THE SPECIAL CLOSURE OF POLYNOMIAL MAPS AND GLOBAL NON-DEGENERACY

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Abstract. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $F^{-1}(0)$ is finite. We analyze the connections between the multiplicity of F, the Newton polyhedron of F and the set of special monomials with respect to F, which is a notion motivated by the integral closure of ideals in the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. In particular, we characterize the polynomial maps whose set of special monomials is maximal.

1. Introduction

The effective computation of numerical invariants attached to functions, maps or ideals of $\mathcal{A}(\mathbb{K}^n)$ or $\mathbb{K}[x_1, \dots, x_n]$ is a fundamental problem in singularity theory, where $\mathcal{A}(\mathbb{K}^n)$ denotes the ring of analytic function germs $(\mathbb{K}^n, 0) \to \mathbb{K}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. One paradigmatic example in this direction is the article of Kouchnirenko [13], where the computation of the Milnor number of an analytic germ $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0 is carried out in terms of the Newton polyhedron of g. By considering global Newton polyhedra (see Definition 2.4), in [13, Théorème II] an analogous expression is obtained for the total Milnor number of a polynomial function $f \in \mathbb{C}[x_1, \dots, x_n]$ with a finite number of singularities. This number is defined as

$$\mu_\infty(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}[x_1, \dots, x_n]}{\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle}.$$

It is known that $\mu_\infty(f)$ equals the sum of the Milnor numbers of the respective germs of f at each singular point of f (see for instance [6] or [7, p. 150]). This is an example where the computation of the colength $\mu(I) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]/I$ of a given ideal $I \subseteq \mathbb{C}[x_1, \dots, x_n]$ generated by n elements plays a special role in singularity theory (see also [11]). In practice, $\mu(I)$ is computed by using Gröbner basis (see for instance [7, 11]). If I is an ideal of $\mathbb{C}[x_1, \dots, x_n]$ generated by the polynomials $F_1, \dots, F_n$, then $\mu(I)$ is interpreted as the number of roots, counting multiplicities, of the system $F_1 = \cdots = F_n = 0$. This point of view is fundamental for the application of techniques coming from combinatorial convexity and algebraic geometry to the computation of the colength of an ideal. There are two essential results in this direction. One is the bound of Bernstein-Khovanskii-Kouchnirenko about the number of roots in $(\mathbb{C} \setminus \{0\})^n$ of a system $F_1 = \cdots = F_n = 0$, where each F_i is a Laurent polynomial

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(see for instance [7, p. 346]). This bound is given by the mixed volume of the set of Newton polyhedra of these Laurent polynomials. The other result is the bound given by Li-Wang [17] where a similar result is established for the number of roots in $\mathbb{C}^n$ of polynomial systems.

There is another result that has been also part of the motivation of our work. This is the article of Saia [22] where there is proven a characterization of the class of ideals I of $\mathcal{A}(\mathbb{K}^n)$ whose integral closure $\overline{I}$ is generated by monomials, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (we refer to [21] for a different approach to this problem). These ideals can be expressed in terms of non-degeneracy conditions with respect to the the local Newton polyhedron of I and gives rise to the definition of Newton non-degenerate ideal (see [3, 4, 22]).

Let us denote the ring $\mathcal{A}(\mathbb{C}^n)$ by $\mathcal{O}_n$. Let us fix a polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$. Inspired by the characterization of integral elements over an ideal of $\mathcal{O}_n$ by means of analytic inequalities proven by Lejeune and Teissier [16] (see Remark 3.2), in this article we study the set of polynomials $h \in \mathbb{K}[x_1, \dots, x_n]$ such that there exist constants $C, M > 0$ for which $|h(x)| \leqslant C\|F(x)\|$ for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. When h satisfies this condition then we say that h is *special with respect to* F and we refer to the set of such polynomials as the special closure of F (see Definition 3.1), which we denote by $\mathrm{Sp}(F)$. We prove that these polynomials satisfy a fundamental condition of preservation of multiplicity with respect to F (see Theorem 3.8) that reminds the Rees' Multiplicity Theorem in the context of local algebra (see for instance [15, p. 222]). In view of this, the set $\mathrm{Sp}(F)$ can be considered as a counterpart in $\mathbb{K}[x_1, \dots, x_n]$ of the notion of integral closure of ideals in a local ring.

Once we fix coordinates in $\mathbb{K}^n$, we can consider the set $\mathbf{S}(F)$ formed by the exponents $k \in \mathbb{Z}^n_{\geqslant 0}$ such that the monomial x^k is special with respect to F. In [5] we gave some techniques for the estimation of the region $\mathbf{S}(F)$ that lead to lower estimates to the Łojasiewicz exponent at infinity of polynomial maps. We recall that the positivity of this exponent is related with the injectivity of polynomial maps (see [5, 14]). If $\widetilde{\Gamma}_+(F)$ is the global Newton polyhedron of F, then $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$, and we characterize when equality holds. We observe that, once $\mathbf{S}(F)$ is computed, then any polynomial $h \in \mathbb{C}[x_1, \dots, x_n]$ whose monomials have the exponents contained in $\mathbf{S}(F)$ automatically belongs to $\mathrm{Sp}(F)$. Therefore, in this article we study fundamental aspects relating three basic objects attached to a given polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$: the global Newton polyhedron $\widetilde{\Gamma}_+(F)$, the special closure of F and the multiplicity of F (when $\mathbb{K} = \mathbb{C}$).

The article is organized as follows. Section 2 is dedicated to expose some preliminary definitions. In Section 3 we introduce the set $\mathrm{Sp}(F)$ (see Definition 3.1) and we study some of its fundamental properties, specially its influence in the computation of the multiplicity of a complex analytic map $\mathbb{C}^n \to \mathbb{C}^n$. In Section 4 we address the problem of determining when $\mathbf{S}(F)$ fills the whole global Newton polyhedron $\widetilde{\Gamma}_+(F)$. In Corollary 4.10 we show that, if F is convenient (that is, if $\widetilde{\Gamma}_+(F)$ has non-empty intersection with each coordinate axis in a point different from the origin), then $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$ if and only if F is Newton non-degenerate at infinity (see Definition 2.5). As we remark in Examples 4.1 and 4.2, the hypothesis of $\widetilde{\Gamma}_+(F)$ being convenient can not be removed. Corollary 4.10 is preceded by a more general result about the characterization of Newton non-degeneracy at infinity of polynomial maps $\mathbb{K}^n \to \mathbb{K}^p$ (Theorem 4.9).

2. Preliminary definitions

In this section we show some preliminary concepts that we need in order to expose our results.

Definition 2.1. Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$. We say that $\widetilde{\Gamma}_+$ is a *global Newton polyhedron*, or a *Newton polyhedron at infinity*, if there exists some finite subset $A \subseteq \mathbb{Z}^n_{\geqslant 0}$ such that $\widetilde{\Gamma}_+$ is equal to the convex hull in $\mathbb{R}^n$ of $A \cup \{0\}$. In this case we also write $\widetilde{\Gamma}_+ = \widetilde{\Gamma}_+(A)$ and we say that $\widetilde{\Gamma}_+$ is the global Newton polyhedron *determined by* A.

Let us denote by $\langle\, ,\rangle$ the standard scalar product in $\mathbb{R}^n$. Given a non-empty compact subset $P \subseteq \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, we define

$$\begin{aligned}
\ell(v,P) &= \min\{\langle v,k\rangle : k \in P\}\\
\mathrm{m}(v,P) &= \max\{\langle v,k\rangle : k \in P\}\\
\Delta(v,P) &= \{k \in P : \langle v,k\rangle = \ell(v,P)\}.
\end{aligned}$$

Any set $\Delta(v,P)$, for some $v \in \mathbb{R}^n$, $v \neq 0$, is called a *face* of P. We will also say that $\Delta(v,P)$ is *the face of P supported by v*. Let us remark that $\mathrm{m}(v,P) = -\ell(-v,P)$, for all $v \in \mathbb{R}^n$, $v \neq 0$.

Definition 2.2. Let us fix a global Newton polyhedron $\widetilde{\Gamma}_+$ in $\mathbb{R}^n$. The *dimension* of a face Δ of $\widetilde{\Gamma}_+$, denoted by $\dim(\Delta)$, is defined as the minimum among the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. The faces of $\widetilde{\Gamma}_+$ of dimension 0 are called *vertices* of $\widetilde{\Gamma}_+$ and the faces of $\widetilde{\Gamma}_+$ of dimension $n-1$ are called *facets* of $\widetilde{\Gamma}_+$. We denote by $\widetilde{\Gamma}$ the union of the faces Δ of $\widetilde{\Gamma}_+$ with $0 \notin \Delta$ and we will refer to $\widetilde{\Gamma}$ as the *global boundary of* $\widetilde{\Gamma}_+$. We define the *dimension* of $\widetilde{\Gamma}_+$, denoted by $\dim(\widetilde{\Gamma}_+)$, as the maximum of $\dim(\Delta)$, where Δ varies in the set of faces of $\widetilde{\Gamma}_+$ not containing the origin.

Let $v \in \mathbb{Z}^n$. We say that v is *primitive* when $v \neq 0$ and v is the vector of smallest length of the set of vectors of $\mathbb{Z}^n$ of the form λv, for some $\lambda > 0$. We denote by $\mathcal{F}(\widetilde{\Gamma}_+)$ the family of primitive vectors $v \in \mathbb{Z}^n$ such that $\dim \Delta(v,\widetilde{\Gamma}_+) = n-1$ and we denote by $\mathcal{F}_0(\widetilde{\Gamma}_+)$ the set of vectors $v \in \mathcal{F}(\widetilde{\Gamma}_+)$ such that $0 \notin \Delta(v,\widetilde{\Gamma}_+)$. Then $\dim(\widetilde{\Gamma}_+) = n-1$ if and only if $\mathcal{F}_0(\widetilde{\Gamma}_+) \neq \emptyset$.

