SURVEYING THE SPIRIT OF ABSOLUTE SUMMABILITY ON MULTILINEAR OPERATORS AND HOMOGENEOUS POLYNOMIALS

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Abstract. We draw a fundamental compendium of the most valuable results of the theory of summing linear operators and detail those that are not shared by known multilinear and polynomial extensions of absolutely summing linear operators. The lack of such results in the theory of non-linear summing operators justifies the introduction of a class of polynomials and multilinear operators that satisfies at once all related non-linear results. Surprisingly enough, this class, defined by means of a summing inequality, happens to be the well known ideal of composition with a summing operator.

1. Introduction

Our aim in this paper is to provide a promenade through the most significant theorems that come from the theory of absolutely summing linear operators, highlighting those non linear notions of summability that lack one or more of their non-linear analogs. The fact that known multilinear or polynomial classes given by summing inequalities do not completely fit the whole compendium of results from the linear theory, will be evidenced. Far from carrying with a disappointing feeling, we mix some of the known notions of non-linear summability in order to get a new one that improve their behavior. Factorable strongly p-summing multilinear operators and homogeneous polynomials are defined to the full extent of absolutely p-summing linear operators. This apparently new class of summing polynomials/multilinear operators is a subclass of strongly p-summing multilinear operators, that stand apart from previous generalizations as they keep a big amount of the fundamental properties, as a natural Pietsch Factorization type theorem or weak compactness. Factorable strongly p-summing homogeneous polynomials also satisfy a factorization theorem in the spirit of Pietsch, are weakly compact and a polynomial belongs to the class if and only if its second adjoint (in the sense of Aron and Schottenloher) is in the class. Actually, an homogeneous polynomial is factorable strongly p-summing if and only if its associated multilinear map is factorable p-summing or, equivalently, its linearization is absolutely p-summing. This brings deep strengths that are not shared by former classes of summing polynomials as dominated or strongly summing polynomials. In addition, this proves that factorable strongly p-summing polynomial/multilinear mappings coincide indeed with those mappings that can be obtained as the composition of an absolutely p-summing linear operator with a polynomial/multilinear one. Then, it is worth mentioning that we are not really introducing a new class, but rediscovering the well known ideal of composition with absolutely p-summing operators by characterizing it by means of a summing inequality.

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The interest of the paper remains on the fact that it closes the longstanding research line whose main aim was the pursuance of better classes of polynomial and multilinear mappings whose behavior was as close as possible to the classical absolutely p-summing linear operators. This search, that at some stage seemed to be a never-ending story, has been in the core of the theory of non-linear absolutely summing mappings since Pietsch proposed his celebrated research program on the subject in 1983 [42]. This is why it is crucial for our purpose to go over classical properties with a fine toothcomb even if their proofs use nothing but classical techniques. To check that these properties are fulfilled by the distinguished subclass of factorable strongly summing multilinear mappings does not require in most cases new ideas, but the reader does not have to forget the two-fold purpose of the paper: to present a compendium of results, some of them having non-linear analogs in certain classes of non-linear summing mappings and some other lacking such analogs, and to stablish the multilinear/polynomial class that shares all the basic essentials of absolutely p-summing linear operators. Although it is not strictly necessary —because of their well-known classical nature— some of the proofs have been included for the sake of completeness.

The seeds of absolutely summing linear operators come from A. Grothendieck's work in the 1950s, but it was A. Pietsch [41] who cultivated the land in 1966-67 and got bumper crops that have fed analysts for many decades. J. Lindenstrauss and A. Pełczyński [27] clarified Grothendieck's ideas without the use of tensor products and were also responsible for the reformulation of Grothendieck's inequality, which is still a fundamental result of Banach Space Theory and Mathematical Analysis in general (see [43]). For a detailed approach to the linear theory of absolutely summing operators we refer to the excellent books of J. Diestel, H. Jarchow and A. Tonge [19] and A. Defant and K. Floret [18]. Nowadays absolutely summing operators is a current subject in books of Banach Space Theory (see, for instance, [2, 22]). The deep influence that the theory of absolutely summing operators has borne justifies the big effort that has been made, since Pietsch's proposal [42], to try and generalize the linear theory to non-linear operators. However, extending summability properties to a non-linear context has been proved difficult and intriguing. For instance, there are several extensions of absolutely p-summing linear operators to the multilinear setting that have been considered in the literature. Besides its intrinsic interest, the multilinear theory of absolutely summing operators has shown important connections, including applications to Quantum Information Theory (see [32]). This proliferation of classes of summing multilinear maps has lead to the appearance of works that compare different approaches (see [15, 40]). The first challenging task when dealing with multilinear operators is probably to identify the class of multilinear operators that best inherits the spirit of the absolutely summing linear operators. According to [37, 39] one of the most natural extensions of the notion of absolutely p-summing linear operators to the multilinear setting is the notion of strongly p-summing multilinear operators, due to V. Dimant ([20]). This class lifts to the multilinear framework most of the main properties of absolutely p-summing linear operators: Grothendieck's Theorem, Pietsch Domination Theorem, Inclusion Theorem. However, as we will see, a natural version of the Pietsch Factorization Theorem does not hold for this class.

The good behavior of multilinear extensions has found no echo when considering extensions of absolutely summing operators to polynomials. In this non linear setting, several attempts have been made but all of them have found rough edges to succeed in. This is the case of p-dominated homogeneous polynomials, for which a Pietsch type factorization theorem has been pursuit (see [28, 31, 9, 14]) and succeeded just when the domain is separable. Related factorization schemes for homogeneous maps and polynomials can be found in [1] and [45]. Recently, the second and third authors [44] have isolated the class of p-dominated polynomials that satisfy a Pietsch type factorization theorem: the factorable p-dominated polynomials. However, even if this makes a big difference with p-dominated polynomials, they still lack good properties as evidenced by the feeling that factorable p-dominated polynomial do not define a composition ideal or, equivalently, the linearization of a factorable p-dominated polynomial may not be absolutely p-summing.

This paper is organized as follows. Section 2 contains a compendium of the main results on linear and non linear summability and, their validity for different classes of nonlinear summing operators is analyzed. In Section 3, and inspired in the recent paper [44] of the second and third author, we combine the notion of strongly p-summability and that of factorable p-domination to generate a new notion that inherits the main properties of each individual class including a natural factorization theorem that follows the spirit of the linear theory. The resulting class is formed by the factorable strongly p-summing multilinear operators. As a consequence we have weak compactness, as in the linear case. In Section 4 we deal with homogeneous polynomials, proving that a polynomial is factorable strongly p-summing if and only if its linearization is absolutely p-summing. This yieds to identify the class of factorable strongly p-summing multilinear operators with the class formed by composition with absolutely p-summing linear operators. The connection between m-homogeneous polynomials and m-linear operators is then established: an m-homogeneous polynomial is factorable strongly p-summing if and only if its associated symmetric m-linear map is factorable strongly p-summing. These results yield to obtain in Section 5 proper generalizations of fundamental properties related to summability for linear operators to multilinear maps and homogeneous polynomials. Among other results, we show that a Dvoretzky-Rogers type theorem, a Lindenstrauss–Pełczyński type theorem or a Grothendieck type theorem work for factorable strongly summability. Finally, in Section 6 we show that the sequence formed by the ideals of factorable strongly summing homogeneous polynomials and factorable strongly summing multilinear operators is coherent and compatible with the ideal of absolutely summing linear operators.

