# DIFFERENTIABILITY OF $L^p$ OF A VECTOR MEASURE AND APPLICATIONS TO THE BISHOP-PHELPS-BOLLOBÁS PROPERTY

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Abstract. We study the properties of Gâteaux, Fréchet, uniformly Fréchet and uniformly Gâteaux smoothness of the space $L^p(m)$ of scalar $p$-integrable functions with respect to a positive vector measure $m$ with values in a Banach lattice. Applications in the setting of the Bishop-Phelps-Bollobás property (both for operators and bilinear forms) are also given.

## 1. Introduction and preliminaries

Convexity and smoothness are Banach space properties that involve metric inequalities and geometric concepts. When the attention is centered in Banach lattices (and more concretely, in Banach function spaces), better characterizations can be obtained, and new properties can be added to the study. The main result on smoothness of Banach function spaces is known since the eighties, when Kutzarova and Troyanski proved that any order continuous Banach function space with a weak unit admits an equivalent lattice uniformly Gâteaux smooth norm (see [21, Theorem 3.5 and Corollary 3.7]). Recently, some results providing new powerful tools have been introduced in the study of this topic, both from the general Banach space theory point of view and from the setting of the spaces of integrable functions (see [4]).

In this work, we are interested in obtaining more information about smoothness properties of Banach function spaces. In particular, we will study the case of the order continuous spaces $L^p(m)$, consisting of scalar functions which are $p$-integrable with respect to a (countably additive) vector measure taking values in a Banach lattice. Although this class is interesting by itself, it must be said that our aim is to use the results obtained about this in order to find information on general Banach function spaces, *via* the well-known representation theorem that establishes that for a given $1 < p < \infty$ each order continuous $p$-convex Banach function space $E(\mu)$ over a finite measure can be represented as an $L^p(m)$ space for a positive vector measure $m$ (see [23, Proposition 3.30]; sometimes a renorming is needed). In this direction, it has been studied in [20] when the convexity properties of a Banach function space $E(\mu)$ are preserved or even improved when the $p$-convexification $E(\mu)_{[1/p]}$ of $E(\mu)$ is considered. As application of our results, we obtain some more examples of couples of Banach spaces with the Bishop-Phelps-Bollobás property for operators and bilinear forms.

Let us recall some definitions and terminology. Given a real Banach space $(X, \|\cdot\|)$, we denote by $B_X$ and $S_X$ the unit ball and the unit sphere of $X$, respectively, and by $X^*$ its topological dual. The space $X$, or its norm $\|\cdot\|$, is said to be *Gâteaux smooth*, or simply

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*smooth*, if $\|\cdot\|$ is Gâteaux differentiable at every $x \in S_X$, that is, if there exists a (necessarily unique) functional $\|\cdot\|'(x) \in S_{X^*}$ such that

$$\lim_{t\to 0} \frac{\|x+th\|-1}{t} = \|\cdot\|'(x)(h), \quad \text{for each } \ h \in X. \tag{1}$$

The space $X$ (or its norm $\|\cdot\|$) is called *Fréchet smooth* if it is Gâteaux smooth and for every $x \in S_X$, the limit (1) is uniform with respect to $h \in B_X$, that is, if

$$\lim_{t\to 0} \sup \left\{ \left| \frac{\|x+th\|-1}{t} - \|\cdot\|'(x)(h) \right| : \ h \in B_X \right\} = 0.$$

If the space $X$ is smooth and for every $h \in S_X$, the limit in (1) is uniform with respect to $x \in S_X$, that is, if

$$\lim_{t\to 0} \sup \left\{ \left| \frac{\|x+th\|-1}{t} - \|\cdot\|'(x)(h) \right| : \ x \in S_X \right\} = 0,$$

then we say that $X$ (or its norm) is *uniformly Gâteaux smooth*. Finally, we say that $X$ is *uniformly Fréchet smooth* if it is Gâteaux smooth and the limit in (1) is uniform with respect to $(x,h) \in S_X \times S_X$.

It is clear that uniformly Fréchet smoothness $\Rightarrow$ uniformly Gâteaux smoothness $\Rightarrow$ smoothness and that uniformly Fréchet smoothness $\Rightarrow$ Fréchet smoothness $\Rightarrow$ smoothness, but the converse implications do not hold true in general, even up to renorming. It is well-known (see e.g. [14, Theorem 13.25]) that if the space $X$ (respectively $X^*$) is weakly compactly generated, then $X$ admits an equivalent Gâteaux smooth norm (respectively Fréchet smooth norm). Another classical result (see e.g. [14, Theorem 9.14]) establishes that Banach spaces having an equivalent uniformly Fréchet smooth norm are exactly super-reflexive spaces. The class of uniformly Gâteaux smooth renormable spaces was characterized in [13] (see also [12]), where it was shown that $X$ admits an equivalent uniformly Gâteaux smooth norm if, and only if, $X$ is a subspace of a Hilbert-generated space (recall that a Banach space $Y$ is said to be *Hilbert-generated* if there exist a Hilbert space $H$ and a bounded linear operator $T : H \to Y$ with dense range). For more information on these properties, and their interplay with Banach space geometry, we refer to the monographs [10] and [14].

Let $(\Omega, \Sigma, \mu)$ be a finite measure space. A Banach space $E(\mu)$ is said to be a *Banach function space* over $(\Omega, \mu)$ if $E(\mu)$ is a linear subspace of $L^1(\mu)$ such that:

(i) if $f \in L^0(\mu)$ and $|f| \le |g|$ $\mu$-a.e. for some $g \in E(\mu)$, then $f \in E(\mu)$ and $\|f\|_{E(\mu)} \le \|g\|_{E(\mu)}$, and

(ii) for every set $A \in \Sigma$ the characteristic function of $A$, $\chi_A$, belongs to $E(\mu)$.

In this case, $E(\mu)$ is a Banach lattice when endowed with the $\mu$-a.e. order.

Now, let $X$ be a Banach space and $m : \Sigma \to X$ be a (countably additive) vector measure. For each $x^*$ in $X^*$ we write $\langle m, x^*\rangle$ to denote the scalar measure given by the formula

$$\langle m, x^*\rangle(A) := \langle m(A), x^*\rangle, \quad A \in \Sigma.$$

A *Rybakov control measure for* $m$ is a measure of the form $\mu = |\langle m, x_0^*\rangle|$ with $x_0^* \in B_{X^*}$ (where $|\langle m, x_0^*\rangle|$ stands for the variation of the measure $\langle m, x_0^*\rangle$, which is necessarily finite since $m$ is supposed to be countably additive) satisfying that $\mu(A) = 0$ if, and only if, $\|m\|(A) = 0$ (here $\|m\|$ denotes the semivariation of $m$). A Rybakov control measure for $m$ always exists for a suitable $x_0^* \in B_{X^*}$ (see [11, p. 268]). A $\Sigma$-measurable function $f : \Omega \to \mathbb{R}$ is said to be *integrable with respect to* $m$ if:

(i) for each $x^* \in X^*$ , $f$ is integrable with respect to the scalar measure $\langle m, x^*\rangle$, and
(ii) for each $A \in \Sigma$ there exists a (unique) vector $\int_A f \mathrm{d}m \in X$ such that

$$\int_A f \mathrm{d}\langle m, x^*\rangle = \left\langle \int_A f \mathrm{d}m, x^* \right\rangle, \quad \text{for all} \quad x^* \in X^*.$$