Let us suppose that $\dim(\widetilde{\Gamma}_+) = n-1$. Since $\widetilde{\Gamma}_+$ is the convex hull of a finite subset of $\mathbb{R}^n$, then $\mathcal{F}(\widetilde{\Gamma}_+)$ is finite and non-empty and any face of $\widetilde{\Gamma}_+$ can be expressed as an intersection $\cap_{v\in J}\Delta(v,\widetilde{\Gamma}_+)$, for some subset $J \subseteq \mathcal{F}(\widetilde{\Gamma}_+)$ (see [9, p. 33]).

We say that $\widetilde{\Gamma}_+$ is *convenient* if, for any $i \in \{1,\dots,n\}$, there exists some $r > 0$ such that $re_i \in \widetilde{\Gamma}_+$, where $\{e_1,\dots,e_n\}$ denotes the canonical basis in $\mathbb{R}^n$. If $\widetilde{\Gamma}_+$ is convenient, then it is immediate to see that $\mathcal{F}(\widetilde{\Gamma}_+) = \mathcal{F}_0(\widetilde{\Gamma}_+) \cup \{e_1,\dots,e_n\}$.

Lemma 2.3. *Let $\widetilde{\Gamma}_+ \subseteq \mathbb{R}^n_{\geqslant 0}$ be a global Newton polyhedron. Let $w = (w_1,\dots,w_n) \in \mathbb{Z}^n$, $w \neq 0$. Then the following conditions are equivalent:*

(a) $0 \notin \Delta(w, \widetilde{\Gamma}_+)$
(b) $\ell(w, \widetilde{\Gamma}_+) < 0$.

If we assume that $\widetilde{\Gamma}_+$ *is convenient, then the above conditions are equivalent to*

(c) $\min_i w_i < 0$.

Proof. It follows easily by using the corresponding definitions. □

Let us fix coordinates $(x_1, \dots, x_n)$ in $\mathbb{K}^n$. If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$ then we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k .

Definition 2.4. Let $f \in \mathbb{K}[x_1, \dots, x_n]$, $f \neq 0$. Let us suppose that f is written as $f = \sum_k a_k x^k$. The *support* of f, denoted by $\mathrm{supp}(f)$, is defined as $\mathrm{supp}(f) = \{k \in \mathbb{Z}^n_{\geqslant 0} : a_k \neq 0\}$. If A denotes any subset of $\mathbb{R}^n_{\geqslant 0}$, then we denote by f_A the sum of all terms $a_k x^k$ such that $k \in A \cap \mathrm{supp}(f)$. If $\mathrm{supp}(f) \cap A = \emptyset$, then we set $f_A = 0$. We define the *global Newton polyhedron of* f as $\widetilde{\Gamma}_+(f) = \widetilde{\Gamma}_+(\mathrm{supp}(f))$. If $f = 0$, then we set $\mathrm{supp}(f) = \widetilde{\Gamma}_+(f) = \emptyset$.

Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. The *support of* F is defined as $\mathrm{supp}(F) = \mathrm{supp}(F_1) \cup \cdots \cup \mathrm{supp}(F_p)$. We write $\widetilde{\Gamma}_+(F_1, \dots, F_p)$ or $\widetilde{\Gamma}_+(F)$ to denote the convex hull of $\widetilde{\Gamma}_+(F_1) \cup \cdots \cup \widetilde{\Gamma}_+(F_p)$. We refer to $\widetilde{\Gamma}_+(F)$ as the *global Newton polyhedron of* F. We say that F is *convenient* when $\widetilde{\Gamma}_+(F)$ is convenient.

We also set $F_A = ((F_1)_A, \dots, (F_p)_A)$, for any subset $A \subseteq \mathbb{R}^n$. If S is any finite subset of $\mathbb{K}[x_1, \dots, x_n]$, then $\mathrm{supp}(S)$ and $\widetilde{\Gamma}_+(S)$ are defined analogously.

Definition 2.5. Let $F = (F_1, \dots, F_s) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. We say that F is *Newton non-degenerate at infinity*, or *globally non-degenerate*, when

$$\big\{x \in \mathbb{K}^n : (F_1)_\Delta(x) = \cdots = (F_p)_\Delta(x) = 0\big\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\} \tag{1}$$

for all faces Δ of $\widetilde{\Gamma}_+(F)$ not containing the origin.

Our motivation to introduce the above notion comes from the articles of Kouchninreko [13], Saia [22] and Yoshinaga [24]. In Section 4 we will characterize this property.

3. Special monomials with respect to a polynomial map

Let us fix $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $\mathbb{K}^* = \mathbb{K} \smallsetminus \{0\}$. In this section we explore the concept of special polynomial with respect to a polynomial map $\mathbb{K}^n \to \mathbb{K}^p$. This notion was introduced in [5] and is directly related with the notion of Łojasiewicz exponent at infinity of a polynomial map (see Lemma 3.5). If $x = (x_1, \dots, x_n) \in \mathbb{C}^n$, then we write $\|x\| = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.

Let us suppose that (P_x) denotes a condition depending on $x \in \mathbb{K}^n$. We say that the condition (P_x) *holds for all* $\|x\| \gg 1$ when there exists a constant $M > 0$ such that (P_x) holds for all $x \in \mathbb{K}^n$ for which $\|x\| \geqslant M$. Analogously, we say that (P_x) *holds for all* $\|x\| \ll 1$ when there exists some open neighbourhood of $0 \in \mathbb{K}^n$ such that (P_x) holds for all $x \in U$.

Along this section we fix coordinates $(x_1, \dots, x_n)$ in $\mathbb{K}^n$.

Definition 3.1. [5] Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. An element $h \in \mathbb{K}[x_1, \dots, x_n]$ is said to be *special with respect to* F when there exists some constant $C > 0$ such that

$$|h(x)| \leqslant C\|F(x)\| \tag{2}$$

for all $\|x\| \gg 1$. We denote by $\mathrm{Sp}(F)$, or by $\mathrm{Sp}(F_1, \dots, F_p)$, the set of all polynomials $h \in \mathbb{K}[x_1, \dots, x_n]$ such that h is special with respect to F. We will refer to $\mathrm{Sp}(F)$ as the *special closure of* F.

If $F^{-1}(0)$ is compact and $h \in \mathbb{K}[x_1, \dots, x_n]$, then it follows from the above definition that $h \in \mathrm{Sp}(F)$ if and only if the function $\frac{|h(x)|}{\|F(x)\|}$ is bounded near infinity.

Remark 3.2. The above definition is motivated by the fundamental results of Lejeune and Teissier in [16] characterizing the integral closure of ideals. In our context we consider an analytic inequality in a neighbourhood of infinity instead of a neighbourhood of the origin. We recall that if I denotes an ideal of a ring R, then an element $h \in I$ is said to be *integral over* I when h satisfies a relation of the form $h^r + a_1 h^{r-1} + \cdots + a_{r-1}h + a_r = 0$, for some integer $r \geqslant 1$, where $a_i \in I^i$, for all $i = 1, \dots, r$. The set of integral elements over I forms an ideal of R, denoted by $\overline{I}$, and is called the *integral closure of* I (see [10, 12, 15, 23]). In [16], Lejeune and Teissier proved that, if $R = \mathcal{O}_n$ and $I = \langle g_1, \dots, g_s \rangle$ is any ideal of R, then $\overline{I}$ is formed by those function germs $h \in \mathcal{O}_n$ such that there exists some constant $C > 0$ such that

$$|h(x)| \leqslant C \sup_i |g_i(x)| \tag{3}$$

for all $\|x\| \ll 1$. As remarked by Gaffney in [10, p. 317], the algebraic definition of the integral closure of an ideal gives a theory sensitive to complex phenomena. Motivated by [10, Proposition 4.2] and the mentioned result of Lejeune and Teissier, if $I = \langle g_1, \dots, g_s \rangle$ is an ideal of $\mathcal{A}(\mathbb{R}^n)$, then we define the *integral closure of* I, which we denote by $\overline{I}$, as the set of those function germs $h \in \mathcal{A}(\mathbb{R}^n)$ such that relation (3) holds in some neighbourhood of 0 in $\mathbb{R}^n$.

Lemma 3.3. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $\varphi : \mathbb{K}^* \to \mathbb{K}^n$ be a continuous map such that $\lim_{t\to 0} \|\varphi(t)\| = +\infty$. Let $h \in \mathrm{Sp}(F)$ such that $h(\varphi(t)) \neq 0$, for all $|t| \ll 1$, $t \neq 0$. Then*

$$\lim_{t\to 0} \frac{\|F(\varphi(t))\|}{|h(\varphi(t))|} > 0.$$

Proof. This follows as an immediate application of the definition of $\mathrm{Sp}(F)$. □

Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. We remark that $F_1, \dots, F_p \in \mathrm{Sp}(F)$ and that $\mathrm{Sp}(F)$ is a subgroup of $\mathbb{K}[x_1, \dots, x_n]$ with respect to addition. It is immediate to see that $h \in \mathrm{Sp}(F)$ if and only if $\mathrm{Sp}(F) = \mathrm{Sp}(F, h)$.