2. Background: linear and multilinear summability

If $1 \leq p < \infty$ and X, Y are Banach spaces, a continuous linear operator $u : X \to Y$ is absolutely p-summing ($u \in \Pi_p(X;Y)$) if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} \|u(x_j)\|^p\right)^{1/p} \leq C \left(\sup_{\varphi \in B_{X^*}} \sum_{j=1}^{m} |\varphi(x_j)|^p\right)^{1/p}$$

for all $x_1, ..., x_m \in X$ and all positive integers m. The infimum of all C that satisfy the above inequality defines a norm, denoted by $\pi_p(u)$, and $(\Pi_p(X, Y), \pi_p)$ is a Banach space. The cornerstones of the theory of absolutely summing linear operators are the following theorems:

- (Dvoretzky-Rogers theorem) If $p \geq 1$, then $\Pi_p(X;X) = \mathcal{L}(X;X)$ if and only if $\dim X < \infty$.
- (Grothendieck's theorem) Every continuous linear operator from ℓ_1 to ℓ_2 is absolutely 1-summing.
- (Lindenstrauss–Pełczyński theorem) If X and Y are infinite-dimensional Banach spaces, X has an unconditional Schauder basis and $\Pi_1(X;Y) = \mathcal{L}(X;Y)$ then $X = \ell_1$ and Y is a Hilbert space.
- (Pietsch Domination theorem) If X and Y are Banach spaces, a continuous linear operator $u : X \to Y$ is absolutely p-summing if and only if there exist a constant $C \geq 0$ and a Borel probability measure μ on the closed unit ball of the dual of X, $(B_{X^*}, \sigma(X^*, X))$, such that

$$\|u(x)\| \leq C \left(\int_{B_{X^*}} |\varphi(x)|^p \, d\mu\right)^{\frac{1}{p}} \tag{1}$$

for all $x \in X$.

- (Inclusion theorem) If $1 \leq p \leq q < \infty$, then every absolutely p-summing operator is absolutely q-summing.
- (Pietsch Factorization theorem) A continuous linear operator $u : X \to Y$ is absolutely p-summing if, and only if, there exist a regular Borel probability measure μ on B_{X^*}, a closed subspace X_p of $L_p(\mu)$ and a continuous linear operator $\widehat{u} : X_p \to Y$ such that
$$j_p \circ i_X(X) \subset X_p \text{ and } \widehat{u} \circ j_p \circ i_X = u,$$
where $i_X : X \to C(B_{X^*})$ and $j_p : C(B_{X^*}) \to L_p(\mu)$ are the canonical inclusions. Moreover, every absolutely p-summing linear operator is weakly compact.

From now on $p \in [1, \infty)$ and $X, X_1, ..., X_n, Y$ are Banach spaces over the same scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is *p-dominated* if there is a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} \| T(x_j^1, ..., x_j^n) \|^{\frac{p}{n}} \right)^{n/p} \leq C \left(\sup_{\varphi \in B_{X_1^*}} \sum_{j=1}^{m} \left| \varphi(x_j^1) \right|^p \right)^{1/p} \cdots \left(\sup_{\varphi \in B_{X_n^*}} \sum_{j=1}^{m} \left| \varphi(x_j^n) \right|^p \right)^{1/p}$$

for all $x_j^k \in X_j$, all $m \in \mathbb{N}$ and $(j, k) \in \{1, ..., m\} \times \{1, ..., n\}$. This concept is essentially due to Pietsch (see [3, 28]) and lifts several important properties of the original linear ideal of absolutely summing operators to the multilinear framework. The terminology "p-dominated", coined by M.C. Matos, is motivated by the following Pietsch-Domination type theorem:

Theorem 2.1 (Pietsch, Geiss [24])**.** *A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is p-dominated if and only if there exist $C \geq 0$ and regular probability measures μ_j on the Borel σ-algebras of $B_{X_j^*}$ endowed with the weak star topologies such that*

$$\| T(x_1, ..., x_n) \| \leq C \prod_{j=1}^{n} \left(\int_{B_{X_j^*}} |\varphi(x_j)|^p \, d\mu_j(\varphi) \right)^{1/p}$$

for every $x_j \in X_j$ and $j = 1, ..., n$.

Corollary 2.2. *If $1 \leq p \leq q < \infty$, then every p-dominated multilinear operator is q-dominated.*

The notion of p-semi-integral operator is another possible multilinear generalization of the class of absolutely summing linear operators. If $p \geq 1$, a continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is *p-semi-integral* if there exists a $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} \| T(x_j^1, ..., x_j^n) \|^p \right)^{1/p} \leq C \left(\sup_{(\varphi_1, ..., \varphi_n) \in B_{X_1^*} \times \cdots B_{X_n^*}} \sum_{j=1}^{m} | \varphi_1(x_j^1) ... \varphi_n(x_j^n) |^p \right)^{1/p}$$

for every $m \in \mathbb{N}$, $x_j^k \in X_k$ with $k = 1, ..., n$ and $j = 1, ..., m$.

This class dates back to the research report [3] of R. Alencar and M.C. Matos. As in the case of p-dominated multilinear operators, a Pietsch Domination theorem is valid in this context:

Theorem 2.3. *A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is p-semi-integral if and only if there exist $C \geq 0$ and a regular probability measure μ on the Borel $\sigma-$algebra $\mathcal{B}(B_{X_1^*} \times \cdots \times B_{X_n^*})$ of $B_{X_1^*} \times \cdots \times B_{X_n^*}$ endowed with the product of the weak star topologies*

$\sigma(X_l^*, X_l)$, $l = 1, ..., n$, *such that*

$$\| T(x_1, ..., x_n) \| \leq C \left(\int_{B_{X_1^*} \times \cdots \times B_{X_n^*}} | \varphi_1(x_1)...\varphi_n(x_n) |^p \, d\mu(\varphi_1, ..., \varphi_n) \right)^{1/p}$$

for all $x_j \in X_j$, $j = 1, ..., n$.

Corollary 2.4. *If* $1 \leq p \leq q < \infty$, *every* p*-semi-integral multilinear operator is* q*-semi-integral.*

This class is strongly connected to the class of p-dominated multilinear operators. For example, in [15] it is shown that every p-semi integral n-linear operator is np-dominated.

