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Additional Information

The characteristic subspace lattice of a linear transformation

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Abstract

Given a square matrix $A \in M_n(\mathbb{F})$, the lattices of the hyperinvariant ($\mathrm{Hinv}(A)$) and characteristic ($\mathrm{Chinv}(A)$) subspaces coincide whenever $\mathbb{F} \neq GF(2)$. If the characteristic polynomial of A splits over $\mathbb{F}$, A can be considered nilpotent. In this paper we investigate the properties of the lattice $\mathrm{Chinv}(J)$ when $\mathbb{F} = GF(2)$ for a nilpotent matrix J. In particular, we prove it to be self-dual.

Keywords: Hyperinvariant subspaces, characteristic subspaces, lattices .
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1. Introduction

Let $\mathbb{F}^n$ be the n-dimensional vector space over a field $\mathbb{F}$, and $A \in M_n(\mathbb{F})$ a square matrix corresponding to an endomorphism of $\mathbb{F}^n$ in a fixed basis. A vector subspace $V \subseteq \mathbb{F}^n$ is called invariant with respect to the endomorphism if $AV \subseteq V$. The subspace V is hyperinvariant if it is invariant for every matrix $T \in Z(A)$ (i.e. commuting with A). Weakening the latter condition, if it is only satisfied for every nonsingular matrix T commuting with A, the subspace is called characteristic. Obviously

$$\mathrm{Hinv}(A) \subseteq \mathrm{Chinv}(A) \subseteq \mathrm{Inv}(A),$$

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where $\mathrm{Hinv}(A)$, $\mathrm{Chinv}(A)$ and $\mathrm{Inv}(A)$ denote the lattices of hyperinvariant, characteristic and invariant subspaces, respectively.

For an arbitrary field $\mathbb{F}$, the lattice $\mathrm{Inv}(A)$ is studied in [3], where it is proven to be self-dual, and characterizations of some other properties are given, for instance when it is distributive or Boolean, among others. A full description of $\mathrm{Hinv}(A)$ when $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ can be found in [5], where it is proven to be a distributive and self-dual lattice, and tight bounds for its cardinality are provided. Concerning $\mathrm{Chinv}(A)$, if the characteristic polynomial of A splits over $\mathbb{F}$ and $\mathrm{card}(\mathbb{F}) > 2$, $\mathrm{Chinv}(A) = \mathrm{Hinv}(A)$ ([1]). When $\mathrm{card}(\mathbb{F}) = 2$, $\mathrm{Chinv}(A)$ and $\mathrm{Hinv}(A)$ in general do not coincide. Morevover, if all of the eigenvalues of A are in $\mathbb{F}$, the study of $\mathrm{Hinv}(A)$ and $\mathrm{Chinv}(A)$ can be reduced to the case where A has a unique eigenvalue (see, for instance [1], [2] and [5]). Therefore, if the characteristic polynomial of A splits over $\mathbb{F}$, we can assume A to be a nilpotent matrix.

If A is a nilpotent matrix, and $\mathrm{card}(\mathbb{F}) = 2$, Shoda's Theorem (see for instance [2]) characterizes the existence of characteristic non hyperinvariant subspaces. General conditions for their existence, as well as some examples, can be found in [1, 2]. A construction to explicitly obtain all of the characteristic non hyperinvariant subspaces of A is given in [7].

Our aim in this paper is to analyze basic properties of the lattice of the characteristic subspaces $\mathrm{Chinv}(A)$ of a nilpotent matrix A when $\mathbb{F} = GF(2)$. In particular we will prove that it is a self-dual lattice.

The paper is organized as follows: In section 2 we introduce the notation and basic results. We present here the structure of the characteristic non-hyperinvariant subspaces of A as obtained in [7]. In section 3 we analyze the properties of the lattice $\mathrm{Chinv}(A)$. In particular, we give an anti-isomorphism from $\mathrm{Chinv}(A)$ to $\mathrm{Chinv}(A)$, hence proving that the lattice is self-dual.

2. Preliminaries

Throughout the paper we will assume that $\mathbb{F} = GF(2)$ and $A = J$ a nilpotent Jordan matrix. Given a set of vectors $\{v_1, \ldots, v_t\} \subset \mathbb{F}^n$, we represent by $\mathrm{span}\{v_1, \ldots, v_t\}$ the vector subspace of linear combinations of the vectors $\{v_1, \ldots, v_t\}$. If E, F are vector subspaces of $\mathbb{F}^n$, the notation $E \cong F$ means that they are isomorphic.