We denote by $\mathbf{S}(F)$, or by $\mathbf{S}(F_1, \dots, F_p)$, the set of those $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$ such that the monomial $x_1^{k_1} \cdots x_n^{k_n}$ is special with respect to F. We remark that the set $\mathbf{S}(F)$ depends on the fixed coordinate system. If $\mathbf{S}(F) \neq \emptyset$, then it is obvious that there exists some $M > 0$ such that

$$F^{-1}(0) \cap \{x \in \mathbb{K}^n : \|x\| \geqslant M\} \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}. \tag{4}$$

If $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ then we denote $\min_i w_i$ by w_0. Hence we define $\mathbb{R}^n_0 = \{w \in \mathbb{R}^n : w_0 < 0\}$. If $A \subseteq \mathbb{R}^n$, then we denote by $\mathrm{Conv}(A)$ the convex hull of A.

If $\phi : (\mathbb{K}^*, 0) \to \mathbb{K}^n$ is an analytic map germ, then we denote by $\mathrm{ord}(\phi)$ the least exponent appearing in the Laurent expansion of ϕ around 0. The next result is a first step towards the study of the relations between the sets $\mathrm{Sp}(F)$ and $\widetilde{\Gamma}_+(F)$.

Lemma 3.4. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Then $\mathbf{S}(F) = \mathrm{Conv}(\mathbf{S}(F)) \cap \mathbb{Z}^n_{\geqslant 0}$ and $\mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)$, for all $h \in \mathrm{Sp}(F)$. In particular $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F)$.*

Proof. Let $k_1, k_2 \in \mathbf{S}(F)$ and let $\lambda \in [0,1]$ such that $\lambda k_1 + (1-\lambda)k_2 \in \mathbb{Z}^n_{\geqslant 0}$. Then, applying the definition of $\mathbf{S}(F)$, we obtain that

$$|x^{\lambda k_1 + (1-\lambda)k_2}| = |(x^{k_1})^{\lambda}(x^{k_2})^{(1-\lambda)}| \leqslant C\|F(x)\|^{\lambda}\|F(x)\|^{1-\lambda} = C\|F(x)\|,$$

for some constant $C > 0$ and all $\|x\| \gg 1$. Hence $\lambda k_1 + (1-\lambda)k_2 \in \mathbf{S}(F)$. In particular $\mathbf{S}(F) = \mathrm{Conv}(\mathbf{S}(F)) \cap \mathbb{Z}^n_{\geqslant 0}$.

Let $h \in \mathrm{Sp}(F)$. Then there exist constants $C, M > 0$ such that

$$|h(x)| \leqslant C\|F(x)\| \tag{5}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Let us fix a vector $w \in \mathbb{R}^n_0$. Let us consider the analytic path $\varphi_w : \mathbb{K}^* \to \mathbb{K}^n$ given by $\varphi_w(t) = (t^{w_1}, \dots, t^{w_n})$. Since $w_0 < 0$, we have $\lim_{t \to 0} \|\varphi_w(t)\| = +\infty$. Composing both sides of (5) with φ_w we obtain that

$$|h(\varphi_w(t))| \leqslant C\|F(\varphi_w(t))\| \tag{6}$$

for all $|t| \ll 1$. Therefore $\mathrm{ord}(h \circ \varphi_w) \geqslant \mathrm{ord}(F \circ \varphi_w)$. Let us observe that $\mathrm{ord}(h \circ \varphi_w) = \ell(w, h)$ and $\mathrm{ord}(F \circ \varphi_w) = \ell(w, \mathrm{supp}(F)) \geqslant \ell(w, \widetilde{\Gamma}_+(F))$. Then $\ell(w, h) \geqslant \ell(w, \widetilde{\Gamma}_+(F))$, for all $w \in \mathbb{R}^n_0$. Thus $\mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)$. □

In Section 4 we will study the problem of characterizing the equality $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.

If $F : \mathbb{K}^n \to \mathbb{K}^p$ is a polynomial map, the *Łojasiewicz exponent at infinity of F* is defined as the supremum of those $\alpha \in \mathbb{R}$ such that there exists a positive constant $C > 0$ such that

$$\|x\|^{\alpha} \leqslant C\|F(x)\|$$

for all $\|x\| \gg 1$ (we refer to the article of Krasiński [14] for a detailed survey about Łojasiewicz exponents at infinity). It is known that $\mathcal{L}_\infty(F)$ exists when $F^{-1}(0)$ is compact and that $\mathcal{L}_\infty(F)$ is a rational number in this case. Moreover $\mathcal{L}_\infty(F) > 0$ if and only if F is a proper map (see [14]). Let $r, s \in \mathbb{Z}_{\geqslant 0}, s \neq 0$. Given a polynomial map $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ and an integer $s \in \mathbb{Z}_{\geqslant 0}$, then we denote by F^s the polynomial map given by $(F_1^s, \dots, F_p^s)$.

Lemma 3.5. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map such that $\mathcal{L}_\infty(F) > 0$. Then*

$$\mathcal{L}_\infty(F) = \sup\left\{\frac{r}{s} : r, s \in \mathbb{Z}_{\geqslant 0},\, s \neq 0,\, x_i^r \in \operatorname{Sp}(F^s), \text{ for all } i = 1, \dots, n\right\}.$$

Proof. Given a point $x \in \mathbb{K}^n$, we have

$$\|F(x)\|^s \leqslant p^{s/2} \max_i |F_i(x)|^s = p^{s/2} \max_i |F_i^s(x)| \leqslant p^{s/2} \|F^s(x)\|.$$

By using this fact, it is straightforward to see that the condition $\|x\|^{r/s} \leqslant C\|F(x)\|$, for some constant $C > 0$ and all $\|x\| \gg 1$, is equivalent to saying that $x_i^r \in \operatorname{Sp}(F^s)$, for all $i = 1, \dots, n$. Hence the result follows. □

Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. If $r, s \in \mathbb{Z}_{\geqslant 1}$, we define

$$\operatorname{Sp}(F^{s/r}) = \{h \in \mathbb{K}[x_1, \dots, x_n] : h^r \in \operatorname{Sp}(F^s)\}. \tag{7}$$

It is immediate to check that the above definition only depends on the fraction s/r. Therefore we obtain that, if $\mathcal{L}_\infty(F) > 0$, then

$$\mathcal{L}_\infty(F) = \frac{1}{\inf\left\{\frac{s}{r} : r, s \in \mathbb{Z}_{\geqslant 1},\, x_1, \dots, x_n \in \operatorname{Sp}(F^{s/r})\right\}}.$$

In [5] we obtained some results about the estimation of the region $\mathbf{S}(F)$ with the objective of determining lower bounds for the Łojasiewicz exponent at infinity of a given polynomial map. Let $S, S' \subseteq \mathbb{K}[x_1, \dots x_n]$. We define $SS' = \{fg : f \in S,\, g \in S'\}$. If $r \in \mathbb{Z}_{\geqslant 0}$, then we denote by S^r the set $\{h_1 \cdots h_r : h_i \in S, \text{ for all } i = 1, \dots, r\}$.

Lemma 3.6. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Then*

(a) $\operatorname{Sp}(F^r)^s \subseteq \operatorname{Sp}(F^{rs})$, *for all $r, s \in \mathbb{Z}_{\geqslant 0}$*

(b) *If $\theta, \theta' \in \mathbb{Q}_{>0}$, then* $\operatorname{Sp}(F^\theta)\operatorname{Sp}(F^{\theta'}) \subseteq \operatorname{Sp}(F^{\theta+\theta'})$.

Proof. The result follows easily using the corresponding definitions. □

Given a point $z \in \mathbb{C}^n$ and $r > 0$, we denote by $B(z; r)$ the open ball in $\mathbb{C}^n$ of center z and radius r.

Lemma 3.7. *Let $F : \mathbb{K}^n \to \mathbb{K}^n$ be a polynomial map such that $F^{-1}(0)$ is compact and let $h \in \mathbb{K}[x_1, \dots, x_n]$, $h \neq 0$. Then there exist $M, \delta > 0$, such that*

(a) $\|F(x)\| \neq 0$, *for all $x \in \mathbb{K}^n$ with $\|x\| \geqslant M$.*

(b) $\|(F + h\alpha)(x)\| \neq 0$, *for all $x \in \mathbb{K}^n$ such that $\|x\| = M$ and for all $\alpha \in B(0; \delta)$.*

Proof. Given a non-negative real number M, we define $C_M = \sup\{|h(x)| : \|x\| = M\}$. Since $F^{-1}(0)$ is compact and $h \neq 0$, there exists some $M > 0$ such that $C_M > 0$ and $\|F(x)\| \neq 0$, for all $x \in \mathbb{K}^n$ with $\|x\| \geqslant M$. Let us consider $\delta_1 = \min\{\|F(x)\| : \|x\| = M\}$. Let $\delta = \frac{\delta_1}{2C_M}$. Then, if $\alpha \in \mathbb{K}^n$ verifies that $\|\alpha\| < \delta$, then

$$\|(F + h\alpha)(x)\| \geqslant \|F(x)\| - \|\alpha\||h(x)| \geqslant \delta_1 - \delta C_M = \delta_1 - \frac{\delta_1}{2} = \frac{\delta_1}{2} > 0,$$

for all $x \in \mathbb{K}^n$ such that $\|x\| = M$, and hence the result follows. □

If $F:\mathbb{C}^n\to\mathbb{C}^n$ is a polynomial map, then we denote by $\mathbf{I}(F)$ the ideal of $\mathbb{C}[x_1,\dots,x_n]$ generated by the component functions of F. Let us suppose that $F^{-1}(0)$ is finite. We define the *multiplicity of* F, denoted by $\mu(F)$, as

$$\mu(F)=\dim_{\mathbb{C}}\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{I}(F)}.$$

It is well known (see for instance [7, p. 150]) that

$$\dim_{\mathbb{C}}\frac{\mathbb{C}[x_1,\dots,x_n]}{\mathbf{I}(F)}=\sum_{x\in F^{-1}(0)}\dim_{\mathbb{C}}\frac{\mathcal{O}_{n,x}}{\mathbf{I}_x(F)},\tag{8}$$

where $\mathcal{O}_{n,x}$ denotes the ring of analytic function germs $(\mathbb{C}^n,x)\to\mathbb{C}$ and $\mathbf{I}_x(F)$ is the ideal of $\mathcal{O}_{n,x}$ generated by the germs at x of the elements of $\mathbf{I}(F)$, that is, $\mathbf{I}_x(F)=\mathbf{I}(F)\mathcal{O}_{n,x}$. If $x\in F^{-1}(0)$, then we denote by $\mu_x(F)$ the dimension as a complex vector space of $\mathcal{O}_{n,x}/\mathbf{I}_x(F)$.

