.

Now, taknig $r = k_0$, we obtain the following system of inequalities

$$\begin{array}{ll} \alpha_{r+1}(x,t)^2 & \leq \alpha_r(x,t)\alpha_{r+2}(x,t), \quad r \geq 2, \\ \alpha_{r-1}(x,t)^2 & \leq \alpha_{r-2}(x,t)\alpha_r(x,t), \quad r \geq 4, \\ \alpha_{r+2}(x,t)^2 & \leq \alpha_r(x,t)\alpha_{r+4}(x,t), \quad r \geq 2, \\ \alpha_{r-2}(x,t)^2 & \leq \alpha_{r-4}(x,t)\alpha_r(x,t), \quad r \geq 6. \end{array} \tag{3.8}$$

Now, since $\alpha_{k_0} = 0$, Equations (3.8) imply that $\alpha_k = 0$ for all $k > r \geq 3$ and that $\alpha_k = 0$ for all $3 \leq k < r$. □

Basically, this theorem (given by Robert Pawula in his Ph.D. thesis, see [23]) states that either we have infinite non-zero terms in Equation (3.6), or we have only the two first terms; there is no in-between case. However, even in the case when there are infinite non-zero terms, we may obtain valuable information from truncating Equation (3.7) (see [21, Sec. 4.6]). The case where only the two first terms appear is known as the Fokker-Planck (forward) Equation (see [1, SS. 7.3.1]), and it has the form

$$\partial_t f(x,t) = -\partial_x\{\alpha_1(x,t)\, f(x,t)\} + \frac{1}{2}\partial_x^2\{\alpha_2(x,t)\, f(x,t)\}$$

in the scalar case. As we shall see in the next section, this equation is of vital importance in the study of a special case of SDEs.

Remark. By rewriting the previous equation as

$$\partial_t f(x,t) + \partial_x\{\alpha_1(x,t)\, f(x,t)\} = \sum_{k=2}^{\infty} \frac{(-1)^k}{k!}\partial_x^k \left\{\alpha_k(x,t)\, f(x,t)\right\}, \tag{3.9}$$

we can see, by applying the Equation (3.6) to the stochastic process defined as the solution of SDE (3.1), that its first PDF is in the form of a conservation law (see [8, S. 3.4]), such as the Liouville-Gibbs, or continuity equation for an RDE (see Section 2.1), but with an added term that acts as a diffusion in the system. We will explore this idea deeper in the next section. Pawula [23] actually proves that, if the "noise" term $\mathbf{Y}(t)$ is a m.s. differentiable, stationary Gaussian process, then the right-hand-side of Equation (3.9) vanishes.

3.2 Itô Diffusions

Diffusion processes have been successfully applied in mathematical modeling problems arising in many areas such as electronic and electrical engineering, theoretical and applied physics, biology, chemistry and economics due to the fact that they verify very interesting mathematical properties, such as being Gaussian and Markovian (see [3, Ch. 7] for the basics on Itô diffusions and Markovian processes).

Diffusion processes may be written in the following way

$$\begin{cases} \mathrm{d}\mathbf{X}(t) = \mathbf{g}(\mathbf{X}(t),t)\mathrm{d}t + \mathbf{G}(\mathbf{X}(t),t)\mathrm{d}\mathbf{B}(t), & t > t_0, \\ \mathbf{X}(t_0) = \mathbf{X}_0, \end{cases} \tag{3.10}$$

where $\mathbf{g} \in C^1(\mathbb{R}^n \times [t_0, +\infty), \mathbb{R}^n)$, $\mathbf{G} \in C^1(\mathbb{R}^n \times [t_0, +\infty), \mathbb{R}^{n\times m})$ and $\mathbf{B} : [t_0, \infty) \to \mathbb{R}^m$ is an $m-$dimensional Wiener process. We recall that its components, $B_j(t)$, $j \in \{1, \dots, m\}$, have the properties (see Def. 1.5)

$$\begin{cases} \mathbb{E}[\Delta B_j(t)] &= \mathbb{E}[B_j(t+\Delta t) - B_j(t)] = 0, \quad t \geq t_0, \\ \mathbb{E}[\Delta B_j(t)\Delta B_i(t)] &= 2D_{ij}\Delta t, \quad i,\, j = 1, \dots, m, \quad t \geq t_0. \end{cases}$$

Our main goal is to calculate the **transition density function**, denoted by $f(\mathbf{x}, t \,|\, \mathbf{x}_0, t_0) =: f_{\mathbf{x}_0}(\mathbf{x}, t)$, given the initial density $f(\mathbf{x}, t_0) = \psi(\mathbf{x})$. It can be easily seen that all $\alpha_{\mathbf{a}} = 0$ for $|\mathbf{a}| \geq 3$, therefore, we will calculate $\alpha_{\mathbf{a}}$ for $|\mathbf{a}| \leq 2$. For the sake of clarity, hereinafter we will drop the multi-index notation and use the usual subindex notation.

3.2.1 Fokker-Planck Equation

By rewriting Equation (3.10) in difference, or incremental, form, we get for each component $i \in \{1, \dots, n\}$

$$\begin{aligned} \Delta X_i(t) &= X_i(t+\Delta t) - X_i(t) \\ &= g_i(\mathbf{X}(t), t)\Delta t + \sum_{k=1}^{m} G_{ik}(\mathbf{X}(t), t)\Delta B_k(t) + o(\Delta t), \end{aligned}$$

where $o(\Delta t)$ represents the terms such that $\frac{o(\Delta t)}{\Delta t} \xrightarrow{\Delta t \to 0} 0$.

Also, we have for $i,\, j \in \{1, \dots, n\}$

$$\begin{aligned} \Delta X_i(t)\Delta X_j(t) &= g_i(\mathbf{X}(t), t) g_j(\mathbf{X}(t), t)\Delta t^2 \\ &+ g_i(\mathbf{X}(t), t)\Delta t \sum_{k=1}^{m} G_{jk}(\mathbf{X}(t), t)\Delta B_k(t) \\ &+ g_j(\mathbf{X}(t), t)\Delta t \sum_{k=1}^{m} G_{ik}(\mathbf{X}(t), t)\Delta B_k(t) \\ &+ \sum_{k,\,l=1}^{m} G_{jk}(\mathbf{X}(t), t) G_{il}(\mathbf{X}(t), t)\Delta B_k(t)\Delta B_l(t) + o(\Delta t). \end{aligned} \tag{3.11}$$

Using the fact that $\Delta\mathbf{B}(t)$ in independent of $\mathbf{X}(t)$, we obtain on the one hand

$$\begin{aligned} \alpha_i(\mathbf{x}, t) &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}[\Delta X_i(t) \,|\, \mathbf{X}(t) = \mathbf{x}] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} (g_i(\mathbf{X}(t), t)\Delta t + \sum_{k=1}^{m} G_{ik}(\mathbf{X}(t), t) \overbrace{\mathbb{E}[\Delta B_k(t)]}^{=0} + o(\Delta t)) \\ &= g_i(\mathbf{X}(t), t). \end{aligned} \tag{3.12}$$

On the other hand, taking into account the calculations from Equation (3.11), we see clearly that the first three terms of the sum will end up being 0, due to the limit and the definition of the Wiener process. Therefore, we see in the last term

$$\begin{aligned}\alpha_{ij}(\mathbf{x},t) &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}[\Delta X_i(t)\Delta \mathbf{X}_j(t)\,|\,\mathbf{X}(t)=\mathbf{x}]\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\sum_{k,\,l=1}^{m} G_{jk}(\mathbf{X}(t),t)G_{il}(\mathbf{X}(t),t)\underbrace{\mathbb{E}[\Delta B_k(t)\Delta B_l(t)]}_{2D_{kl}\Delta t}+o(\Delta t)\\ &= 2\sum_{k,\,l=1}^{m} G_{jk}(\mathbf{X}(t),t)G_{il}(\mathbf{X}(t),t)D_{kl} = 2(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{ij},\end{aligned} \tag{3.13}$$

where $\mathbf{D}$ is an $m\times m$ matrix, usually called the **diffusion tensor**. Therefore, the Fokker-Planck equation verified by the density of the solution stochastic process of the SDE (3.10) is

$$\partial_t f_{\mathbf{x}_0}(\mathbf{x},t)+\sum_{i=1}^{n}\partial_{x_i}\{g_i(\mathbf{x},t)\,f_{\mathbf{x}_0}(\mathbf{x},t)\}-\frac{1}{2}\sum_{i,\,j=1}^{n}\partial^2_{x_i\,x_j}\{(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{ij}(\mathbf{x},t)\,f_{\mathbf{x}_0}(\mathbf{x},t)\}=0. \tag{3.14}$$

Furthermore, assuming known the density of the initial condition in our stochastic IVP (3.10) and imposing boundary conditions such as $\lim_{\|\mathbf{x}\|\to\infty} f_{\mathbf{x}_0}(\mathbf{x},t)=0$ for all $t\geq t_0$, we obtain a boundary value problem with initial condition. Precisely we have

$$\begin{cases}\partial_t f_{\mathbf{x}_0}(\mathbf{x},t)+\sum_{i=1}^{n}\partial_{x_i}\{g_i(\mathbf{x},t)\,f_{\mathbf{x}_0}(\mathbf{x},t)\}-\sum_{i,\,j=1}^{n}\partial^2_{x_i\,x_j}\{(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{ij}(\mathbf{x},t)\,f_{\mathbf{x}_0}(\mathbf{x},t)\}=0,\\ f_{\mathbf{x}_0}(\pm\infty,t):=\lim_{\|\mathbf{x}\|\to\infty} f_{\mathbf{x}_0}(\mathbf{x},t)=0,\quad t\geq t_0\\ f_{\mathbf{x}_0}(\mathbf{x},t_0)=f_0(\mathbf{x}),\quad \mathbf{x}\in\mathbb{R}^n.\end{cases} \tag{3.15}$$

This problem, formed by a second order parabolic PDE, is very hard to solve explicitly and will need of numerical approximation in most practical cases. Furthermore, we do not have a general solution form, as we did in Liouville-Gibbs' equation. However, the examples discussed in the present project will be simple and instructive to give an idea of the usefulness of this equation.

3.2.2 Stationary and Moment Equations

We can also use the Fokker-Planck equation to compute the stationary density, that is, the density of the solution stochastic process when $t\to+\infty$. When the SDE (3.10) is **time-homogeneous** (see [3, Ch. 7]), that is, if $\mathbf{g}(\mathbf{X}(t),t)=\mathbf{g}(\mathbf{X}(t))$ and $\mathbf{G}(\mathbf{X}(t),t)=\mathbf{G}(\mathbf{X}(t))$,

the stationary density, denoted by f_s, can be found by solving

$$\begin{cases} \displaystyle\sum_{i=1}^{n} \partial_{x_i}\{g_i(\mathbf{x})\, f_s(\mathbf{x})\} - \sum_{i,\,j=1}^{n} \partial^2_{x_i\, x_j}\{(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{ij}(\mathbf{x})\, f_s(\mathbf{x})\} = 0, \\ f_s(\pm\infty) := \displaystyle\lim_{\|\mathbf{x}\|\to\infty} f_s(\mathbf{x}) = 0, \quad t \geq t_0 \\ \displaystyle\int_{\mathbb{R}^n} f_s(\mathbf{x})\mathrm{d}\mathbf{x} = 1, \end{cases} \tag{3.16}$$

whenever the solution exists. [3, Th. 8.4.3] shows that, under certain hypotheses, the non-homogeneous case can be changed by a time-homogeneous case with the same probability law.

Finally, we can calculate the moments of our SDE using the Fokker-Planck equation (3.14) ([1, Ch. 7]). This is very important due to the fact that solution stochastic processes of the SDE problem (3.10) are Gaussian, whose moments are determined by the mean and variance, or the first and centered second moments, respectively. However, the equation we are going to study now permits to access the k-th moment of the stochastic process without calculating the first and second moments previously. Let $\mathbf{X}(t)$ be the solution vector stochastic process of the Itô diffusion problem (3.10).