Let $J \in M_n(GF(2))$ be a nilpotent Jordan matrix. We write $\alpha = (\alpha_1, \ldots, \alpha_m)$ for its Segre characteristic; that is to say, $m = \dim\ker(J)$ and $\alpha_1 \geq \cdots \geq \alpha_m$ are the orders of the Jordan blocks. We fix a Jordan basis for J and denote by $u_1, \ldots, u_m$ the generators of the Jordan chains,

$$u_j, Ju_j, \ldots, J^{\alpha_j - 1}u_j, \quad 1 \leq j \leq m.$$

We write $V^1, \ldots, V^m$ for the corresponding monogenic subspaces,

$$V^j = \mathrm{span}\{u_j, Ju_j, \ldots\}.$$

They satisfy that $(GF(2))^n = V^1 \oplus \cdots \oplus V^m$.

For a vector $w \in (GF(2))^n$, $w \neq 0$, its *exponent* $p = \exp(w) \geq 1$ and its *depth* $q = \operatorname{depth}(w)$ are defined by

$$\begin{aligned} w \in \ker J^p, &\quad w \notin \ker J^{p-1}, \\ w \in \operatorname{Im} J^q, &\quad w \notin \operatorname{Im} J^{q+1}. \end{aligned}$$

In particular, $\exp(J^k u_j) = \alpha_j - k$ and $\operatorname{depth}(J^k u_j) = k$.

We understand the lattice $\operatorname{Chinv}(J)$ as

$$\operatorname{Chinv}(J) = \operatorname{Hinv}(J) \cup (\operatorname{Chinv}(J) \setminus \operatorname{Hinv}(J)).$$

The hyperinvariant subspaces have been characterized in [5] and [2], and the characteristic non-hyperinvariant subspaces in [7]. We recall now both results.

Let $J \in M_n(GF(2))$ be a nilpotent Jordan matrix and $\alpha = (\alpha_1, \ldots, \alpha_m)$ its Segre characteristic. Given a partition $(k_1, \ldots, k_m)$ such that

$$0 \leq k_j \leq \alpha_j, \tag{1}$$

we denote by $V^j_{k_j}$ the vector subspace spanned by the last k_j vectors of the corresponding Jordan chain:

$$V^j_{k_j} = \operatorname{span}\{J^{\alpha_j - k_j} u_j, \ldots, J^{\alpha_j - 1} u_j\},$$

and set

$$V(k_1, \ldots, k_m) = V^1_{k_1} \oplus \cdots \oplus V^m_{k_m}, \tag{2}$$

(we take $V^j_{k_j} = 0$ if $k_j \leq 0$).

Theorem 2.1 (Gohberg & al. [5]). *The subspaces in* $\operatorname{Hinv}(J)$ *are of the form:*

$$V(k_1, \ldots, k_m),$$

with

$$k_1 \geq \cdots \geq k_m \geq 0, \tag{3}$$

$$\alpha_1 - k_1 \geq \cdots \geq \alpha_m - k_m \geq 0. \tag{4}$$

In particular, if $\alpha_{j+1} = \alpha_j$*, then* $k_{j+1} = k_j$*.*

The tuples $(k_1, \ldots, k_m)$ satisfying conditions (1), (3) and (4) will be called *hyper-tuples*. They can be visualized as decreasing both in exponent and depth.

Example 2.2. For $\alpha = (4, 2, 2, 1)$, the possible non-trivial hyper-tuples are: $(1, 0, 0, 0)$, $(1, 1, 1, 0)$, $(1, 1, 1, 1)$, $(2, 0, 0, 0)$, $(2, 1, 1, 0)$, $(2, 1, 1, 1)$, $(2, 2, 2, 1)$, $(3, 1, 1, 0)$, $(3, 1, 1, 1)$, $(3, 2, 2, 1)$.

We recall next an explicit construction of the characteristic non hyperinvariant subspaces, which has been given in [7]. According to Shoda's theorem (see for instance [2]), there exists $X \in \operatorname{Chinv}(J) \setminus \operatorname{Hinv}(J)$ if and only if there exist at least two Jordan blocks of *unique order* (i.e., no other block has the same order) which differ in more than 1. We will refer to this property as the "Shoda condition".