The following result shows that we cannot expect to lift coincidence results of the linear case to dominated multilinear operators:

Theorem 2.5 (Jarchow, Palazuelos, Pérez-García and Villanueva [26])**.** *For every* $n \geq 3$ *and every* $p \geq 1$ *and every infinite dimensional Banach space* X *there exists a continuous* n*-linear operator* $T : X \times \cdots \times X \to \mathbb{K}$ *that fails to be* p*-dominated.*

Since p-semi-integral n-linear operators are np-dominated, we have:

Corollary 2.6. *For every* $n \geq 3$, *every* $p \geq 1$ *and every infinite dimensional Banach space* X *there exists a continuous* n*-linear operator* $T : X \times \cdots \times X \to \mathbb{K}$ *that fails to be* p*-semi-integral.*

So, in view of the previous result, it is obvious that we cannot expect a Grothendieck type theorem for dominated or semi-integral operators. In this direction, the classes of multiple summing multilinear operators ([4, 29]), strongly multiple summing multilinear operators ([8]) and strongly summing multilinear operators ([20]) are other possible generalizations, with a quite better behavior if we are interested in lifting coincidence theorems, like Grothendieck's theorem. But, as a matter of fact, none of these classes lifts all the main properties of absolutely summing linear operators to the multilinear setting.

In [44], a variant of the notion of p-dominated polynomials which satisfy (in a very natural form) a Pietsch factorization type theorem, is introduced. A continuous n-homogeneous polynomial $P : X \to Y$ is *factorable* p*-dominated* if there is a $C \geq 0$ such that for every $x_j^i \in X$, and scalars λ_j^i, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$ and all positive integers m_1, m_2, we have

$$\left(\sum_{j=1}^{m_1} \left\| \sum_{i=1}^{m_2} \lambda_j^i P\left(x_j^i\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in B_{X^*}} \left(\sum_{j=1}^{m_1} \left| \sum_{i=1}^{m_2} \lambda_j^i \varphi\left(x_j^i\right)^n \right|^p \right)^{\frac{1}{p}}.$$

The natural multilinear version of the notion of "factorable p-dominated polynomials" seems to be:

Definition 2.7. *A continuous* n*-linear operator* $T : X_1 \times \cdots \times X_n \to Y$ *is factorable* p*-dominated if there is a constant* $C \geq 0$ *such that for every* $x_{k,j}^i \in X_k$, *and scalars* λ_j^i, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$ *and all positive integers* m_1, m_2, *we have*

$$\left(\sum_{j=1}^{m_1} \left\| \sum_{i=1}^{m_2} \lambda_j^i T\left(x_{1,j}^i, ..., x_{n,j}^i\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\substack{\varphi_k \in B_{X_k^*} \\ k=1,...,n}} \left(\sum_{j=1}^{m_1} \left| \sum_{i=1}^{m_2} \lambda_j^i \varphi_1\left(x_{1,j}^i\right) \cdots \varphi_n\left(x_{n,j}^i\right) \right|^p \right)^{\frac{1}{p}}.$$

These notions have some connection with the idea of weighted summability, sketched in [38]. It is likely that this class has a nice factorization theorem (like its polynomial version) but a simple calculation shows that any factorable p-dominated multilinear operator is p-semi-integral and thus we have:

Proposition 2.8. *For every $n \geq 3$ and every $p \geq 1$ and every infinite dimensional Banach space X there exists a continuous n-linear operator $T : X \times \cdots \times X \to \mathbb{K}$ that fails to be factorable p-dominated. A fortiori, regardless of the Banach space Y, there exists a continuous n-linear operator $T : X \times \cdots \times X \to Y$ that fails to be factorable p-dominated.*

So, since we are looking for classes that also lift coincidence results to the multilinear setting, the class of factorable p-dominated multilinear operators is not what we are searching.

A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is *strongly p-summing* if there exists a constant $C \geq 0$ such that

$$\left(\sum_{j=1}^{m} \| T(x_j^1, ..., x_j^n) \|^p\right)^{1/p} \leq C \left(\sup_{\phi \in B_{\mathcal{L}(X_1,...,X_n;\mathbb{K})}} \sum_{j=1}^{m} | \phi(x_j^1, ..., x_j^n) |^p\right)^{1/p}. \tag{2}$$

for every $m \in \mathbb{N}$, $x_j^k \in X_k$ with $k = 1, ..., n$ and $j = 1, ..., m$.

The class of strongly p-summing multilinear operators is due to V. Dimant [20] and according to [37, 39] it is perhaps the class that best translates to the multilinear setting the properties of the original linear concept. For example, a Grothendieck type theorem and a Pietsch-Domination type theorem are valid:

Theorem 2.9 (Grothendieck-type theorem [20])**.** *Every continuous n-linear operator $T : \ell_1 \times \cdots \times \ell_1 \to \ell_2$ is strongly 1-summing.*

Theorem 2.10 (Pietsch Domination type theorem [20])**.** *A continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is strongly p-summing if, and only if, there are a probability measure μ on $B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}$, with the weak-star topology, and a constant $C \geq 0$ so that*

$$\|T(x_1, ..., x_n)\| \leq C \left(\int_{B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}} |\varphi(x_1 \otimes \cdots \otimes x_n)|^p \, d\mu(\varphi)\right)^{\frac{1}{p}} \tag{3}$$

for all $(x_1, ..., x_n) \in X_1 \times \cdots \times X_n$.

Corollary 2.11. *If $p \leq q$ then every strongly p-summing multilinear operator is strongly q-summing.*

It is not hard to prove that a Dvoretzky-Rogers Theorem is also valid for this class:

Theorem 2.12 (Dvoretzky-Rogers type theorem)**.** *Every continuous n-linear operator $T : X \times \cdots \times X \to X$ is strongly p-summing if, and only if, $\dim X < \infty$.*

A property fulfilled by the class of absolutely summing operators which is not lifted to the multilinear framework by the notion of strong summability is the weak compactness. In fact, it is well known that every absolutely p-summing linear operator is weakly compact, but Carando and Dimant have shown that there exist strongly p-summing multilinear operators that fail to be weakly compact [16]. This result implies that a natural version of the Pietsch Factorization Theorem is not valid for strongly summing multilinear operators, as we will see below.

Suppose that the following factorization theorem holds: $T : X_1 \times \cdots \times X_n \to Y$ is strongly p-summing if and only if there is a regular Borel probability measure μ on $B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}$, with the weak-star topology, a closed subspace Z_p of $L_p\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}, \mu\right)$ and a continuous linear operator $\widehat{T} : Z_p \to Y$ such that

$$j_p \circ i_{X_1 \times \cdots \times X_n}(X_1 \times \cdots \times X_n) \subset Z_p \text{ and } \widehat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n} = T,$$

where

$$i_{X_1 \times \cdots \times X_n} : X_1 \times \cdots \times X_n \to C\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}\right)$$

is the canonical n-linear map $i_{X_1\times\cdots\times X_n}(x_1,...,x_n)(\varphi)=\varphi(x_1\otimes\cdots\otimes x_n)$ and

$$j_p : C\left(B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_n)^*}\right)\to L_p\left(B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_n)^*},\mu\right)$$

is the canonical linear inclusion.