Let us consider an arbitrary function h depending on $\mathbf{X}(t)$ and t, that is, $h(\mathbf{X}, t)$, which is C^2 over any finite interval of $\mathbf{X}$ and t. Now, let us consider a finite forward increment over time

$$\Delta h := h(\mathbf{X} + \Delta\mathbf{X}, t + \Delta t) - h(\mathbf{X}, t).$$

The Taylor expansion around $(\mathbf{X}, t)$, up to second order, is

$$\Delta h = \partial_t h\, \Delta t + \sum_{i=1}^{n} \partial_{x_j} h\, \Delta X_j + \frac{1}{2} \sum_{i,j=1}^{n} \partial^2_{x_i\, x_j} h\, \Delta X_i\, \Delta X_j + o(\Delta X \Delta X^{\mathsf{T}}) + o(\Delta t). \tag{3.17}$$

We have shown that

$$\begin{aligned} \mathbb{E}[\Delta X_i \,|\, \mathbf{X}] &= g_i(\mathbf{X}, t)\Delta t + o(\Delta t), \\ \mathbb{E}[\Delta X_i\, \Delta X_j \,|\, \mathbf{X}] &= 2(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{i\,j}\Delta t + o(\Delta t). \end{aligned}$$

Therefore, taking the expectation in Equation (3.17), conditioned to $\mathbf{X}(t) = \mathbf{x}$, we obtain:

$$\mathbb{E}[\Delta h \,|\, \mathbf{X}(t)] = \partial_t h \Delta t + \sum_{j=1}^{n} g_j(\mathbf{X}(t), t)\partial_{x_j} h \Delta t + \sum_{i,j=1}^{n} (\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{i\,j}\partial^2_{x_i\, x_j} h \Delta t + o(\Delta t).$$

Now, we use the well-known fact that, for any two random variables, $\mathbb{E}[\mathbb{E}[\mathbf{Y} \,|\, \mathbf{X}]] = \mathbb{E}[\mathbf{Y}]$. Therefore, applying the expectation operator to the previous equation, dividing by Δt, taking the limits when $\Delta t \to 0$, and interchanging the differentiation and expectation operators, we obtain:

$$\frac{\mathrm{d}\mathbb{E}[h]}{\mathrm{d}t} = \mathbb{E}[\partial_t h] + \sum_{j=1}^{n} \mathbb{E}[g_j(\mathbf{X}(t), t)\partial_{x_j} h] + \sum_{i,j=1}^{n} \mathbb{E}[(\mathbf{G}\mathbf{D}\mathbf{G}^{\mathsf{T}})_{i\,j}\partial^2_{x_i\, x_j} h]. \tag{3.18}$$

Now, if we set $h(\mathbf{X}, t) = X_1^{k_1} \dots X_n^{k_n}$, Equation (3.18) will be the **moment equation**, which is very useful when studying nonlinear SDEs, or when the Fokker Planck equation is too difficult to study without numerical simulations.

3.3 Example

In order to illustrate the use of the Fokker-Planck equation, we will study a SDE where the drift term $\mathbf{g}$ and the diffusion term $\mathbf{G}$ are given in the following way

$$g_j(\mathbf{x}, t) = a_j x_j, \quad \mathbf{G}(\mathbf{x}, t) = \mathbf{I} \in \mathbb{R}^{n \times n},$$

respectively. In this case, Equation (3.14) has the form

$$\partial_t f_{\mathbf{x}_0}(\mathbf{x}, t) = -\sum_{i=1}^{n} a_i \partial_{x_i} \{x_i f_{\mathbf{x}_0}(\mathbf{x}, t)\} + \sum_{i\,j=1}^{n} D_{i\,j}\, \partial^2_{x_i\, x_j} f_{\mathbf{x}_0}(\mathbf{x}, t). \tag{3.19}$$

Now, the solution must verify the initial condition

$$f_{\mathbf{x}_0}(\mathbf{x}, t_0) = \prod_{j=1}^{n} \delta(x_j - x_{j\,0}),$$

where δ denotes the Dirac delta distribution; and vanishing boundary conditions at infinity, that is,

$$\lim_{\|\mathbf{x}\| \to 0} f_{\mathbf{x}_0}(\mathbf{x}, t) = 0.$$

Equation (3.19) is a parabolic PDE with $n + 1$ independent variables. We are going to take the Fourier transform in the spatial variables and use its relationship to the characteristic function in order to lower the order of the PDE and obtain a general, explicit form of the solution of this class of Fokker-Planck equations.

It is easy to show, using the relation between the characteristic function of a stochastic process and its Fourier transform, that

$$\mathcal{F}[\partial_t f](\mathbf{u}) = \partial_t \Phi, \quad \mathcal{F}[\partial_{x_j}(x_j f)](\mathbf{u}) = -u_j \partial_{u_j} \Phi, \quad \mathcal{F}[\partial^2_{x_j\, x_i} f](\mathbf{u}) = -u_j u_i \Phi.$$

Applying these relations to Equation (3.19), we obtain the following first-order PDE

$$\partial_t \Phi(\mathbf{u}, t) = \sum_{i=1}^{n} a_i\, u_i\, \partial_{u_i} \Phi(\mathbf{u}, t) - \sum_{i,\,j=1}^{n} D_{i\,j}\, u_j\, u_i\, \Phi(\mathbf{u}, t).$$

As we did in Chapter 2, when solving the Liouville-Gibbs equation, we can make use of the characteristic equations to obtain the solution. In particular, we have the following

system

$$\frac{\mathrm{d}t}{\mathrm{d}s}(\mathbf{r},s)=1,\quad t(\mathbf{r},0)=t_0,$$

$$\frac{\mathrm{d}u_1}{\mathrm{d}s}(\mathbf{r},s)=-a_1\,u_1,\quad u_i(\mathbf{r},0)=r_1,$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}s}(\mathbf{r},s)=-a_2\,u_2,\quad u_2(\mathbf{r},0)=r_2,$$

$$\vdots$$

$$\frac{\mathrm{d}u_n}{\mathrm{d}s}(\mathbf{r},s)=-a_n\,u_n,\quad u_n(\mathbf{r},0)=r_n,$$

whose solutions are easily computed. They are, for each $i\in\{1,\dots,n\}$,

$$u_i(\mathbf{r},s)=r_i\mathrm{e}^{-a_i\,s},\tag{3.20}$$

where $(\mathbf{r},s)$ are the local coordinates in the characteristic equations. However, as we can see they are easily solved for. Moving to the last equation of the characteristic equations system

$$\frac{\mathrm{d}\Phi}{\mathrm{d}s}(\mathbf{r},s)=-\sum_{i,\,j=1}^{n}D_{i\,j}\,u_j\,u_i\,\Phi(\mathbf{u},t)=-\Phi(\mathbf{u},t)\sum_{i,\,j=1}^{n}D_{i\,j}r_ir_j\mathrm{e}^{-(a_i+a_j)\,s}.\tag{3.21}$$

Let us denote $\mathcal{I}=\{(i,\,j)\in\{1,\dots,n\}^2\,|\,a_i+a_j\neq 0\}$. For $i,\,j\notin\mathcal{I}$, let us consider the limit of $a_i+a_j+\varepsilon=\varepsilon$, when $\varepsilon\to 0$. Now, solving the ODE by separation of variables we obtain

$$\Phi(\mathbf{r},s)\exp\left[-\sum_{\substack{i,\,j=1\\(i,\,j)\in\mathcal{I}}}^{n}D_{i\,j}u_i\,u_j\frac{1}{a_i+a_j}-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{\substack{i,\,j=1\\(i,\,j)\notin\mathcal{I}}}^{n}D_{i\,j}u_iu_j\right]=\tilde{c}(\mathbf{r}),$$

where $u_i=u_i(\mathbf{r},s)$ are given in (3.20), and $\tilde{c}$ is a constant (respect to s) of integration that appears in solving Equation (3.21)[2]. Now, solving for the characteristic coordinates in every equation, we get

$$\Phi(\mathbf{u},t)=\psi\left(u_1\mathrm{e}^{a_1\,\tau},\dots,u_n\mathrm{e}^{a_n\,\tau}\right)\exp\left[\sum_{\substack{i,\,j=1\\(i,\,j)\in\mathcal{I}}}^{n}D_{i\,j}u_i\,u_j\frac{1}{a_i+a_j}+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{\substack{i,\,j=1\\(i,\,j)\notin\mathcal{I}}}^{n}D_{i\,j}u_iu_j\right],$$

where $\tau:=(t-t_0)$. Now, the initial condition for Equation (3.21) is

$$\Phi(\mathbf{u},t_0)=\mathrm{e}^{\mathrm{i}\sum_{i=1}^{n}u_j\,x_{j\,0}}.$$

[2]The integral when solving the ODE and the limit can be changed by the monotone convergence theorem.

Therefore,

$$\psi(u_1,\ldots,u_n)=\exp\left[-\sum_{\substack{i,\,j=1\\(i,\,j)\in\mathcal{I}}}^{n}D_{i\,j}u_i\,u_j\frac{1}{a_i+a_j}-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\sum_{\substack{i,\,j=1\\(i,\,j)\notin\mathcal{I}}}^{n}D_{i\,j}u_iu_j+\mathrm{i}\sum_{i=1}^{n}u_j\,x_{j\,0}\right].$$

Finally, we obtain $\Phi(\mathbf{u},t)=$

$$\exp\left[\mathrm{i}\sum_{j=1}^{n}u_j\mathrm{e}^{a_j\,\tau}\,x_{j\,0}+\sum_{\substack{i,\,j=1\\(i,\,j)\in\mathcal{I}}}^{n}u_i\,u_j\frac{D_{i\,j}}{a_i+a_j}\left(1-\mathrm{e}^{(a_i+a_j)\,\tau}\right)+\lim_{\varepsilon\to 0}\sum_{\substack{i,\,j=1\\(i,\,j)\notin\mathcal{I}}}^{n}\frac{D_{i\,j}u_iu_j}{\varepsilon}\left(1-\mathrm{e}^{\varepsilon\,\tau}\right)\right].$$

This function clearly defines an n-dimensional Gaussian stochastic process, provided its covariance matrix function is definite positive[3]. Therefore, the solution $f_{\mathbf{x}_0}(\mathbf{x},t)$, which can be obtained by taking the inverse Fourier transform, can also be obtained by the following relation

$$f_{\mathbf{x}_0}(\mathbf{x},t)=\frac{1}{\sqrt{(2\pi)^n\,\det\Lambda}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\Lambda^{-1}(\mathbf{x}-\mathbf{m})\right],$$

where the components of $\mathbf{m}$ and Λ are, respectively,

$$m_j=x_{j\,0}\mathrm{e}^{a_j(t-t_0)},\quad \Lambda_{i\,j}=\begin{cases}-\dfrac{2D_{i\,j}}{a_i+a_j}\left(1-\mathrm{e}^{(a_i+a_j)\,(t-t_0)}\right), & (i,\,j)\in\mathcal{I},\\[2ex] -\displaystyle\lim_{\varepsilon\to 0}\frac{2D_{i\,j}}{\varepsilon}\left(1-\mathrm{e}^{\varepsilon\,(t-t_0)}\right), & (i,\,j)\notin\mathcal{I},\end{cases}$$

As we will see in the following chapter, this example is applicable in linear SDEs, although the calculations in practical cases may be quite tedious.

To conclude, we want to emphasize the applicability of the System (3.8) since, as in the case of the Liouville-Gibbs Equation (2.8), even when the explicit form of the solution stochastic process is not available, a great amount of information about the stochastic process is obtainable. Also, by following an analogous reasoning to the last part of Section 2.2, we can see that Equation (3.7), and Equation (3.14) in the particular case of Itô diffusions, are applicable in the case of having random coefficients and even general stochastic processes in the drift term of the SDE (see [1, Example 7.3]).

[3]That is, it has real, positive eigenvalues.

Chapter 4

Applications

This chapter is addressed to showcase some relevant models, formulated via differential equations with uncertainty, that appear in different scientific realms. As in practice model parameters are usually fixed using sampling and/or measurements, so containing uncertainties, we will assume that they are random variables instead of deterministic constants. This approach leads to formulate such models by means of RDEs. Also, when a system cannot be completely isolated from external influences, the non-determined forces or effects can be modeled as noise; that is, a random fluctuation varying in time and/or space. This fact leads to the formulation of SDEs. Then, we will take advantage of the main results exhibited in Chapter 2 and 3 to determine an explicit expression to the 1-PDF of the solution stochastic process of each randomized model, and afterwards we will compute the mean and the variance of the solution by integrating the corresponding 1-PDF using expression (1.1).

To show a full overview of the different scenarios treated in Chapter 2, first we will deal with the case that randomness only appears via the initial condition (see Sections 4.1 and 4.2) where formula (2.13) applies. Secondly, we shall consider the general scenario where uncertainties appear in both initial conditions and coefficients (see Section 4.3). In this case, we will take advantage of expressions (2.17)–(2.18). Furthermore, one of the examples is original work, to be published soon.