We denote by Ω the set of indexes corresponding to blocks of unique order:

$$\Omega := \{1 \leq i_1 < \cdots < i_l \leq m : \text{ only one Jordan block has order } \alpha_{i_j}\}.$$

Let us consider a tuple of the form

$$b = (b_{i_1}, \ldots, b_{i_t}), \quad t \geq 2, \quad \{i_1, \ldots, i_t\} = \Omega_t \subset \Omega,$$

with $1 \leq i_1 < i_2 < \cdots < i_t \leq m$. The tuple $b = (b_{i_1}, \ldots, b_{i_t})$ is said to be a *char-tuple* associated to Ω_t if

$$\begin{aligned} b_{i_1} > b_{i_2} > \cdots > b_{i_t} > 0, \\ \alpha_{i_1} - b_{i_1} > \alpha_{i_2} - b_{i_2} > \cdots > \alpha_{i_t} - b_{i_t} \geq 0. \end{aligned}$$

Given a char-tuple $b = (b_{i_1}, \ldots, b_{i_t})$ associated to Ω_t, two families of vector subspaces can be associated to b, in order to describe the characteristic non-hyperinvariant subspaces:

1. A *hyperinvariant subspace* Y is associated to b if it is of the form:

   $$\begin{aligned} Y = V(&k_1, \ldots, k_{i_1-1}, b_{i_1} - 1, k_{i_1+1}, \ldots \\ &\ldots, k_{i_2-1}, b_{i_2} - 1, k_{i_2+1}, \ldots, k_{i_t-1}, b_{i_t} - 1, k_{i_t+1}, \ldots, k_m), \end{aligned}$$

   and the following subspace is also hyperinvariant:

   $$\begin{aligned} V(&k_1, \ldots, k_{i_1-1}, b_{i_1}, k_{i_1+1}, \ldots \\ &\ldots, k_{i_2-1}, b_{i_2}, k_{i_2+1}, \ldots, k_{i_t-1}, b_{i_t}, k_{i_t+1}, \ldots, k_m). \end{aligned}$$

   Observe that the required conditions are (see Theorem 2.1) $k_{i_j-1} \geq b_{i_j}$ and $\alpha_{i_j} - b_{i_j} \geq \alpha_{i_j+1} - k_{i_j+1}, \ j = 1, \ldots, t$.
2. Define $z_1, \ldots, z_t$ as

   $$z_j = J^{\alpha_{i_j} - b_{i_j}} u_{i_j}, \qquad 1 \leq j \leq t.$$

   The subspace Z is called a *minext subspace* associated to b if:

   a) $z \in Z \Rightarrow z = z_{j_1} + \cdots + z_{j_p}, \ 1 \leq j_1 < j_2 < \ldots < j_p \leq i_t, \ p \leq t.$
   b) $z_j \notin Z, \text{ for } j = 1, \ldots, t.$
   c) Each z_j appears as a summand of some $z \in Z$, i.e.

   $$\dim\,(\operatorname{span}\{z_1, \ldots, \tilde{z}_j, \ldots, z_t\} + Z) = t, \qquad \forall j = 1, \ldots, t. \tag{5}$$

Notice that, by construction, $z_j \notin Y$ and $z_j \notin Z$ for $1 \leq j \leq t$, and Y, Z as above. Moreover,

$$z_1, \ldots, z_t \notin Z \oplus Y.$$

In fact, the subspace Z plays the role of a direct "extension" of Y such that the sum $Z \oplus Y$ is still characteristic but non-hyperinvariant ([7]).

Finally, a characterization of the subspaces $\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$ is given in the next result.

Theorem 2.3 ([7])**.** *A subspace $X \in \mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$ if and only if $X = Z \oplus Y$ for some Z and Y defined as above; i.e., if and only if there exists a char-tuple such that Z and Y are, respectively, a minext and a hyperinvariant subspaces associated to it.*

Remark 2.4. Notice that in the above theorem the subspaces Z and Y can not be zero.