Since j_p is absolutely p-summing (and thus weakly compact), then we conclude that the set $j_p\left(i_{X_1\times\cdots\times X_n}(B_{X_1}\times\cdots\times B_{X_n})\right)$ is relatively weakly compact in Z_p. Since $\widehat{T}$ is continuous and linear, then $T(B_{X_1},...,B_{X_n})=\widehat{T}\left(j_p\left(i_{X_1\times\cdots\times X_n}(B_{X_1}\times\cdots\times B_{X_n})\right)\right)$ is relatively weakly compact in Y and thus T is weakly compact, but this is not true in general ([16]).

In this paper we combine the idea of factorable summability from [44] with the notion of strongly p-summing multilinear operators and we show that the new class we introduce recovers all these lacks suffered by the former multilinear extensions. Indeed we will show that this class coincides with the class of composition of multilinear mappings with absolutely summing operators.

3. Factorable strongly p-summing multilinear operators

The following definition is inspired in ideas from [44], adapted to the notion of strongly summing multilinear operators:

Definition 3.1. *A continuous n-linear operator $T : X_1\times\cdots\times X_n\to Y$ is factorable strongly p-summing if there is a constant $C\geq 0$ such that for every $x_{k,j}^i\in X_k$, $1\leq j\leq m_1$, $1\leq i\leq m_2$ and all positive integers m_1,m_2, we have*

$$\left(\sum_{j=1}^{m_1}\left\|\sum_{i=1}^{m_2}T\left(x_{1,j}^i,...,x_{n,j}^i\right)\right\|^p\right)^{\frac{1}{p}}\leq C\sup_{\|\varphi\|\leq 1}\left(\sum_{j=1}^{m_1}\left|\sum_{i=1}^{m_2}\varphi\left(x_{1,j}^i,...,x_{n,j}^i\right)\right|^p\right)^{\frac{1}{p}}.$$

where the supremum is taken over all the continuous n-linear functionals $\varphi : X_1\times\cdots\times X_n\to\mathbb{K}$ of norm less or equal than 1. The class of all factorable strongly p-summing n-linear operators $T : X_1\times\cdots\times X_n\to Y$ is denoted by $\mathcal{L}_{FSt,p}(X_1,\ldots,X_n;Y)$ and endowed with the norm $\|\cdot\|_{FSt,p}$, where $\|T\|_{FSt,p}$ is given by the infimum of all constant C fulfilling the above inequality.

Note that if T is factorable strongly p-summing then making $m_2=1$ we have

$$\left(\sum_{j=1}^{m_1}\left\|T\left(x_{1,j}^1,\ldots,x_{n,j}^1\right)\right\|^p\right)^{1/p}\leq C\sup_{\|\varphi\|\leq 1}\left(\sum_{j=1}^{m_1}\left|\varphi\left(x_{1,j}^1,...,x_{n,j}^1\right)\right|^p\right)^{\frac{1}{p}},$$

i.e., T is strongly p-summing. In particular, whenever $n=1$, $\mathcal{L}_{FSt,p}(X_1;Y)=\Pi_p(X_1;Y)$ is the class of all absolutely p-summing operators from X_1 to Y.

The ideal property is straightforward. It is also trivial that every scalar-valued n-linear operator is factorable strongly p-summing. Straightforward calculations show that this class forms a Banach multi-ideal.

As we will see in Section 5, this class preserves the nice properties of the class of strongly summing multilinear operators and has extra desirable properties: weak compactness and a factorization theorem.

Theorem 3.2 (Pietsch-Domination type theorem)**.** *A continuous n-linear operator $T : X_1\times\cdots\times X_m\to Y$ is factorable strongly p-summing if and only if there is a regular probability measure μ on $B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_n)^*}$, endowed with the weak-star topology, and a constant $C\geq 0$, such that*

$$\left\|\sum_{i=1}^{m}T\left(x_1^i,...,x_n^i\right)\right\|\leq C\left(\int_{B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_n)^*}}\left|\sum_{i=1}^{m}\varphi\left(x_1^i,...,x_n^i\right)\right|^p d\mu\left(\varphi\right)\right)^{\frac{1}{p}}.$$

for every m *and every* $x_k^i \in X_k$, $1 \leq k \leq n$, $1 \leq i \leq m$.

Proof. The notion of factorable strongly p-summing multilinear operator is precisely the concept of RS-abstract p-summing (see [11, 36, 39]) for

$$R : B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*} \times (X_1 \times \cdots \times X_n)^{\mathbb{N}} \times \{0\} \to [0, \infty)$$

given by

$$R\left(\varphi, \left(x_1^1, ..., x_n^1\right), ..., \left(x_1^m, ..., x_n^m\right), 0\right) = \left|\sum_{i=1}^{m} \varphi\left(x_1^i \otimes \cdots \otimes x_n^i\right)\right|$$

and

$$S : \mathcal{L}\left(X_1, ..., X_n; Y\right) \times (X_1 \times \cdots \times X_n)^{\mathbb{N}} \times \{0\} \to [0, \infty)$$

given by

$$S\left(T, \left(x_1^1, ..., x_n^1\right), ..., \left(x_1^m, ..., x_n^m\right), 0\right) = \left\|\sum_{i=1}^{m} T\left(x_1^i, ..., x_n^i\right)\right\|.$$

Since R and S satisfy the hypotheses of the general Pietsch Domination Theorem, the result follows straightforwardly. □

Theorem 3.3 (Pietsch-Factorization type theorem)**.** *A continuous* n*-linear operator* $T : X_1 \times \cdots \times X_n \to Y$ *is factorable strongly* p*-summing if and only if there exist a regular probability measure* μ *on* $B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}$, *endowed with the weak-star topology, a constant* $C \geq 0$, *a closed subspace* Z_p *of* $L_p\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}, \mu\right)$ *and a continuous linear operator* $\widehat{T} : Z_p \to Y$ *such that*

$$j_p \circ i_{X_1 \times \cdots \times X_n}(X_1 \times \cdots \times X_n) \subset Z_p \text{ and } \widehat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n} = T,$$

where

$$i_{X_1 \times \cdots \times X_n} : X_1 \times \cdots \times X_n \to C\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}\right)$$

is the canonical n*-linear map* $i_{X_1 \times \cdots \times X_n}(x_1, ..., x_n)(\varphi) = \varphi(x_1 \otimes \cdots \otimes x_n)$ *and*

$$j_p : C\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}\right) \to L_p\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}, \mu\right)$$

is the canonical linear inclusion.

