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A class of four parametric with- and without-memory root finding methods

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In this paper, we have constructed a derivative-free weighted eighth-order iterative method with and without memory for solving nonlinear equations. This method is an optimal method as it satisfies the Kung-Traub conjecture. We have used four accelerating parameters, a univariate and a multivariate weight function at the second and third step of the method, respectively. This method is converted into with-memory method by approximating the parameters using Newton's interpolating polynomials of appropriate degree to increase the order of convergence to 15.51560 and the efficiency index is nearly two. Numerical comparison of our methods is done with the recent methods of respective domain.

KEYWORDS
iterative methods, Kung-Traub conjecture, with-memory method

1 | INTRODUCTION

Finding solution of nonlinear equations $f(x) = 0$ is an important problem in various branches of science and engineering. In the past decades, without-memory methods were considered enough for the approximation of roots. However, in the recent years, without-memory methods which are extendable to with-memory methods without insertion of any extra functional evaluation by using Newton's interpolating polynomials have gained attention. These iterative methods offer a choice to achieve higher convergence order and increased efficiency. A first attempt of this kind is known from that of Traub[1] who gave the first with-memory scheme by modifying Steffensen's iterative scheme[2]

$$
\begin{aligned}
w_n &= x_n + p_n f(x_n), p_n \neq 0,\\
x_{n+1} &= x_n - \frac{f(x_n)}{f[x_n, w_n]}, n \geq 0,
\end{aligned}
$$

where x_0, p_0 are given and p_n is a self-accelerating parameter given by

$$
p_n \approx -\frac{1}{N_1'(x_n)},
$$

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where $N_1(x)$ is the first degree Newton's interpolating polynomial given by

$$N_1(x) = f(x_n) + (x - x_n)f[x_n, x_{n-1}].$$

Order of convergence of this method is 2.41. Some three-step optimal eighth-order iterative methods have been developed recently containing four parameters.[3-5]

In this paper, we construct a new four-parametric three-step with and without-memory family of methods with high efficiency index. The scheme is constructed by modifying optimal fourth-order two-step King's method[6] to an optimal eighth-order three-step derivative-free method by the insertion of a new substep using four parameters and two weight functions of one and two variables. Some special cases from the new scheme are compared with the methods of respective domain using standard nonlinear test functions.

2 | DERIVATIVE-FREE THREE-STEP OPTIMAL EIGHTH-ORDER CLASS INVOLVING WEIGHT FUNCTIONS OF ONE AND TWO VARIABLES

The main purpose of this section is to construct an eighth-order derivative-free family of iterative methods by using four parameters and two weight functions depending on one and two variables. This class can be modified in order to be extended as a family of iterative schemes with memory. The order of convergence is increased by means of accelerating parameters. The optimal fourth-order two-step King's method[6] is

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \\ x_{n+1} &= y_n - \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad \gamma \in \mathbb{R}. \end{aligned}$$

By adding a Newton-type third step, we have

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, n \geq 0, \\ z_n &= y_n - \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2) f(y_n)} \frac{f(y_n)}{f'(x_n)}, \quad \gamma \in \mathbb{R}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}. \end{aligned}$$

The values of the first derivative $f'(x_n)$ involved in the iterative expression are approximated by

$$f'(x_n) \approx f[x_n, w_n] + bf(x_n),$$

at the first step,

$$f'(x_n) \approx \frac{f[y_n, w_n] + bf(w_n) + q(y_n - w_n)(y_n - x_n)}{S(u_n)}$$

at the second step, and

$$f'(x_n) \approx \frac{f[y_n, z_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + d(z_n - w_n)(z_n - y_n)(z_n - x_n)}{J(u_n, t_n)},$$

at the third one, where $u_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$ and $\gamma \in \mathbb{R}$. In addition, a, b, d, and q are real free parameters, $S(u_n)$ and $J(u_n, t_n)$ are weight functions.

By applying the above procedure, the resulting three-step derivative-free family of iterative methods without memory is

$$\begin{aligned} w_n &= x_n + af(x_n), n \geq 0, \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + bf(w_n)}, \\ z_n &= y_n - S(u_n) \frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2) f(y_n)} \frac{f(y_n)}{f[y_n, w_n] + bf(w_n) + q(y_n - w_n)(y_n - x_n)}, \\ x_{n+1} &= z_n - J(u_n, t_n) \frac{f(z_n)}{P_n}, \end{aligned} \tag{1}$$

where $u_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$, $\gamma \in \mathbb{R}$, $P_n = f[y_n, z_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + d(z_n - w_n)(z_n - y_n)(z_n - x_n)$, a, b, q, and d are real free parameters and weight functions $S(u_n)$ and $J(u_n, t_n)$ are chosen in such a way that the members of the class are of optimal order eight. To show the order of convergence of class (1), the following result is proven.

Theorem 1. *Let us suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a real-valued nonlinear sufficiently differentiable function and σ is a real simple zero of the function in the open interval I. If the initial guess x_0 is close enough to σ, then all the members of the class of iterative methods (1) have optimal order eight, if weight functions $S(u_n)$ and $J(u_n, t_n)$ satisfy conditions*

$$S(0) = 1, \ \ S'(0) = -1, \ \ S''(0) = -2$$

and

$$J(0,0) = 1, \ \ J_u(0,0) = 0, \ \ J_t(0,0) = 0, \ \ J_{u,u}(0,0) = 0 \ \text{ and } \ J_{u,u,u}(0,0) = 0.$$

Then, the error equation of the family is

$$\begin{aligned} e_{n+1} = {} & \frac{1}{(f'(\sigma))^2}(c_2+b)^2\left(1+a\left(f'(\sigma)\right)^4\right) \\ & \left(2\gamma a\left(f'(\sigma)\right)^2\left(b^2+2bc_2+c_2^2\right)+f'(\sigma)\left(2\gamma b^2+2bc_2+4\gamma bc_2-c_3+2c_2^2+2\gamma c_2^2+q\right)\right) \\ & \Big(-d+f'(\sigma)c_4+qc_2-f'(\sigma)c_2c_3+2f'(\sigma)c_2^3+2bc_2^2f'(\sigma)+2\gamma f'(\sigma)c_2^3+4f'(\sigma)bc_2^2 \\ & +2c_2\left(f'(\sigma)\right)^2+2\gamma c_2^3a\left(f'(\sigma)\right)^2+4\gamma bc_2^2a\left(f'(\sigma)\right)^2+2c_2\gamma b^2a\left(f'(\sigma)\right)^2\Big)e_n^8+O\left(e_n^9\right), \end{aligned} \tag{2}$$

where $c_k = \frac{f^{(k)}(\sigma)}{k!f'(\sigma)}$, $k \geq 2$.