Example 2.5. Let $J \in M_{31}(GF(2))$ be a nilpotent Jordan matrix with Segre characteristic $\alpha = (12, 7, 4, 4, 3, 1)$. Then,

$$\Omega = \{1, 2, 5, 6\}.$$

Taking $\Omega_3 = \{1, 5, 6\}$, the tuple $b = (10, 2, 1)$ is a char-tuple associated to Ω_3. In this case there is only one hyperinvariant subspace associated to b, namely,

$$Y = V(9, 5, 2, 2, 1, 0).$$

Moreover, for

$$z_1 = J^2 u_1, \quad z_2 = J u_5 \quad, z_3 = J^0 u_6 = u_6,$$

there are only two minext subspaces Z associated to b:

$$\begin{cases} \mathrm{span}\{z_1 + z_2 + z_3\} \\ \mathrm{span}\{z_1 + z_2, z_2 + z_3\} \end{cases}$$

Therefore,

$$\begin{cases} X_1 = \mathrm{span}\{z_1 + z_2 + z_3\} \oplus V(9, 5, 2, 2, 1, 0), \\ X_2 = \mathrm{span}\{z_1 + z_2, z_2 + z_3\} \oplus V(9, 5, 2, 2, 1, 0), \end{cases}$$

are characteristic non-hyperinvariant subspaces.

3. Properties of the lattice Chinv(J)

A lattice is a partially order set where each pair of elements X_1, X_2 has a meet $(X_1 \cap X_2)$ and a join $(X_1 + X_2)$. By the definition of a characteristic subspace, if $X_1, X_2 \in \mathrm{Chinv}(J)$, then $X_1 \cap X_2 \in \mathrm{Chinv}(J)$ and $X_1 + X_2 \in \mathrm{Chinv}(A)$. Therefore, $\mathrm{Chinv}(J)$ is a lattice with inclusion as order, intersection as meet and linear sum as join. In particular, $\mathrm{Chinv}(J)$ is a sublattice of $\mathrm{Inv}(J)$. Given a lattice L, a linear application $\phi : L \longrightarrow L$ is an *anti-isomorphism* if it is an isomorphism which reverses the order. Therefore, $\phi(X_1 \cap X_2) = \phi(X_1) + \phi(X_2)$ and $\phi(X_1 + X_2) = \phi(X_1) \cap \phi(X_2)$.

Remark 3.1. a) Notice that $\text{Chinv}(J) \setminus \text{Hinv}(J)$ is not a lattice. For instance, let X_1, X_2 be the characteristic non-hyperinvariant subspaces given in Example 2.5. Then, $X_1 \cap X_2 = V(9,5,2,2,1,0)$, which is hyperinvariant, therefore, it is not in $\text{Chinv}(J) \setminus \text{Hinv}(J)$.

b) Observe that given $V_1 = V(k_1, \dots, k_m), V_2 = V(k'_1, \dots, k'_m)$ as in (2), then

$$V_1 \cap V_2 = V(\min\{k_1, k'_1\}, \dots, \min\{k_m, k'_m\}).$$

In particular, we remark that if $V_1, V_2 \in \text{Hinv}(J)$ are nontrivial subspaces, they have nontrivial intersections.

We recall next some general definitions:

Definition 3.2. Let $L(A)$ be a lattice of subspaces of $\mathbb{F}^n$ with zero element $\{0\}$ and unit element $\mathbb{F}^n$. We say that

1. $L(A)$ is *distributive* if for every $X_1, X_2, X_3 \in L(A)$ the following identity is satisfied
$$(X_1 + X_2) \cap X_3 = (X_1 \cap X_3) + (X_2 \cap X_3). \tag{6}$$
and $L(A)$ is *modular* if (6) holds whenever $X_1 \subseteq X_3$.
2. $L(A)$ is *complemented* if for every $X_1 \in L(A)$ there exist $X_2 \in L(A)$ such that
$$X_1 \cap X_2 = \{0\} \quad \text{and} \quad X_1 \oplus X_2 = \mathbb{F}^n.$$
3. $L(A)$ is a *Boolean algebra* if it is distributive and complemented.
4. $L(A)$ is *finite* if it has a finite number of elements.
5. $L(A)$ is *self-dual* if there exist an anti-isomorphism from $L(A)$ to $L(A)$.

For the lattice $\text{Hinv}(J)$ we have the following results.

Proposition 3.3 ([4]). *Let $J \in M_n(GF(2))$ be a nilpotent Jordan matrix and $\alpha = (\alpha_1, \dots, \alpha_m)$ its Segre characteristic. Then,*

1. $\text{Hinv}(J)$ *is distributive. In particular,* $\text{Hinv}(J)$ *is modular.*
2. $\text{Hinv}(J)$ *is complemented if and only if $\alpha = (1, \dots, 1)$.*
3. $\text{Hinv}(J)$ *is finite.*
4. $\text{Hinv}(J)$ *is self-dual.*

Let us analyze these properties on $\text{Chinv}(J)$.