Let $f:U\to\mathbb{R}^n$ be a continuous function, where $U\subseteq\mathbb{R}^n$ is a connected open set. Let us suppose that $y\in U$ is an isolated zero of f. Then we denote by $\mathrm{ind}_y(f)$ the topological index of f at y (see for instance [8, 19]). If $G:\mathbb{R}^n\to\mathbb{R}^n$ denotes a polynomial map such that $G^{-1}(0)$ is finite, then we denote by $\mathrm{ind}(G)$ the sum of the indices $\mathrm{ind}_y(G)$, where y varies in $G^{-1}(0)$.

Let us consider the bijection $\sigma:\mathbb{C}^n\to\mathbb{R}^{2n}$ given by the map $\sigma(a_1+\mathbf{i}b_1,\dots,a_n+\mathbf{i}b_n)=(a_1,b_1,\dots,a_n,b_n)$, for all $a_1,b_1,\dots,a_n,b_n\in\mathbb{R}$. If $F:\mathbb{C}^n\to\mathbb{C}^n$ is a polynomial map, then we denote by $F_{\mathbb{R}}$ the underlying map $\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ obtained from F under the identification σ. We recall that, by a result of Palamodov [20], if x denotes an isolated zero of F and $y=\sigma(x)$, then

$$\mu_x(F)=\mathrm{ind}_y(F_{\mathbb{R}})\tag{9}$$

(see for instance [1, Section 5] or [8, Section 2]). Applying (9) and relation (8), we conclude that, if $F^{-1}(0)$ is finite, then

$$\mu(F)=\mathrm{ind}(F_{\mathbb{R}}).\tag{10}$$

If $F:\mathbb{K}^n\to\mathbb{K}^p$ is a polynomial map, then we say that F is *finite* when the zero set of F is finite.

Theorem 3.8. *Let $F:\mathbb{C}^n\to\mathbb{C}^n$ be a finite polynomial map and let $h\in\mathbb{C}[x_1,\dots,x_n]$, $h\neq 0$. Then the following conditions are equivalent:*

(a) *h is special with respect to F;*
(b) *there exists some $\delta>0$ such that for all $\alpha\in B(0;\delta)$, the map $F+h\alpha$ is a finite polynomial map and $\mu(F)=\mu(F+h\alpha)$.*

Proof. Let us prove (a) $\Rightarrow$ (b). Let us suppose that h is special with respect to F. Since $F^{-1}(0)$ is finite, there exist positive constants C and M such that

$$|h(x)|\leqslant C\|F(x)\|\quad\text{and}\quad\|F(x)\|\neq 0,\quad\text{for all }\|x\|\geqslant M.\tag{11}$$

Let us fix a vector $\alpha \in \mathbb{C}^n$. Then we define the homotopy $H_\alpha : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$ given by $H_\alpha(t,x) = F(x) + th(x)\alpha$, for all $(t,x) \in [0,1] \times \mathbb{C}^n$. If $t \in [0,1]$, then we consider the map $H_{\alpha,t} : \mathbb{C}^n \to \mathbb{C}^n$ given by $H_{\alpha,t}(x) = H_\alpha(t,x)$, for all $x \in \mathbb{C}^n$.

Let $t \in [0,1]$ and let $x \in \mathbb{C}^n$ such that $\|x\| \geqslant M$. Then

$$\|H_\alpha(t,x)\| \geqslant \|F(x)\| - t\|\alpha\||h(x)| \geqslant \|F(x)\|(1 - t\|\alpha\|C),$$

for all $\alpha \in \mathbb{C}^n$. Moreover $1 - t\|\alpha\|C = 0$ if and only if $t = \frac{1}{\|\alpha\|C}$. Therefore, if we assume that $\|\alpha\| < \frac{1}{C}$, then $\|H_\alpha(t,x)\| > 0$, for all $t \in [0,1]$ and for all $x \in \mathbb{C}^n$ such that $|x\| \geqslant M$. In particular, the zero set of $F + th\alpha$ is finite and

$$\mathrm{ind}(F_\mathbb{R}) = \mathrm{ind}\big((F + th\alpha)_\mathbb{R}\big)$$

for all $t \in [0,1]$ and all $\alpha \in B(0; \frac{1}{C})$, by the invariance of the index by homotopies (see for instance [6, p. 220], [18, Theorem 2.1.2] or [19, p. 144]). In particular $\mathrm{ind}(F_\mathbb{R}) = \mathrm{ind}((F+h\alpha)_\mathbb{R})$ and hence $\mu(F) = \mu(F + h\alpha)$, by (10).

Let us prove (b) $\Rightarrow$ (a) by contradiction. Let us fix a $\delta > 0$ such that $F + h\alpha$ is a finite polynomial map and $\mu(F) = \mu(F + h\alpha)$, for all $\alpha \in B(0;\delta)$, and let us suppose that h is not special with respect to F. It follows from Definition 3.1 that there exists a sequence $\{x_m\}_{m \geqslant 1}$ in $\mathbb{C}^n$ such that

$$\|x_m\| \geqslant m \qquad \text{and} \qquad |h(x_m)| > m\|F(x_m)\| \tag{12}$$

for all $m \geqslant 1$. In particular $h(x_m) \neq 0$, for all $m \geqslant 1$, and

$$\lim_{m\to\infty} \frac{F(x_m)}{h(x_m)} = 0. \tag{13}$$

Since F is a finite polynomial map, by Lemma 3.7 we can choose positive constants M and δ_1 such that $\|F(x)\| \neq 0$ for all $\|x\| \geqslant M$ and

$$\|(F + h\alpha)(x)\| \neq 0 \tag{14}$$

for all $x \in \mathbb{C}^n$ with $\|x\| = M$ and all $\alpha \in B(0;\delta_1)$.

By (12) and (13) there exists an $m_0 \in \mathbb{Z}_{\geqslant 1}$ such that $\|x_{m_0}\| > M$ and $\|F(x_{m_0})\|/|h(x_{m_0})| < \min\{\delta, \delta_1\}$. Let us consider the point

$$\alpha_0 = -\frac{F(x_{m_0})}{h(x_{m_0})}.$$

Since $\|\alpha_0\| < \delta$, the map $F + h\alpha_0$ has finite zero set and $\mu(F) = \mu(F + h\alpha_0)$, by hypothesis.

Let us consider the homotopy $H : [0,1] \times \mathbb{C}^n \to \mathbb{C}^n$, defined by $H(t,x) = (F + ht\alpha_0)(x)$, for all $(t,x) \in [0,1] \times \mathbb{C}^n$. The map H satisfies the following conditions:

(i) $H(0,x) = F(x)$ and $H(1,x) = (F + h\alpha_0)(x)$, for all $x \in \mathbb{C}^n$;
(ii) $H(t,x) \neq 0$, for all $\|x\| = M$ and for all $t \in [0,1]$, by (14).

Let U denote the open ball $B(0;M)$ in $\mathbb{C}^n$. We observe that $F^{-1}(0) \subseteq U$. Then, applying the invariance of the index by homotopies ([19, p. 144]) we obtain that

$$\mu(F) = \sum_{y \in (F+h\alpha_0)^{-1}(0) \cap U} \mu_y(F + h\alpha_0). \tag{15}$$

By the definition of α_0 we have $(F+h\alpha_0)(x_{m_0}) = 0$. Then $\mu_{x_{m_0}}(F+h\alpha_0)$ is a positive number. Moreover $x_{m_0} \notin U$. Therefore, by (15), we deduce the following:

$$\mu(F) < \sum_{y \in (F+h\alpha_0)^{-1}(0) \cap U} \mu_y(F + h\alpha_0) + \mu_{x_{m_0}}(F + h\alpha_0) \leqslant \mu(F + h\alpha_0),$$

which is a contradiction. □

Remark 3.9. Let I be an ideal of $\mathcal{O}_n$ and let $h \in \mathcal{O}_n$. Let us suppose that I has finite colength. By the Rees' Multiplicity Theorem [15, p. 222], h is integral over I if and only if $e(I) = e(I + \langle h \rangle)$, where $e(I)$ denotes the Samuel multiplicity of I. If we assume that I is generated by n elements, say $g_1, \dots, g_n$, then $e(I+\langle h \rangle) = e(g_1+\alpha_1 h, \dots, g_n+\alpha_n h)$, for generic $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ (see [15, Theorem 8.6.6 and Proposition 11.2.1]). Therefore, we can consider Theorem 3.8 as a version for finite polynomial maps of the Rees' Multiplicity Theorem.