Finally, we will consider two interesting cases dealing with real models with a stochastic forcing term given by a white noise process (see Chapter 3). Our goal is to use, and solve, the Fokker Planck Equation (4.24) in these particular cases. In particular, we will consider a case in which we will be able to compare between the case with random parameters and identically zero force term, and one with deterministic parameters but with a white noise process force term.

4.1 Undamped Oscillator with Random Parameters

Certain physical processes, such as the dynamics of a mass-spring system or an electric current in a LC (Inductor-Capacitor) electronic circuit, can be modeled by a second-order ODE with initial conditions. In the mass-spring system, initial conditions represent the initial position and velocity of the mass. In most physical experiments, parameters are measured directly in the system and measurement errors appear often. Therefore, it is more natural to treat the initial conditions as random variables rather than deterministic constants. Analogously, the initial electric current and voltage in a LC circuit model can be treated as random variables.

Let us consider the IVP modeling the dynamics of an undamped linear oscillator

$$\begin{cases} \ddot{X}(t) + \omega^2 X(t) = 0, \\ X(0) = X_0, \dot{X}(0) = \dot{X}_0, \end{cases} \tag{4.1}$$

where $X_0, \dot{X}_0$ are random variables. We will assume that the joint PDF of the initial conditions, $f_0(x_0, \dot{x}_0)$, is known. The (deterministic) parameter $\omega^2 > 0$ represents the frequency of the oscillator.

To take advantage of the results shown in Chapter 2, we introduce the change of variable $\mathbf{X}(t) = (X_1(t), X_2(t))^\mathsf{T} = (X(t), \dot{X}(t))^\mathsf{T}$ to transform model (4.1) into the following first-order random IVP,

$$\begin{cases} \dot{\mathbf{X}}(t) = \mathbf{A}\mathbf{X}(t), \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}, \\ \mathbf{X}(0) = \mathbf{X_0}. \end{cases}$$

Observe that according to (2.1), $\mathbf{g}(t, \mathbf{X}(t)) = \mathbf{A}\mathbf{X}(t)$, $\mathbf{g} = (g_1, g_2)$, being $g_1(t, \mathbf{x}) = x_2$ and $g_2(t, \mathbf{x}) = -\omega^2 x_1$, $\mathbf{x} = (x_1, x_2)$, and $\mathbf{X_0} := (X_0, \dot{X}_0)^\mathsf{T}$. Therefore

$$\nabla_{\mathbf{x}} \cdot \mathbf{g}(t, \mathbf{x}) = \frac{\partial g_1(t, \mathbf{x})}{\partial x_1} + \frac{\partial g_2(t, \mathbf{x})}{\partial x_2} = 0, \quad \forall (t, \mathbf{x}) \in]0, \infty[\times \mathbb{R}^2.$$

As a consequence, the corresponding Liouville-Gibbs PDE becomes

$$\begin{cases} \dfrac{\partial f(t, \mathbf{x})}{\partial t} + x_2 \dfrac{\partial f(t, \mathbf{x})}{\partial x_1} - \omega^2 x_1 \dfrac{\partial f(t, \mathbf{x})}{\partial x_2} = 0, \\ f(0, \mathbf{x}) = f_0(x_0, \dot{x}_0). \end{cases} \tag{4.2}$$

To compute $f(t, \mathbf{x}) = f(t, x, \dot{x})$, we first solve the random IVP obtaining $\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}_0$, and then we solve for the random initial condition,

$$\mathbf{X}_0 = \mathrm{e}^{-\mathbf{A}t}\mathbf{X}(t) = \begin{bmatrix} X_0 \\ \dot{X}_0 \end{bmatrix} = \begin{bmatrix} \cos(\omega t) & -\frac{1}{\omega}\sin(\omega t) \\ \omega \sin(\omega t) & \cos(\omega t) \end{bmatrix} \begin{bmatrix} X(t) \\ \dot{X}(t) \end{bmatrix}.$$

Therefore, according to (2.13) the solution of Liouville-Gibbs IVP (4.1) writes

$$\begin{aligned} f(t, x, \dot{x}) &= f_0(x_0, \dot{x}_0)\mathrm{e}^{-\int_0^t 0 \, \mathrm{d}s} \\ &= f_0\left(x\cos(\omega t) - \tfrac{\dot{x}}{\omega}\sin(\omega t), \omega x \sin(\omega t) + \dot{x}\cos(\omega t)\right). \end{aligned} \tag{4.3}$$

Notice that by integrating this later expression, we can compute the 1-PDF for the position and the velocity

$$f(t,x)=\int_{-\infty}^{+\infty} f(t,x,\dot{x})\,\mathrm{d}\dot{x},\quad f(t,\dot{x})=\int_{-\infty}^{+\infty} f(t,x,\dot{x})\,\mathrm{d}x, \tag{4.4}$$

respectively.

In order to perform numerical simulations, we have chosen $\omega=2$ and the following Gaussian distributions for the initial conditions, $X_0\sim \mathrm{N}(0;1)$ and $\dot{X}_0\sim \mathrm{N}(-1;1)$. Assuming that they are independent, and if $f_{X_0}(x_0)$ and $f_{\dot{X}_0}(\dot{x}_0)$ denote their respective PDF's, we know that

$$f_0(x_0,\dot{x}_0)=f_{X_0}(x_0)f_{\dot{X}_0}(\dot{x}_0)=\frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}\left(x_0^2+(\dot{x}_0+1)^2\right)}.$$

Then, according to (4.3)

$$\begin{aligned} f(t,x,\dot{x}) &= f_0\left(x\cos(\omega t)-\tfrac{\dot{x}}{\omega}\sin(\omega t),\omega x\sin(\omega t)+\dot{x}\cos(\omega t)\right)\\ &= \tfrac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}\left(x\cos(t\omega)-\frac{\dot{x}\sin(t\omega)}{\omega}\right)^2-\frac{1}{2}(x\omega\sin(t\omega)+\dot{x}\cos(t\omega)+1)^2}. \end{aligned} \tag{4.5}$$

By applying (4.4), we obtain the 1-PDF of position,

$$f(t,x)=\frac{\exp\left(-\frac{(a\omega+\sin(t\omega))^2}{(\omega^2-1)\cos(2t\omega)+\omega^2+1}\right)}{\sqrt{2\pi}\sqrt{\frac{\sin^2(t\omega)}{\omega^2}+\cos^2(t\omega)}}. \tag{4.6}$$

Fig. 4.1 shows its graphical representation. We can observe that the peaks in the 1-PDF correspond to maximum amplitude in the oscillations. In the right panel of this plot, we show the approximations of the 1-PDF obtained after applying Monte Carlo simulations with $10,000$ simulations. The superiority of the Liouville-Gibbs method is clear because it provides more accurate results. Although theoretically similar results may be obtained using Monte Carlo at expense of increasing the number of simulations, this approach is demanding and the computational burden becomes unaffordable in comparison with the use of the Liouville-Gibbs approach. At this point, it is convenient to point out that this same model has been studied in [1, Ch. 6 (Example 6.1)], but using the random variable transformation method. However, the advantage of using the Liouville-Gibbs technique relies upon the fact that we do not need to look for an appropriate injective transformation whose Jacobian is distinct from zero to calculate the 1-PDF

Fig. 4.2 shows the joint 1-PDF of position and velocity in the phase space for fixed values of time, $t=0$ and $t=1.3$. They have been computed by expression (4.5). The red curve shows the orbit corresponding to the mean of the solution. The black dot corresponds, in each case, with the point of the orbit at $t=0$ and $t=1.3$. Notice that the mean in the phase diagram (position, $X(t)$, and velocity, $\dot{X}(t)$) can be computed via

$$\mathbb{E}[X(t)]=\int_{-\infty}^{+\infty} xf(t,x)\,\mathrm{d}x,\quad \mathbb{E}[\dot{X}(t)]=\int_{-\infty}^{+\infty} xf(t,\dot{x})\,\mathrm{d}\dot{x},$$

where $f(t,x)$ and $f(t,\dot{x})$ are given by (4.4), or directly,

$$\begin{aligned}\mathbb{E}[\mathbf{X}(t)] &= \mathrm{e}^{\mathbf{A}t}\mathbb{E}[\mathbf{X}_0] = \begin{bmatrix} \frac{1}{2}\mathrm{e}^{-\mathrm{i}t\omega}+\frac{1}{2}\mathrm{e}^{\mathrm{i}t\omega} & \frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}t\omega}}{2\omega}-\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t\omega}}{2\omega} \\ \frac{1}{2}\mathrm{i}\mathrm{e}^{\mathrm{i}t\omega}\omega-\frac{1}{2}\mathrm{i}\mathrm{e}^{-\mathrm{i}t\omega}\omega & \frac{1}{2}\mathrm{e}^{-\mathrm{i}t\omega}+\frac{1}{2}\mathrm{e}^{\mathrm{i}t\omega} \end{bmatrix} \begin{bmatrix} \mathbb{E}[X_0] \\ \mathbb{E}[\dot{X}_0] \end{bmatrix} \\ &= \begin{bmatrix} -\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}t\omega}}{2\omega}+\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t\omega}}{2\omega} \\ -\frac{1}{2}\mathrm{e}^{-\mathrm{i}t\omega}-\frac{1}{2}\mathrm{e}^{\mathrm{i}t\omega} \end{bmatrix} = \begin{bmatrix} \mathbb{E}[X(t)] \\ \mathbb{E}[\dot{X}(t)] \end{bmatrix},\end{aligned} \tag{4.7}$$

where we have substituted $\mathbb{E}[X_0]=0$ and $\mathbb{E}[\dot{X}_0]=-1$.

Finally, observe that in this example the solution stochastic process can be expressed as $\mathbf{X}(t)=\mathbf{M}(t)\mathbf{X}_0$, where $\mathbf{M}(t)=\mathrm{e}^{\mathbf{A}t}$ and $\mathbf{X}_0=(X_0,\dot{X}_0)^\mathsf{T}\sim \mathrm{N}(\boldsymbol{\mu}_{\mathbf{X}_0};\boldsymbol{\Sigma}_{\mathbf{X}_0})$ being $\boldsymbol{\mu}_{\mathbf{X}_0}=[0,-1]^\mathsf{T}$ and $\boldsymbol{\Sigma}_{\mathbf{X}_0}=\mathbf{I}_2$ (identity matrix of size 2, since X_0 and $\dot{X}_0$ are independent Gaussian random variables having each of them unit variance). Then, according to the well-known properties of a linear transformation of a Gaussian random vector, the solution stochastic process is also Gaussian (specifically it corresponds to a binormal distribution), $\mathbf{X}(t)\sim \mathrm{N}(\boldsymbol{\mu}_{\mathbf{X}(t)};\boldsymbol{\Sigma}_{\mathbf{X}(t)})$, where

$$\boldsymbol{\mu}_{\mathbf{X}(t)}=\mathbf{M}(t)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \mathrm{e}^{\mathbf{A}t}\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{i}\mathrm{e}^{-\mathrm{i}t\omega}}{2\omega}+\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}t\omega}}{2\omega} \\ -\frac{1}{2}\mathrm{e}^{-\mathrm{i}t\omega}-\frac{1}{2}\mathrm{e}^{\mathrm{i}t\omega} \end{bmatrix}$$

and

$$\boldsymbol{\Sigma}_{\mathbf{X}(t)}=\mathbf{M}(t)\mathbf{I}_2\mathbf{M}(t)^\mathsf{T} = \begin{bmatrix} \frac{\omega^2+\left(\omega^2-1\right)\cos(2t\omega)+1}{2\omega^2} & -\frac{\left(\omega^2-1\right)\sin(2t\omega)}{2\omega} \\ -\frac{\left(\omega^2-1\right)\sin(2t\omega)}{2\omega} & \frac{1}{2}\left(\omega^2-\left(\omega^2-1\right)\cos(2t\omega)+1\right) \end{bmatrix}.$$

As a consequence, although initially both position and velocity are independent they become statistically dependent as time goes on. These features are also observed in the shape of the phase diagram depicted in Fig. 4.2.