Proof. Let us define the error at the nth step as $e_n = x_n - \sigma$. By using Taylor series, we expand $f(x)$ about the real root σ as

$$f(x_n) = f'(\sigma)\left(e_n + c_2e_n^2 + c_3e_n^3 + \dots + O\left(e_n^9\right)\right), \tag{3}$$

where $c_k = \frac{f^{(k)}(\sigma)}{k!f'(\sigma)}$, $k \geq 2$. By using Taylor series, the error term $e_{n,w} = w_n - \sigma = e_n + af(x_n)$ is

$$e_{n.w} = (1+af'(\sigma))e_n + af'(\sigma)c_2e_n^2 + \dots + O\left(e_n^9\right).$$

Thus,

$$f(w_n) = f'(\sigma)\left[(1+af'(\sigma))e_n + af'(\sigma)c_2e_n^2 + \dots + O\left(e_n^9\right)\right].$$

Therefore,

$$\begin{aligned} f[x_n, w_n] + bf(w_n) = {} & f'(\sigma) + f'(\sigma)\left[b(1+af'(\sigma)) + c_2(2+af'(\sigma))\right]e_n \\ & + f'(\sigma)\left[f'(\sigma)a(3+af'(\sigma)) + (c_3+bc_2) + 3c_3 + c_2\left(b+ac_2^2f'(\sigma)\right)\right]e_n^2 + \ \dots \ + O\left(e_n^9\right). \end{aligned} \tag{4}$$

By using (3) and (4), we have

$$\frac{f(x_n)}{f[x_n, w_n] + bf(w_n)} = e_n - (c_2+b)(1+af'(\sigma))e_n^2 + \dots + O\left(e_n^9\right).$$

Now, by denoting $e_{n,y} = y_n - \sigma$, the error at the second step is

$$e_{n,y} = (c_2+b)(1+af'(\sigma))e_n^2 + \dots + O\left(e_n^9\right).$$

Therefore,

$$f(y_n) = c_1\left[(c_2+b)(1+af'(\sigma))e_n^2 + \ \dots \ + O\left(e_n^9\right)\right].$$

Then, if $Q = f[y_n, w_n] + bf(w_n) + q(y_n - w_n)(y_n - x_n)$, we have

$$\begin{aligned} Q = {} & f'(\sigma) + f'(\sigma)(c_2+b)(1+af'(\sigma))e_n \\ & + f'(\sigma)\Big[2c_2^2af'(\sigma) + c_3 + 2af'(\sigma)c_3 + c_3a^2\Big(\left(f'(\sigma)\right)^2 + c_2^2 + 2bc_2 + 4bc_2af'(\sigma) \\ & + \left(f'(\sigma)\right)^2bc_2a^2 + q(1+af'(\sigma))\Big)\Big]e_n^2 + \ \dots \ + O\left(e_n^9\right). \end{aligned}$$

Therefore,

$$\begin{aligned}
e_{n,z} &= -(c_2+b)(1+af'(\sigma))(-1+S(0))e_n^2 \\
&+ \Big(-2bc_2 - 2c_2^2 af'(\sigma) + 3af'(\sigma)c_3 - b^2 - b^2 a\big(f'(\sigma)\big)^2 - 2b^2 af'(\sigma) - 2c_2^2 + 2c_3 + c_3 a^2\big(f'(\sigma)\big)^2 \\
&\quad - a^2\big(f'(\sigma)\big)^2 c_2^2 - bc_2 a^2\big(f'(\sigma)\big)^2 - 2S'(0)c_2^2 af'(\sigma) - S'(0)a^2\big(f'(\sigma)\big)^2 c_2^2 - 2bc_2 af'(\sigma) \\
&\quad - 2S'(0)b^2 af'(\sigma) - S'(0)a^2b^2\big(f'(\sigma)\big)^2 - 2S''(0)c_3 + S''(0)c_2^2 - 2S'(0)bc_2 - 4S'(0)bc_2 af'(\sigma) \\
&\quad - 2S'(0)bc_2a^2\big(f'(\sigma)\big)^2 - S'(0)c_2^2 - S'(0)b_2 - 2S''(0)bc_2 af'(\sigma) - S''(0)\big(f'(\sigma)\big)^2 bc_2a^2 - 3S''(0)af'(\sigma)c_3 \\
&\quad - S''(0)c_3p^2\big(f'(\sigma)\big)^2\Big) e_n^3 + \cdots + O\left(e_n^9\right).
\end{aligned}$$