Lemma 3.4. *Let $J \in M_n(GF(2))$ be a nilpotent Jordan matrix and $\alpha = (\alpha_1, \dots, \alpha_m)$ its Segre characteristic. Assume that the Shoda condition is satisfied. Then,*

1. $\text{Chinv}(J)$ *is not distributive, but it is modular.*
2. $\text{Chinv}(J)$ *is not complemented.*
3. $\text{Chinv}(J)$ *is finite.*

Proof. 1. We give a counterexample. Let $\alpha = (8, 6, 4)$. Let

$$Y = V(6, 4, 2),$$

$$X_1 = \text{span}\{z_1 + z_2 + z_3\} \oplus V(5, 4, 3),$$

$$X_2 = \text{span}\{z_1 + z_2, z_2 + z_3\} \oplus V(5, 4, 3),$$

where $z_1 = J^2 u_1$, $z_2 = J u_2$ and $z_3 = u_3$. Then, $X_1, X_2, Y \in \text{Chinv}(J)$ and

$$(X_1 + X_2) \cap Y = V(6, 5, 4) \cap Y = V(6, 4, 2) \neq \\ \neq (X_1 \cap Y) + (X_2 \cap Y) = V(5, 4, 2).$$

The property of $\text{Chinv}(J)$ being modular follows from the fact that the lattice $\text{Inv}(J)$ is modular ([3]).

2. As in this case $\alpha_1 > 1$, $\text{Hinv}(J)$ is not complemented. Therefore, there exists a subspace $X_1 \in \text{Hinv}(J)$ not complemented in $\text{Hinv}(J)$.
   Assume that X_1 is complemented in $\text{Chinv}(J)$. Then, there exists a subspace $X_2 \in \text{Chinv}(J)$ such that $X_1 \cap X_2 = \{0\}$ and $X_1 \oplus X_2 = (GF(2))^n$. Observe that $X_2 \in \text{Chinv}(J) \setminus \text{Hinv}(J)$. By Theorem 2.3, there exists a char-tuple such that if Y is a hyperinvariant subspace and Z a minext subspace associated to it, $X_2 = Z \oplus Y$.
   But this implies that $X_1 \cap Y \subset X_1 \cap X_2 = \{0\}$, what is a contradiction because $X_1 \cap Y \neq \{0\}$ (see Remark 2.4 and Remark 3.1.b). This proves that $\text{Chinv}(J)$ is never complemented.
3. Given α, the number of char-tuples is finite (they are a particular type of hyper-tuples, and this is a finite number ([5])). Moreover, given a char-tuple, the number of minext subspaces is finite because the minext subspaces are linear subspaces of a finite dimension space over a finite field $GF(2)$ and the number of hyperinvariant subspaces associated to this char-tuple are finite too because the number of hyper-tuples is finite (see [4]). Therefore, the order of $\text{Chinv}(J)$ is always finite.

□

Remark 3.5. As $\text{Chinv}(J)$ is neither distributive nor complemented, $\text{Chinv}(J)$ is not a Boolean lattice.

In what follows we prove that $\text{Chinv}(J)$ is self-dual.

Given a subset S of $\mathbb{F}^n$, we denote by $\text{Ann}(S)$ the annihilator of S:

$$\text{Ann}(S) = \{u \in \mathbb{F}^n |\ u \cdot v = 0, \quad \forall v \in S\},$$

where '$\cdot$' is the standard scalar product of the components of the vectors, with respect to the canonical basis (notice that if $\mathbb{F} = GF(2)$, the scalar product is a bilinear form, non positive definite).

We will also find annihilators of subsets with respect to subspaces of $\mathbb{F}^n$ instead of with respect to the whole space. In that case, we will specify the subspace in the notation. Given a vector subspace $V \subset \mathbb{F}^n$,

$$\mathrm{Ann}(S, V) = \{u \in V |\ u \cdot v = 0, \quad \forall v \in S\}.$$

In particular, $\mathrm{Ann}(S, \mathbb{F}^n) = \mathrm{Ann}(S)$.