4.2 Damped Oscillator with Random Parameters

As an application of the result for the n-order linear ODE with randomness in the initial conditions seen at the end of Chapter 2, we will study the damped linear oscillator assuming that both its initial position and velocity are random variables. Specifically, we will consider the following random IVP

$$\begin{cases} m\ddot{X}(t)+c\dot{X}(t)+kX(t)=0, \quad t>0, \\ X(0)=X_0, \quad \dot{X}(0)=\dot{X}_0, \end{cases} \tag{4.8}$$

where m is the mass of the oscillator, c is called the viscous damping coefficient and k is a constant that depends on the oscillator, [24, Sec. 6.1]. We will denote by f_0 the joint PDF of the initial random vector $(X_0,\dot{X}_0)$. Using the same classical reasoning exhibited

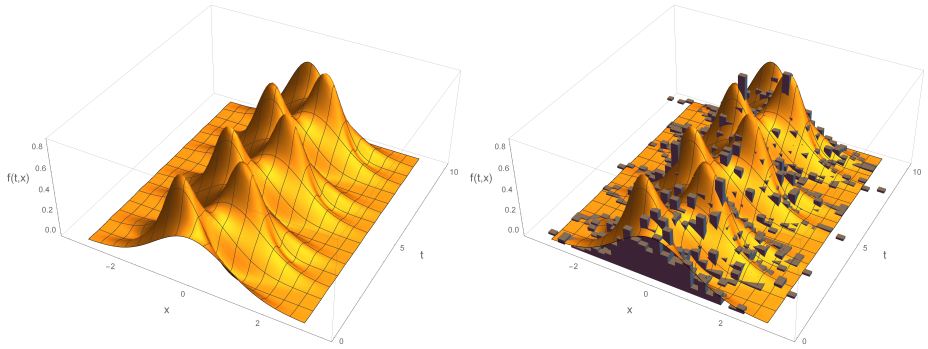


Figure 4.1: Both plots correspond to the random IVP (4.1) taking $\omega = 2$, which implies an oscillation period of $T = \pi$ units of time. Left panel: 1-PDF, $f(t, x)$ given by (4.6), obtained by applying (4.4) to expression (4.3). Observe that the maxima of PDF's are reached at the points $(\frac{\pi}{2}k, \pm\frac{1}{2})$, $k = 0, 1, 2, \ldots$, which correspond when the oscillator reaches the maximum amplitude in its oscillations. Right panel: Approximation to the 1-PDF given by a histogram (black bars) obtained by Monte Carlo sampling in the initial conditions of IVP (4.1). We have taken 10,000 simulations using the built-in random sampling tools by software Mathematica®. The solution in the right panel is overlayed to make the comparison easier. The superiority of Liouville-Gibbs approach is apparent against Monte Carlo simulations. Example of the random linear oscillator developed in Section 4.1.

in the analysis of the damped linear oscillator, the IVP (4.8) can be written in the form (2.1) where $\mathbf{g}(t, \mathbf{X}(t)) = \mathbf{A}\mathbf{X}(t)$ being

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix}, \tag{4.9}$$

$\mathbf{X}(t) = (X_1(t), X_2(t))^{\mathsf{T}} = (X(t), \dot{X}(t))^{\mathsf{T}}$. As a consequence, $\mathbf{g} = (g_1, g_2)$, with $g_1(t, \mathbf{x}) = x_2$ and $g_2(t, \mathbf{x}) = -\frac{k}{m}x_1 - \frac{c}{m}x_2$, , $\mathbf{x} = (x_1, x_2)$, and $\mathbf{X_0} := (X_0, \dot{X}_0)^{\mathsf{T}}$. Therefore, now

$$\nabla_{\mathbf{x}} \cdot \mathbf{g}(t, \mathbf{x}) = \frac{\partial g_1(t, \mathbf{x})}{\partial x_1} + \frac{\partial g_2(t, \mathbf{x})}{\partial x_2} = -\frac{c}{m}, \qquad \forall (t, \mathbf{x}) \in [0, \infty[\times\mathbb{R}^2,$$

and then the Liouville-Gibbs PDE can be written as

$$\begin{cases} \dfrac{\partial f(t, \mathbf{x})}{\partial t} + x_2 \dfrac{\partial f(t, \mathbf{x})}{\partial x_1} - \left(\dfrac{k}{m}x_1 + \dfrac{c}{m}x_2\right) \dfrac{\partial f(t, \mathbf{x})}{\partial x_2} = \dfrac{c}{m} f(t, \mathbf{x}), \\ f(0, \mathbf{x}) = f_0(x_0, \dot{x}_0). \end{cases}$$

It is well-known that the solution of this model, $\mathbf{X}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{X}_0$, can be expressed in different forms depending on the eigenvalues $\lambda_1 = \frac{-c+\sqrt{c^2-4km}}{2m}$, $\lambda_2 = \frac{-c-\sqrt{c^2-4km}}{2m}$ of matrix $\mathbf{A}$ defined in (4.9). It leads to three different physical behaviours of the oscillator, namely,

1. If $c^2 > 4km$, the oscillator will be overdamped. In such case, $0 > \lambda_1 > \lambda_2$. The system returns to its steady state (equilibrium state) without oscillating.

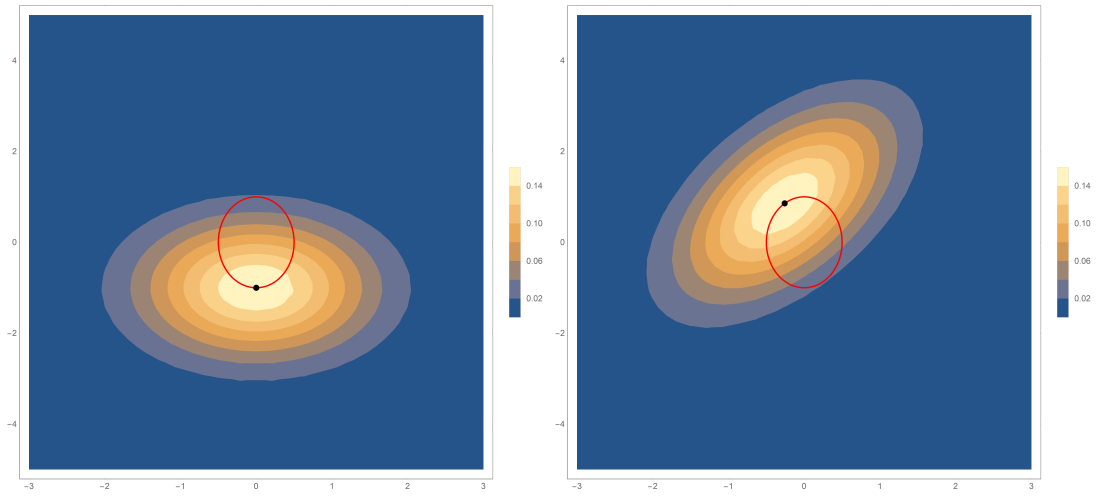


Figure 4.2: Left: Joint PDF of position and velocity at $t = 0$. Right: Joint PDF of position and velocity at $t = 1.3$. As we can see, the fact that the level curves of the random IVP (4.1) are ellipses, is in full agreement with properties of a binormal distribution. In both panels, the red curve represents the mean curve of the solution which is given by (4.7). Example of the random linear oscillator developed in Section 4.1.

2. If $c^2 = 4km$, the oscillator will be critically damped. In such case, $0 > \lambda_1 = \lambda_2 = -\frac{c}{2m}$. The system returns to its equilibrium state as quickly as possible without oscillating, although overshoot (1 oscillation) may occur.

3. If $c^2 < 4km$, the oscillator will be underdamped. In such case, we will have two conjugate complex roots. The system oscillates, but the amplitude gradually decays to 0.

For illustrative purposes only, hereinafter, we will consider the third case which is usually referred to as the damped linear oscillator. In such a case the solution is given by $\mathbf{Y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{Y}_0$ and then

$$\begin{aligned}\mathbf{Y}_0 &= \mathrm{e}^{-\mathbf{A}t}\mathbf{Y}(t)\\ &= \begin{bmatrix} \mathrm{e}^{\frac{ct}{m}}\left(\cos\left(\frac{\sqrt{4km-c^2}t}{m}\right) - \frac{c\sin\left(\frac{\sqrt{4km-c^2}t}{m}\right)}{\sqrt{4km-c^2}}\right) & -\frac{\mathrm{e}^{\frac{ct}{m}}m\sin\left(\frac{\sqrt{4km-c^2}t}{m}\right)}{\sqrt{4km-c^2}} \\ \frac{\mathrm{e}^{\frac{ct}{m}}k\sin\left(\frac{\sqrt{4km-c^2}t}{m}\right)}{\sqrt{4km-c^2}} & \mathrm{e}^{\frac{ct}{m}}\cos\left(\frac{\sqrt{4km-c^2}t}{m}\right) \end{bmatrix}\mathbf{Y}(t).\end{aligned}$$

Therefore, using (2.21), and identifying $a_n(t) = m$, $a_{n-1}(t) = c$ and $a_{n-2}(t) = k$ in (2.19) (with $n = 2$), we obtain the 1-PDF of the solution stochastic process after re-substituting

$(y_1, y_2) = (x, \dot{x})$

$$f_0\left(x\mathrm{e}^{\frac{ct}{m}}\left(\cos\left(\frac{t\sqrt{4km-c^2}}{m}\right)-\frac{c\sin\left(\frac{t\sqrt{4km-c^2}}{m}\right)}{\sqrt{4km-c^2}}\right)-\dot{x}\frac{m\mathrm{e}^{\frac{ct}{m}}\sin\left(\frac{t\sqrt{4km-c^2}}{m}\right)}{\sqrt{4km-c^2}},\right.$$
$$\left.x\,\frac{k\mathrm{e}^{\frac{ct}{m}}\sin\left(\frac{t\sqrt{4km-c^2}}{m}\right)}{\sqrt{4km-c^2}}+\dot{x}\mathrm{e}^{\frac{ct}{m}}\cos\left(\frac{t\sqrt{4km-c^2}}{m}\right)\right)\mathrm{e}^{\frac{ct}{m}} = f(t,x,\dot{x}). \tag{4.10}$$

To carry out numerical simulations, we will assume that the initial conditions are independent Gaussian random variables, $X_0 \sim \mathrm{N}(0; 0.0625)$, $\dot{X}_0 \sim \mathrm{N}(-1; 0.25)$. Also, $m = 1.4$, $k = 5$ and $c = 0.3$. In Fig. 4.3, we show the marginal 1-PDF of position, $f(t, x)$, on the time interval $t \in [0, 10]$, using

$$f(t,x) = \int_{-\infty}^{+\infty} f(t,x,\dot{x})\,\mathrm{d}\dot{x}, \tag{4.11}$$

where $f(t, x, \dot{x})$ is given by (4.10).

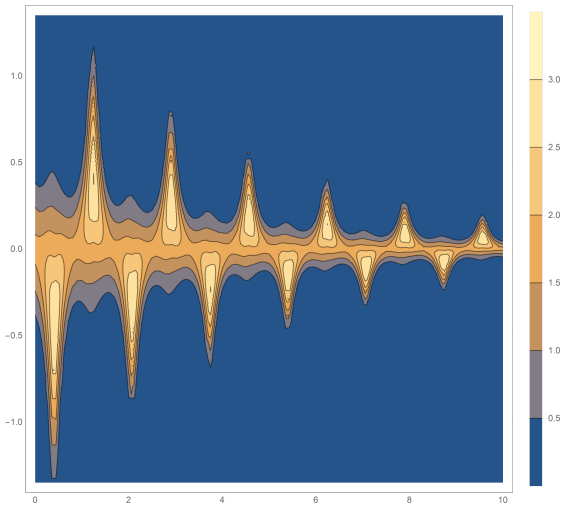


Figure 4.3: Level curves of the 1-PDF of position, $f(t, x)$ for $t \in [0, 10]$. For each t, we see that the PDF takes its highest values when the oscillator reaches its maximum amplitude and that the amplitude gradually decreases. This is a different, but equivalent representation to Fig. 4.1 in our new setting oscillator model. Example of the random damped linear oscillator in Section 4.2. Function $f(t, x)$ has been computed using the "Global Adaptive" built-in integration method by software Mathematica®.