For $1 \leq p < \infty$, let $1 < p' \leq \infty$ be the conjugate exponent given by $1/p + 1/p' = 1$. The space $L^p(m)$ is the Banach function space over $(\Omega, \Sigma, \mu)$, where $\mu$ is a Rybakov control measure for $m$, consisting of those (equivalence classes of) functions $f : \Omega \to \mathbb{R}$ such that $|f|^p$ is integrable with respect to $m$, endowed with the (lattice) norm

$$\|f\|_{L^p(m)} := \left\||f|^p\right\|_{L^1(m)}^{1/p} = \sup\left\{ \left( \int_\Omega |f|^p \mathrm{d}|\langle m, x^*\rangle| \right)^{1/p} : \ x^* \in B_{X^*} \right\}.$$

For a given $1 < p < \infty$ the most important properties and results of the well-known Lebesgue spaces, $L^p(\mu)$, are also satisfied in our spaces $L^p(m)$. This is, for instance, the case of *Hölder's inequality* (see [23, p. 133])

$$\|fg\|_{L^1(m)} \leq \|f\|_{L^p(m)} \|g\|_{L^{p'}(m)}, \quad f \in L^p(m), \ g \in L^{p'}(m),$$

and the *Lebesgue dominated convergence theorem* (see [23, Theorem 3.7]).

In the sequel, we shall always assume that $X$ is a Banach lattice and that the (countably additive) vector measure $m : \Sigma \to X$ is positive, that is, the range of $m$ is included in the positive cone of $X$. In this case, the norm of $L^p(m)$ can be computed by the easier formula (see [23, Lemma 3.13]),

$$\|f\|_{L^p(m)} = \left\| \int_\Omega |f|^p \mathrm{d}m \right\|^{1/p}, \quad \text{for each} \quad f \in L^p(m).$$

The reader can find the unexplained information on vector measures and integration of scalar functions with respect to such measures in the monographs [11] and [23, Chapter 3], respectively.

Observe that, according to the aforementioned results in [21], the space $L^p(m)$, has an equivalent uniformly Gâteaux smooth lattice norm whenever $p \geq 1$ (and therefore, by the result in [13], this space is a subspace of a Hilbert-generated space). In [16, Theorem 3.1], it is shown that $L^p(m)^*$ is weakly compactly generated for $p > 1$. Consequently, $L^p(m)$ admits an equivalent Fréchet smooth norm as well. However, as it is noticed in the recent paper [4, Example 1], the canonical norm of $L^p(m)$ is not smooth in general. There, it is also shown that if $m : \Sigma \to X$ is a positive vector measure and $p > 1$, then under a metric assumption on the unit ball of $L^p(m)^*$, the (Gâteaux) smoothness of the space $X$ is transferred directly to $L^p(m)$, when the canonical norm of this space is considered. Although the requirement on the ball of $L^p(m)^*$ holds very often, the general question about the necessity of such assumption is posed in that paper (see [4, Problem (Q1)]). In this work, we give a positive answer to this question. More precisely, we show that if $X$ is Gâteaux smooth and $p > 1$, then the space $L^p(m)$, endowed with its canonical norm, is Gâteaux smooth too. We also obtain similar results for the properties of Fréchet, uniformly Gâteaux and uniformly Fréchet smoothness. The uniformly Gâteaux smooth case is restricted to the order continuous Banach function space over a finite measure setting. As a consequence of these results we deduce that if $L^1(m)$ has any of the above properties of smoothness, then for every $p > 1$ the space $L^p(m)$ shares the same property, answering in particular [4, Problem (Q2)]. In the final part of the work we provide new examples of pairs of spaces having the Bishop-Phelps-Bollobás property for operators (and also for bilinear forms). In particular, we extend in the setting of vector valued measures a result that ensures that, if $1 < p, q < \infty$ and $\mu$ is a scalar measure, then the couple $(L^p(\mu), L^q(\nu))$ satisfies the Bishop-Phelps-Bollobás property (in both cases, linear and bilinear).

## 2. Gâteaux and Fréchet smoothness of $L^p(m)$

The main result of this section reads as follows.

**Theorem 2.1.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < p < \infty$. If the norm of $X$ is smooth (Fréchet smooth), then so is the canonical norm on $L^p(m)$.*

As we mentioned in Introduction, the non parenthetic part of this theorem solves affirmatively Problem (Q1) in [4]. An important ingredient in the proof of this result is the following proposition, which will be also useful in the next section.

**Proposition 2.2.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ a positive vector measure and $1 < p < \infty$. Then the mapping $\varphi : L^p(m) \to X$ defined by the formula*

$$\varphi(f) = \int_\Omega |f|^p \,\mathrm{d}m, \quad f \in L^p(m),$$

*satisfies the following properties:*

(i) *$\varphi$ is Gâteaux differentiable on all of $L^p(m)$ and for every $f, h \in L^p(m)$ we have*

$$\varphi'(f)(h) = p \int_\Omega \mathrm{sign}(f)|f|^{p-1} h \,\mathrm{d}m.$$

(ii) *There exist $C_p > 0$ and $s > 0$ such that*

$$\|\varphi'(f)(h) - \varphi'(g)(h)\| \le C_p \|f - g\|^s_{L^p(m)} \|h\|_{L^p(m)},$$

*whenever $f, g \in B_{L^p(m)}$ and $h \in L^p(m)$. In particular $\varphi$ is Fréchet differentiable. Moreover we can take $s = 1$ if $p \ge 2$ and $s = p - 1$ otherwise.*

(iii) *$\varphi$ is $2p$-Lipschitzian on the unit ball of $L^p(m)$.*

In the proof of this proposition, we shall use the following elementary inequalities (for convenience we write $0^0 = 0$).

**Lemma 2.3.** *For each $a, b \in [0, \infty)$ we have*

$$(a+b)^r \le a^r + b^r \le 2^{1/(1-r)}(a+b)^r, \qquad 0 \le r < 1, \tag{2}$$

$$|a^r - b^r| \le |a - b|^r, \qquad 0 \le r \le 1, \tag{3}$$

$$a^r + b^r \le (a+b)^r \le 2^{r-1}(a^r + b^r), \qquad r \ge 1, \tag{4}$$

$$|a^r - b^r| \le r(a^{r-1} + b^{r-1})|a - b|, \qquad r \ge 1. \tag{5}$$

*Proof of Proposition 2.2.* (i) Let us pick $f, h \in L^p(m)$, and let $(t_n)$ be any sequence of non-zero real numbers such that $t_n \to 0$. Since the function $|\cdot|^p$ is differentible on all of $\mathbb{R}$ we get

$$\lim_n \frac{|f(\omega) + t_n h(\omega)|^p - |f(\omega)|^p}{t_n} = p \,\mathrm{sign}(f(\omega))|f(\omega)|^{p-1} h(\omega),$$

for all $\omega \in \Omega$. On the other hand, for every $n \in \mathbb{N}$, formula (5) yields

$$\left| \frac{|f(\omega)+t_n h(\omega)|^p - |f(\omega)|^p}{t_n} \right| \leq p\big(|f(\omega)|^{p-1} + |t_n|^{p-1}|h(\omega)|^{p-1}\big)|h(\omega)|,$$

for each $\omega \in \Omega$. As $f, h \in L^p(m)$ we have $|f(\omega)|^{p-1} + |t_n|^{p-1}|h(\omega)|^{p-1} \in L^{p'}(m)$, and Hölder's inequality guarantees the integrability of the last function in the previous inequality. Thus, according to Lebesgue's dominated convergence theorem, it follows that

$$\lim_n \int_\Omega \frac{|f+t_n h|^p - |f|^p}{t_n} \mathrm{d}m = p \int_\Omega \operatorname{sign}(f)|f|^{p-1} h \, \mathrm{d}m.$$