Proof. Suppose that T is factorable strongly p-summing. Let μ be the measure given by the Pietsch Domination Theorem (Theorem 3.2) applied to T. Let W_p be the subspace of $L_p\left(B_{(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_n)^*}, \mu\right)$ given by the linear span of $j_p \circ i_{X_1 \times \cdots \times X_n}(X_1 \times \cdots \times X_n)$. Define the linear operator $\widehat{T} : W_p \to Y$ by

$$\widehat{T}(z) = \sum_{i=1}^{n} \lambda_i T\left(x_1^i, ..., x_n^i\right)$$

for

$$z = \sum_{i=1}^{n} \lambda_i \langle \cdot, \left(x_1^i \otimes \cdots \otimes x_n^i\right)\rangle \in W_p.$$

Note that $\widehat{T}$ is well-defined. In fact, if

$$z_1 = \sum_{i=1}^{m_1} \lambda_i \langle \cdot, \left(x_1^i \otimes \cdots \otimes x_n^i\right)\rangle \text{ and } z_2 = \sum_{i=1}^{m_2} \alpha_i \langle \cdot, \left(y_1^i \otimes \cdots \otimes y_n^i\right)\rangle$$

coincide in W_p, then considering

$$w := \sum_{i=1}^{m_1} \lambda_i \langle \cdot, \left(x_1^i \otimes \cdots \otimes x_n^i\right) \rangle - \sum_{i=1}^{m_2} \alpha_i \langle \cdot, \left(y_1^i \otimes \cdots \otimes y_n^i\right) \rangle,$$

we have $w = 0$ almost everywhere in W_p, i.e.,

$$\int_{B_{(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^*}} \left| \sum_{i=1}^{m_1} \lambda_i \varphi\left(x_1^i \otimes \cdots \otimes x_n^i\right) - \sum_{i=1}^{m_2} \alpha_i \varphi\left(y_1^i \otimes \cdots \otimes y_n^i\right) \right|^p d\mu(\varphi) = 0.$$

Thus, from the domination theorem,

$$\begin{aligned} &\left\| \sum_{i=1}^{m_1} \lambda_i T\left(x_1^i, ..., x_n^i\right) - \sum_{i=1}^{m_2} \alpha_i T\left(y_1^i, ..., y_n^i\right) \right\| \\ &\leq C \left(\int_{B_{(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^*}} \left| \sum_{i=1}^{m_1} \lambda_i \varphi\left(x_1^i \otimes \cdots \otimes x_n^i\right) - \sum_{i=1}^{m_2} \alpha_i \varphi\left(y_1^i \otimes \cdots \otimes y_n^i\right) \right|^p d\mu(\varphi) \right)^{\frac{1}{p}} = 0 \end{aligned}$$

and we conclude that

$$\widehat{T}(z_1) - \widehat{T}(z_2) = \sum_{i=1}^{m_1} \lambda_i T\left(x_1^i, ..., x_n^i\right) - \sum_{i=1}^{m_2} \alpha_i T\left(y_1^i, ..., y_n^i\right) = 0.$$

Note also that for $z = \sum_{i=1}^{m} \lambda_i \langle \cdot, \left(x_1^i \otimes \cdots \otimes x_n^i\right) \rangle \in W_p$ we have

$$\begin{aligned} \left\| \widehat{T}(z) \right\| &= \left\| \sum_{i=1}^{m} \lambda_i T\left(x_1^i, ..., x_n^i\right) \right\| \\ &\leq C \left(\int_{B_{(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n)^*}} \left| \sum_{i=1}^{m} \lambda_i \varphi\left(x_1^i \otimes \cdots \otimes x_n^i\right) \right|^p d\mu(\varphi) \right)^{\frac{1}{p}} \\ &= C \left\| z \right\|_{L_p(\mu)} \end{aligned}$$

and $\widehat{T}$ is continuous. It is obvious that from the very definition of $\widehat{T}$ we have $\widehat{T} \circ j_p \circ i_{X_1 \times \cdots \times X_n} = T$. Now we extend $\widehat{T}$ to $Z_p = \overline{W_p}$. The converse is immediate. □

4. Factorable strongly p-summing polynomials

The m-fold symmetric tensor product of X is the linear span of all tensors of the form $x \otimes \cdots \otimes x$, $x \in X$, and is denoted by $\otimes^{m,s} X$. This space is endowed with the s-projective tensor norm, defined as

$$\pi_s(z) = \inf\{\sum_{j=1}^{k} |\lambda_j| \|x_j\|^n : k \in \mathbb{N}, z = \sum_{j=1}^{k} \lambda_j x_j \otimes \cdots \otimes x_j\},$$

for $z \in \otimes_{m,s} X$. Let $\hat{\otimes}_{\pi_s}^{m,s} X$ denote the completion of $\otimes_{\pi_s}^{m,s} X$.

Given $P \in \mathcal{P}(^m X; Y)$, the linearization of P is the unique linear operator $P_{L,s} : \hat{\otimes}_{\pi_s}^{m,s} X \to Y$ such that $P_{L,s}(x \otimes \cdots \otimes x) = P(x)$ for all $x \in X$. Ryan [46] proved that the correspondence $P \leftrightarrow P_{L,s}$ establishes a isometric isomorphism between the space $\mathcal{P}(^m X)$, endowed with the usual sup norm, and the strong dual of $\hat{\otimes}_{\pi_s}^{m,s} X$. Another map associated to $P \in \mathcal{P}(^m X; Y)$ is the unique continuous symmetric m-linear mapping $\check{P}$ that satisfies $\check{P}(x, \ldots, x) = P(x)$, for all $x \in X$. It is well known that $\|\check{P}\| \leq c(m, X)\|P\|$ for all $P \in \mathcal{P}(^m X)$, where $c(m, X)$ is the m-th polarization constant of X. For the general theory of homogeneous polynomials we refer to [21] and [33].

Concomitantly to multilinear mappings, factorable strongly p-summing homogeneous polynomials can be introduced. Our aim is to prove that both classes coincide in the sense that a polynomial is factorable strongly p-summing if and only if its associated symmetric multilinear mapping is factorable strongly p-summing. Moreover, we will see the deep relationship between factorable strong summability and absolute summability by proving that, for an homogeneous polynomial, it is equivalent that the polynomial is factorable strongly summing to that its linearization is an absolutely summing operator. To attain this purpose, we will show that both, factorable strongly p-summing polynomials and factorable strongly p-summing multilinear operators, form composition ideals. It is worth mentioning that the linearization of a dominated polynomial is not necessarily absolutely summing.