4.3 The random logistic model

In this section, which is original work accepted to be published in the book Computational Mathematics and Applications (Springer), we study the randomized version of the logistic differential equation, which plays a very important role in modeling systems whose behavior is characterized by a first stage with rapid growth followed by a slower growth until reaching stabilization. Particularly, this model is very useful in biology when describing the dynamics of a population whose individuals have limited available resources. The logistic model has also been used to describe the diffusion (demand) of a certain technology (mobile phones, use of electronic commerce, etc.) characterized by a fast initial growth and whose long-term growth is slow down due to the extensive use of the given technology by all the individuals of the population, saturating its demand. Here, we will consider a fully randomized formulation of the logistic model via the following IVP

$$\begin{cases} X'(t) = X(t)(A - BX(t)), \quad t > t_0, \\ X(t_0) = X_0, \end{cases} \tag{4.12}$$

where X_0, A and B are assumed to be absolutely continuous r.v's. Here, A denotes the random reproductive parameter of the population (also termed growth rate), B is the ratio between the growth rate A and the maximum, or asymptotic, population (also termed random carrying capacity) and X_0 the initial number of individuals. For the sake of generality, in our subsequent theoretical analysis we assume that the initial input random vector (X_0, A, B) has a joint PDF, $f_0(x_0, a, b)$, which is assumed to be known. According to the notation introduced for the extended IVP (2.15), in this case $n = 1$ and $m = 2$, being $\mathbf{Y}(t) = (X(t), \mathbf{A}) = (X(t), A, B)$, $\mathbf{G}(t, \mathbf{Y}(t)) = (g(t, X(t)), 0, 0)$ and $\mathbf{X}_0 = (X_0, \mathbf{A}) = (X_0, A, B)$. Besides, observe that $g(t, x) = x(a - bx)$, thus $\nabla_x \cdot g(t, x, a, b) = a - 2bx$. Therefore, the Liouville-Gibbs PDE (2.16) becomes

$$\begin{cases} \dfrac{\partial f(t,x)}{\partial t} + x(a - bx)\dfrac{\partial f(t,x)}{\partial x} = -(a - 2bx)f(t, x), \quad t > t_0, \\ f(t_0, x) = f_0(x_0, a, b). \end{cases}$$

On the other hand, it is well known that the solution of random IVP (4.12) is given by

$$X(t) = \frac{\frac{A}{B}X_0}{X_0 + \mathrm{e}^{-A(t-t_0)}(\frac{A}{B} - X_0)} = h(t, X_0, \mathbf{A}). \tag{4.13}$$

Let us fix $t > t_0$ and denote $X(t) = X$. Solving (4.13) for X_0, that is, obtaining the inverse of h as a function of X_0 gives

$$X_0 = \frac{\frac{A}{B}X}{X + \mathrm{e}^{A(t-t_0)}(\frac{A}{B} - X)} = h^{-1}(t, X, \mathbf{A}).$$

In order to apply expression (2.17), we first need to calculate the following integral

$$\begin{aligned} \int_{t_0}^{t} \nabla_x \cdot g(s, h(t, x_0, \mathbf{a}))\,\mathrm{d}s &= \int_{t_0}^{t} \nabla_x \cdot g(s, h(t, x_0, a, b))\,\mathrm{d}s \\ &= -a(t - t_0) + \ln\left(\frac{a^2}{(bx_0 + \mathrm{e}^{a(t-t_0)}(a - bx_0))^2}\right). \end{aligned}$$

Now, using (2.17), we obtain

$$f(t,x;\mathbf{a}) = f(t,x;a,b) = f_0\left(\frac{\frac{a}{b}x}{x+\mathrm{e}^{a(t-t_0)}(\frac{a}{b}-x)},a,b\right)\frac{a^2\mathrm{e}^{a(t-t_0)}}{(bx+\mathrm{e}^{a(t-t_0)}(a-bx))^2}. \quad (4.14)$$

Finally, we marginalize with respect to $\mathbf{A}=(A,B)$ using (2.18), and we then obtain the 1-PDF of the solution stochastic process of random IVP (4.12)

$$f(t,x) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t,x;a,b)\,\mathrm{d}a\,\mathrm{d}b. \quad (4.15)$$

To illustrate our previous theoretical conclusions, we will assume the following distributions for the random inputs: $X_0 \sim \mathrm{N}|_{[0,1]}(0.05, 0.0001)$ (a Gaussian distribution truncated on the interval $[0,1]$), and Uniform distributions for A and B, specifically $A \sim \mathrm{Un}(0.3, 0.5)$ and $B \sim \mathrm{Un}(2,3)$. Assuming that X_0, A and B are independent, the factor f_0 in (4.14) can be expressed as

$$f_0\left(\frac{\frac{a}{b}x}{x+\mathrm{e}^{a(t-t_0)}(\frac{a}{b}-x)},a,b\right) = f_{X_0}\left(\frac{\frac{a}{b}x}{x+\mathrm{e}^{a(t-t_0)}(\frac{a}{b}-x)}\right)f_A(a)f_B(b), \quad (4.16)$$

being f_{X_0}, f_A and f_B the PDF's of X_0, A and B, respectively. In Fig. 4.4, we show the 1-PDF of the solution $X(t)$ for different values of t. This graphic has been performed using expressions (4.14)–(4.16). We can observe as the value of the mean and the variance increase until stabilization. This behaviour is in full agreement with the graphical representation shown in Fig. 4.5. Table 4.1 collects the values of the mean and the variance of the solution stochastic process, $\mathbb{E}[X(t)]$, at the same instants $t = 0, 1, 5, 10, 20, 40$ shown in Fig. 4.4. These figures have been calculated by expression (1.1) (with $k = 1$) and applying (4.14)–(4.16). Observe that these numerical values agree with the ones observed in Fig. 4.4 and Fig. 4.5. As we can see in Table 4.1, the limit of the mean of the solution stochastic process converges to 0.1624 approximately. This fact can be rigorously checked in a different way. From (4.13), observe that $X(t) \xrightarrow[t\to\infty]{} A/B$, and applying the random variable transformation method [1, Ch. 2] it can be seen that the PDF of the limit r.v., $Z := A/B$, is given by

$$f_Z(z) = \int_{-\infty}^{+\infty} f_A(\xi z)f_B(\xi)\,\mathrm{d}\xi,$$

then

$$\mathbb{E}[A/B] = \mathbb{E}[Z] = \int_{-\infty}^{+\infty} z f_Z(z)\,\mathrm{d}z.$$

One of the most useful applications of determining an expression of the 1-PDF, $f(t,x)$, is the computation of confidence intervals for a specific confidence level $(1-\alpha)\times 100\%$ for each fixed time $\hat{t}$. These intervals are computed by determining a suitable value $k_{\hat{t}} > 0$ such that

$$1-\alpha = \mathbb{P}\left[\{\omega\in\Omega : X(\hat{t},\omega)\in[L(\hat{t}),U(\hat{t})]\}\right] = \int_{L(\hat{t})}^{U(\hat{t})} f(\hat{t},x)\,\mathrm{d}x, \quad (4.17)$$

	$t=0$	$t=2$	$t=20$	$t=40$	$t=100$
$\mathbb{E}[X(t)]$	0.05	0.0802289	0.162014	0.162117	0.162383
$\mathbb{V}[X(t)]$	9.93×10^{-5}	2.05×10^{-4}	9.17×10^{-4}	9.13×10^{-4}	9.13×10^{-4}

Table 4.1: Mean and variance of the solution stochastic process of IVP (4.12) at different time instants. We observe that the mean tends to 0.1624 while the variance stabilizes around 9.13×10^{-4}. Example of the random logistic differential equation developed in Section 4.3.

being $L(\hat{t}) := \mu_X(\hat{t}) - k_{\hat{t}}\,\sigma_X(\hat{t})$, $U(\hat{t}) := \mu_X(\hat{t}) + k_{\hat{t}}\,\sigma_X(\hat{t})$, and $\mu_X(\hat{t}) = \mathbb{E}[X(\hat{t})]$ and $\sigma_X(\hat{t}) = \sqrt{\mathbb{V}[X(\hat{t})]}$ the mean and the standard deviation, respectively.

In our computations, we have taken $\alpha = 0.05$ to construct 95% confidence intervals. Results are shown in Fig. 4.5. In this plot we can observe that the diameters of the confidence intervals tend to stabilize as $t \to \infty$. This feature is in full agreement with the stabilization of variance previously shown.

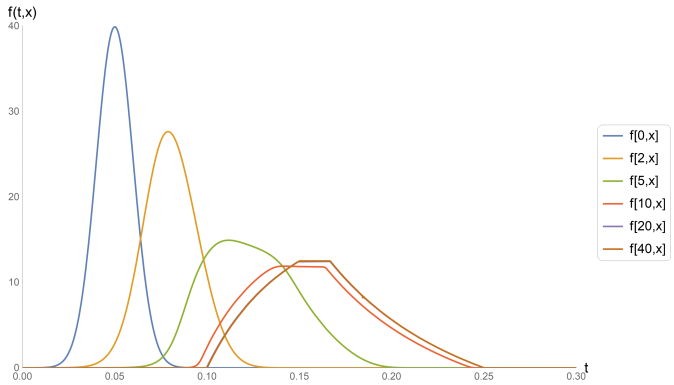


Figure 4.4: Time evolution of the 1-PDF of the solution stochastic process of random logistic equation (4.12) for different time instants. It can be seen how variance grows until it stabilizes. In fact, the difference between the PDF corresponding to $t = 20$ and $t = 40$ is barely observable because both PDF's overlap having as mean value 0.1624. Example of the random logistic differential equation developed in Section 4.3. Function $f(t, x)$, given by Equation (4.15), has been computed using the "Multidimensional Rule" built-in method by software Mathematica®.

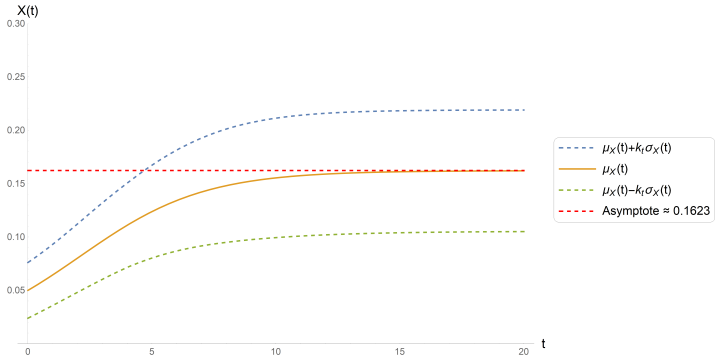


Figure 4.5: Time evolution of the mean, $\mu_X(t)$, and the confidence interval, $\mu_X(t) \pm k_t\sigma_X(t)$, where $k_t > 0$ and $\sigma_X(t)$ is the standard deviation of the solution stochastic process $X(t)$. The coefficient $k_t > 0$ has been determined, for each $t > 0$, so that the confidence interval captures 95% of probability according to (4.17). Example of the random logistic differential equation developed in Section 4.3. We have obtained the values k_t by solving the non-linear equations that appear in (4.17) using the numerical root finder tool and the "Multidimensional Rule" built-in integration method by software Mathematica®.

4.4 Damped Oscillator with White Noise Excitation

In this case, we are going to study the linear damped oscillator with deterministic parameters, but with a random excitation input. The equation of motion of the damped mass-spring oscillator with a white noise stochastic excitation is

$$\begin{cases} \ddot{X}(t) + 2\beta\dot{X}(t) + \omega_0^2 X(t) = W(t), \quad t > 0, \\ X(0) = x_0, \quad \dot{X}(0) = \dot{x}_0, \end{cases} \tag{4.18}$$

where $W(t)$ is a Gaussian white noise process with mean zero and correlation function given by $\Gamma(t,s) = 2D\,\delta(t-s)$; and $0 < \beta^2 < \omega_0^2$. This equation has an explicit form for its solution stochastic process, which can be obtained adapting the theory of deterministic linear ODEs as we did in the previous problems. The solution is quite complicated and, since we are focused on obtaining and solving the Fokker-Planck Equation, we will not consider the explicit form of its solution. Let us build the Fokker-Planck Equation for this problem. In this problem, we will take $t_0 = 0$, $x_0 = 0$, $\dot{x}_0 = -1$. In vector form, the equation in (4.18) is

$$\mathrm{d}\mathbf{X}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{bmatrix} \mathbf{X}(t) + \mathrm{d}\mathbf{B}(t), \tag{4.19}$$

where $\mathrm{d}\mathbf{B}(t) = (0, \mathrm{d}B(t))^\mathsf{T}$ and $\mathbf{X}(t) = (X(t), \dot{X}(t))^\mathsf{T}$. Therefore, by choosing $B(t) \sim \mathrm{N}(0,t)$, we easily identify

$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},$$

therefore, the corresponding coefficients for the Fokker-Planck equation are

$$\alpha_1(x, \dot{x}, t) = \dot{x}, \quad \alpha_2(x, \dot{x}, t) = -\omega_0^2\, x - 2\beta\dot{x}, \quad \alpha_{11} = \alpha_{12} = \alpha_{21} = 0, \quad \alpha_{22} = 1.$$

The Fokker-Planck equation reads

$$\partial_t f_{(x_0,\dot{x}_0)} = -\dot{x}\,\partial_x f_{(x_0,\dot{x}_0)} + \partial_{\dot{x}}\{(\omega_0^2\,x - 2\beta\dot{x})\,f_{(x_0,\dot{x}_0)}\} + \frac{1}{2}\,\partial_{\dot{x}}^2 f_{(x_0,\dot{x}_0)},$$

where $f_{(x_0,\dot{x}_0)}(x,\dot{x},t) := f(x,\dot{x},t\,|\,x_0,\dot{x}_0,0)$. Now, we can make a change of variables such that the Fokker Planck equation is of the type seen in Example 3.3 and use the result obtained in that section in order to make calculations easier.