Since this equality holds for every $f, h \in L^p(m)$ and every sequence $t_n \to 0$ with $t_n \neq 0$ we deduce that $\varphi$ is Gâteaux differentiable at every $f \in L^p(m)$ (note that the directional derivatives are linear and bounded in $h$ since $\varphi$ is continuous and convex on $f$), and

$$\varphi'(f)(h) = p \int_\Omega \operatorname{sign}(f)|f|^{p-1} h \, \mathrm{d}m, \quad \text{for all} \quad h \in L^p(m).$$

(ii) If $f, g \in L^p(m)$, we denote by $A_0$ and $A_1$ the complementary measurable sets given by

$$A_0 := \big\{\omega \in \Omega : \operatorname{sign}(f(\omega)) = \operatorname{sign}(g(\omega))\big\} \quad \text{and} \quad A_1 := \big\{\omega \in \Omega : \operatorname{sign}(f(\omega)) \neq \operatorname{sign}(g(\omega))\big\}.$$

Then, for every $h \in L^p(m)$ we have

$$\begin{aligned} \|\varphi'(f)(h) - \varphi'(g)(h)\| &\leq \left\| \int_{A_0} \big(\operatorname{sign}(f)|f|^{p-1} - \operatorname{sign}(g)|g|^{p-1}\big) h \, \mathrm{d}m \right\| \\ &+ \left\| \int_{A_1} \big(\operatorname{sign}(f)|f|^{p-1} - \operatorname{sign}(g)|g|^{p-1}\big) h \, \mathrm{d}m \right\|. \end{aligned}$$

We shall estimate the two expressions of the right-hand side of this inequality, that we denote by $I_0$ and $I_1$, respectively. In order to do it, we consider three cases: $p > 2$, $1 < p < 2$ and $p = 2$. From now on, we assume that $f, g, h \in B_{L^p(m)}$.

*Case 1:* $p > 2$. We shall prove that the result holds with $C_p := 2^{p-2}\big((p-1)K_p^{1/p'} + 1\big)$, where $K_p$:=1 if $p \geq 3$ and $K_p$:=$2^{1/(3-p)}$ otherwise.

$(I_0)$ Applying inequality (5) with $r{:=}p{-}1>1$, Hölder's inequality, (2) and (4) and, again, Hölder's inequality with $s:=p/p'$ and $s'=(p-1)/(p-2)$, we obtain

$$\begin{aligned}
I_0 &\leq \left\| \int_{A_0} \big||f|^{p-1}-|g|^{p-1}\big||h|\,\mathrm{d}m \right\| \\
&\leq (p-1)\left\| \int_{A_0} \big(|f|^{p-2}+|g|^{p-2}\big)\cdot\big|f-g\big||h|\,\mathrm{d}m \right\| \\
&\leq (p-1)\left\| \int_{A_0} \big(|f|^{p-2}+|g|^{p-2}\big)^{p'}|h|^{p'}\,\mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_{A_0} \big|f-g\big|^{p}\,\mathrm{d}m \right\|^{1/p} \\
&\leq (p-1)\left\| \int_{A_0} K_p\big(|f|+|g|\big)^{p'(p-2)}|h|^{p'}\,\mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_{A_0} \big|f-g\big|^{p}\,\mathrm{d}m \right\|^{1/p} \\
&\leq (p-1)K_p^{1/p'}\left\| \int_{A_0} \big(|f|+|g|\big)^{p}\,\mathrm{d}m \right\|^{(p-2)/p} \cdot \left\| \int_{A_0} |h|^{p}\,\mathrm{d}m \right\|^{1/p} \cdot \|f-g\|_{L^p(m)} \\
&\leq (p-1)K_p^{1/p'}\big(\|f\|_{L^p(m)}+\|g\|_{L^p(m)}\big)^{p-2}\cdot\|f-g\|_{L^p(m)} \\
&\leq (p-1)K_p^{1/p'}2^{p-2}\cdot\|f-g\|_{L^p(m)} = M_p\|f-g\|_{L^p(m)},
\end{aligned}$$

where $M_p:=(p-1)K_p^{1/p'}2^{p-2}>0$.

$(I_1)$ Using (4) with $r:=p-1>1$ and Hölder's inequality we get

$$\begin{aligned}
I_1 &\leq \left\| \int_{A_1} \big(|f|+|g|\big)^{p-1}|h|\,\mathrm{d}m \right\| \leq \left\| \int_{A_1} \big(|f|+|g|\big)^{p'(p-1)}\,\mathrm{d}m \right\|^{1/p'} \cdot \|h\|_{L^p(m)} \\
&\leq \left\| \int_{A_1} \big(|f|+|g|\big)^{p}\,\mathrm{d}m \right\|^{1/p'} = \|(f-g)\chi_{A_1}\|_{L^p(m)}^{p/p'} \leq \|f-g\|_{L^p(m)}^{p/p'}.
\end{aligned}$$

Adding up the last two inequalities it follows that

$$\begin{aligned}
\|\varphi'(f)(h)-\varphi'(g)(h)\| &\leq M_p\|f-g\|_{L^p(m)}+\|f-g\|_{L^p(m)}^{p/p'} \\
&\leq \Big(M_p+\|f-g\|_{L^p(m)}^{p-2}\Big)\,\|f-g\|_{L^p(m)} \\
&\leq \big(M_p+2^{p-2}\big)\,\|f-g\|_{L^p(m)} = C_p\|f-g\|_{L^p(m)}.
\end{aligned}$$

*Case 2:* $1<p<2$. We shall prove that in this case the inequality works with the constant $C_p:=2^{1/(2-p)}+1$.

$(I_0)$ Because of (3) with $0<r:=p-1<1$ and Hölder's inequality we have

$$\begin{aligned}
I_0 &\leq \left\| \int_{A_0} \big||f|^{p-1}-|g|^{p-1}\big||h|\,\mathrm{d}m \right\| \leq \left\| \int_{A_0} \big||f|-|g|\big|^{p-1}|h|\,\mathrm{d}m \right\| \\
&\leq \left\| \int_{A_0} \big||f|-|g|\big|^{(p-1)p'}\,\mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_{A_0} |h|^{p}\,\mathrm{d}m \right\|^{1/p} \leq \|f-g\|_{L^p(m)}^{p/p'} = \|f-g\|_{L^p(m)}^{p-1}.
\end{aligned}$$

$(I_1)$ Applying (2) with $0 < r := p-1 < 1$ and Hölder's inequality it follows that

$$\begin{aligned} I_1 &\leq \left\| \int_{A_1} \left(|f|^{p-1} + |g|^{p-1}\right)|h| \, \mathrm{d}m \right\| \leq 2^{1/(2-p)} \left\| \int_{A_1} (|f| + |g|)^{p-1} |h| \, \mathrm{d}m \right\| \\ &\leq 2^{1/(2-p)} \left\| \int_{A_1} (|f| + |g|)^{(p-1)p'} \, \mathrm{d}m \right\|^{1/p'} \left\| \int_{A_1} |h|^p \, \mathrm{d}m \right\|^{1/p} \\ &= 2^{1/(2-p)} \|(f-g)\chi_{A_1}\|_{L^p(m)}^{p/p'} \cdot \|h\chi_{A_1}\|_{L^p(m)} \leq 2^{1/(2-p)} \|f-g\|_{L^p(m)}^{p/p'} \\ &= 2^{1/(2-p)} \|f-g\|_{L^p(m)}^{p-1}. \end{aligned}$$