Definition 4.1. *A continuous n-homogeneous polynomial $P : X \to Y$ is factorable strongly p-summing if there is a $C \geq 0$ such that for every $x_j^i \in X$, and scalars λ_j^i, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$ and all positive integers m_1, m_2, we have that*

$$\left(\sum_{j=1}^{m_1} \left\| \sum_{i=1}^{m_2} \lambda_j^i P(x_j^i) \right\|^p \right)^{1/p} \leq C \sup_{\|q\| \leq 1, q \in \mathcal{P}(^m X)} \left(\sum_{j=1}^{m_1} \left| \sum_{i=1}^{m_2} \lambda_j^i q(x_j^i) \right|^p \right)^{1/p} .$$

The class of all factorable strongly p-summing m-homogeneous polynomials from X to Y is denoted by $\mathcal{P}_{FSt,p}(^m X; Y)$ and endowed with the norm $\|\cdot\|_{FSt,p}$ given by the infimum of all constants C fulfilling the above inequality.

It is clear that factorable p-dominated polynomials are factorable strongly p-summing. An easy calculation shows the following ideal property:

Proposition 4.2. *If $P \in \mathcal{P}_{FSt,p}(^m X; Y)$ and $u : G \to X$, $v : Y \to Z$ are continuous linear operators then $v \circ P \circ u \in \mathcal{P}_{FSt,p}(^m G; Z)$ and $\|v \circ P \circ u\|_{FSt,p} \leq \|v\| \cdot \|P\|_{FSt,p} \|u\|^m$.*

It is not difficult to complete Proposition 4.2 and show that factorable strongly n-homogeneous polynomials form an ideal of polynomials (for the definition of ideal of polynomials we refer to [5]).

Dimant [20] introduced the class of *strongly p-summing* m-homogeneous polynomials from X to Y as those m-homogeneous polynomials $P : X \to Y$ that satisfy that there exists $K > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$,

$$(\sum_{j=1}^{n} \|P(x_j)\|^p)^{1/p} \leq K \sup_{\|q\| \leq 1, q \in \mathcal{P}(^m X)} (\sum_{j=1}^{n} |q(x_j))|^p)^{1/p}.$$

In [20, Proposition 3.2] it is proved that if the linearization $P_{L,s}$ of $P \in \mathcal{P}(^m X; Y)$ is absolutely p-summing then p is strongly p-summing. However, the converse is not true (see [16, Example 3.3]). The reason, as for p-dominated polynomials, is that not every strongly p-summing polynomial is weakly compact. So, once again, the lack of connection with weak compactness turns out to be a deep inconvenience in the way that strongly p-summing polynomials generalize absolutely p-summing linear operators. Even if a domination holds also for strongly p-summing polynomials [20, Proposition 3.2], no factorization theorem is expected. Let us prove a factorization theorem for factorable strongly p-summing polynomials. We first need a domination theorem, that is obtained as a particular case of [11, Theorem 2.2]. We denote by $\delta : X \to C(B_{\mathcal{P}(^m X)})$ the m-homogeneous polynomial given by $\delta(x) := \delta_x : B_{\mathcal{P}(^m X)} \to \mathbb{K}$, where $\delta_x(P) := P(x)$. Considering that the space of continuous m-homogeneous polynomials is a dual space (see [46]), its closed unit ball $B_{\mathcal{P}(^m X)}$ is a weak-star compact set.

Theorem 4.3 (Pietsch-Domination type theorem)**.** *Let $P \in \mathcal{P}(^m X; Y)$. Then P is factorable strongly p-summing if and only if there exists a regular Borel probability measure μ*

on $B_{\mathcal{P}(^mX)}$, endowed with the weak-star topology, such that

$$\|\sum_{i=1}^{k}\lambda^i P(x^i)\| \leq C(\int_{B_{\mathcal{P}(^mX)}} |\sum_{i=1}^{k}\lambda^i q(x^i)|^p \, d\mu)^{1/p}$$

for all $x^1,\dots,x^k \in X$ and $\lambda^1,\dots,\lambda^k \in \mathbb{K}$.

Proof. It is a particular case of [11, Theorem 2.2] analogous to the proof of Theorem 3.2. □

We shall need the following result to prove the sufficiency of the Factorization Theorem. When dealing with polynomials, the main difficulty to obtain a factorization theorem is to prove that the linear operator that closes the diagram is well defined. We will see that the class of factorable strongly p-summing polynomials fits perfectly to recover this handicap. Besides, Proposition 4.4 will be the key for our purposes to obtain that factorable strongly p-summing homogeneous polynomials form a composition ideal. We include the proof for the sake of completeness.

Proposition 4.4. *If $Q \in \mathcal{P}(^mG;X)$ and $u : X \to Y$ is an absolutely p-summing linear operator, then $u \circ Q \in \mathcal{P}_{FSt,p}(^mG;Y)$ and $\|u\circ Q\|_{FSt,p} \leq \pi_p(u)\|Q\|$.*

Proof. Let m_1, m_2 be positive integers, $x_j^i \in X$, and scalars λ_j^i, $1 \leq j \leq m_1$, $1 \leq i \leq m_2$. Then,

$$\begin{aligned}(\sum_{j=1}^{m_1}\|(\sum_{i=1}^{m_2}\lambda_j^i u\circ Q(x_j^i)\|^p)^{1/p} &= (\sum_{j=1}^{m_1}\|u(\sum_{i=1}^{m_2}\lambda_j^i Q(x_j^i)\|^p)^{1/p}\\ &\leq \pi_p(u) \sup_{\|x^*\|\leq 1, x^*\in X^*}(\sum_{j=1}^{m_1}|\langle x^*, \sum_{i=1}^{m_2}\lambda_j^i Q(x_j^i)\rangle|^p)^{1/p}\\ &\leq \pi_p(u)\|Q\| \sup_{\|x^*\|\leq 1, x^*\in X^*}(\sum_{j=1}^{m_1}|\sum_{i=1}^{m_2}\lambda_j^i\langle x^*, Q/\|Q\|(x_j^i)\rangle|^p)^{1/p}\\ &\leq \pi_p(u)\|Q\| \sup_{\|q\|\leq 1, q\in\mathcal{P}(^mG)}(\sum_{j=1}^{m_1}|\sum_{i=1}^{m_2}\lambda_j^i q(x_j^i)|^p)^{1/p}.\end{aligned}$$

□

Theorem 4.5 (Pietsch-Factorization type theorem)**.** *Let $P \in \mathcal{P}(^mX;Y)$. Then P is factorable strongly p-summing if and only if there exists a regular Borel probability measure μ on $B_{\mathcal{P}(^mX)}$, a closed subspace G_p of $L_p(\mu)$ and a continuous linear operator $v_0 : G_p \to Y$ such that $j_p \circ \delta(X) \subset G_p$ and $v_0 \circ j_p \circ \delta = P$, where $j_p : C(B_{\mathcal{P}(^mX)}) \to L_p(B_{\mathcal{P}(^mX)}, \mu)$ is the canonical inclusion.*