We can express the values of α_1 and α_2 as a bilinear form in the $\{x,\,\dot{x}\}$ base. That is,

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -\omega_0^2 & -2\beta\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}x\\ \dot{x}\end{bmatrix}.$$

However, in order to use what we saw in 3.3, we must find a new basis formed by the eigenvalues of matrix $\mathbf{A}$. That is, we must find an invertible matrix $\mathbf{C}$ such that

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \mathbf{C}\begin{bmatrix}x\\ \dot{x}\end{bmatrix} \Rightarrow \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix} := \mathbf{C}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \mathbf{C}\mathbf{A}\mathbf{C}^{-1}\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix},$$

where λ_1 and λ_2 are the eigenvalues of matrix $\mathbf{A}$, and $(y_1, y_2)^{\mathsf{T}}$ denotes the base formed by eigenvectors. It is easily checked that the eigenvalues and their corresponding eigenvectors are, respectively

$$\lambda_1 = -\beta + \mathrm{i}\sqrt{\omega_0^2 - \beta^2},\ \lambda_2 = -\beta - \mathrm{i}\sqrt{\omega_0^2 - \beta^2}, \quad \mathbf{v}_1 = \begin{bmatrix}1\\ \lambda_1\end{bmatrix},\ \mathbf{v}_2 = \begin{bmatrix}1\\ \lambda_2\end{bmatrix},$$

where i denotes the imaginary unit and $\mathbf{v}_i$ is the eigenvector corresponding to the eigenvalue λ_i, $i = 1, 2$. Now, we can obtain our matrices $\mathbf{C}$, $\mathbf{C}^{-1}$, which are given by

$$\mathbf{C}^{-1} = \begin{bmatrix}1 & 1\\ \lambda_1 & \lambda_2\end{bmatrix}, \quad \mathbf{C} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix}\lambda_2 & -1\\ -\lambda_1 & 1\end{bmatrix}$$

After rewriting the Fokker-Planck equation in the new variables, we obtain

$$\partial_t f_{\mathbf{y}_0}(y_1,y_2,t) = -\sum_{i=1}^{2}\lambda_i\,\partial_{y_i}(y_i\,f_{\mathbf{y}_0}(y_1,y_2,t)) + \sum_{i,\,j=1}^{2}\sigma_{i\,j}\partial_{y_i,y_j}^2 f_{\mathbf{y}_0}(y_1,y_2,t),$$

where

$$[\sigma_{i\,j}] = \mathbf{C}\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\mathbf{C}^{\mathsf{T}} = \frac{1}{4(\beta^2 - \omega_0^2)}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}.$$

The associated initial condition is

$$f_{\mathbf{y}_0}(y_1,y_2,0) = \delta(y_1 - y_{1\,0})\delta(y_2 - y_{2\,0}), \quad \begin{bmatrix}y_{1\,0}\\ y_{2\,0}\end{bmatrix} = \mathbf{C}\begin{bmatrix}x_0\\ \dot{x}_0\end{bmatrix},$$

and the boundary conditions are

$$f_{\mathbf{y}_0}(y_1, \pm\infty, t) = 0, \quad f_{\mathbf{y}_0}(\pm\infty, y_2, t) = 0.$$

Now we can use the result obtained in Section 3.3. Let us calculate the mean and covariance matrices with respect to the y_1, y_2 variables. We have

$$\tilde{\mathbf{m}} = (y_{1\,0}\mathrm{e}^{\lambda_1 t}, y_{2\,0}\mathrm{e}^{\lambda_2 t})^{\mathsf{T}}, \quad \tilde{\Lambda}_{i\,j} = -\frac{\sigma_{i\,j}}{\lambda_i + \lambda_j}(1 - \mathrm{e}^{(\lambda_i+\lambda_j)t}),$$

Now, by taking the inverse transformation, and using initial conditions, we obtain

$$\mathbf{m} = \mathbf{C}^{-1}\begin{bmatrix} y_{1\,0}\mathrm{e}^{\lambda_1 t} \\ y_{2\,0}\mathrm{e}^{\lambda_2 t} \end{bmatrix}.$$

The explicit expression of the mean vector function could have also been obtained if we had solved the moment equation of (4.18).

Also, the covariance matrix is given by

$$[\Lambda_{ij}] = \mathbf{C}^{-1}\tilde{\Lambda}_{ij}(\mathbf{C}^{-1})^{\mathsf{T}}.$$

Finally, the joint PDF of the solution stochastic process of (4.19) is given by

$$f_{\mathbf{x}_0}(x, \dot{x}, t) = \frac{1}{2\pi\sqrt{\det\Lambda}}\mathrm{e}^{-\frac{1}{2}(\mathbf{x}-\mathbf{m})^{\mathsf{T}}\Lambda^{-1}(\mathbf{x}-\mathbf{m})}, \tag{4.20}$$

where $\mathbf{m}$ and $\Lambda = [\Lambda_{ij}]$ are obtained following the previous calculations.

Also, we can obtain the marginal densities at any time. To obtain the marginal PDF of position, we integrate with respect to the velocity variable. To obtain the PDF of velocity, we integrate with respect to the position variable; that is

$$f_{\mathbf{x}_0}(x, t) = \int_{-\infty}^{+\infty} f_{\mathbf{x}_0}(x, \dot{x}, t)\mathrm{d}\dot{x}, \quad f_{\mathbf{x}_0}(\dot{x}, t) = \int_{-\infty}^{+\infty} f_{\mathbf{x}_0}(x, \dot{x}, t)\mathrm{d}x. \tag{4.21}$$

The explicit, general expression of the solution is extremely complicated and tedious even in this case, where the diffusion tensor $\mathbf{D}$ is a very simple, constant matrix. Also, it does not add any important information aside of the information that can be visually seen in the graphs of the solution. Therefore, we have decided not to include the explicit expression.

For the numerical simulations, we are going to consider the following values for the parameters of the SDE (4.18)

$$x_0 = 0, \quad \dot{x}_0 = -1, \quad \beta = 0.214286, \quad \omega_0 = 1.88982.$$

These values were chosen so that the oscillator under study has the same parameters as the one studied in Section 4.2, that is $\omega_0 = \frac{k}{m} = 1.88982$, $\beta = \frac{c}{m} = 0.214286$. This implies that it is an underdamped oscillator. The mean vector is given by

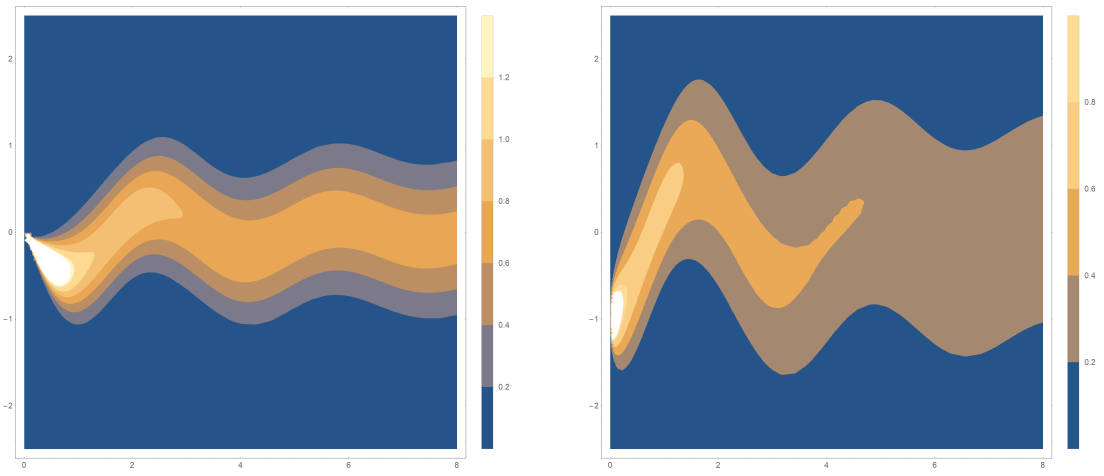
$$\begin{aligned}\mathbf{m}(t) &= \begin{bmatrix} -\frac{e^{-\beta t}\sin\left(t\sqrt{\omega_0^2-\beta^2}\right)}{\sqrt{\omega_0^2-\beta^2}} \\ -e^{-\beta t}\left(\cos\left(t\sqrt{\omega_0^2-\beta^2}\right) - \frac{\beta\sin\left(t\sqrt{\omega_0^2-\beta^2}\right)}{\sqrt{\omega_0^2-\beta^2}}\right) \end{bmatrix} \\ &= \begin{bmatrix} -0.532585\,\mathrm{e}^{-0.214286\,t}\sin(1.87763\,t) \\ -\mathrm{e}^{-0.214286\,t}(\cos(1.87763t) - 0.114125\,\sin(1.87763\,t)) \end{bmatrix}.\end{aligned} \tag{4.22}$$

It is interesting to note that, although at the beginning of the system $t = 0$, the position and velocity stochastic processes are independent, and have zero variance, their diffusion grows monotonically over time and after a while it stabilizes. However, their dependence is an oscillatory function decaying to 0. Figures 4.6a-4.6b show the marginal densities of the solution PDF (4.20), Figure 4.7 shows the evolution of variance and covariance of position and velocity stochastic processes and Figures 4.8a-4.8d show the dynamics of the joint PDF.

Due to the fact that the solution stochastic process of (4.19) is a Gaussian process, its covariance and/or correlation shows the dependency of the position with respect to velocity, and vice versa. In particular, Figure 4.7 shows 2 important facts. The first fact is that the variance of both position and velocity tend to stabilize. The second fact is that, for large times, position and velocity are completely independent. This allows for the computation of the stationary state of the system. By using the explicit expression of the mean vector function (4.22), we conclude that the stationary state of the undamped stochastic oscillator system (4.19) is a random variable Z, following the distribution

$$Z \sim \mathrm{N}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0.326667 & 0\\ 0 & 1.16667\end{bmatrix}\right).$$

Another interesting and important fact is that, although the white noise process in this problem has mean 0, the dynamics of the damped stochastic oscillator (4.19) is very different from the one with identically null excitation and random parameters. We recall that we are dealing with an underdamped oscillator. Therefore, its oscillation amplitude should become smaller until it eventually stops in a point (the equilibrium point). However, the white noise excitation adds diffusion to the oscillator faster than it stabilizes, although the mean dynamics do tend to stabilize in the equilibrium point. This fact can be deduced from Figures 4.6a, 4.6b and 4.7. Compare with Figure 4.3.



(a) Marginal PDF of space.

(b) Marginal PDF of velocity.

Figure 4.6: Marginal densities given by the left and right equations of (4.21), respectively, in the interval (0,8]. As said before, variance grows monotonically, but then appears to stabilize at time $t = 3$, approximately. Calculations have been performed by Mathematica®.