Combining the last two inequalities we get

$$\|\varphi'(f)(h) - \varphi'(g)(h)\| \leq C_p \|f-g\|_{L^p(m)}^{p-1}.$$

*Case 3:* $p = 2$. Following the lines in the previous cases, spliting the integral into over the sets $A_0$ and $A_1$ and applying Hölder's inequality (with $p = p' := 2$) it follows that

$$\|\varphi'(f)(h) - \varphi'(g)(h)\| \leq 2\|f-g\|_{L^2(m)}.$$

(iii) If $p > 1$ and $f, g \in B_{L^p(m)}$, then using (5) with $r{:=}p > 1$, (4) with $r{:=}p'$, and Hölder's inequality we get

$$\begin{aligned} \|\varphi(f) - \varphi(g)\| &\leq \left\| \int_\Omega \big| |f|^p - |g|^p \big| \, \mathrm{d}m \right\| \leq \left\| \int_\Omega p\left(|f|^{p-1} + |g|^{p-1}\right) \cdot |f-g| \, \mathrm{d}m \right\| \\ &\leq p \left\| \int_\Omega \left(|f|^{p-1} + |g|^{p-1}\right)^{p'} \mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_\Omega |f-g|^p \, \mathrm{d}m \right\|^{1/p} \\ &\leq p \left\| \int_\Omega 2^{p'-1}\left(|f|^{p'(p-1)} + |g|^{p'(p-1)}\right) \mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_\Omega |f-g|^p \, \mathrm{d}m \right\|^{1/p} \\ &= p2^{1/p} \left\| \int_\Omega \left(|f|^p + |g|^p\right) \mathrm{d}m \right\|^{1/p'} \cdot \left\| \int_\Omega |f-g|^p \, \mathrm{d}m \right\|^{1/p} \\ &\leq 2p\|f-g\|_{L^p(m)}. \end{aligned}$$

So $\varphi$ is $2p$-Lipschitzian on the unit ball of $L^p(m)$ and the result is proved. □

We are ready to prove Theorem 2.1.

*Proof of Theorem 2.1.* Let $\Phi : L^p(m) \to [0, +\infty)$ be the function defined by the formula

$$\Phi(f) = \|\varphi(f)\|, \quad f \in L^p(m).$$

At first, we shall show that if $X$ is smooth, then $\Phi$ is Gâteaux differentiable at every $f \in S_{L^p(m)}$. We already know that $\varphi$ is Gâteaux differentiable on $L^p(m)$. Pick $h \in L^p(m)$ and $t \in \mathbb{R}$ with $t \neq 0$. Using the triangle inequality we get

$$\begin{aligned} \big|\Phi(f+th) - \Phi(f) - (\|\varphi(f) + t\varphi'(f)(h)\| - \|\varphi(f)\|)\big| &= \big|\|\varphi(f+th)\| - \|\varphi(f) + t\varphi'(f)(h)\|\big| \\ &\leq \|\varphi(f+th) - \varphi(f) - t\varphi'(f)(h)\|. \end{aligned}$$

Thus,

$$\left| \frac{\Phi(f+th) - \Phi(f)}{t} - \frac{\|\varphi(f) + t\varphi'(f)(h)\| - \|\varphi(f)\|}{t} \right| \leq \left\| \frac{\varphi(f+th) - \varphi(f)}{t} - \varphi'(f)(h) \right\|.$$

Since, by hypothesis, the norm $\|\cdot\|$ on $X$ is Gâteaux smooth, we have

$$\frac{\|\varphi(f)+t\varphi'(f)(h)\|-\|\varphi(f)\|}{t} \longrightarrow \langle \|\cdot\|'(\varphi(f),\varphi'(f)(h)\rangle \text{ as } t\to 0.$$

Consequently,

$$\frac{\Phi(f+th)-\Phi(f)}{t} \longrightarrow \langle \|\cdot\|'(\varphi(f),\varphi'(f)(h)\rangle \text{ as } t\to 0.$$

Therefore since the directional derivative is linear and bounded in $h$ (because, for instance, $\Phi$ is continuous and convex around $f$) then it is Gâteaux differentiable at $f$. Now, bearing in mind that $\|\cdot\|_{L^p(m)}=\alpha\circ\Phi$, where $\alpha(t)=|t|^{1/p}$, it easily follows that $\|\cdot\|_{L^p(m)}$ is Gâteaux differentiable on $L^p(m)\setminus\{0\}$, and

$$\|\cdot\|'_{L^p(m)}(f)(h)=p^{-1}\Phi'(f)(h)=p^{-1}\left\langle \|\cdot\|'(\varphi(f)),\varphi'(f)(h)\right\rangle. \tag{6}$$

The statement about Fréchet smoothness is now easy. Indeed, by Proposition 2.2 (ii) the mapping $\varphi$ is Fréchet differentiable at $f$. Thus, if the norm on $X$ is Fréchet differentiable, according to the chain rule [17, Theorem 69] it follows that the function $\|\cdot\|_{L^p(m)}=(\varphi\circ\|\cdot\|)^{1/p}$ is Fréchet differentiable at every $f\in L^p(m)\setminus\{0\}$. □

**Remark 2.4.** According to Šmulyan Lemma (see e.g. [14, Corollary 7.22]) it follows that if $X$ is Gâteaux smooth then for a given $f\in S_{L^p(m)}$ there exists a unique norm one functional $x_f^*\in X^*$ such that $x_f^*(\varphi(f))=1$, and moreover, $x_f^*=\|\cdot\|'(\varphi(f))$. Therefore, from (6) it follows that if $f,h\in S_{L^p(m)}$, then

$$\|\cdot\|'_{L^p(m)}(f)(h)=p^{-1}\langle x_f^*,\varphi'(f)(h)\rangle=\int_\Omega \operatorname{sign}(f)|f|^{p-1}h\,\mathrm{d}\langle m,x_f^*\rangle.$$

As a consequence of Theorem 2.1, we obtain a positive answer to the second question posed at the end of [4]. Recall that for a given vector measure (not necessarily positive) $m:\Sigma\to X$, the formula

$$m_0(A)=\chi_A,\quad A\in\Sigma,$$

defines another (countably additive) positive vector measure, $m_0:\Sigma\to L^1(m)$. It is well-known (see e.g. [23, Proposition 3.28 (i), Proposition 3.30]) that $L^p(m)$ is isometrically isomorphic to $L^p(m_0)$. Therefore, we obtain the following result, which improves [4, Corollary 3.1] since in this case we do not need the assumption that $L^1(m)$ has the Fatou property.

**Corollary 2.5.** *Let $X$ be a Banach space, $m:\Sigma\to X$ be a vector measure and $1<p<\infty$. If the space $L^1(m)$ is Gâteaux smooth (Fréchet smooth), then so is $L^p(m)$.*

Notice that, there exist *non-trivial* examples of spaces $L^1(m)$ with good properties of smoothness. For instance, consider, for each $1<r<\infty$, the positive vector measure $m_r:\Sigma\to L^r([0,1])$ given by

$$m_r(A)=\chi_A,\quad A\in\Sigma.$$

Since $L^1(m_r)$ is isometrically isomorphic to $L^r([0,1])$, and this space is Fréchet smooth, so is $L^1(m_r)$. However, we want to point out that our result gives new examples. This is for instance the case of the following space related with the measure associated to the Volterra operator.