Substituting $S(0) = 1, S'(0) = -1$ and $S''(0) = -2$ in the above expression, we have

$$\begin{aligned}
e_{n,z} = \frac{1}{f'(\sigma)} \Big[&\Big(1 + a\big(f'(\sigma)\big)^2\Big)(c_2+b)\Big(2\gamma\big(f'(\sigma)\big)^2 b^2 a + 4\gamma\big(f'(\sigma)\big)^2 c_2 ba + 2\gamma\big(f'(\sigma)\big)^2 c_2^2 a + 2\gamma\big(f'(\sigma)\big)^2 \\
&+ 2f'(\sigma)bc_2 + 4\gamma f'(\sigma)bc_2 + 2\gamma f'(\sigma)c_2^2 - f'(\sigma)c_3 + 2f'(\sigma)c_2^2 + q\Big)\Big] e_n^4 + \cdots + O\left(e_n^9\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
f(z_n) = f'(\sigma)\Big[&\Big(1 + a\big(f'(\sigma)\big)^2\Big)(c_2+b)\Big(2\gamma\big(f'(\sigma)\big)^2 b^2 a + 4\gamma\big(f'(\sigma)\big)^2 c_2 ba + 2\gamma\big(f'(\sigma)\big)^2 c_2^2 a + 2\gamma\big(f'(\sigma)\big)^2 \\
&+ 2f'(\sigma)bc_2 + 4\gamma f'(\sigma)bc_2 + 2\gamma f'(\sigma)c_2^2 - f'(\sigma)c_3 + 2f'(\sigma)c_2^2 + q\Big) e_n^4 + \cdots + O\left(e_n^9\right)\Big].
\end{aligned}$$

Thus, by taking $J(0,0) = 1$ and $J_u(0,0) = 0$, we have the error term

$$\begin{aligned}
e_{n+1} = -\frac{1}{2f'(\sigma)}\Big[&J_{u,u}(0,0)(c_2+b)^3(1+af'(\sigma))^4\Big(2\gamma\big(f'(\sigma)\big)^2 b^2 a + 4\gamma\big(f'(\sigma)\big)^2 c_2 ba \\
&+ 2\gamma\big(f'(\sigma)\big)^2 c_2^2 a + 2\gamma\big(f'(\sigma)\big)^2 + 2f'(\sigma)bc_2 + 4\gamma f'(\sigma)bc_2 + 2\gamma f'(\sigma)c_2^2 \\
&- f'(\sigma)c_3 + 2f'(\sigma)c_2^2 + q\Big)\Big] e^6 + \cdots + O\left(e_n^9\right).
\end{aligned}$$

Now, with $J_t(0,0) = 0, J_{u,u}(0,0) = 0$ and $J_{u,u,u}(0,0) = 0$, finally the error equation is

$$\begin{aligned}
e_{n+1} = &\frac{1}{((f'(\sigma))^2}(c_2+b)^2(1+af'(\sigma))^4 \\
&\Big(2\gamma a\big(f'(\sigma)\big)^2\left(b^2 + 2bc_2 + c_2^2\right) + f'(\sigma)\left(2\gamma b^2 + 2bc_2 + 4\gamma bc_2 - c_3 + 2c_2^2 + 2\gamma c_2^2 + q\right)\Big) \\
&\Big(-d + f'(\sigma)c_4 + qc_2 - f'(\sigma)c_2c_3 + 2f'(\sigma)c_2^3 + 2bc_2^2 f'(\sigma) + 2\gamma f'(\sigma)c_2^3 + 4f'(\sigma)bc_2^2 \\
&\quad + 2c_2\big(f'(\sigma)\big)^2 + 2\gamma c_2^3 a\big(f'(\sigma)\big)^2 + 4\gamma bc_2^2 a\big(f'(\sigma)\big)^2 + 2c_2\gamma b^2 a\big(f'(\sigma)\big)^2\Big) e_n^8 + O\left(e_n^9\right).
\end{aligned}$$

□

Let us remark that, as the error equation has $(c_2 + b)^2$ and $(1 + af'(\sigma))^4$ as factors, it can be seen that this method is extendable to with memory.

3 | CONSTRUCTION OF THE FOUR-PARAMETRIC WITH-MEMORY ITERATIVE CLASS

To increase the order of convergence of the family of iterative methods without-memory (1), we modify it in order to include memory. Extension can be done by approximating the parameters at each step by using Newton's interpolating polynomials. It can be noted that the coefficient of e_n^8 in the error equation (2) disappears if $a = \frac{-1}{f'(\sigma)}, b = -c_2, q = f'(\sigma)c_3$ and $d = f'(\sigma)c_4$, being $c_k = \frac{f^{(k)}(\sigma)}{k!f'(\sigma)}, k \geq 2$. For construction of methods with memory, free parameters a, b, q, and d are calculated by formulas

$$\begin{aligned}
a \approx a_n &= -\frac{1}{N_4'(x_n)} = \frac{-1}{\widetilde{f'}(\sigma)}, n = 1, 2, \dots, \\
b \approx b_n &= -\frac{N_5''(w_n)}{2N_5'(w_n)} = -\frac{\widetilde{f''}(\sigma)}{2\widetilde{f'}(\sigma)}, \\
q \approx q_n &= \frac{N_6'''(y_n)}{6} = \frac{\widetilde{f'''}(\sigma)}{6}, \\
d \approx d_n &= \frac{N_7^{iv}(z_n)}{24} = \frac{\widetilde{f^{iv}}(\sigma)}{24},
\end{aligned} \tag{5}$$

where $\widetilde{f'}(\sigma)$, $\widetilde{f''}(\sigma)$, $\widetilde{f'''}(\sigma)$ and $\widetilde{f^{iv}}(\sigma)$ are approximations to $f'(\sigma)$, $f''(\sigma)$, $f'''(\sigma)$, and $f^{iv}(\sigma)$, respectively. These estimations are made by using $N_4(x_n), N_5(w_n), N_6(y_n)$ and $N_7(z_n)$, Newton's interpolating polynomials of degree four, five, six and seven respectively defined by:

$$\begin{aligned}
N_4(\zeta) &= N_4(\zeta; x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}), \\
N_5(\zeta) &= N_5(\zeta; w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}), \\
N_6(\zeta) &= N_6(\zeta; y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}), \\
N_7(\zeta) &= N_7(\zeta; z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}),
\end{aligned}$$

for any $n \geq 2$. The explicit representation of $N_4(\zeta), N_5(\zeta), N_6(\zeta)$ and $N_7(\zeta)$ is given below:

$$\begin{aligned}
N_4(\zeta; x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}) = {} & f(x_n) + f[x_n, z_{n-1}](\zeta - x_n) + f[x_n, z_{n-1}, y_{n-1}](\zeta - x_n)(\zeta - z_{n-1}) \\
& + f[x_n, z_{n-1}, y_{n-1}, w_{n-1}](\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1}) \\
& + f[x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}](\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1})(\zeta - w_{n-1}).
\end{aligned} \tag{6}$$