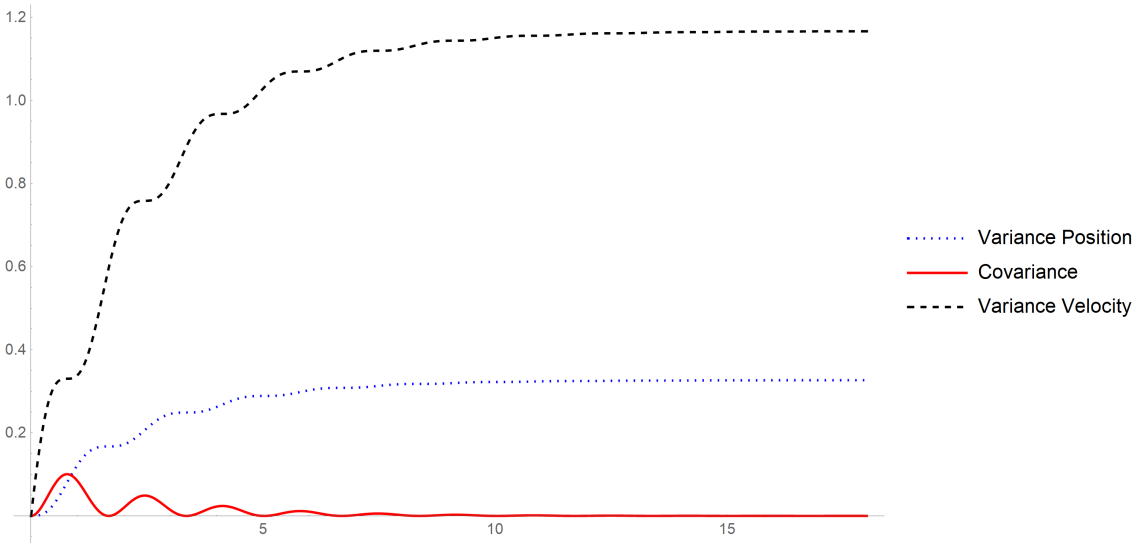
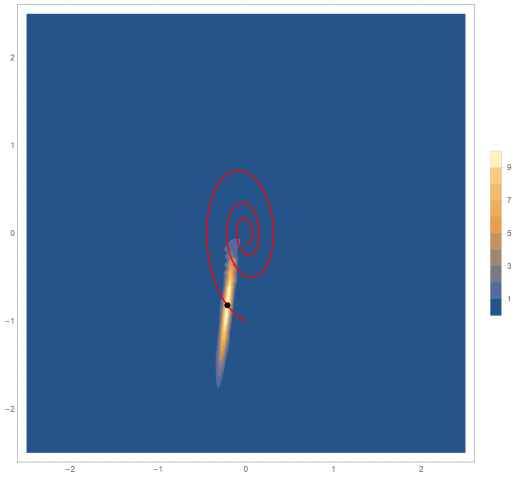
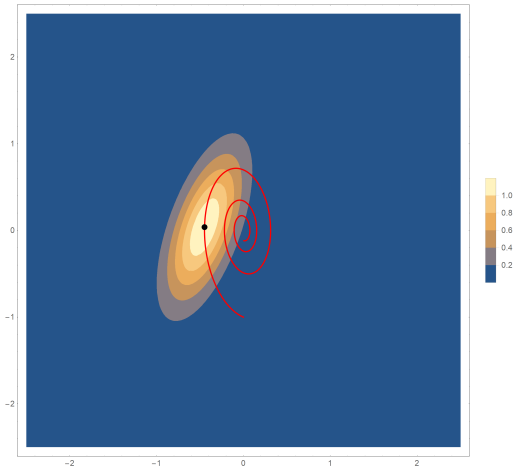


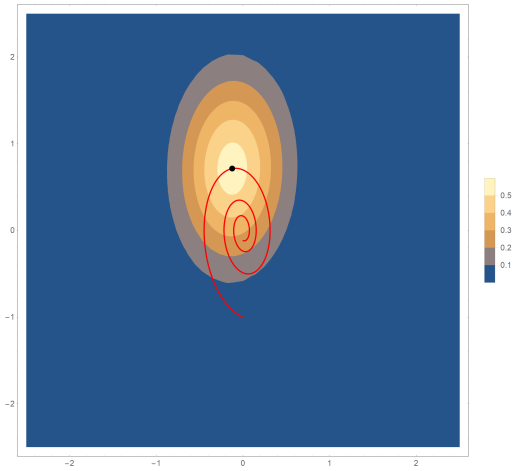
Figure 4.7: Time evolution in the [0,18] time interval of the variance function of position (blue, dotted) and velocity (black, dashed), given by the first and second diagonal elements of matrix Λ, respectively; as well as the covariance function, given by the elements $\Lambda_{12} = \Lambda_{21}$ (red, solid). This plot shows what was said in the previous figure. Both variances stabilize at 0.326667 for position and 1.16667 for velocity from $t = 10$ and beyond. Also the position and velocity become independent. Calculations performed by Mathematica®.



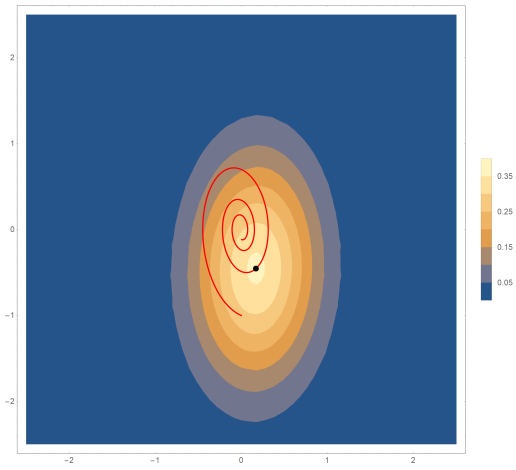
(a) Joint PDF of the solution stochastic process $\mathbf{X}$, of (4.19), at time $t = 0.23$. We can observe there are some instabilities in the upper part of the contour. For lower times the program does not give a reliable graph.



(b) Joint PDF of the solution stochastic process $\mathbf{X}$, of (4.19), at time $t = 0.8$.



(c) Joint PDF of the solution stochastic process $\mathbf{X}$, of (4.19), at time $t = 1.5$.



(d) Joint PDF of the solution stochastic process $\mathbf{X}$, of (4.19), at time $t = 3$.

Figure 4.8: The figures above show the dynamics of the joint PDF of the solution stochastic process $\mathbf{X}$ over time. The horizontal axis corresponds to position, X; whereas the vertical axis corresponds to velocity, $\dot{X}$. The red curve is the mean vector function, given in (4.22), and the black dot is the point of the curve at the respective time values. We can see a big difference with the random case: The base of the PDF gets wider as time goes by. This fact can also be seen in the increase of variance with respect to time in both space and velocity. We can see there is a monotonic increase in the diffusion of the density of probability (that is, the function gets wider as it turns) while in the random case there was an overall conservation of the density (it turned, but the shape was conserved). Calculations have been performed by Mathematica®.

4.5 Bang-Bang Stochastic Controller

A bang-bang controller is a feedback controller that changes abruptly between two states. Some well known examples of bang-bang controllers are water heaters or some air conditioners. By setting a desired temperature, the controller will run at maximum power until the desired temperature is reached. Then it will turn off until temperature is slightly below our target before turning on again at maximum power. However, there are many factors that may act as "noise" for the sensors of a bang-bang controller. For example if room temperature is very close to the desired temperature, any thermal change such as rapidly opening and closing a window or coughing could make the controller go on and off several times in a brief period of time.

Let us consider the mathematical description of a bang-bang controller with a white noise excitation.

$$\begin{cases} \mathrm{d}X(t) = -\beta \operatorname{sign}(X(t))\mathrm{d}t + \mathrm{d}B(t), \quad t > 0, \\ X(0) = x_0, \end{cases} \tag{4.23}$$

where $\operatorname{sign}(\cdot)$ denotes the sign function, β is a positive real value and we assume our Wiener process to be

$$B(t) \sim \mathrm{N}(0, 2D\,t),$$

where $D > 0$ represents the diffusion strength of the Wiener process. The Fokker Planck System of the Bang-Bang controller (4.23) is

$$\begin{cases} \partial_t f_{x_0}(x,t) = \beta \partial_x \{\operatorname{sign}(x) f_{x_0}(x,t)\} + D\partial_x^2 f_{x_0}(x,t), \quad t > 0,\, x \in [-M, M], \\ f_{x_0}(x,0) = \delta(x - x_0), \quad x \in [-M, M], \\ f_{x_0}(\pm M, t) = 0, \quad t \geq 0, \end{cases} \tag{4.24}$$

where x_0 is the initial state of the system and $|M| > |x_0|$ is an arbitrary constant. We will use the method of separation of variables. Let

$$f_{x_0}(x,t) = Y(x)T(t).$$

Substituting in the PDE of (4.24) and ordering properly, we have

$$\frac{1}{D}\frac{T'(t)}{T(t)} = -\lambda, \quad \frac{\beta \operatorname{sign}(x)}{D}\frac{Y'(x)}{Y(x)} + \frac{Y''(x)}{Y(x)} = -\lambda. \tag{4.25}$$

The first equation can be easily solved. We obtain

$$T(t) = c\,\mathrm{e}^{-\lambda D\,t},$$

where c is an integration constant. Now, let us use the definition of the sign function. That is, the equation for Y is given by the two following equations

$$\begin{aligned} Y''(x) + \tfrac{\beta}{D}Y'(x) + \lambda Y(x) = 0, \quad x > 0, \\ Y''(x) - \tfrac{\beta}{D}Y'(x) + \lambda Y(x) = 0, \quad x < 0. \end{aligned} \tag{4.26}$$

Also, at the origin we force continuity by the condition

$$Y(0^+) = \lim_{x\to 0^+} Y(x) = \lim_{x\to 0^-} Y(x) = Y(0^-).$$

Now, for the derivative condition, we integrate the right equation of (4.25) in a small interval around the origin, and then take the limit, obtaining

$$\dot{Y}(0^+) + \frac{\beta}{D}Y(0^+) = \dot{Y}(0^-) - \frac{\beta}{D}Y(0^-). \tag{4.27}$$

Now, the general solution of Equation (4.26) is

$$Y(x) = \begin{cases} a\mathrm{e}^{\tilde{\lambda}x - \frac{\beta}{2D}x} + b\mathrm{e}^{-\tilde{\lambda}x - \frac{\beta}{2D}x}, & x > 0 \\ c\mathrm{e}^{\tilde{\lambda}x + \frac{\beta}{2D}x} + d\mathrm{e}^{-\tilde{\lambda}x + \frac{\beta}{2D}x}, & x < 0, \end{cases}$$

where

$$\tilde{\lambda} = \sqrt{\frac{\beta^2}{4D^2} - \lambda} = \mathrm{i}\sqrt{\lambda - \frac{\beta^2}{4D^2}}.$$

We have assumed that it is an imaginary number because all the values we will obtain for λ as eigenvalues for the problem will make the inside of the square root a negative number, except for the first one. However, as we will see, we can still use the same reasoning. Figure 4.9 gives more graphical insight about this fact.

Now, using the imposed continuity condition, the derivative condition (4.27) and the boundary condition from (4.24), we obtain the following values for the integration constants

$$\begin{gathered} a + b = c + d, \\ a = -b\,\mathrm{e}^{-2\tilde{\lambda}M}, \quad d = -c\,\mathrm{e}^{-2\tilde{\lambda}M}, \\ b(1 - \mathrm{e}^{-2\tilde{\lambda}M}) = c(1 - \mathrm{e}^{-2\tilde{\lambda}M}), \\ (a - b - c + d)\tilde{\lambda} = -\tfrac{\beta}{2D}(a + b + c + d). \end{gathered} \tag{4.28}$$

Now, two cases must be taken into account. First, let us assume $(1 - \mathrm{e}^{-2\tilde{\lambda}M}) \neq 0$. This clearly implies $b = c$ and, as a consequence, $a = d$. Therefore, we can rewrite the last of the equations in (4.28), obtaining

$$\tilde{\lambda} = \frac{\beta}{2D}\tanh(\tilde{\lambda}M), \tag{4.29}$$

where $\tanh(\cdot)$ denotes the hyperbolic tangent. This equation cannot be solved explicitly. However, we will be able to obtain numerical approximations of its solutions, yielding the values for λ in this case, which we will denote as $\lambda_k^{(1)}$. Figure 4.9 shows the real and imaginary part of Equation (4.29). We can now write down the solution Y in the first case, denoted by $Y^{(1)}$

$$Y^{(1)}(x) = g(\lambda_k^{(1)})\mathrm{e}^{-\frac{\beta|x|}{2D}}\sin\left((|x| - M)\sqrt{\lambda_k^{(1)} - \frac{\beta^2}{4D^2}}\right). \tag{4.30}$$

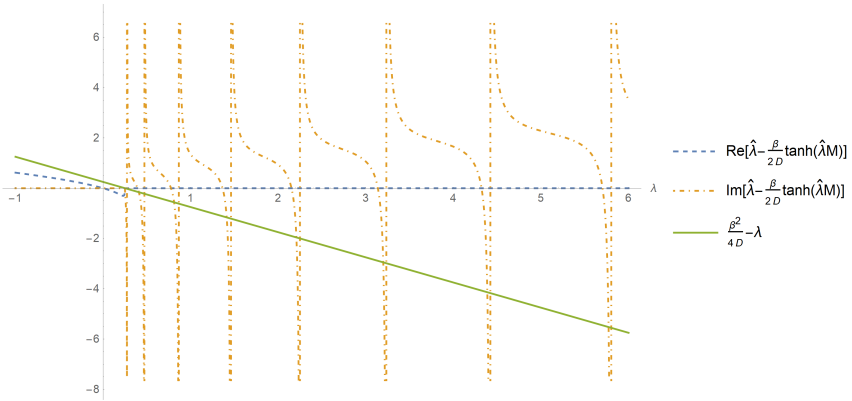


Figure 4.9: Plots of the real and imaginary parts of Equation (4.29), along with the line given by the inside of the square root that defines $\tilde{\lambda}$, as functions of λ. Values used are $\beta = D = 1$, $M = 10$. Roots are given by the intersection of the yellow, dash-dotted, non-vertical (imaginary part, continuity region) curves and the blue curve (real part). Also, this graphs shows why it is worth to consider $\tilde{\lambda}$ as a purely imaginary number. All eigenvalues for the first case are obtained when $\lambda > \frac{\beta}{2D}$, except for the eigenvalue $\lambda = 0$. However, this won't be a problem due to the relationship between the sine of an imaginary number and its hyperbolic sine. Graphics and calculations done by Mathematica®.