**Example 2.6.** Consider, for $1 \le r < \infty$, the Volterra integral operator $V_r : L^r([0,1]) \to L^r([0,1])$ defined by means of the formula

$$V_r(f)(t) = \int_0^t f(u)du, \quad f \in L^r([0,1]), t \in [0,1]. \tag{7}$$

Hence, the *Volterra measure of order* $r$ is the $L^r([0,1])$-valued vector measure defined on $\mathcal{B}([0,1])$, the Borel $\sigma$-algebra in $[0,1]$ associated to the Volterra operator which is given by

$$\nu_r(A) = V_r(\chi_A) = \int_0^t \chi_A(u)du, \quad A \in \mathcal{B}([0,1]). \tag{8}$$

The corresponding space $L^p(\nu_r)$ is nowadays well-known (see for instance [23, Example 3.10]). In particular in [23, Example 3.26], it is shown that, for $1 < r < \infty$,

$$L^r([0,1]) \subsetneq L^1([0,1]) \subsetneq L^1(|\nu_r|) \subsetneq L^1(\nu_r) \subsetneq L^1(\nu_1),$$

with all inclusion being strict. For $1 < p, r < \infty$ the space $L^p(\nu_r)$ is Fréchet smooth.

## 3. Uniform smoothness of $L^p(m)$

In this section, we establish the analogue of Theorem 2.1 for the properties of uniformly Gâteaux and uniformly Fréchet smoothness. Unfortunately, we have not been able to obtain the first case in full generality. In this case we restrict ourselves to positive vector measures with values in an *order continuous Banach function space*. The result concerning uniform Fréchet smoothness can be achieved easily using the Šmulyan characterization of this property (see e.g. [14, Fact 9.7]).

**Theorem 3.1.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < p < \infty$. If the norm of $X$ is uniformly Fréchet smooth, then so is the norm $\|\cdot\|_{L^p(m)}$ on $L^p(m)$.*

*Proof.* Thanks to Theorem 2.1, the norm $\|\cdot\|_{L^p(m)}$ is Fréchet smooth, and bearing in mind the corresponding formula for the derivative (Remark 2.4), for every $f, h \in S_{L^p(m)}$ we have

$$\|\cdot\|'_{L^p(m)}(f)(h) = p^{-1} \left\langle x_f^*, \varphi'(f)(h) \right\rangle,$$

where $x_f^*$ is the unique functional in $S_{X^*}$ such that $x_f^*(\varphi(f)) = \|\varphi(f)\| = 1$. Thus, for each $f, g, h \in S_{L^p(m)}$ we have

$$\begin{aligned} p\left|\|\cdot\|'_{L^p(m)}(f)(h) - \|\cdot\|'_{L^p(m)}(g)(h)\right| &= \left|\left\langle x_f^*, \varphi'(f)(h)\right\rangle - \left\langle x_g^*, \varphi'(g)(h)\right\rangle\right| \\ &\le \left|\left\langle x_f^*, \varphi'(f)(h) - \varphi'(g)(h)\right\rangle\right| + \left|\left\langle x_f^* - x_g^*, \varphi'(g)(h)\right\rangle\right| \\ &\le \|\varphi'(f)(h) - \varphi'(g)(h)\| + \|x_f^* - x_g^*\| \cdot \|\varphi'(g)(h)\|. \end{aligned}$$

On the other hand, Proposition 2.2 (ii) guarantees that

$$\|\varphi'(f)(h) - \varphi'(g)(h)\| \le C_p\|f-g\|^s_{L^p(m)} \quad \text{and} \quad \|\varphi'(g)(h)\| \le C_p\|g\|^s_{L^p(m)} = C_p,$$

for some constants $C_p > 0$ and $s > 0$. Consequently, for every $f, g \in S_{L^p(m)}$ we have

$$\left|\|\cdot\|'_{L^p(m)}(f) - \|\cdot\|'_{L^p(m)}(g)\right| \le p^{-1}C_p\left(\|f-g\|^s_{L^p(m)} + \|x_f^* - x_g^*\|\right). \tag{9}$$

Now, let $(f_n)$ and $(g_n)$ be two sequences in $S_{L^p(m)}$ such that $\|f_n - g_n\|_{L^p(m)} \to 0$. Since, by Proposition 2.2 (iii), the mapping $\varphi$ is $2p$-Lipschitzian on the unit ball of $L^p(m)$ we get $\|\varphi(f_n) - \varphi(g_n)\| \longrightarrow 0$. On the other hand, as the norm $\|\cdot\|$ on $X$ is uniformly Fréchet smooth, according to Šmulyan Lemma it follows that the mapping $S_X \ni x \longmapsto \|\cdot\|'(x) \in S_{X^*}$ is norm-to-norm uniformly continuous, and thus, $\|x^*_{f_n} - x^*_{g_n}\| \longrightarrow 0$. Applying inequality (9), with $f := f_n$ and $g := g_n$, we obtain

$$\big|\|\cdot\|'_{L^p(m)}(f_n) - \|\cdot\|'_{L^p(m)}(g_n)\big| \longrightarrow 0,$$

and a new appeal to Šmulyan Lemma yields that the norm $\|\cdot\|_{L^p(m)}$ is uniformly Fréchet smooth. □

As a consequence of Theorem 3.1, we obtain the following uniformly Fréchet counterpart of Corollary 2.5.

**Corollary 3.2.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < p < \infty$. If the space $L^1(m)$ is uniformly Fréchet smooth, then so is $L^p(m)$.*

Another consequence of Theorem 3.1, the former corollary, and the fact that super-reflexivity is equivalent to the existence of an equivalent uniformly Fréchet smooth renorming, is the following result.

**Corollary 3.3.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < p < \infty$. If either $X$ or $L^1(m)$ is super-reflexive, then so is the space $L^p(m)$.*

Now, we present an application of the former result (see [4, Example 7]).

**Example 3.4.** Let $\mu$ the Lebesgue measure on $[0,1]$ and let $(A_i)_{i\geq 1}$ be a sequence of pairwise disjoint measurable subsets of $[0,1]$. Define the positive $\ell_2$-valued vector measure $m : \Sigma \to \ell_2$ given by

$$m(A) = \sum_{i\geq 1} \mu(A \cap A_i)e_i, \quad A \in \Sigma,$$

where $(e_i)$ is the usual canonical basis of $\ell_2$. It is easy to check that $\nu := |\langle m, x_0^*\rangle|$, where $x_0^* := (2^{-i/2})_{i\geq 1} \in \ell_2$ is a Rybakov control measure for $m$. Since in this case $X = \ell_2$ then for $1 < p < \infty$, the space

$$L^p(m) = \left\{ f \in L^0(\nu) : \sum_{i\geq 1} \Big( \int_{A_i} |f|^p d\mu \Big)^2 < \infty \right\} = \bigoplus_{2p} L^p(\mu_{|A_i}),$$

is super-reflexive.

**Remark 3.5.** The space $L^1(m)$ is not necessarily super-reflexive, even if the Banach lattice $X$ where the measure $m$ takes its values is super-reflexive. Indeed, if $m$ is the Lebesgue measure of $[0,1]$, then $X = \mathbb{R}$ while $L^1(m) = L^1([0,1])$, and this space is not even reflexive. Our results also apply to the case of the vector measure $\nu_r$ associated to the *Volterra operator* introduced above. More concretely, if $1 < r < \infty$ then the space $L^1(\nu_r)$ is not reflexive ([23, Example 3.26(iv)]).

We end this section with the case of uniformly Gâteaux smoothness. The argument employed in Theorem 3.1 does not seem to work now. As we said before we restrict ourselves to positive vector measures with values in an order continuous Banach function space.