Also,

$$\begin{aligned}
N_5(\zeta; w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}) = {} & f(w_n) + f[w_n, x_n](\zeta - w_n) + f[w_n, x_n, z_{n-1}](\zeta - w_n)(\zeta - x_n) \\
& + f[w_n, x_n, z_{n-1}, y_{n-1}](\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1}) \\
& + f[w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}](\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1}) \\
& + f[w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}](\zeta - w_n) \\
& (\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1})(\zeta - w_{n-1}).
\end{aligned} \tag{7}$$

Moreover,

$$\begin{aligned}
N_6(\zeta; y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}) = {} & f(y_n) + f[y_n, w_n](\zeta - y_n) + f[y_n, w_n, x_n](\zeta - y_n)(\zeta - w_n) \\
& + f[y_n, w_n, x_n, z_{n-1}](\zeta - y_n)(\zeta - w_n)(\zeta - x_n) \\
& + f[y_n, w_n, x_n, z_{n-1}, y_{n-1}](\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1}) \\
& + f[y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}](\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1}) \\
& + f[y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}](\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1}) \\
& (\zeta - y_{n-1})(\zeta - w_{n-1}).
\end{aligned} \tag{8}$$

and

$$\begin{aligned} N_7(\zeta; z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}) = {} & f(z_n) + f[z_n, y_n](\zeta - z_n)(\zeta - z_n)(\zeta - y_n) + f[z_n, y_n, w_n] \\ & + f[z_n, y_n, w_n, x_n](\zeta - z_n)(\zeta - y_n)(\zeta - w_n) \\ & + f[z_n, y_n, w_n, x_n, z_{n-1}](\zeta - z_n)(\zeta - y_n)(\zeta - w_n)(\zeta - x_n) \\ & + f[z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}](\zeta - z_n)(\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1}) \\ & + f[z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}](\zeta - z_n) \\ & \quad (\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1}) \\ & + f[z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}] \\ & \quad (\zeta - z_n)(\zeta - y_n)(\zeta - w_n)(\zeta - x_n)(\zeta - z_{n-1})(\zeta - y_{n-1})(\zeta - w_{n-1}). \end{aligned} \tag{9}$$

Hence, by replacing the free parameters a, b, d and q in (1) with self-accelerators a_n, b_n, d_n and q_n, we have the following class of root-solvers with memory,

$$\begin{aligned} w_n &= x_n + a_n f(x_n), n \geq 2, \\ y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + b_n f(w_n)}, \\ z_n &= y_n - S(u_n)\frac{f(x_n) + \gamma f(y_n)}{f(x_n) + (\gamma - 2) f(y_n)} \frac{f(y_n)}{f[y_n, w_n] + b_n f(w_n) + q_n(y_n - w_n)(y_n - x_n)}, \\ x_{n+1} &= z_n - J(u_n, t_n)\frac{f(z_n)}{P_n} \end{aligned} \tag{10}$$

where u_n, t_n and P_n are the same given in (1). Also a_n, b_n, d_n and q_n are described in (5). Now, we will use Herzberger's matrix method[7] to prove that root-solver with memory (10) has order of convergence 15.51560, almost doubling the order of convergence of the original method.

Theorem 2. *Let us suppose that $f : I \subset \mathbb{R} \to \mathbb{R}$ is a real-valued nonlinear sufficiently differentiable function and σ is a real simple zero of the function in the open interval I. Let us also suppose that x_0 and x_1 are the initial guesses satisfactorily close to the zero σ of $f(x)$. If self-accelerating parameters a_n, b_n, d_n and q_n are calculated from (5), then the R-order of convergence of (10) is at least 15.51560 with efficiency index* $15.51560^{\frac{1}{4}} \approx 1.98468$.

Proof. We will determine the R-order of convergence of the class of iterative methods with memory (10) by means of Herzberger's matrix method.[7] Spectral radius of a matrix $B^{(p)} = (t_{ij})$, $1 \leq i, j \leq p$, associated with one step p-point method with memory $x_m = \Psi(x_{m-1}, x_{m-2}, \ldots, x_{m-p})$ is the lower bound of its convergence order. Associated matrix has following elements:

$$\begin{aligned} t_{1,j} &= \text{amount of information used at point } x_{m-j}, j = 1, 2, \ldots, p, \\ t_{i,i-1} &= 1 \text{ for } i = 2, 3, \ldots, p, \\ t_{i,j} &= 0, \text{otherwise}. \end{aligned}$$

The spectral radius of $B_1 \cdot B_2 \cdot \cdots \cdot B_p$ is the lower bound of order of a p-step method $\Psi = \Psi_1 \circ \Psi_2 \circ \ldots \circ \Psi_p$, where matrices B_r corresponds to the iteration step Ψ_r, $1 \leq r \leq p$. From (5) and (10), the construction of matrices is made following the iterative expression of each step, starting from the last one. For

$$x_{n+1} = \Psi_1\left(z_n, y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}\right),$$

we have

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$z_n = \Psi_2\left(y_n, w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}, z_{n-2}\right),$$

gives

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$y_n = \Psi_3 \left(w_n, x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}, z_{n-2}, y_{n-2}\right),$$

generates

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly,

$$w_n = \Psi_4(x_n, z_{n-1}, y_{n-1}, w_{n-1}, x_{n-1}, z_{n-2}, y_{n-2}, w_{n-2})$$

implies

$$B_4 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Then,

$$B^{(4)} = B_1 \cdot B_2 \cdot B_3 \cdot B_4 = \begin{bmatrix} 8 & 8 & 8 & 8 & 8 & 0 & 0 & 0 \\ 4 & 4 & 4 & 4 & 4 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Eigenvalues of $B^{(4)}$ are $\{0, 0, 0, 0, 0, 0, 15.51560977, -0.515609770\}$. Therefore, as the R-order convergence of three-step method (10) is the spectral radius of matrix $B^{(4)}$, $\rho(B^{(4)}) = 15.51560$. □

Let us remark that, from Theorem 2, the R-order of convergence of presented class with memory (10) is 15.51560, with the highest efficiency index $15.51560^{\frac{1}{4}} \approx 1.9847$.