Proof. Assume first that P is factorable strongly p-summing. Let μ be given by Theorem 4.3. Take G_p the completion of the image by j_p of the linear span of $\delta(X)$. Define $v_0(j_p(\sum_{i=1}^k \lambda_i\delta_{x_i})) := \sum_{i=1}^k \lambda_i P(x_i)$. To see that v_0 is well defined, consider that $j_p(\sum_{i=1}^k \lambda_i\delta_{x_i}) = j_p(\sum_{i=1}^l \eta_i\delta_{y_i})$. Then $w := \sum_{i=1}^k \lambda_i\delta_{x_i} - \sum_{i=1}^l \eta_i\delta_{y_i} = 0$ a.e. on $B_{\mathcal{P}(^mX)}$. Hence,

$$\|\sum_{i=1}^{k}\lambda_i P(x_i) - \sum_{i=1}^{l}\eta_i P(y_i)\| \leq \|P\|_{FSt,p}(\int_{B_{\mathcal{P}(^mX)}}|\sum_{i=1}^{k}\lambda_i q(x_i) - \sum_{i=1}^{l}\eta_i q(y_i)|^p \, d\mu)^{1/p} = 0.$$

Thus, $v_0(w) = 0$. That proves that v_0 is well defined. The continuity of v_0 follows from the calculations:

$$\|v_0(z)\| = \|\sum_{i=1}^{k} \lambda_i P(x_i)\| \leq \|P\|_{FSt,p}(\int_{B_{\mathcal{P}(^m X)}} |\sum_{i=1}^{k} \lambda_i q(x_i)|^p d\mu)^{1/p}$$

$$= \|P\|_{FSt,p}\|\sum_{i=1}^{k} \lambda_i \delta_{x_i}\|_{L_p(\mu)} = \|P\|_{FSt,p}\|z\|_{L_p(\mu)}$$

for any $z = j_p(\sum_{i=1}^{k} \lambda_i \delta_{x_i})$. The desired linear operator is just the continuous extension of v_0 to G_p. The converse follows from Proposition 4.4. □

Corollary 4.6. *Let $P \in \mathcal{P}(^m X; Y)$. Then $P \in \mathcal{P}_{FSt,p}(^m X; Y)$ if and only if $P = u \circ Q$, for some continuous m-homogeneous polynomial Q and some absolutely p-summing linear operator u. In that case $\|P\|_{FSt,p} = \inf\{\pi_p(u)\|Q\| : P = u \circ Q\}$.*

Proof. It follows from Theorem 4.5 and Proposition 4.4. □

Corollary 4.6 says that the ideal of all factorable strongly p-summing m-homogeneous polynomials is the composition ideal with all absolutely p-summing linear operators, that is, $\mathcal{P}_{FSt,p} = \Pi_p \circ \mathcal{P}$ (see [10]). An analogous argument for multilinear operators instead of polynomials yields to prove that the ideal of all factorable strongly p-summing m-linear operators is the composition ideal with all absolutely p-summing linear operators, that is, $\mathcal{L}_{FSt,p} = \Pi_p \circ \mathcal{L}$.

Remark 4.7. *In [31] it is shown an example of a continuous m-homogeneous polynomial $P : X \to Y$ and $\phi \in \Pi_r(Y; Z)$ such that $\phi \circ P : X \to Z$ is not r-dominated. By Proposition 4.4, $\phi \circ P$ is factorable strongly r-summing. Therefore, the class of dominated polynomials differs from the class of factorable strongly r-summing polynomials.*

An application of Corollary 4.6 and [10, Proposition 3.2(b)] yields the announced characterization:

Proposition 4.8. *Let $P \in \mathcal{P}(^m X; Y)$. The following are equivalent:*

(1) $P \in \mathcal{P}_{FSt,p}(^m X; Y)$.
(2) $P_{L,s}$ *is absolutely p-summing.*
(3) $\check{P} \in \mathcal{L}_{FSt,p}(^m X; Y)$.

In that case, $\|P\|_{FSt,p} = \pi_p(P_{L,s})$.

5. The wealth of factorable strong p-summability

In this section it is shown that factorable strong p-summability is an excellent non linear frame where linear results for absolute summability are properly generalized to multilinear operators and polynomials. This evidences the interest of this new class as it really reflects the good behavior of absolute summability in the non linear context. Some of these results are established for multilinear operators and some for homogeneous polynomials. However, as a consequence of Proposition 4.8 it is clear that one can pass easily from one to each other.

Proposition 4.8 spreads open the way to lift the classical results for linear operators to the polynomial and multilinear setting. A good example is the following Grothendieck type theorem.

Theorem 5.1 (Grothendieck type theorem)**.** *If $m \geq 1$ is a positive integer, then $\mathcal{P}(^m \ell_1; \ell_2) = \mathcal{P}_{FSt,1}(^m \ell_1; \ell_2)$.*

Proof. Let $P \in \mathcal{P}(^m X;Y)$. Then $P_{L,s} \in \mathcal{L}(\hat{\otimes}^{m,s}_{\pi_s}\ell_1;\ell_2) = \mathcal{L}(\ell_1;\ell_2) = \Pi_1(\ell_1;\ell_2)$. Theorem 4.8 yields the result. □

Using this technique that combines Proposition 4.8 and their linear analogs we can now lift many other classical results (see [19, Theorems 3.15 and 3.17]). Most of the proofs are then omitted.

Proposition 5.2 (Composition Theorem). *If $u \in \Pi_p(X;Y)$ and $P \in \mathcal{P}_{FSt,q}(^m G;X)$ then $u \circ P \in \mathcal{P}_{FSt,r}(^m G;Y)$ for $1/r := \min\{1, 1/p + 1/q\}$.*

Proof. By Theorem 4.8 $P_{L,s}$ is absolutely q-summing. Then $u \circ P_{L,s}$ is r-summing for $1/r := \min\{1, 1/p + 1/q\}$ (see [19, Theorem 2.22]). Since $u \circ P_{L,s} = (u \circ P)_{L,s}$, a second application of Theorem 4.8 yields the result. □

Theorem 5.3 (Extrapolation type theorem). *Let $1 < r < p < \infty$, and let X be a Banach space. If $\mathcal{P}_{FSt,p}(^m X;\ell_p) = \mathcal{P}_{FSt,r}(^m X;\ell_p)$ then $\mathcal{P}_{FSt,p}(^m X;Y) = \mathcal{P}_{FSt,1}(^m X;Y)$ for every Banach space Y.*

Recall that given $1 \le p \le \infty$ and $\lambda > 1$, a Banach space X is said to be an *$\mathcal{L}_{p,\lambda}$-space* if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X for which there is an isomorphism $v : F \to \ell_p^{\dim F}$ with $\|v\| \cdot \|v^{-1}\| < \lambda$.