Note that the functions $\psi_k^{(1)}(x) := \sin\left((|x| - M)\sqrt{\lambda_k^{(1)} - \frac{\beta^2}{4D^2}}\right)$ form an orthogonal set over the interval $[-M, M]$, whose L^2-norm is

$$\|\psi_k^{(1)}\|_{\mathrm{L}^2}^2 = M - \frac{\sin\left(2M\sqrt{\lambda_k^{(1)} - \frac{\beta^2}{4D^2}}\right)}{2\sqrt{\lambda_k^{(1)} - \frac{\beta^2}{4D^2}}}.$$

Let us now consider the second case, where $(1 - \mathrm{e}^{-2\tilde{\lambda}M}) = 0$. Then, taking into account that $\tilde{\lambda}$ is zero or purely imaginary,

$$\tilde{\lambda} = \mathrm{i}k\pi, \iff \lambda_k^{(2)} = \frac{\beta^2}{4D^2} + \left(\frac{k}{M}\pi\right)^2, \quad k \in \mathbb{Z}.$$

In this case, the values for $a\,, b\,, c$ and d and the respective eigenfunctions are easily computed:

$$a = -b \quad c = -d, \quad \psi_k^{(2)}(x) = \sin\left(M\sqrt{\lambda_k^{(2)} - \frac{\beta^2}{4D^2}}\right), \quad \|\psi_k^{(2)}\|_{\mathrm{L}^2}^2 = M.$$

Therefore, the solution of Y in this case, denoted by $Y^{(2)}$, is given by

$$Y^{(2)}(x) = h(\lambda_k^{(1)})\mathrm{e}^{-\frac{\beta|x|}{2D}} \sin\left(x\sqrt{\lambda_k^{(2)} - \frac{\beta^2}{4D^2}}\right). \tag{4.31}$$

We can finally write down the explicit expression of the Fokker-Planck Equation

$$\begin{aligned} f_{x_0}(x,t) &= \mathrm{e}^{-\frac{\beta|x|}{2D}}\left[\sum_{k=1}^{\infty} g_k \mathrm{e}^{-\lambda_k^{(1)}Dt}\sin\left((|x|-M)\sqrt{\lambda_k^{(1)}-\frac{\beta^2}{4D^2}}\right)\right. \\ &+ \left.\sum_{k=1}^{\infty} h_k \mathrm{e}^{-\lambda_k^{(2)}Dt}\sin\left(x\sqrt{\lambda_k^{(2)}-\frac{\beta^2}{4D^2}}\right)\right]. \end{aligned} \tag{4.32}$$

All that is left is to calculate the coefficients $\{g_k\}_k$, $\{h_k\}_k$, given by the L^2-projection of the solution into each $\psi_k^{(1)}$ and $\psi_k^{(2)}$. Note that $\int_{-M}^{M}\psi_k^{(1)}\psi_j^{(2)}\mathrm{d}x = 0$, because of the parity of the functions.

To calculate the first set of coefficients, $\{g_k\}_k$, we consider the solution of (4.24) at time $t = 0$, multiply both sides of the equation by $\psi_k^{(1)}$, integrate over the interval $[-M, M]$ and use the orthogonality condition, obtaining

$$\int_{-M}^{M} f_{x_0}(x,0)\mathrm{e}^{\frac{\beta|x|}{2D}}\psi_k^{(1)}(x)\mathrm{d}x = \mathrm{e}^{\frac{\beta|x_0|}{2D}}\psi_k^{(1)}(x_0) = g_k\|\psi_k^{(1)}\|_{\mathrm{L}^2}^2,$$

therefore,

$$g_k = \mathrm{e}^{\frac{\beta|x_0|}{2D}}\frac{\psi_k^{(1)}(x_0)}{\|\psi_k^{(1)}\|_{\mathrm{L}^2}^2}, \quad \psi_k^{(1)}(x_0) = \sin\left((|x_0|-M)\sqrt{\lambda_k^{(1)}-\frac{\beta^2}{4D^2}}\right).$$

Using the same idea, we can calculate the $\{h_k\}_k$ coefficients, obtaining

$$h_k = \mathrm{e}^{\frac{\beta|x_0|}{2D}}\frac{\psi_k^{(2)}}{\|\psi_k^{(2)}\|_{\mathrm{L}^2}^2}, \quad \psi_k^{(2)}(x_0) = \sin\left(x_0\sqrt{\lambda_k^{(2)}-\frac{\beta^2}{4D^2}}\right).$$

For the numerical simulations, let us take $D = 1$, $\beta = 1$, $M = 10$ and $x_0 = 0$. The first 10 eigenvalues for the first case are

$$\lambda_1^{(1)} = 0,\ \lambda_2^{(1)} = 0.393658,\ \lambda_3^{(1)} = 0.775661,\ \lambda_4^{(1)} = 1.3638, \lambda_5^{(1)} = 2.15145,$$

$$\lambda_6^{(1)} = 3.13748,\ \lambda_7^{(1)} = 4.32129, \lambda_8^{(1)} = 5.7027,\ \lambda_9^{(1)} = 7.2816,\ \lambda_{10}^{(1)} = 9.05.$$

Since we chose $x_0 = 0$, all eigenfunctions for the second case are zero, and, therefore, there is no need to calculate its eigenvalues. Figure 4.10 shows the solution PDF, given by Equation (4.32), for several time instants. Also, it is easily seen that, by taking the limit when $t \to \infty$, the only "surviving" summand of Equation (4.32) is the one corresponding to $\lambda_0^{(1)} = 0$, which is the PDF of the stationary state of the Bang-Bang controller (4.23).

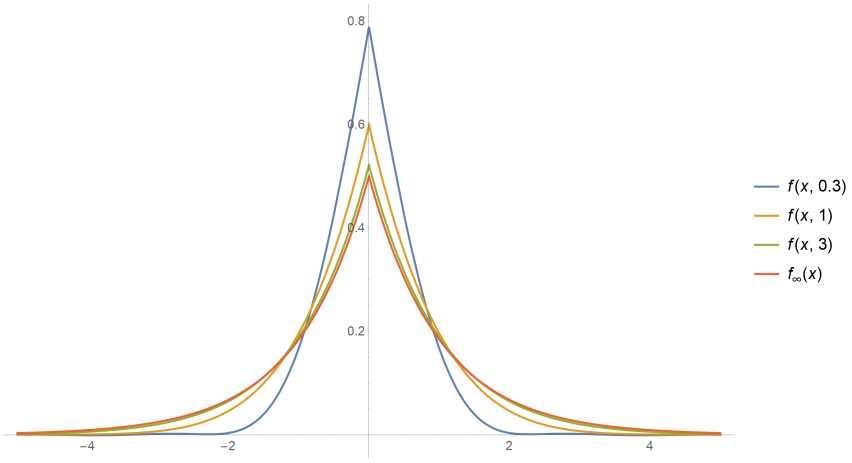


Figure 4.10: Graphs of the PDF, given by Equation (4.32), of the solution stochastic process of Problem (4.23). f_∞ denotes the stationary solution of (4.24). It is seen that the PDFs gradually decay to the stationary solution, with a faster decay at the beginning. Graphics and calculations done with Mathematica®.

It is interesting to note that we cannot assure any kind of convergence of the right-hand side of (4.32) to the true solution due to the fact that the Dirac delta "function", taken as an initial condition, is not really a function. If we substitute the Dirac delta function for a very narrow base normalized rectangular function, such as the PDF of a uniformly distributed random variable on an interval $[-\epsilon, \epsilon]$, we can obtain a solution in the form of a series that converges uniformly (because of homogeneous boundary conditions) to the unique solution of the problem. Then, by taking the limit in ε we will obtain the solution for our initial problem. This is a consequence of the **Hilbert-Schmidt theorem** (see [7, Th. 5.1]), which assures convergence in the L^2 norm in general, and uniform convergence in the homogeneous boundary condition case. However, the final solution obtained by taking the rectangular normalized function in $[-\epsilon, \epsilon]$ is equal to the one obtained by following the steps above in almost every point (see [1, Pg. 196]). Therefore, although we cannot assure its convergence, we can assure that the solution can be written as (4.32). Also, integrating the solution (4.32) for several values of time gives a comforting fact: they all integrate 1. If they did not integrate 1, the solution obtained would not be valid since all PDFs must integrate 1 over its domain.

Chapter 5

Conclusion

Throughout this project, we have seen the theory behind the relationship between the solution stochastic process of differential equations with uncertainty and its first probability density function. In particular, we have studied the case of random differential equations, where uncertainty appears as time-constant random variables; and also the case of Itô stochastic differential equations, where randomness appears as a white noise stochastic process forcing term. Both cases are a particular case of the Kinetic Equation, which gives a partial differential equation verified by the first probability density function of an arbitrary stochastic process. To do so, we have relied heavily in the relationship between the characteristic function of a stochastic process and its first probability function.

Mathematical modeling has grown rapidly to become a very active research area, and it will still grow due to lightning-paced advance of technology and computation in, basically, every realm of science and engineering. Therefore, due to the measurements involved in engineering devices and the uncertainty inherent to measurements, among other factors, uncertainty quantification plays a fundamental role when studying real models with real data. For this reason this topic was chosen for this project. It is my hope that I can continue my studies with the writing of a Ph.D. thesis and work in mathematical modeling with uncertainty. I have already been introduced into mathematical modeling research with my tutors and their colleagues and, consequently have been published as coauthor twice; in an article and in a chapter of the *Uncertainties 2020* congress proceedings book (see [25, 26], respectively). Also, I am a coauthor in the book to be published in late 2020, mentioned in Section 4.3. Furthermore, I have presented original work in two international mathematics congresses: *BYMAT 2019* and *Modeling for Engineering and Human Behavior 2019* (see [27]), organized by the Instituto de Matemática Multidisciplinar (UPV).

We can see how most of the subjects studied throughout the InvestMat master's degree have been used and implemented in this project. Specifically, the concepts and results from the subject Random Differential Equations and Applications have been used, and extended, to study the important problem of determining the first probability density function. Moreover, as I said before, I would like to write a Ph.D. thesis on the theory and

applications of ordinary and partial differential equations with uncertainty appearing in real models and dealing with real data. Other subjects such as Fundamentals of Advanced Mathematics, Mathematical Analysis and Applications will be fundamental to deal with the theory of these models. Also, subjects such as Neural Networks and Genetic Algorithms, Fuzzy Topology and Applied Mathematics Seminar will be of great importance when dealing with the real data used in the models, as well as computing and simulating solutions of the problems.

To conclude, I would like to mention that one of the most outstanding aspects of this project is that it can be seen how some of the most abstract and profound areas of mathematics, such as probability theory and mathematical analysis, can be united with state-of-the-art methods in computational mathematics and statistics to create both a rigorous mathematical theory and a useful and applicable set of methods to study real models and systems, taking into account the inherent uncertainty.

Glossary

IVP Initial Value Problem. 5, 6, 15–21, 29, 36, 38–40, 42–44

ODE Ordinary Differential Equation. 15, 19, 21, 23, 32, 36, 38, 45

PDE Partial Differential Equation. 6, 15–17, 20, 23, 26, 29, 31, 36, 39, 42, 51

PDF Probability Density Function. 5, 6, 10, 15–17, 19–22, 24, 26, 27, 35–44, 47–50, 54, 55

RDE Random Differential Equation. 5, 6, 15, 16, 19, 23, 27, 35

SDE Stochastic Differential Equation. 5, 6, 11, 23, 24, 27, 29–31, 33, 35, 47

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