4 | SOME SPECIAL CASES

To satisfy the optimality condition, we choose $S(u_n)$ and $J(u_n, t_n)$ such that the following conditions are satisfied:

$$S(0) = 1, S'(0) = -1, S''(0) = -2$$

and

$$J(0,0) = 1, J_u(0,0) = 0, J_t(0,0) = 0, J_{u,u}(0,0) = 0 \text{ and } J_{u,u,u}(0,0) = 0.$$

By using suitable choices of weight functions, different methods can be given. In particular, we propose two schemes whose iterative expressions are as follows:

Method ZR1: For

$$S(u_n) = 1 - u_n - u_n^2,$$
$$J(u_n, t_n) = 1 + t_n u_n,$$

we have

$$\begin{aligned}
w_n &= x_n + a_n f(x_n), \quad n \geq 2,\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + b_n f(w_n)},\\
z_n &= y_n - \left(1 - u_n - u_n^2\right) \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f[y_n, w_n] + b_n f(w_n) + q_n(y_n - w_n)(y_n - x_n)},\\
x_{n+1} &= z_n - (1 + t_n u_n)\frac{f(z_n)}{P_n},
\end{aligned}$$

being $a_n = -\frac{1}{N_4'(x_n)}$, $b_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}$, $q_n = \frac{N_6'''(y_n)}{6}$, $d_n = \frac{N_7^{(iv)}(z_n)}{24}$, $u_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$ and $P_n = f[y_n, z_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + d_n(z_n - w_n)(z_n - y_n)(z_n - x_n)$.

Method ZR2: For

$$\begin{aligned}
S(u_n) &= \frac{1}{1 + u_n},\\
J(u_n, t_n) &= \frac{1}{1 + u_n t_n},
\end{aligned}$$

we have

$$\begin{aligned}
w_n &= x_n + a_n f(x_n), \qquad n \geq 2,\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + b_n f(w_n)},\\
z_n &= y_n - \frac{1}{1 + u_n} \frac{f(x_n)}{f(x_n) - 2f(y_n)} \frac{f(y_n)}{f[y_n, w_n] + b_n f(w_n) + q_n(y_n - w_n)(y_n - x_n)},\\
x_{n+1} &= z_n - \frac{1}{1 + u_n t_n} \frac{f(z_n)}{P_n},
\end{aligned}$$

where $a_n = -\frac{1}{N_4'(x_n)}$, $b_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}$, $q_n = \frac{N_6'''(y_n)}{6}$, $d_n = \frac{N_7^{(iv)}(z_n)}{24}$, $u_n = \frac{f(y_n)}{f(x_n)}$, $t_n = \frac{f(z_n)}{f(x_n)}$ and $P_n = f[y_n, z_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + d_n(z_n - w_n)(z_n - y_n)(z_n - x_n)$.

5 | NUMERICAL RESULTS

Here, we compare numerical results of our proposed methods with some existing ones. The computational order of convergence (COC) of iterative methods was defined by Jay[8] and is calculated as follows:

$$COC = \frac{\log |f(x_n)/f(x_{n-1})|}{\log |f(x_{n-1})/f(x_{n-2})|}.$$

We have considered the first three iterations of all proposed and known methods by using fixed floating point arithmetics with 2000 digits of mantissa to measure the accuracy. We denote by σ the actual root of the nonlinear equation to be solved and x_0 is the initial approximation used.

Maple software is used for numerical computations. Now, we make a comparison of our proposed schemes ZR1 and ZR2 with other four-parametric one, designed by Lotfi and Assari[3] (denoted by LA)

$$\begin{aligned}
w_n &= x_n + \gamma_n f(x_n),\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + p_n f(w_n)},\\
z_n &= y_n - \frac{f(y_n)}{f[y_n, x_n] + f[w_n, x_n, y_n](y_n - x_n) + \lambda_n(y_n - x_n)(y_n - w_n)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{Q_n},
\end{aligned}$$

where $\gamma_n = -\frac{1}{N_4'(x_n)}$, $p_n = -\frac{N_5''(w_n)}{2N_5'(w_n)}$, $\lambda_n = \frac{N_6'''(y_n)}{6}$, $\beta_n = \frac{N_7^{(iv)}(z_n)}{24}$ and $Q_n = f[x_n, z_n] + (f[w_n, x_n, y_n] - f[w_n, x_n, z_n] - f[y_n, x_n, z_n])(x_n - z_n) + \beta_n(z_n - x_n)(z_n - w_n)(z_n - y_n)$. With these accelerating parameters, the scheme LA reaches order of convergence 15.51560.

We also compare our schemes with methods M1 and M2 appearing in the work of Cordero et al,[4] whose order of convergence is also 15.51560. The iterative expressions of the method M1 is as follows:

$$\begin{aligned}
w_n &= x_n + \theta_{1,n} f(x_n), n \geq 0,\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + \theta_{2,n} f(w_n)},\\
z_n &= y_n - (1 + 2u_n)(1 - u_n)\frac{f(y_n)}{f[y_n, w_n] + \theta_{2,n} f(w_n) + \theta_{3,n}(y_n - w_n)(y_n - x_n)},\\
x_{n+1} &= z_n - \frac{f(z_n)}{\psi_n},
\end{aligned}$$

being $\theta_{1,n} = -\frac{1}{N_4'(x_n)}$, $\theta_{2,n} = -\frac{N_5''(w_n)}{2N_5'(w_n)}$, $\theta_{3,n} = \frac{N_6'''(y_n)}{6}$, and $\theta_{4,n} = \frac{N_7^{(iv)}(z_n)}{24}$. Moreover, $u_n = \frac{f(y_n)}{f(x_n)}$ and $\psi_n = f[y_n, z_n] + f[z_n, y_n, x_n](z_n - y_n) + f[z_n, y_n, x_n, w_n](z_n - y_n)(z_n - x_n) + \theta_{4,n}(z_n - w_n)(z_n - y_n)(z_n - x_n)$.