4. Newton non-degeneracy at infinity

Our work in this section is motivated by the following observation. If $F : \mathbb{K}^n \to \mathbb{K}^p$ is a polynomial map, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, even if we assume that F is Newton non-degenerate at infinity, then $\mathbf{S}(F)$ can be empty and, if it is non-empty, then it is not true in general that $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$, as Examples 4.1 and 4.2 show (see also Example 4.14). This section is devoted to characterizing the Newton non-degeneracy property of polynomial maps $F : \mathbb{K}^n \to \mathbb{K}^p$ by means of the set of special monomials of F.

If $F : \mathbb{K}^n \to \mathbb{K}^p$ is a polynomial map, then we denote by $\mathbf{S}'(F)$ the set $\mathbf{S}(F, 1)$. That is, $\mathbf{S}'(F)$ is the set of those $k \in \mathbb{Z}^n_{\geqslant 0}$ such that x^k is special with respect to the map $(F, 1) : \mathbb{K}^n \to \mathbb{K}^{p+1}$. Obviously we have $\mathbf{S}(F) \subseteq \mathbf{S}'(F)$ and this inclusion is strict in general, as is shown in Example 4.2. By Lemma 3.4 we have that

$$\mathbf{S}(F) \subseteq \mathbf{S}'(F) \subseteq \widetilde{\Gamma}_+(F, 1) \cap \mathbb{Z}^n_{\geqslant 0} = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}. \tag{16}$$

If $h \in \mathrm{Sp}(F, 1)$, then we will say that h is *quasi-special* with respect to F.

Example 4.1. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be the polynomial defined by $F(x, y) = xy - 1$. The global boundary of $\widetilde{\Gamma}_+(F)$ is given by $\widetilde{\Gamma}(F) = \{(1, 1)\}$. By Lemma 3.4, we have that $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F)$. It is obvious that neither 1 nor xy are special monomials with respect to F. Then $\mathbf{S}(F) = \emptyset$. We also point out that F does not satisfy relation (4). However F is Newton non-degenerate at infinity.

Example 4.2. Let $F : \mathbb{C}^2 \to \mathbb{C}$ be the polynomial map given by $F(x, y) = x^2y^2$. Then, F is Newton non-degenerate at infinity. We have that $\mathbf{S}(F) \neq \emptyset$, since $x^2y^2 \in \mathrm{Sp}(F)$. We observe that $\widetilde{\Gamma}_+(F)$ is the segment joining the origin and the point $(2, 2)$. Then $(1, 1) \in \widetilde{\Gamma}_+(F)$.

We claim that $xy \notin \mathrm{Sp}(F)$. Indeed, let us consider the curve $\varphi : \mathbb{C}^* \to \mathbb{C}^2$ defined as $\varphi(t) = (t^{-1}, t^2)$. It immediate to check that $\lim_{t\to 0} \|\varphi(t)\| = +\infty$ and

$$\lim_{t\to 0} \frac{\|F(\varphi(t))\|}{|xy \circ \varphi(t)|} = \lim_{t\to 0} \frac{|t^2|}{|t|} = 0.$$

Therefore $xy \notin \mathrm{Sp}(F)$, by Lemma 3.3. On the other hand, it is immediate to check that $xy \in \mathrm{Sp}(F, 1)$. Therefore $\mathbf{S}(F) \subsetneq \mathbf{S}'(F)$.

Definition 4.3. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let d denote the maximum of the degrees of the component functions of F, which we will also denote by $\deg(F)$. Let us introduce a new variable x_{n+1}. If $h \in \mathbb{K}[x_1, \dots, x_n]$, $\deg(h) \leqslant d$, then we denote by h^* the homogenization of h of degree d by means of the extra variable x_{n+1}. That is, h^* is the polynomial of $\mathbb{K}[x_1, \dots, x_{n+1}]$ such that

$$h^*(x_1, \dots, x_{n+1}) = x_{n+1}^d h\left(\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}\right), \tag{17}$$

for all $(x_1, \dots, x_{n+1}) \in \mathbb{K}^{n+1}$ such that $x_{n+1} \neq 0$. Thus we denote by $\widetilde{F}$ the polynomial map $\mathbb{K}^{n+1} \to \mathbb{K}^{p+1}$ given by $\widetilde{F} = (F_1^*, \dots, F_p^*, x_{n+1}^d)$.

Although $\widetilde{F}$ is again a polynomial map, we will regard it as a germ of analytic function $(\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$. We recall the counterpart of Definition 2.4 in the context of germs of analytic functions $(\mathbb{K}^n, 0) \to \mathbb{K}$.

Definition 4.4. Let $\mathcal{A}(\mathbb{K}^n)$ denote the ring of analytic function germs $(\mathbb{K}^n, 0) \to \mathbb{K}$. Let $f \in \mathcal{A}(\mathbb{K}^n)$, $f \neq 0$. Let us suppose that the Taylor expansion of f around the origin is given by $f = \sum_k a_k x^k$. The *support* of f, which we will denote by $\mathrm{supp}(f)$, is defined as $\mathrm{supp}(f) = \{k \in \mathbb{Z}^n_{\geqslant 0} : a_k \neq 0\}$. If Δ is a compact subset of $\mathbb{R}^n_{\geqslant 0}$, then we define f_Δ as the sum of all terms $a_k x^k$ such that $k \in \Delta$. If $\Delta \cap \mathrm{supp}(f) = \emptyset$, then we set $f_\Delta = 0$.

The *Newton polyhedron of* f, denoted by $\Gamma_+(f)$, is defined as the convex hull in $\mathbb{R}^n$ of $\{k + v : k \in \mathrm{supp}(f), v \in \mathbb{R}^n_{\geqslant 0}\}$. If $f = 0$, then we set $\mathrm{supp}(f) = \Gamma_+(f) = \emptyset$.

Let $g = (g_1, \dots, g_p) : (\mathbb{K}^n, 0) \to \mathbb{K}^p$ be an analytic map germ. We denote the map $((g_1)_\Delta, \dots, (g_p)_\Delta)$ by g_Δ. The *support of* g is defined as $\mathrm{supp}(g) = \mathrm{supp}(g_1) \cup \cdots \cup \mathrm{supp}(g_p)$. The *Newton polyhedron of* g, denoted by $\Gamma_+(g)$, is the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_p)$. Analogously to Definition 2.5, we say that g is *Newton non-degenerate*, when $g_\Delta^{-1}(0) \subseteq \{x \in \mathbb{K}^n : x_1 = \cdots = x_n = 0\}$, for all compact faces Δ of $\Gamma_+(g)$.

Let I be an ideal of $\mathcal{A}(\mathbb{K}^n)$ and let $g = (g_1, \dots, g_p) : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be an analytic map whose components generate I. Then the *Newton polyhedron of* I is defined as $\Gamma_+(I) = \Gamma_+(g)$. We say that I is *Newton non-degenerate* when g is Newton non-degenerate. It is immediate to see that these notions do not depend on the given generating system of I.

Let π denote the projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ onto the first n coordinates. Let $\mathbf{c}' = (1, \dots, 1) \in \mathbb{R}^{n+1}$ and let $\mathbf{c} = \pi(\mathbf{c}')$. Under the conditions of Definition 4.3, we easily observe that, since each component function of $\widetilde{F}$ is homogeneous of degree d, then any compact face of $\Gamma_+(\widetilde{F})$ is contained in the n-dimensional compact face $\Delta(\mathbf{c}', \Gamma_+(\widetilde{F}))$.

We recall that if $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ then we denote $\min_i w_i$ by w_0 and we defined $\mathbb{R}^n_0 = \{w \in \mathbb{R}^n : w_0 < 0\}$. Let us define

$$\mathbb{R}^{n+1}_* = \left\{(v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}_{>0} : v_{n+1} = 2v_0\right\}.$$

Let us consider the map $\mathbf{w} : \mathbb{R}^{n+1} \to \mathbb{R}^n$ given by

$$\mathbf{w}(v) = (v_1 - v_{n+1}, \dots, v_n - v_{n+1}) = (v_1, \dots, v_n) - v_{n+1}\mathbf{c}, \tag{18}$$

for all $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}$. We observe that if $v \in \mathbb{R}^{n+1}_*$, then the minimum of the coordinates of $\mathbf{w}(v)$ is equal to $-v_0$, since $v_{n+1} = 2v_0$. Therefore $\mathbf{w}(\mathbb{R}^{n+1}_*) \subseteq \mathbb{R}^n_0$.

Given a vector $w = (w_1, \dots, w_n) \in \mathbb{R}^n_0$, we define

$$\mathbf{v}(w) = (w_1 - 2w_0, \dots, w_n - 2w_0, -2w_0) = (w, 0) - 2w_0\mathbf{c}'. \tag{19}$$

We observe that $\mathbf{v}(w) \in \mathbb{R}^{n+1}_*$, for all $w \in \mathbb{R}^n_0$. Then we have constructed a map $\mathbf{v} : \mathbb{R}^n_0 \to \mathbb{R}^{n+1}_*$. It is an easy exercise to check that $\mathbf{v} : \mathbb{R}^n_0 \to \mathbb{R}^{n+1}_*$ is a bijection and $\mathbf{v}^{-1} = \mathbf{w}|_{\mathbb{R}^{n+1}_*}$. Then, from (18), we have that $\pi(v) = \mathbf{w}(v) + v_{n+1}\mathbf{c}$, for all $v \in \mathbb{R}^{n+1}_{\geqslant 0}$.