**Theorem 3.6.** *Let $E(\mu)$ be an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$, $m : \Sigma \to E(\mu)$ a positive vector measure and $1 < p < \infty$. If the norm of $E(\mu)$ is uniformly Gâteaux smooth, then so is the norm $\|\cdot\|_{L^p(m)}$ on $L^p(m)$.*

The proof in this case relies on the following lemma, which provides a characterization of the uniform Gâteaux smoothness for the class of order continuous Banach function spaces. In order to simplify the notation, since our space $E(\mu)$ will be Gâteaux smooth, for each $x \in S_{E(\mu)}$ let us denote by $x^*$ the derivative $\|\cdot\|'(x)$ (that is, $x^*$ is the unique norm one functional in $E(\mu)^*$ norming the vector $x$).

**Lemma 3.7.** *Let $E(\mu)$ be an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$. Then, $E(\mu)$ is uniformly Gâteaux smooth if, and only if, $E(\mu)$ is Gâteaux smooth and for every $x \in S_{E(\mu)}$ and every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\big|\langle |x_1^* - x_2^*|, x\rangle\big| < \varepsilon \quad \textit{whenever} \quad x_1, x_2 \in S_{E(\mu)} \quad \textit{and} \quad \|x_1 - x_2\| < \delta.$$

The proof of this lemma is based on the fact that if the space $E(\mu)$ is order continuous, then the (topological) dual space $E(\mu)^*$ coincides isometrically with the Köthe dual space $E(\mu)'$, which can be identified with the set of functionals defined by integrals given by a function (see [23, Proposition 2.16, Remark 3.8] and the references therein), and a Šmulyan type criterion for uniformly Gâteaux smoothness (see e.g. [15, Lemma 2.5]), which ensures that a Banach space $X$ is uniformly Gâteaux smooth if, and only if, $X$ is Gâteaux smooth and for every $h \in B_X$, the mapping $S_X \ni x \longmapsto \|\cdot\|'(x)(h) \in \mathbb{R}$ is uniformly continuous.

*Proof of Lemma 3.7.* Only the left-to-right implication needs a proof. So, suppose that $E(\mu)$ is uniformly Gâteaux smooth and fix $x \in S_{E(\mu)}$ and $\varepsilon > 0$. Then there is a $\delta > 0$ such that

$$\big|\langle x_1^* - x_2^*, x\rangle\big| < \varepsilon,$$

for all pairs $x_1, x_2 \in E(\mu)$ satisfying that $\|x_1 - x_2\| < \delta$ (recall that, for $i = 1, 2$, the functionals $x_i^*$ are identified with the derivatives $\|\cdot\|'(x_i)$). Pick any two vectors $x_1, x_2 \in E(\mu)$ with $\|x_1 - x_2\| < \delta$. Since the space $E(\mu)$ is order continuous, the topological dual space $E(\mu)^*$ can be identified with the Köthe dual, $E(\mu)'$. More concretely the functionals $x_1^*, x_2^* \in E(\mu)^*$ are associated with two ($\Sigma$-measurable) functions $g_1, g_2 \in E(\mu)'$ by means of the formula

$$\langle x_i^*, x\rangle = \int_\Omega x \cdot g_i \,\mathrm{d}\mu, \quad x \in E(\mu),\ i = 1, 2.$$

With this notation let $A := \{w \in \Omega : (g_1 - g_2)(w) \geq 0\} \in \Sigma$, and write $A^c = \Omega \setminus A$ and $\Theta := \chi_A - \chi_{A^c}$. Since $\big|(x_1 - x_2)\Theta\big| = \big|x_1 - x_2\big|$, it is clear that

$$\big\|(x_1 - x_2)\Theta\big\| = \|x_1 - x_2\|.$$

Now, for every $y \in S_{E(\mu)}$ we denote by $g$ the function in $E(\mu)'$ which is identified with $y^*$ (the unique norm-one functional in $E(\mu)^*$ norming $y$). Then by using again the order continuity of $E(\mu)$ one obtains that

$$\langle y^*, y\rangle = \int_\Omega y \cdot g \,\mathrm{d}\mu = \int_\Omega \Theta y \cdot \Theta g \,\mathrm{d}\mu.$$

In other words, the functional $(\Theta y)^* \in E(\mu)^*$ is identified with the function $\Theta g \in E(\mu)'$.

Finally, using the uniform Gâteaux smoothness of $E(\mu)$ with the pair of vectors $\widehat{x_i} = \Theta x_i$ $(i = 1, 2)$, that satisfy $\|\widehat{x_1} - \widehat{x_2}\| < \delta$, we get

$$\begin{aligned} \varepsilon > \big|\langle \widehat{x}_1^* - \widehat{x}_2^*, x\rangle\big| &= \big|\langle (\Theta x_1 - \Theta x_2)^*, x\rangle\big| = \left|\int_\Omega x \cdot (\Theta g_1 - \Theta g_2)\,\mathrm{d}\mu\right| \\ &= \left|\int_\Omega x \cdot |g_1 - g_2|\,\mathrm{d}\mu\right| = \big|\langle |x_1^* - x_2^*|, x\rangle\big|, \end{aligned}$$

as we wanted to show. □

*Proof of Theorem 3.6.* Since $E(\mu)$ is uniformly Gâteaux smooth it is, in particular, Gâteaux smooth, and Theorem 2.1 ensures that the space $L^p(m)$ is Gâteaux smooth. Therefore we only have to prove that for every $h \in S_{L^p(m)}$, the function $S_{L^p(m)} \ni f \longmapsto \|\cdot\|'_{L^p(m)}(f)(h) \in \mathbb{R}$, is uniformly continuous. For each function $f \in S_{L^p(m)}$ we denote by $x_f^*$ the unique norm-one functional in $E(\mu)^*$ norming the vector $\varphi(f) = \int_\Omega |f|^p dm \in E(\mu)$. Bearing in mind Remark 2.4 we have

$$\begin{aligned} \big|\|\cdot\|'_{L^p(m)}(f)(h) - \|\cdot\|'_{L^p(m)}(g)(h)\big| &= p^{-1}\big|\langle x_f^*, \varphi'(f)(h)\rangle - \langle x_g^*, \varphi'(g)(h)\rangle\big| \\ &\le p^{-1}\big(|\langle x_f^* - x_g^*, \varphi'(f)(h)\rangle| + |\langle x_g^*, \varphi'(f)(h) - \varphi'(g)(h)\rangle|\big). \end{aligned} \tag{10}$$

Fix $\varepsilon > 0$ and $h \in S_{L^p(m)}$. Using Proposition 2.2 (ii) we find a constant $C_p > 0$ such that

$$\big|\big\langle x_g^*, \varphi'(f)(h) - \varphi'(g)(h)\big\rangle\big| \le \big\|\varphi'(f)(h) - \varphi'(g)(h)\big\| \le C_p\big\|f - g\big\|^s_{L^p(m)}, \tag{11}$$

for some $s \in \{1, p-1\}$ and all $f, g \in B_{L^p(m)}$. On the other hand, since $E(\mu)$ is uniformly Gâteaux smooth, by using the previous lemma we have that, for the fixed element $y = \varphi(h) \in E(\mu)$, there is a $\delta_1 > 0$ such that, if $x_1, x_2 \in S_{E(\mu)}$ satisfy the inequality $\|x_1 - x_2\| < \delta_1$, then the corresponding norm-one functionals $x_1^*, x_2^*$ attaining their norms, satisfy that $\big|\langle |x_1^* - x_2^*|, \varphi(h)\rangle\big| = \langle |x_1^* - x_2^*|, \varphi(h)\rangle < (\varepsilon/2)^p$. Let us take $\delta > 0$ such that

$$0 < \delta \le \min\left\{\left(\frac{p\varepsilon}{2C_p}\right)^{1/s}, \frac{\delta_1}{2p}\right\}. \tag{12}$$