Regarding scheme M2 from the work of Cordero et al,[4] its iterative expression involves the same notations as M1 and it can be expressed as

$$\begin{aligned}
w_n &= x_n + \theta_{1,n} f(x_n), n \geq 0,\\
y_n &= x_n - \frac{f(x_n)}{f[x_n, w_n] + \theta_{2,n} f(w_n)},\\
z_n &= y_n - (1 - u_n)\frac{f(x_n)}{f(x_n) - 2f(y_n)}\frac{f(y_n)}{f[y_n, w_n] + \theta_{2,n} f(w_n) + \theta_{3,n}(y_n - w_n)(y_n - x_n)}.\\
x_{n+1} &= z_n - \frac{f(z_n)}{\psi_n}.
\end{aligned}$$

We consider three nonlinear standard test functions for the sake of comparison. For all compared methods, we have considered $\gamma_0 = \theta_1 = a_0 = 0.01, p_0 = \theta_2 = b_0 = 0.1, \lambda_0 = \theta_3 = q_0 = 0.01$, and $\beta_0 = \theta_4 = d_0 = 0.01$. When the schemes use memory, these initial values allows us to calculate starting values required to initialize the process for accelerating parameters and, then, continuing the iterative process with self-improving values of the parameters. When the schemes do not change the values of the parameters, the iterative methods are without memory having eighth-order of convergence and they are denoted by LA_8, $M1_8$, $M2_8$, $ZR1_8$, and $ZR2_8$.

Example 1. Function $f_1(x) = e^{x^2 + x\cos(x) - 1}\sin(\pi x) + x\log(x\sin(x) + 1)$ has two real roots at $\sigma = 0$ and -1.2829, we seek to approximate the null solution with $x_0 = 0.6$ as initial estimation. When numerical tests are made on $f_1(x)$ by using iterative schemes without memory (see Table 1), the best results in terms of precision are obtained by schemes $ZR1_8$ and $M2_8$. In Table 2, the errors obtained by their with-memory schemes ZR1 and ZR2 are showing the smallest error. In all cases, the COC coincides with the theoretical value.

Example 2. In function $f_2(x) = e^{-x^2}(x - 2)(x^6 + x^3 + 1)$, the only real root is $\sigma = 2$, which is efficiently reached by all proposed and known methods; however, the precision reached by iterative schemes with memory is much bigger than the error of their without-memory partners. In Tables 3 and 4, it is shown that the best performance without using memory is obtained by methods ZR1 and M2; when memory is used, the exact error is reduced by a factor of almost 1/5, being the more precise results given by our proposed methods ZR1 and ZR2.

TABLE 1 Comparison table for $f_1(x)$ using methods without memory

$f_1(x) = e^{x^2+x\cos(x)-1}\sin(\pi x) + x\log(x\sin(x)+1)$, $x_0 = 0.6$.	**LA_8**	**$M1_8$**	**$M2_8$**	**$ZR1_8$**	**$ZR2_8$**
$\lvert x_1 - \sigma\rvert$	6.639×10^{-2}	8.537×10^{-4}	1.193×10^{-3}	7.193×10^{-4}	1.639×10^{-3}
$\lvert x_2 - \sigma\rvert$	4.331×10^{-19}	7.754×10^{-24}	9.837×10^{-25}	1.564×10^{-25}	1.958×10^{-22}
$\lvert x_3 - \sigma\rvert$	1.579×10^{-148}	3.566×10^{-184}	2.123×10^{-193}	7.795×10^{-199}	7.937×10^{-174}
COC	7.999	8.000	7.999	8.000	8.000

Abbreviation: COC, computational order of convergence.

TABLE 2 Comparison table for $f_1(x)$ using methods with memory

$f_1(x) = e^{x^2+x\cos(x)-1}\sin(\pi x) + x\log(x\sin(x)+1), x_0 = 0.6$	LA	M1	M2	ZR1	ZR2
$\lvert x_1 - \sigma\rvert$	6.639×10^{-3}	8.537×10^{-4}	1.193×10^{-3}	7.193×10^{-4}	1.639×10^{-3}
$\lvert x_2 - \sigma\rvert$	1.440×10^{-37}	2.675×10^{-48}	4.512×10^{-49}	6.253×10^{-49}	2.828×10^{-49}
$\lvert x_3 - \sigma\rvert$	4.079×10^{-569}	5.575×10^{-734}	1.177×10^{-737}	1.332×10^{-744}	2.089×10^{-749}
COC	15.33	15.40	15.15	15.43	15.29

Abbreviation: COC, computational order of convergence.

TABLE 3 Comparison table for $f_2(x)$ using methods without memory

$f_2(x) = e^{-x^2}(x-2)(x^6+x^3+1), x_0 = 1.8$	LA_8	$M1_8$	$M2_8$	$ZR1_8$	$ZR2_8$
$\lvert x_1 - \sigma\rvert$	1.185×10^{-6}	1.508×10^{-6}	5.633×10^{-7}	2.594×10^{-7}	7.558×10^{-7}
$\lvert x_2 - \sigma\rvert$	2.116×10^{-47}	1.760×10^{-45}	3.197×10^{-50}	1.213×10^{-52}	9.952×10^{-50}
$\lvert x_3 - \sigma\rvert$	2.189×10^{-373}	6.039×10^{-357}	3.439×10^{-396}	2.786×10^{-415}	8.992×10^{-393}
COC	7.999	7.999	7.999	7.999	8.000

Abbreviation: COC, computational order of convergence.

