ERRATUM TO: STRONG EXTENSIONS FOR q-SUMMING OPERATORS ACTING IN p-CONVEX BANACH FUNCTION SPACES FOR $1 \le p \le q$". WEAK* COMPATNESS OF THE CLOSED UNIT BALL OF A KÖTHE DUAL SPACE

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ABSTRACT. Let X be a saturated Banach function space and denote by X' its Köthe dual. In the paper [1] referenced in the title, emulating what happens with the weak* topology of the topological dual of X, it is used that the closed unit ball $B_{X'}$ of X' is compact for the topology $\sigma(X',X)$ on X' defined by the elements of X. The purpose of this note is to clarify that this fact could be not true in general if X is not σ-order continuous.

Banach function space, Köthe dual, weak* compactness, σ-order continuity 46B50 and 46E30 and 46B42

1. Notation

Let (Ω, Σ, μ) be a σ-finite measure space and denote by $L^0(\mu)$ the space of real measurable functions defined on Ω, where functions which are equal μ-a.e. are identified. Consider a *saturated Banach function space* X, that is, a Banach space contained in $L^0(\mu)$ with norm $\|\cdot\|_X$ satisfying

(i) $f \in L^0(\mu)$, $g \in X$ and $|f| \le |g|$ μ-a.e. implies $f \in X$ with $\|f\|_X \le \|g\|_X$.
(ii) There is no $A \in \Sigma$ with $\mu(A) > 0$ such that $f\chi_A = 0$ μ-a.e. for all $f \in X$.

In particular X is a Banach lattice with the μ-a.e. pointwise order. The space X is called *σ-order continuous* if for every $(f_n) \subset X$ with $f_n \downarrow 0$ μ-a.e. it follows that $\|f_n\|_X \downarrow 0$. If $\|f_n\|_X \uparrow \|f\|_X$ whenever $0 \le f_n \uparrow f$ μ-a.e. with $f \in X$ then X is said to be *order semi-continuous*.

The *σ-order continuous part* X_a of X is defined as the largest σ-order continuous closed solid subspace of X and can be described as

$$X_a = \big\{ f \in X : |f| \ge f_n \downarrow 0 \ \text{ implies } \ \|f_n\|_X \downarrow 0 \big\}.$$

Note that X_a could be the trivial space as in the case of $X = L^\infty(\mu)$ when μ is nonatomic. The space X_a is a saturated Banach function space with the norm of X if and only if there exists some $f \in X_a$ such that $f > 0$ μ-a.e. Such a function f is called a *weak unit*.

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The *Köthe dual* X' of X is defined as the space of functions $g \in L^0(\mu)$ such that $\int |fg| \, d\mu < \infty$ for all $f \in X$. The space X' is a saturated Banach function space with norm

$$\|g\|_{X'} = \sup_{f \in B_X} \Big| \int fg \, d\mu \Big|$$

where B_X denotes the closed unit ball of X. Note that X is always contained in its Köthe bidual X'' and $\|f\|_{X''} \leq \|f\|_X$ for all $f \in X$. It is known that $\|f\|_{X''} = \|f\|_X$ for all $f \in X$ if and only if X is order semi-continuous.

The space X' can be identified with a closed subspace of the topological dual X^* of X via the linear isometry $\eta \colon X' \to X^*$ given by $\langle \eta(g), f \rangle = \int fg \, d\mu$ for all $g \in X'$ and $f \in X$. The map η is surjective if and only if X is σ-order continuous.

For issues related to Banach function spaces see for instance [4, Ch. 15], considering the function norm ρ defined there as $p(f) = \|f\|_X$ if $f \in X$ and $p(f) = \infty$ in other case.

2. $\sigma(X', X)$-compactness for $B_{X'}$

Let X be a saturated Banach function space and consider the weak* topology $\sigma(X', X)$ defined by X on its Köthe dual X', that is, the Hausdorff locally convex topology induced by the family of seminorms $\{p_f\}_{f \in X}$ on X' given by $p_f(g) = |\int fg \, d\mu|$ for $g \in X'$. Through all the recently published paper [1] referenced in the title, the fact that the closed unit ball $B_{X'}$ of X' is $\sigma(X', X)$-compact is used, but *this is not in general true*. We only have to take $X = L^\infty(\mu)$ for which $X' = L^1(\mu)$ and note that, by the Banach-Bourbaki theorem, $B_{L^1(\mu)}$ is not compact for the topology $\sigma(L^1(\mu), L^\infty(\mu))$.

However $B_{X'}$ is $\sigma(X', X)$-compact whenever X is σ-order continuous, as in this case, since B_{X^*} is identified with $B_{X'}$ via the map $\eta^{-1} \colon X^* \to X'$ which is $\sigma\big(X^*, X\big)$-$\sigma\big(X', X\big)$-continuous, we can apply the Banach-Alaoglu theorem. Therefore, requiring X to have the natural condition of σ-order continuity all the results in [1] hold.

A characterization of the weak* compactness of $B_{X'}$ follows as a particular case of a known result for Riesz spaces, see [3, Theorem 82G and Proposition 82B]. Namely, $B_{X'}$ is $\sigma\big(X', X\big)$-compact if and only if

$$\|f_n\|_{X''} \downarrow 0 \text{ whenever } (f_n) \subset X \text{ with } f_n \downarrow 0 \ \mu\text{-a.e.},$$

that is, if and only if $X \subset (X'')_a$.

In the main results of the paper [1] the space X is also supposed to be order semi-continuous. Note that in this case, since $\|\cdot\|_X = \|\cdot\|_{X''}$ on X, the weak* compactness of $B_{X'}$ is equivalent to X being σ-order continuous.

Finally note that we can obtain compactness for $B_{X'}$ if we consider the topology on X' defined by the elements of X_a. Indeed, in the case when X_a has a weak unit we have that X_a is super order dense in X (see for instance [2, Lema 2.1]) and so it can be proved that $(X_a)' = X'$ with equal norms. Then, since X_a is σ-order continuous, it follows that $B_{X'}$ is $\sigma\big(X', X_a\big)$-compact.

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References

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