Theorem 5.4 (Lindenstrauss–Pełczyński type theorem). *Let $1 \le p \le 2$ and $2 < q < \infty$. If X is a Banach space and Y is a subspace of an $\mathcal{L}_{p,\lambda}$-space, then $\mathcal{P}_{FSt,q}(^m X;Y) = \mathcal{P}_{FSt,2}(^m X;Y)$.*

We have already proved that Domination/Factorization Theorems are fulfilled in the multilinear and polynomial classes of factorable strongly p-summing maps. As a straightforward consequence of the Factorization Theorem 4.5 we get

Theorem 5.5. *Any factorable strongly p-summing polynomial is weakly compact.*

An alternative way to prove it is the following: by Theorem 4.8 the linearization of a factorable strongly p-summing polynomial P is absolutely p-summing and hence weakly compact. By [46] this is equivalent to the weak compactness of P. The same holds for the case of multilinear operators.

The Domination Theorem 3.2 also yields to the following inclusion theorem.

Proposition 5.6 (Inclusion Theorem). *If $1 \le p \le q < \infty$ then every factorable strongly p-summing polynomial is factorable strongly q-summing.*

The forthcoming lemmas 5.7, 5.10 and its consequences show that, besides its good properties, the classes of factorable strongly p-summing multilinear operators and polynomials have a coherent size.

Lemma 5.7. *If every continuous n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is factorable strongly p-summing, then every continuous linear operator $u_j : X_j \to Y$ is absolutely p-summing for every $j = 1, ..., n$.*

Proof. Since $\Pi_p \circ \mathcal{L} = \mathcal{L}_{FSt,p}$, it follows from [10, Lemma 3.4]. □

The following two theorems are immediate consequences of the previous lemma and of the respective linear results (see also [10, Propositions 5.3 and 5.5]):

Theorem 5.8 (Dvoretzky-Rogers type theorem). *Let Y be a Banach space. Every continuous n-linear operator $T : Y \times \cdots \times Y \to Y$ is factorable strongly p-summing if, and only if,* $\dim Y < \infty$.

Theorem 5.9 (Lindenstrauss–Pełczyński type theorem)**.** *Let m be a positive integer. If X and Y are infinite-dimensional Banach spaces, X has an unconditional Schauder basis and $\mathcal{L}_{FSt,1}(^mX;Y) = \mathcal{L}(^mX;Y)$ then $X = \ell_1$ and Y is a Hilbert space.*

For polynomials we have a natural version of Lemma 5.7:

Lemma 5.10. *If every continuous n-homogeneous polynomial $P : X \to Y$ is factorable strongly p-summing, then every continuous linear operator $u : X \to Y$ is absolutely p-summing.*

Proof. Let $u : X \to Y$ be a continuous linear operator and $\varphi \in X^*$ be a non-null linear functional and $a \in X$ be so that $\varphi(a) = 1$. Then $P(x) := u(x)\varphi(x)^{n-1}$ is factorable strongly p-summing. Thus $\check{P}$ is factorable strongly p-summing. From the proof of Lemma 5.7 we conclude that the linear operator $v : X \to Y$ defined by $v(x) = \check{P}(a, ..., a, x)$ is absolutely p-summing. But v is a linear combination of $u(a)\varphi$ and u; since $u(a)\varphi$ is absolutely p-summing it follows that u is absolutely p-summing. □

An immediate consequence of the previous lemma is that the analogs of theorems 5.8 and 5.9 work for polynomials. For instance:

Theorem 5.11 (Dvoretzky-Rogers type theorem for polynomials)**.** *Let Y be a Banach space. Every continuous n-homogeneous polynomial $P : Y \to Y$ is factorable strongly p-summing if, and only if, $\dim Y < \infty$.*

Given $P \in \mathcal{P}(^mX;Y)$ let us consider its transpose $P^t : Y^* \to \mathcal{P}(^mX)$ given by $P^t(y^*) := y^* \circ P$. Note that P^t is a continuous linear operator. Let $P^{tt} : \mathcal{P}(^mX)^* \to Y^{**}$ be the transpose of P^t. It is well known (see [19, Theorem 2.21]) that, if $Y = H$ is a Hilbert space then a continuous linear operator is absolutely 1-summing whenever its transpose is absolutely p-summing for some $1 \leq p < \infty$. Let us see that the analogous result is true for polynomials.

Proposition 5.12. *Let H be a Hilbert space and $P \in \mathcal{P}(^mX;H)$. If $P^t \in \Pi_p(\hat{\otimes}^{m,s}_{\pi_s} X;H)$ for some $1 \leq p < \infty$ then $P \in \mathcal{P}_{FSt,1}(^mX;H)$.*

Proof. From the equality $P^t_{L,s} = \delta \circ P^t$, where $\delta : \mathcal{P}(^mX) \to \left(\hat{\otimes}^{m,s}_{\pi_s} X\right)^*$ is the canonical isomorphism, it follows that $P^t_{L,s}$ is absolutely p-summing and then $P_{L,s}$ is absolutely 1-summing. By Theorem 4.8 we conclude that P is factorable strongly 1-summing. □

Proposition 5.13. *Let $P \in \mathcal{P}(^mX;Y)$. Then $P \in \mathcal{P}_{FSt,p}(^mX;Y)$ if and only if $P^{tt} \in \Pi_p(\mathcal{P}(^mX)^*;Y^{**})$.*

Proof. It is a consequence of Theorem 4.8, the fact that $P^{tt}_{L,s} = P^{tt} \circ \delta^t$ and the analogous well known property for linear operators (see [19, Proposition 2.19]). □

6. Coherence and compatibility

Let us denote the ideal of factorable strongly p-summing n-homogeneous polynomials by $\mathcal{P}^n_{FSt,p}$, whereas $\mathcal{L}^n_{FSt,p}$ denotes the ideal of factorable strongly p-summing n-linear operators. The notions of coherent and compatible ideals of polynomials were introduced by Carando, Dimant and Muro [17] in order to evaluate what polynomial approaches preserve the spirit of a given operator ideal. Standard calculations show that $\left(\mathcal{P}^n_{FSt,p}\right)^{\infty}_{n=1}$ is coherent and compatible with Π_p. Very recently, in [35], the notions of coherence and compatibility were extended to pairs of ideals of polynomials and multi-ideals. It is also possible to show that $\left(\mathcal{P}^n_{FSt,p}, \mathcal{L}^n_{FSt,p}\right)^{\infty}_{n=1}$ is coherent and compatible with Π_p.

We have shown that $\mathcal{P}^n_{FSt,p}$ coincides with the composition ideal with the absolutely p-summing operators. However, we cannot apply [35, Theorem 5.7] to get the coherence and compatibility as the topology involved in that result comes from the multilinear operators space norm and it does not coincide with $\|\cdot\|_{FSt,p}$ (see [10]). Despite of this, standard calculations also allow to get the following:

Theorem 6.1. *The sequence* $\left(\mathcal{P}^n_{FSt,p}, \mathcal{L}^n_{FSt,p}\right)_{n=1}^{\infty}$ *is coherent and compatible with* Π_p.

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