TABLE 4 Comparison table for $f_2(x)$ using methods with memory

$f_2(x) = e^{-x^2}(x-2)(x^6+x^3+1), x_0 = 1.8$	LA	M1	M2	ZR1	ZR2
$\lvert x_1 - \sigma\rvert$	1.185×10^{-6}	1.508×10^{-6}	5.663×10^{-7}	2.594×10^{-7}	7.558×10^{-7}
$\lvert x_2 - \sigma\rvert$	7.208×10^{-97}	9.283×10^{-96}	6.033×10^{-102}	7.648×10^{-103}	1.564×10^{-99}
$\lvert x_3 - \sigma\rvert$	3.191×10^{-1498}	1.335×10^{-1479}	7.590×10^{-1578}	1.278×10^{-1592}	8.037×10^{-1592}
COC	15.53	15.51	15.54	15.59	15.52

Abbreviation: COC, computational order of convergence.

TABLE 5 Comparison table for $f_3(x)$ and methods without memory

$f_3(x) = x^5 + x^4 + \frac{1}{x^2+1} - \frac{5}{2}x^2, x_0 = 1.5$	LA_8	$M1_8$	$M2_8$	$ZR1_8$	$ZR2_8$
$\lvert x_1 - \sigma\rvert$	2.831×10^{-2}	6.761×10^{-2}	1.974×10^{-2}	3.346×10^{-2}	1.038×10^{-2}
$\lvert x_2 - \sigma\rvert$	1.900×10^{-9}	1.497×10^{-5}	1.326×10^{-10}	9.698×10^{-9}	2.064×10^{-13}
$\lvert x_3 - \sigma\rvert$	1.425×10^{-66}	1.129×10^{-33}	8.615×10^{-76}	2.011×10^{-61}	1.693×10^{-98}
COC	7.963	7.472	7.944	7.990	7.937

Abbreviation: COC, computational order of convergence.

Example 3. Function $f_3(x) = x^5 + x^4 + \frac{1}{x^2+1} - \frac{5}{2}x^2$ has three real roots, −0.566312746, 1, and 0.620346251, but the desired one in the numerical tests is $\sigma = 1$. When the methods without memory are used (see Table 5), the best results (lowest exact error) have been obtained by $ZR2_8$ and $M2_8$. Regarding the performance of iterative schemes with memory (see Table 6), the lowest errors have been obtained by the proposed methods ZR1 and ZR2, although the reduction of the error had a factor of 1/4, approximately.

Example 4 (Continuous stirred tank reactor (CSTR)).

TABLE 6 Comparison table for $f_3(x)$ and methods with memory

	$f_3(x)=x^5+x^4+\frac{1}{x^2+1}-\frac{5}{2}x^2, x_0=1.5$				
	LA	**M1**	**M2**	**ZR1**	**ZR2**
$\lvert x_1-\sigma\rvert$	2.831×10^{-2}	6.761×10^{-2}	3.607×10^{-2}	3.346×10^{-2}	1.038×10^{-2}
$\lvert x_2-\sigma\rvert$	3.345×10^{-19}	6.912×10^{-13}	5.721×10^{-17}	1.802×10^{-17}	1.924×10^{-25}
$\lvert x_3-\sigma\rvert$	5.898×10^{-290}	3.448×10^{-188}	1.611×10^{-253}	1.510×10^{-261}	3.509×10^{-389}
COC	15.99	15.79	15.98	15.92	15.98

Abbreviation: COC, computational order of convergence.

TABLE 7 Comparison table for $f_4(x)$ and methods without memory

	$f_4(x)=x^4+11.50x^3+47.49x^2+86.0325x+51.23266875, x_0=-1.2$				
	LA_8	**$M1_8$**	**$M2_8$**	**$ZR1_8$**	**$ZR2_8$**
$\lvert x_1-\sigma\rvert$	5.546×10^{-5}	7.586×10^{-4}	5.204×10^{-5}	1.270×10^{-4}	3.529×10^{-6}
$\lvert x_2-\sigma\rvert$	2.2504×10^{-33}	2.372×10^{-22}	1.643×10^{-33}	1.643×10^{-30}	1.213×10^{-42}
$\lvert x_3-\sigma\rvert$	1.654×10^{-260}	2.205×10^{-170}	1.626×10^{-261}	1.304×10^{-237}	2.366×10^{-334}
COC	7.999	7.999	7.999	7.999	7.999

Abbreviation: COC, computational order of convergence.

TABLE 8 Comparison table for $f_4(x)$ and methods with memory

	$f_4(x)=x^4+11.50x^3+47.49x^2+86.0325x+51.23266875, x_0=-1.2$				
	LA	**M1**	**M2**	**ZR1**	**ZR2**
$\lvert x_1-\sigma\rvert$	5.546×10^{-5}	7.586×10^{-4}	5.204×10^{-5}	1.270×10^{-4}	3.529×10^{-6}
$\lvert x_2-\sigma\rvert$	3.627×10^{-67}	2.408×10^{-48}	5.847×10^{-67}	9.266×10^{-61}	1.168×10^{-85}
$\lvert x_3-\sigma\rvert$	4.063×10^{-1062}	2.591×10^{-760}	3.771×10^{-1058}	5.963×10^{-959}	2.434×10^{-1357}
COC	15.99	15.99	15.99	15.99	15.99

Abbreviation: COC, computational order of convergence.

Let us consider an isothermal continuous stirred tank reactor (CSTR). Components A & R are fed to the reactor at rates of Q and q-Q respectively. The following reaction scheme develops in the reactor:

$$A+R\rightarrow B$$
$$B+R\rightarrow C$$
$$C+R\rightarrow D$$
$$D+R\rightarrow E.$$

The equation for the transfer function of the reactor is given as

$$K_C\frac{2.98(x+2.25)}{x^4+11.50x^3+47.49x^2+86.0325x+51.23266875}=-1,$$

where K_C is the gain of the proportional controller. The control system is stable for values of K_C that yields roots of the transfer function having negative real part. If we choose $K_C=0$, we get the poles of the open-loop transfer function as roots of the nonlinear equation

$$f_4(x)=x^4+11.5x^3+47.49x^2+86.0325x+51.23266875=0 \tag{11}$$

given as -1.45, -2.85 (double root), and -4.35. For the numerical tests, we consider $\sigma=-1.45$ and the initial guess $x_0=-1.2$.