If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_{\geqslant 0}$, then we denote by $|k|$ the sum $k_1 + \cdots + k_n$.

Lemma 4.5. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $d = \deg(F)$. Let us suppose that $v = (v_1, \dots, v_n, v_{n+1}) \in \mathbb{R}^{n+1}_{\geqslant 0}$. Then*

$$\ell(v, \Gamma_+(\widetilde{F})) = dv_{n+1} + \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)). \tag{20}$$

Proof. By the construction of $\widetilde{F}$, we see that any element of $\operatorname{supp}(\widetilde{F}) \smallsetminus \{de_{n+1}\}$ can be written as $(k, d - |k|)$, where k belongs to $\operatorname{supp}(F)$. Then we obtain

$$\begin{aligned}
\ell(v, \Gamma_+(\widetilde{F})) &= \min\left\{\langle (k, k_{n+1}), v\rangle : (k, k_{n+1}) \in \Gamma_+(\widetilde{F})\right\} \\
&= \min\left\{\langle (k, k_{n+1}), (\pi(v), v_{n+1})\rangle : (k, k_{n+1}) \in \operatorname{supp}(\widetilde{F}) \cup \{de_{n+1}\}\right\} \\
&= \min\left\{\langle (k, d - |k|), (\mathbf{w}(v) + v_{n+1}\mathbf{c}, v_{n+1})\rangle :\ k \in \operatorname{supp}(F) \cup \{0\}\right\} \\
&= \min\left\{\langle k, \mathbf{w}(v) + v_{n+1}\mathbf{c}\rangle + (d - |k|)v_{n+1} :\ k \in \operatorname{supp}(F) \cup \{0\}\right\} \\
&= \min\left\{\langle k, \mathbf{w}(v)\rangle + v_{n+1}\langle k, \mathbf{c}\rangle + (d - |k|)v_{n+1} :\ k \in \operatorname{supp}(F) \cup \{0\}\right\} \\
&= dv_{n+1} + \min\left\{\langle k, \mathbf{w}(v)\rangle :\ k \in \operatorname{supp}(F) \cup \{0\}\right\} \\
&= dv_{n+1} + \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)).
\end{aligned}$$

□

As a consequence the above lemma, we obtain the following relation between the set of faces of $\widetilde{\Gamma}_+(F)$ and the set of faces of $\Gamma_+(\widetilde{F})$.

Corollary 4.6. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $d = \deg(F)$. Let $v \in \mathbb{R}^n_{>0}$. Then*

$$\Delta(v, \Gamma_+(\widetilde{F})) = \left\{(k, d - |k|) : k \in \Delta(\mathbf{w}(v), \widetilde{\Gamma}_+(F))\right\}. \tag{21}$$

In particular, we have that $de_{n+1} \notin \Delta(v, \Gamma_+(F^*))$ if and only if $0 \notin \Delta(\mathbf{w}(v), \widetilde{\Gamma}_+(F))$.

Proof. Let us fix a vector $v \in \mathbb{R}^{n+1}_{>0}$ and let $(k, k_{n+1}) \in \Gamma_+(\widetilde{F})$. We have that $(k, k_{n+1}) \in \Delta(v, \Gamma_+(\widetilde{F}))$ if and only if $\langle (k, k_{n+1}), v) \rangle = \ell(v, \Gamma_+(\widetilde{F}))$. By Lemma 4.5, this is equivalent to

$$\begin{aligned} 0 &= \langle k, \pi(v) \rangle + (k_{n+1} - d)v_{n+1} - \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)) \\ &= \langle k, \mathbf{w}(v) + v_{n+1}\mathbf{c} \rangle + (k_{n+1} - d)v_{n+1} - \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)) \\ &= \langle k, \mathbf{w}(v) \rangle - \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)) + (k_1 + \cdots + k_{n+1} - d)v_{n+1}. \end{aligned} \tag{22}$$

Let us observe that equality (22) holds if and only if $k_1 + \cdots + k_n + k_{n+1} = d$ and $k \in \Delta(\mathbf{w}(v), \widetilde{\Gamma}_+(F))$, since $k_1 + \cdots + k_n + k_{n+1} \geqslant d$ and $\langle k, \mathbf{w}(v) \rangle - \ell(\mathbf{w}(v), \widetilde{\Gamma}_+(F)) \geqslant 0$, for all $(k, k_{n+1}) \in \Gamma_+(\widetilde{F})$. Hence equality (21) follows. $\square$

Remark 4.7. Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $d = \deg(F)$. Corollary 4.6 shows that the projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ induces a bijection between the set of compact faces of $\Gamma_+(\widetilde{F})$ not containing the point $de_{n+1} \in \mathbb{R}^{n+1}$ and the faces of $\widetilde{\Gamma}_+(F)$ not containing the origin. We also observe that, if Δ is a compact face of $\Gamma_+(\widetilde{F})$ not containing the point $de_{n+1} \in \mathbb{R}^{n+1}$, then

$$(F_i{}^*)_\Delta = \big((F_i)_{\pi(\Delta)}\big)^* \tag{23}$$

for all $i = 1, 2, \ldots, p$, where the superscript * denotes the homogenization of degree d defined in (17).

In the next result we show a relation between the notions of special closure of polynomial maps and the integral closure of ideals.

Proposition 4.8. *Let $F = (F_1, \ldots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $d = \deg(F)$. Let I denote the ideal of $\mathcal{A}(\mathbb{K}^n)$ generated by the component functions of $\widetilde{F}$. Let $h \in \mathbb{K}[x_1, \ldots, x_n]$ such that $\deg(h) \leqslant d$ and let h^* denote the homogenization of h defined in (17). Then $h \in \mathrm{Sp}(F, 1)$ if and only if $h^* \in \overline{I}$.*

Proof. Let us see first the *only if* part. Let us suppose that $h \in \mathrm{Sp}(F, 1)$. Then there exist constants $C, M > 0$ such that

$$|h(x)| \leqslant C\|(F(x), 1)\| \tag{24}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Let us consider the following subsets

$$\begin{aligned} V &= \{(z_1, \ldots, z_{n+1}) \in \mathbb{K}^n : |z_i| < 1, \text{ for all } i = 1, \ldots, n\} \\ V_i &= \{(z_1, \ldots, z_{n+1}) \in V \cap (\mathbb{K}^*)^n : |z_i| \geqslant M|z_{n+1}|\}, \ i = 1, \ldots, n \\ W &= \{(z_1, \ldots, z_{n+1}) \in V \cap (\mathbb{K}^*)^n : |z_i| < M|z_{n+1}|, \text{ for all } i = 1, \ldots, n\}. \end{aligned}$$

We have that $V \cap (\mathbb{K}^*)^n = V_1 \cup \cdots \cup V_n \cup W$. Let us see that there exists some constant $D > 0$ such that $|h^*(z)| \leqslant D\|\widetilde{F}(z)\|$, for all $z \in V$.

Let us fix an index $i \in \{1, \dots, n\}$ and let $(z_1, \dots, z_{n+1}) \in V_i$. Let us consider the point $x = (\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}})$. We observe that

$$\|x\| \geqslant \max_j \frac{|z_j|}{|z_{n+1}|} \geqslant \frac{|z_i|}{|z_{n+1}|} \geqslant M.$$

Therefore, by (24), we obtain

$$\left| h\left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}\right) \right| \leqslant C \left\| \left(F\left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}\right), 1\right) \right\|$$

Multiplying both sides of the above relation by $|z_{n+1}|^d$ we conclude that

$$|h^*(z_1, \dots, z_{n+1})| \leqslant C\|\widetilde{F}(z_1, \dots, z_{n+1})\|.$$

Now, let us suppose that $z \in W$ and that h is written as $h = \sum_k a_k x^k$. Therefore

$$\begin{aligned} |h^*(z_1, \dots, z_{n+1})| &= \left| z_{n+1}^d h\left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}\right) \right| = \left| \sum_k a_k z_1^{k_1} \cdots z_n^{k_n} z_{n+1}^{d-|k|} \right| \\ &\leqslant \sum_k |a_k| M^{k_1} |z_{n+1}|^{k_1} \cdots M^{k_n} |z_{n+1}|^{k_n} z_{n+1}^{d-|k|} \\ &= \left(\sum_k |a_k| M^{|k|} \right) |z_{n+1}|^d \leqslant C' \|(F_1^*(z), \dots, F_p^*(z), z_{n+1}^d)\| = C'\|\widetilde{F}(z)\| \end{aligned}$$

where $C' = \sum_k |a_k| M^{|k|}$.

Let $D = \max\{C, C'\}$. Then we have proven that $|h^*(z)| \leqslant D\|\widetilde{F}(z)\|$, for all $z \in V \cap (\mathbb{K}^*)^n$. Hence, the same inequality holds for all $z \in V$, by the continuity of the functions involved in this inequality.

Let us see the *if* part. Let us suppose that $h^* \in \overline{I}$. Then, there exists a constant $C > 0$ and an open neighbourhood U of 0 in $\mathbb{K}^{n+1}$ such that

$$|h^*(z)| \leqslant C\|\widetilde{F}(z)\| \tag{25}$$

for all $z \in U$. Let $M > 0$ such that $(e^{-\|x\|}x_1, \dots, e^{-\|x\|}x_n, e^{-\|x\|}) \in U$ whenever $x \in \mathbb{K}^n$ and $\|x\| \geqslant M$. In particular, from relation (25), we obtain that

$$|h^*(e^{-\|x\|}x_1, \dots, e^{-\|x\|}x_n, e^{-\|x\|})| \leqslant C\|\widetilde{F}(e^{-\|x\|}x_1, \dots, e^{-\|x\|}x_n, e^{-\|x\|})\|, \tag{26}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$.