Fix $f, g \in S_{L^p(m)}$ such that $\|f-g\|_{L^p(m)} < \delta$, and let $x_1 = \varphi(f) \in E(\mu)$ and $x_2 = \varphi(g) \in E(\mu)$. Using Proposition 2.2 (iii) we get

$$\big\|x_1 - x_2\big\| = \big\|\varphi(f) - \varphi(g)\big\| \le 2p\big\|f - g\big\|_{L^p(m)} < \delta_1.$$

Therefore, if we denote by $x_f^*$ and $x_g^*$ the norm-one functionals norming respectively $x_1 = \varphi(f)$ and $x_2 = \varphi(g)$, because of Lemma 3.7 we get

$$\big\langle |x_f^* - x_g^*|, \varphi(h)\big\rangle < \big(\varepsilon/2^{1+1/p'}\big)^p. \tag{13}$$

Hence using the positivity of the measure $m$ and taking into account that $|f|^{p-1} \in S_{L^{p'}(m)}$, as an application of Hólder's inequality and (13) we obtain

$$\begin{aligned} \big|\big\langle x_f^* - x_g^*, \varphi'(f)(h)\big\rangle\big| &= \left|\left\langle p\int_\Omega \operatorname{sign}(f)|f|^{p-1}h\,\mathrm{d}m, x_f^* - x_g^*\right\rangle\right| \\ &\le p\int_\Omega |f|^{p-1}|h|\,\mathrm{d}\langle m, |x_f^* - x_g^*|\rangle \\ &\le p\big\|x_f^* - x_g^*\big\|^{1/p'}\left(\int_\Omega |h|^p\,\mathrm{d}\langle m, |x_f^* - x_g^*|\rangle\right)^{1/p} \\ &= p\big\|x_f^* - x_g^*\big\|^{1/p'}\big(\langle |x_f^* - x_g^*|, \varphi(h)\rangle\big)^{1/p} \\ &\le p\,2^{1/p'}\big(\langle |x_f^* - x_g^*|, \varphi(h)\rangle\big)^{1/p} < p\,\varepsilon/2. \end{aligned} \tag{14}$$

Finally, putting (14) and (11) in (10) and bearing in mind (12) we have

$$\big|\|\cdot\|'_{L^p(m)}(f)(h) - \|\cdot\|'_{L^p(m)}(g)(h)\big| \le \frac{1}{p}\left(p\frac{\varepsilon}{2} + C_p\delta^s\right) \le \varepsilon.$$

The proof is done. □

Now, we establish the uniformly Gâteaux smooth counterpart of Corollaries 2.5 and 3.2. Although in Theorem 3.6, the measure $m$ must take values in an order continuous Banach function space, we do not require such assumption now. The reason is that $L^1(m)$ is always an order continuous Banach function space (see [23, Proposition 3.28]) over a Rybakov measure for $m$.

**Corollary 3.8.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < p < \infty$. If $L^1(m)$ is uniformly Gâteaux smooth, then so is the space $L^p(m)$.*

As a combination of the previous results we obtain the following corollaries.

**Corollary 3.9.** *Let $E(\mu)$ be an order continuous Banach function space over a finite measure space $(\Omega, \Sigma, \mu)$, $m : \Sigma \to E(\mu)$ be a positive vector measure and $1 < p < \infty$. If $E(\mu)$ is simulteanously Fréchet and uniformly Gâteaux smooth, then so is the space $L^p(m)$.*

**Corollary 3.10.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < p < \infty$. If $L^1(m)$ is simulteanously Fréchet and uniformly Gâteaux smooth, then so is the space $L^p(m)$.*

## 4. Applications

In this final section we apply our results in order to obtain some examples in the setting of the well-known *Bishop-Phelps-Bollobás property* and its corresponding *bilinear form* version. In [6], Bishop and Phelps proved that the set of norm-attaining functionals on a Banach space is dense in its dual space. Some years later, Bollobás [7] gave a quantitate version of this theorem, known now as the *Bishop-Phelps-Bollobás theorem*. In [1], the corresponding version for operators was studied. In particular, it was shown there that the version of the theorem fails in general when we pass from functionals to operators, and hence the *Bishop-Phelps-Bollobás property* (BPBP for short) was introduced. More recently, the corresponding version for bilinear forms has been studied in [9] and [3]. Recall that a couple $(X, Y)$ of Banach spaces satisfies the Bishop-Phelps-Bollobás property for operators (BPBP for operators, for short) if, for each $\varepsilon > 0$ there exist $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0} \beta(t) = 0$ such that for every operator $T \in S_{\mathcal{L}(X,Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist $u_0 \in S_X$ and $S \in S_{\mathcal{L}(X,Y)}$ satisfying

$$\|S(u_0)\| = 1, \quad \|u_0 - x_0\| < \beta(\varepsilon), \quad \text{ and } \quad \|S - T\| < \varepsilon.$$

Analogously, the pair $(X, Y)$ enjoys the Bishop-Phelps-Bollobás property for bilinear forms (BPBP for bilinear forms, for short) if, for each $\varepsilon > 0$ there exist $\eta(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ with $\lim_{t\to 0} \beta(t) = 0$ such that for every bounded bilinear form $A \in S_{\mathcal{L}^2(X\times Y)}$, if $(x_0, y_0) \in S_X \times S_Y$ is such that $|A(x_0, y_0)| > 1 - \eta(\varepsilon)$, then there exist $(u_0, v_0) \in S_X \times S_Y$ and $B \in S_{\mathcal{L}(X\times Y)}$ satisfying

$$|B(u_0, v_0)| = 1, \quad \max\{\|u_0 - x_0\|, \|v_0 - y_0\|\} < \beta(\varepsilon), \quad \text{ and } \quad \|S - T\| < \varepsilon.$$

We start with the case of BPBP for bilinear forms. In [3, Theorem 2.2]), it is shown that if $X$ is a uniforly convex Banach space, then for every Banach space $Y$, the couple $(X, Y)$ enjoys this property. Tnus, if $X$ is any Banach space, $m : \Sigma \to X$ is a vector measure and $1 < p < \infty$, then the pairs $(\ell^p, L^q(m))$ and $(L^p(\mu), L^q(m))$ have the BPBP for bilinear forms for $1 \le q < \infty$. Now, we address our attention to the case $p = 1$. In [3], it is proved that for a given Banach space $Y$, the pair $(\ell^1, Y)$ has the BPBP for bilinear forms if, and only if, the pair $(Y, Y^*)$ satisfies an isometric property, called the *apprximamate hyperplane series property* (AHSP), and that this property is fulfilled if the Banach space $Y$ is uniformly Fréchet smooth. Therefore, Theorem 3.1 yield the following corollaries.

**Corollary 4.1.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < q < \infty$. If the norm of $X$ is uniformly Fréchet smooth, then $(\ell^1, L^q(m))$ has the BPBP for bilinear forms.*

**Corollary 4.2.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < q < \infty$. If the norm of $L^1(m)$ is uniformly Fréchet smooth, then $(\ell^1, L^q(m))$ has the BPBP for bilinear forms.*

For the case of the pair $(L^1(\mu), Y)$ —where $\mu$ is a $\sigma$-finite scalar measure—, a similar result is obtained in [2], but in the setting of Asplund spaces. More concretely, it is proved there that if $Y$ is an Asplund space, then the pair $(L^1(\mu), Y)$ has the BPBP for bilinear forms if, and only if, the pair $(Y, Y^*)$ has the AHSP. Since the presence of an equivalent Fréchet smooth renorming implies Asplundness we have the following results.