In case of parametric eighth-order schemes (see Table 7), the best results are provided by $ZR2_8$ and $M2_8$ methods, holding in all cases a computational order that agrees with the theoretical one. When methods with memory are used, we can see in Table 8 that the most precise solution is obtained by ZR2 scheme, followed in a big distance by the LA method. The process of using memory in the accelerating parameters has decreased the error in a factor of 1/4, approximately. The estimation of the order of convergence again agrees with the theoretical one.

6 | CONCLUSIONS

We have proposed new three-step four-parametric family of iterative methods by using two weight functions. We have taken test functions of different types and have applied our newly proposed methods on them. We also have compared our results with recent three-step four-parametric methods. In case of schemes without memory, our methods ZR1 and ZR2 give much better accuracy and COC for test functions f_1, f_2, f_3, and f_4 as compared to the methods LA, M1, and M2. In case of extension with memory of our methods, efficiency index is increased from 1.6817 to 1.9847 and we obtain much better results and high COC. As a future extension, we can analyze the stability of the proposed family of methods using complex dynamics similar to the works of Chun and Neta[9] and Gdawiec.[10]

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

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Fiza Zafar completed her bachelor's degree and master's degree in applied mathematics with a gold medal at Bahauddin Zakariya University, Multan, Pakistan. She obtained her PhD in mathematics in 2010 from the Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan for the dissertation “Some Generalizations of Ostrowski Inequalities and Their Applications to Numerical Integration and Special Means.” During these years, she has worked in the fields of calculus of variation, numerical solution of nonlinear equations, and numerical integration and has significant number of international impact factor publications in these areas. Recently, she has completed a one-year postdoctoral research fellowship with Professor Juan Ramon Torregrosa and

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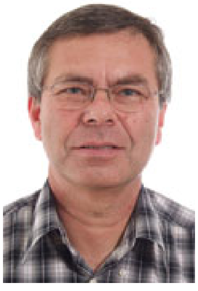
Currently, she is working as a tenured assistant professor at Bahauddin Zakariya University, Multan, Pakistan. She has supervised 21 MS and 2 PhD theses. She has also been listed as a Dynamic Woman Scientist of Pakistan and Productive Young Scientist of Pakistan under 40 by Pakistan Council for Science and Technology. She is a guest editor for a special issue of Algorithms (MDPI), an active reviewer for many renowned scientific publishers like Elsevier, Springer, Taylor & Francis, and Hindawi Publishing Corporation, and has been a speaker at several national and international conferences.

Alicia Cordero Since her PhD dissertation in 2003, "Cadenas de Órbitas Periódicas en la Variedad S2xS1," supervised by Professors J.A. Martínez (Universitat de València) and P. Vindel (Universitat Jaume I de Castellón UJI), her research career had two branches, which have finally converged: her doctoral thesis, and the subsequent research until 2007, was focused on continuous dynamic systems and decomposition into round handles of the three-dimensional surface S2xS1, with applications to celestial mechanics. From 2005, and simultaneously with the previous one, she started a new line of research in numerical analysis, specifically in iterative techniques for solving equations and nonlinear systems. The design and analysis of the convergence of iterative methods is a field of research that has developed very rapidly in recent years. The first works in this line, which dealt with the extension to nonlinear systems of specific methods for equations, were published in 2006 and 2007 and have been widely cited. During this time, she has coadvised four PhD in numerical analysis, framed in the Doctorate with Mention of Quality "Multidisciplinary Mathematics" of the Department of Applied Mathematics of the Polytechnic University of Valencia (UPV).
The mentioned research has also led to her role as guest editor in several special issues in international journals as Journal of Applied Mathematics and Abstract and Applied Analysis, both published by Hindawi, Journal of Computational and Applied Mathematics (Elsevier), Algorithms (MDPI). Likewise, since 2015, she is an editor of the journal Discrete Dynamics in Nature and Society and the journal Algorithms (MDPI). On the other hand, she is a reviewer of different indexed journals indexed in JCR.
Regarding research grants, she has participated in more than ten research projects of the Spanish Government, regional grants by Generalitat Valenciana, and in international research projects at Dominican Republic, which aims to create a computer center that provides mathematical coverage in numerical modeling and simulation, basic and applied research in the region.



Juan R. Torregrosa is a full professor of mathematics at the Universitat Politècnica de València (UPV). He has researched in functional analysis, specifically in geometrical properties of Banach spaces, in matrix analysis, namely, in the theory of completion of partial matrices and in totally positive matrices. Since 2006, and initially simultaneously with the previous ones, he started a line of research on numerical analysis, in particular, in iterative methods for solving nonlinear equations and systems, topic that has developed very rapidly in recent years. The first works in this area, which deal with the extension to nonlinear systems of specific methods for equations, were published in 2006 and 2007 and have been widely cited. From the point of view of publications, his work has been seen to be reflected in 182 publications in journals indexed in JCR, 15 publications in journals not indexed in JCR, 151 papers in national and international conferences, and collaboration in 7 monographs on different research topics.
During this time, he has advised nine PhD in matrix analysis and numerical analysis, the most of them, framed in the Doctorate with Mention of Quality "Multidisciplinary Mathematics" of the Department of Applied Mathematics of the UPV. He has also led 5 master dissertation and more than 50 final dissertation degree.
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Anneqa Rafi completed her bachelor's degree (2014) and master's degree (2016) in computational mathematics by securing second position from the Centre for Advanced Studies in Pure and Applied Mathematics, Bahauddin Zakariya University, Multan, Pakistan. Her MS dissertation titled "Optimal Three Step Four Parametric With- and Without-Memory Iterative Method with Convergence Planes" was supervised by Dr Fiza Zafar. She is currently working as an elementary school educator for science and mathematics at Government Girls Elementary School, Multan, Pakistan.

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