Since $\widetilde{F}$ is a homogeneous polynomial map of degree d, we have:

$$\widetilde{F}(e^{-\|x\|}x_1, \dots, e^{-\|x\|}x_n, e^{-\|x\|}) = e^{-\|x\|d}\widetilde{F}(x_1, \dots, x_n, 1)$$

for all $x \in \mathbb{K}^n$. Hence, relation (26) implies that

$$e^{-\|x\|d}|h(x_1, \dots, x_n)| \leqslant Ce^{-\|x\|d}\|\widetilde{F}(x_1, \dots, x_n, 1)\| = Ce^{-\|x\|d}\|(F_1(x), \dots, F_n(x), 1)\| \tag{27}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. In particular, canceling $e^{-\|x\|d}$ in all members of (27) we obtain that $h \in \mathrm{Sp}(F, 1)$.

□

Theorem 4.9. *Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map and let $d = \deg(F)$. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity;*
(b) *$\widetilde{F} = (F_1^*, \dots, F_p^*, x_{n+1}^d) : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ is Newton non-degenerate;*
(c) *$\mathbf{S}'(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.*

Proof. Let $d = \deg(F)$. Let us prove (a) $\Rightarrow$ (b). Let Δ be a compact face of $\Gamma_+(\widetilde{F})$ and let $\mathbf{p} = (0, \dots, 0, d) \in \mathbb{R}^{n+1}$. If $\mathbf{p} \in \Delta$, then $(x_{n+1}^d)_\Delta = x_{n+1}^d$ and thus $(\widetilde{F}_\Delta)^{-1}(0) \subseteq \{x \in \mathbb{K}^{n+1} : x_1 \cdots x_{n+1} = 0\}$. Let us suppose that $\mathbf{p} \notin \Delta$. Then $\pi(\Delta)$ is a face of $\widetilde{\Gamma}_+(F)$ not containing the origin, by Corollary 4.6. Let $y = (y_1, \dots, y_{n+1}) \in (\mathbb{K}^*)^{n+1}$ be a point such that

$$(F_1^*)_\Delta(y) = \cdots = (F_p^*)_\Delta(y) = 0.$$

Let us fix an index $i \in \{1, \dots, p\}$. By (23) and the definition of homogenization of degree d, we have

$$({F_i}^*)_\Delta(y) = \left((F_i)_{\pi(\Delta)}\right)^* (y) = y_{n+1}^d (F_i)_{\pi(\Delta)} \left(\frac{y_1}{y_{n+1}}, \dots, \frac{y_n}{y_{n+1}}\right) = 0.$$

In particular, we find that the point $\frac{1}{y_{n+1}}(y_1, \dots, y_n)$ is solution of the system of equations

$$(F_1)_{\pi(\Delta)}(x) = \cdots = (F_p)_{\pi(\Delta)}(x) = 0,$$

which is a contradiction, since we assume that F is Newton non-degenerate at infinity.

Let us prove (b) $\Rightarrow$ (c). By Lemma 3.4, we have that $\mathbf{S}'(F)$ is contained in $\widetilde{\Gamma}_+(F, 1)$, which is equal to $\widetilde{\Gamma}_+(F)$. Then it suffices to see that $\widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0} \subseteq \mathbf{S}'(F)$.

Let us fix a point $k = (k_1, \dots, k_n) \in \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$. Let I denote the ideal of $\mathcal{A}(\mathbb{K}^{n+1})$ generated by the component functions of $\widetilde{F}$. Since $\widetilde{F}$ is Newton non-degenerate, the integral closure $\overline{I}$ of I is generated by all monomials $x_1^{\nu_1} \cdots x_{n+1}^{\nu_{n+1}}$ of $\mathbb{K}[x_1, \dots, x_{n+1}]$ such that $(\nu_1, \dots, \nu_{n+1})$ belongs to $\Gamma_+(\widetilde{F})$, by the main result of [22] (see also [2]). Since $(k_1, \dots, k_n, d - |k|) \in \Gamma_+(\widetilde{F})$, then $x_1^{k_1} \cdots x_n^{k_n} x_{n+1}^{d-|k|}$ belongs to $\overline{I}$, which is to say that $x^k \in \mathrm{Sp}(F, 1)$, by Proposition 4.8.

Let us prove the implication (c) $\Rightarrow$ (a). Let us suppose that $\mathbf{S}'(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$. Let Δ be a face of $\widetilde{\Gamma}_+(F)$ not containing the origin. In particular, Δ is supported by a vector $w \in \mathbb{Z}^n$ such that $w_0 < 0$ and $\ell(w, \widetilde{\Gamma}_+(F)) < 0$. We remark that $\ell(w, \widetilde{\Gamma}_+(F)) \leqslant \ell(w, \mathrm{supp}(F))$ and equality holds when $0 \notin \Delta(w, \widetilde{\Gamma}_+(F))$. Let us suppose that the system

$$(F_1)_\Delta(x) = \cdots = (F_p)_\Delta(x) = 0$$

has a solution $q = (q_1, \dots, q_n) \in (\mathbb{K}^*)^n$. Let us consider the curve $\varphi : \mathbb{K}^* \to \mathbb{K}^n$ given by $\varphi(t) = (q_1 t^{w_1}, \dots, q_n t^{w_n})$, for all $t \in \mathbb{K}^*$. Since $w_0 < 0$, we have $\lim_{t \to 0} \|\varphi(t)\| = +\infty$. For all $i \in \{1, \dots, n\}$, let $G_i = F_i - (F_i)_\Delta$. Let $r = \ell(w, \widetilde{\Gamma}_+(F))$. Then

$$\begin{aligned} F_i(\varphi(t)) &= (F_i)_\Delta(q_1 t^{w_1}, \dots, q_n t^{w_n}) + G_i(q_1 t^{w_1}, \dots, q_n t^{w_n}) \\ &= t^r (F_i)_\Delta(q_1, \dots, q_n) + G_i(q_1 t^{w_1}, \dots, q_n t^{w_n}) \end{aligned}$$

for all $i = 1, \dots, n$. Given an index $i \in \{1, \dots, n\}$, if $\mathrm{supp}(F_i) \cap \Delta \neq \emptyset$, then we have that $\ell(w, G_i) > \ell(w, (F_i)_\Delta) = \ell(w, F_i) = \ell(w, \widetilde{\Gamma}_+(F))$. Otherwise, $G_i = F_i$ and $\ell(w, F_i) >$

$\ell(w, \widetilde{\Gamma}_+(F))$. In any case $\ell(w, G_i) > r$, for all $i \in \{1, \dots, p\}$. Let $G = (G_1, \dots, G_p)$ and let $s = (s_1, \dots, s_n)$ denote a vertex of the face Δ. Then

$$\begin{aligned}
\lim_{t \to 0} \frac{\|(F(\varphi(t), 1)\|}{|x^s \circ \varphi(t)|} &= \lim_{t \to 0} \frac{\|(t^r F_\Delta(q) + G(\varphi(t)), 1)\|}{|q_1^{s_1} \cdots q_n^{s_n} t^{\langle s, w\rangle}|} \\
&= \lim_{t \to 0} \frac{\|(G(\varphi(t)), 1)\|}{|q_1^{s_1} \cdots q_n^{s_n}||t|^r} \\
&= \frac{1}{|q_1^{s_1} \cdots q_n^{s_n}|} \lim_{t \to 0} \frac{\|(G(\varphi(t)), 1)\|}{|t|^r} = 0
\end{aligned}$$

where the last equality follows from the fact that $r < 0$ and $\ell(w, G_i) > r$, for all $i \in \{1, \dots, p\}$. Hence we have a contradiction, by Lemma 3.3. $\square$

By Example 4.1 (see also Example 4.14), in the above theorem we can not replace the condition $\mathbf{S}'(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$ by $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$ unless we assume that F is convenient, as we will see in the next result.

Corollary 4.10. *Let $F : \mathbb{K}^n \to \mathbb{K}^p$ be a convenient polynomial map. Then the following conditions are equivalent:*

(a) *F is Newton non-degenerate at infinity.*
(b) $\mathbf{S}(F) = \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$.
(c) $\mathrm{Sp}(F) = \{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\}$.

Proof. Let us prove (a) $\Rightarrow$ (b). Let us assume that the map F is Newton non-degenerate at infinity. Since F is convenient, for each $i \in \{1, \dots, n\}$, there exists some positive integer r_i such that $x_i^{r_i} \in \mathrm{Sp}(F, 1)$, by Theorem 4.9. Let $r_0 = \min\{r_1, \dots, r_n\}$. Then, there exists a constant $C > 0$ such that

$$\|x\|^{r_0} \leqslant C \max\{\|F(x)\|, 1\}. \tag{28}$$

for all $\|x\| \gg 1$. Obviously we can assume that $C > 1$. Let us suppose that $x \in \mathbb{K}^n$ is such that $\|x\| > C$ and relation (28) holds. If $\|F(x)\| < 1$, then

$$C^{r_0} < \|x\|^{r_0} \leqslant C \max\{\|F(x)\|, 1\} \leqslant C,$$

which is a contradiction. Then we have that $\|F(x)\| \geqslant 1$, for all $|x| \gg 1$. This implies that $\mathrm{Sp}(F) = \mathrm{Sp}(F, 1)$ and thus the result follows, by Theorem 4.9.