**Corollary 4.3.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < q < \infty$. If the norm of $X$ is uniformly Fréchet smooth, then $(L^1(\mu), L^q(m))$ has the BPBP for bilinear forms.*

**Corollary 4.4.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < q < \infty$. If the norm of $L^1(m)$ is uniformly Fréchet smooth, then $(L^1(\mu), L^q(m))$ has the BPBP for bilinear forms.*

These results can be extended to the general vector measure setting. Indeed, since the dual space $X^*$ is uniformly convex if the space $X$ is uniformly Fréchet smooth, we get

**Corollary 4.5.** *Let $X_1$ be a uniformly Fréchet smooth Banach lattice, $m_1 : \Sigma_1 \to X_1$ be a positive vector measure, $X_2$ be a Banach space, $m_2 : \Sigma_2 \to X_2$ be a vector measure, and $1 < p_i < \infty$ for $i = 1, 2$. Then $(L^{p_1}(m_1)^*, L^{p_2}(m_2))$ has the BPBP for bilinear forms.*

**Corollary 4.6.** *For $i = 1, 2$ let $X_i$ be Banach spaces, $m_i : \Sigma_i \to X_i$ be vector measures and $1 < p_i < \infty$. If the norm of $L^1(m_1)$ is uniformly Fréchet smooth, then $(L^{p_1}(m_1)^*, L^{p_2}(m_2))$ has the BPBP for bilinear forms.*

**Remark 4.7.** (i) The space $X_2$ does not need to be uniformly Fréchet smooth. (ii) Since the pair $(L^1(\mu), L^1(\mu)^*)$ does not have the AHSP (see [3, Proposition 4.8]) then $(\ell^1, L^1(\mu))$ does not have the BPBP for bilinear forms. Hence, the result cannot be extended for $p_i = 1$ $(i = 1, 2)$. (iii) For $1 < p_1 < \infty$ and $p_2 = 1$ the result is also true. (iv) Finally, for $p_1 = 1$ and $1 < p_2 < \infty$ we do not know if the result is true.

The counterpart of the previous corollaries for the BPBP case is now a direct consequence of the natural identification of the space of bounded bilinear forms defined on $X \times Y$ with the space of bounded linear maps from $X$ into $Y^*$. Indeed, this identification means that if the pair $(X, Y)$ has the BPBP for bilinear forms then the pair $(X, Y^*)$ has the corresponding BPBP (now for operators).Therefore:

**Corollary 4.8.** *Let $X$ be a Banach lattice, $m : \Sigma \to X$ be a positive vector measure and $1 < q < \infty$. If the norm of $X$ is uniformly Fréchet smooth, then $(\ell^1, L^q(m))$ and $(L^1(\mu), L^q(m))$ have the BPBP for bilinear forms.*

**Corollary 4.9.** *Let $X$ be a Banach space, $m : \Sigma \to X$ be a vector measure and $1 < q < \infty$. If the norm of $L^1(m)$ is uniformly Fréchet smooth, then $(\ell^1, L^q(m))$ and $(L^1(\mu), L^q(m))$ have the BPBP for bilinear forms.*

And also the corresponding vector valued case:

**Corollary 4.10.** *Let $X_1$ be a uniformly Fréchet smooth Banach lattice, $m_1 : \Sigma_1 \to X_1$ a positive vector measure, $X_2$ a Banach space, $m_2 : \Sigma_2 \to X_2$ a vector measure, and $1 < p_i < \infty$ for $i = 1, 2$. Then $(L^{p_1}(m_1)^*, L^{p_2}(m_2)^*)$ has the BPBP.*

**Corollary 4.11.** *For $i = 1, 2$ let $X_i$ be Banach spaces, $m_i : \Sigma_i \to X_i$ vector measures and $1 < p_i < \infty$. If the norm of $L^1(m_1)$ is uniformly Fréchet smooth then $(L^{p_1}(m_1)^*, L^{p_2}(m_2)^*)$ has the BPBP.*

Some other results can be obtained. Namely, since the pairs $(c_0, Y)$, $(C(K), Y)$ —being $K$ a compact Hausdorff topological space— and $(L^\infty(\mu), Y)$ have the BPBP for all uniformly convex Banach space $Y$ (see [18, 19]), we have

**Corollary 4.12.** *Let $X$ be a Banach lattice (resp. a Banach space) , $Y$ a Banach space, $m : \Sigma \to X$ a positive vector measure (resp. a vector measure) and $1 < p < \infty$. If the norm of $X$ (resp. $L^1(m)$) is uniformly Fréchet smooth, then $(c_0, L^p(m)^*)$, $(C(K), L^p(m)^*)$ and $(L^\infty(\mu), L^p(m)^*)$ have the BPBP.*

In a similar way, since for the Asplund case one has that both $(X, \mathcal{A})$ —being $\mathcal{A}$ a uniform algebra— and $(X, C_0(L))$ —where $L$ is a locally compact Hausdorff topological space—, have the BPBP (see [5, 8]), we obtain

**Corollary 4.13.** *Let $X$ be a Banach lattice (resp. a Banach space) , $Y$ a Banach space, $m : \Sigma \to X$ a positive vector measure (resp. a vector measure) and $1 < p < \infty$. If the norm of $X$ (resp. $L^1(m)$) is uniformly Fréchet smooth, then $(L^p(m), \mathcal{A})$ and $(L^p(m), C_0(L))$ have the BPBP.*

Of course, some examples given in this work recover very well-known examples of pairs of Banach spaces having the BPBP (both for operators and for bilinear forms). This is, for instance , the classical case of $(\ell_q, L^p(\mu))$ where $\mu$ is a scalar $\sigma$-finite measure and $1 < p < \infty$ or, even $(\ell_q, \bigoplus_{2p} L^p(\mu_{|A_i}))$ (see Example 3.4) in both cases for $1 \leq q < \infty$ (and also replacing $\ell_q$ for the corresponding $L^q(\nu)$). However, for $1 < r, p < \infty$, the space $L^p(\nu_r)$ associated to the Volterra operator given in Example 2.6 gives us some new results. We finish this paper with a short list of examples of pairs of spaces having the BPBP (both for operators and for bilinear forms).

**Example 4.14.**
- (i) *The pairs $(\ell^q, L^p(\nu_r))$ and $(L^q(\mu), L^p(\nu_r))$ have the BPBP for bilinear forms.*
- (ii) *The pairs $(L^p(\nu_r)^*, L^q(m))$ and $(L^p(\nu_r)^*, L^q(m)^*)$ have the BPBP for bilinear forms for all vector measure $m$.*
- (iii) *The pairs $(\ell^q, L^p(\nu_r)^*)$, $(L^q(\mu), L^p(\nu_r)^*)$, $(c_0, L^p(\nu_r)^*)$, $(C(K), L^p(\nu_r)^*)$ and $(L^\infty(\mu), L^p(\nu_r)^*)$ have the BPBP.*
- (iv) *The pairs $(L^p(\nu_r), C_0(L))$, $(L^p(\nu_r), \mathcal{A})$, $(L^p(\nu_r), C_0(L))$ and $(L^p(\nu_r), \mathcal{A})$ have the BPBP.*

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