The implication (b) $\Rightarrow$ (a) is an immediate consequence of relation (16) and Theorem 4.9.

Let us prove (b) $\Rightarrow$ (c). By the definition of $\mathbf{S}(F)$ and Lemma 3.4, we have

$$\left\{x^k : k \in \mathbf{S}(F)\right\} \subseteq \mathrm{Sp}(F) \subseteq \left\{h \in \mathbb{K}[x_1, \dots, x_n] : \mathrm{supp}(h) \subseteq \widetilde{\Gamma}_+(F)\right\}. \tag{29}$$

If we assume condition (b), then (c) follows as a direct consequence of (29).

By Lemma 3.4 we have the inclusion $\mathbf{S}(F) \subseteq \widetilde{\Gamma}_+(F) \cap \mathbb{Z}^n_{\geqslant 0}$. Hence, the implication (c) $\Rightarrow$ (b) is immediate. $\square$

Under the conditions of Corollary 4.10, we conclude that $\mathcal{L}_\infty(F) = \min\{\alpha_1, \dots, \alpha_n\}$, where $\alpha_i e_i$ is the point of intersection of the global boundary of $\widetilde{\Gamma}_+(F)$ with the x_i-axis, for all $i = 1, \dots, n$.

Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map. Let us define $\alpha_i = \min\{k_i : k \in \operatorname{supp}(F)\}$, for all $i = 1, \dots, n$, and $\alpha_F = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geqslant 0}$. Given an index $i \in \{1, \dots, n\}$, we observe that $\alpha_i > 0$ if and only if F_j is divisible by $x_i^{\alpha_i}$, for all $j = 1, \dots, p$. Moreover, $\alpha_F = 0$ if F is convenient. We can always express F univocally as $F = x^{\alpha_F} G$, for some polynomial map $G : \mathbb{K}^n \to \mathbb{K}^p$.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$, then we define $\mathbb{Z}^n_{\geqslant \alpha} = \{(k_1, \dots, k_n) \in \mathbb{Z}^n : k_i \geqslant \alpha_i$, for all $i = 1, \dots, n\}$. Given a subset $S \subseteq \mathbb{R}^n$, we also define $\alpha + S = \{\alpha + k : k \in S\}$.

Lemma 4.11. *Let us consider a polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$, let $\alpha = \alpha_F$ and let G be the polynomial map $\mathbb{K}^n \to \mathbb{K}^p$ such that $F = x^\alpha G$. Then*

$$\mathbf{S}(F) = \alpha + \mathbf{S}(G). \tag{30}$$

Proof. If $k \in \mathbf{S}(G)$, then $|x^k| \leqslant C\|G(x)\|$, for some constant $C > 0$ and all $|x\| \gg 1$. If we multiply both sides of this inequality by $|x^\alpha|$, then we immediately obtain that $x^{\alpha+k} \in \operatorname{Sp}(F)$ and hence the inclusion $(\subseteq)$ of (30) follows.

Let us prove the inclusion $(\supseteq)$. If $k = (k_1, \dots, k_n) \in \mathbf{S}(F)$, we will prove first that $k_i \geqslant \alpha_i$, for all $i = 1, \dots, n$. Let $M > 0$ and $C' > 0$ such that

$$|x^k| \leqslant C'\|F(x)\| \tag{31}$$

for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. Let us fix an index $i \in \{1, \dots, n\}$. If $e_1, \dots, e_n$ denote the canonical basis in $\mathbb{K}^n$, then we consider the curve $\gamma : \mathbb{K} \to \mathbb{K}^n$ by $\gamma(t) = M\sum_{j \neq i} e_j + te_i$, for all $t \in \mathbb{K}$. We observe that $\|\gamma(t)\| \geqslant M$, for all $t \in \mathbb{K}$. Then, composing both sides of (31) by $\gamma(t)$ we obtain that

$$|M^{|k|-k_i} t^{k_i}| \leqslant C'|M^{|\alpha|-\alpha_i} t^{\alpha_i}|\|G(\gamma(t))\|$$

for all $t \in \mathbb{K}$. This is equivalent to saying that

$$|t|^{k_i - \alpha_i} \leqslant C'|M^{|\alpha|-\alpha_i-|k|+k_i}|\|G(\gamma(t))\|$$

for all $t \in \mathbb{K}$. Since the composition $G(\gamma(t))$ is a polynomial in t, taking limits when $t \to 0$ in both sides of the above inequality, we deduce that $k_i - \alpha_i \geqslant 0$, that is, $k_i \geqslant \alpha_i$. Hence $\mathbf{S}(F) \subseteq \mathbb{Z}^n_\alpha$.

If $x \in \mathbb{K}^n$, verifies that $\|x\| \geqslant M$ and $x_1 \cdots x_n \neq 0$, then we can divide each member of (31) by $|x^\alpha|$ and we obtain that

$$|x^{k-\alpha}| \leqslant C'\|G(x)\|. \tag{32}$$

Since both sides of (32) are continuous functions of x, we obtain that relation (32) holds for all $x \in \mathbb{K}^n$ such that $\|x\| \geqslant M$. This means that $k - \alpha \in \mathbf{S}(G)$. Then equality (30) is proven. □

To end the article, in Corollary 4.13 we show a consequence of the previous lemma that is particularly useful when $n = 2$. First we will introduce a new definition.

Definition 4.12. Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial. We denote by $\mathrm{C}(F)$ the convex hull of $\mathrm{supp}(F)$. Obviously we have $\mathrm{C}(F) \subseteq \widetilde{\Gamma}_+(F)$. We say that F is *globally non-degenerate* if and only if $(F_\Delta)^{-1}(0) \subseteq \{x \in \mathbb{K}^n : x_1 \cdots x_n = 0\}$, for all face Δ of $\mathrm{C}(F)$ supported by some $w \in \mathbb{R}^n$ such that $w_0 < 0$.

Let $F = (F_1, \dots, F_p) : \mathbb{K}^n \to \mathbb{K}^p$ be a polynomial map, then from the above definition, we observe that if F is globally non-degenerate then F is Newton non-degenerate at infinity. Clearly the converse is not true, but both conditions are equivalent if we assume that F is convenient (see Lemma 2.3).

Corollary 4.13. *Let us consider a polynomial map $F : \mathbb{K}^n \to \mathbb{K}^p$, let $\alpha = \alpha_F$ and let $G : \mathbb{K}^n \to \mathbb{K}^p$ be the polynomial map such that $F = x^\alpha G$. Let us suppose that G is convenient. Then the following conditions are equivalent:*

(a) *F is globally non-degenerate;*
(b) *G is Newton non-degenerate at infinity;*
(c) $\mathbf{S}(F) = \alpha_F + \left(\widetilde{\Gamma}_+(G) \cap \mathbb{Z}^n_{\geqslant 0}\right)$.

Proof. Since G is convenient, then $\ell(w, \mathrm{supp}(G)) = \ell(w, \widetilde{\Gamma}_+(G))$, for all $w \in \mathbb{R}^n_0$. Therefore $\ell(w, F) = \langle w, \alpha \rangle + \ell(w, \widetilde{\Gamma}_+(G))$, for all $w \in \mathbb{R}^n_0$. This shows that if Δ is a face of $\widetilde{\Gamma}_+(G)$ such that $0 \notin \Delta$ and we write $\Delta = \Delta(w, \widetilde{\Gamma}_+(G))$, for some $w \in \mathbb{R}^n_0$, then

$$F_{\Delta(w, \widetilde{\Gamma}_+(F))} = x^\alpha G_\Delta. \tag{33}$$

By Lemma 2.3, the hypothesis of G being convenient implies that the set of faces of $\widetilde{\Gamma}_+(G)$ not passing through the origin is equal to $\{\Delta(w, \widetilde{\Gamma}_+(G)) : w \in \mathbb{R}^n_0\}$. In particular (33) shows the equivalence between (a) and (b).

By Corollary 4.10, we have that (b) is equivalent to saying that $\mathbf{S}(G) = \widetilde{\Gamma}_+(G) \cap \mathbb{Z}^n_{\geqslant 0}$. Using this and Lemma 4.11, the equivalence between (b) and (c) follows. $\square$

Example 4.14. Let us consider the map $F : \mathbb{C}^2 \to \mathbb{C}^2$ given by $F(x, y) = (xy^3, x^3y^2)$. Since $(xy^3, x^3y^2) = xy^2(y, x^2)$, we have $\alpha_F = (1, 2)$ and $G_F = (y, x^2)$. Therefore, by Corollary 4.13, we obtain that $\mathbf{S}(F) = \{x^{k_1}y^{k_2} : (k_1, k_2) \in \widetilde{\Gamma}_+(F), k_1 \geqslant 1, k_2 \geqslant 2\}$, that is $\mathbf{S}(F) = \{(1, 2), (1, 3), (3, 2)\}$. In particular, xy is a monomial whose support belongs to $\widetilde{\Gamma}_+(F) \smallsetminus \mathbf{S}(F)$. We can also check that $xy \notin \mathrm{Sp}(F)$ explicitly: if we consider the curve $\varphi : \mathbb{C}^* \to \mathbb{C}^2$ given by $\varphi(t) = (t^{-1}, t^3)$, then we observe that

$$\lim_{t \to 0} \frac{\|F(\varphi(t))\|}{|h(\varphi(t))|} = \lim_{t \to 0} \frac{\|(t^8, t^3)\|}{|t^2|} = 0.$$

However we have that $xy \in \mathbf{S}'(F)$, by Theorem 4.9.

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