Proposition 3.6. *If Z is a minext subspace associated to a char-tuple $b = (b_{i_1}, \dots, b_{i_t})$ and $\mathcal{Z}_t = \mathrm{span}\{z_1, \dots, z_t\}$, then*

1. $\mathrm{Ann}(Z, \mathcal{Z}_t)$ *is a minext subspace associated to the same char-tuple.*
2. *The* $\mathrm{Ann}(Z)$ *is*

   $$\mathrm{Ann}(Z, \mathcal{Z}_t) \oplus V(\alpha_1, \dots, \alpha_{i_1-1}, \tilde{\alpha}_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_t-1}, \tilde{\alpha}_{i_t}, \alpha_{i_t+1}, \dots, \alpha_m),$$

   where,

   $$V(\alpha_1, \dots, \alpha_{i_1-1}, \tilde{\alpha}_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_t-1}, \tilde{\alpha}_{i_t}, \alpha_{i_t+1}, \dots, \alpha_m) = \\ = V^1 \oplus \dots \oplus V^{i_1-1} \oplus \tilde{V}^{i_1} \oplus V^{i_1+1} \oplus \dots \oplus V^{i_t-1} \oplus \tilde{V}^{i_t} \oplus V^{i_t+1} \oplus \dots \oplus V^m,$$

   with

   $$\tilde{V}^{i_j} = \mathrm{span}\{u_{i_j}, \dots, J^{\alpha_{i_j} - b_{i_j} - 1} u_{i_j}, J^{\alpha_{i_j} - b_{i_j} + 1} u_{i_j}, \dots, J^{\alpha_{i_j} - 1} u_{i_j}\},\ j = 1, \dots, t.$$

Proof.

1. Assume that the minext space Z can be written as

   $$Z = \mathrm{span}\{w_1, \dots, w_d\} \subseteq \mathrm{span}\{z_1, \dots, z_t\} = \mathcal{Z}_t.$$

   Taking $F_i = \mathrm{span}\{z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_t\}$ for $i = 1, \dots, t$, then

   $$\mathrm{Ann}(F_i, \mathcal{Z}_t) = \mathrm{span}\{z_i\}.$$

   Conditions (5b) and (5c) in the definition of Z can be written as:
   - $\mathrm{span}\{z_i\} \nsubseteq Z$.
   - $Z \nsubseteq F_i$.

   Using annihilator properties ([6]),
   - $\mathrm{Ann}(Z, \mathcal{Z}_t) \nsubseteq \mathrm{Ann}(\mathrm{span}\{z_i\}, \mathcal{Z}_t) = F_i$.
   - $\mathrm{Ann}(F_i, \mathcal{Z}_t) = \mathrm{span}\{z_i\} \nsubseteq \mathrm{Ann}(Z, \mathcal{Z}_t)$.

   It means that $\mathrm{Ann}(Z, \mathcal{Z}_t)$ is a minext subspace associated to the same char-tuple as Z.
2. It is straightforward.

□

Corollary 3.7. *Given $\alpha = (\alpha_1, \dots, \alpha_m)$, let $b = (b_{i_1}, \dots, b_{i_t})$ be a char-tuple associated to α. If Z and Y are a minext subspace and an hyperinvariant subspace associated to b, then*

$$\mathrm{Ann}(Z, \mathcal{Z}_t) \subset \mathrm{Ann}(Y).$$

Proof. By the above proposition, $\mathrm{Ann}(Z, \mathcal{Z}_t)$ is a minext subspace associated to b. For $Y = V(k_1, \ldots, b_{i_1} - 1, \ldots, b_{i_t} - 1, \ldots, k_m)$, it is obvious that

$$\mathrm{Ann}(Z, \mathcal{Z}_t) \subset \mathcal{Z}_t = \mathrm{span}\{z_1, \ldots, z_t\} \subset \mathrm{Ann}(Y).$$

□

Let

$$\mathcal{B} = \{u_1, Ju_1, \ldots, J^{\alpha_1 - 1}u_1, \ \ldots \ , u_m, \ldots, J^{\alpha_m - 1}u_m\},$$

be a Jordan basis for $(GF(2))^n$. Let S be the matrix of the change of basis from the basis $\mathcal{B}$ to the basis

$$\mathcal{B}' = \{J^{\alpha_1 - 1}u_1, \ldots, Ju_1, u_1, \ \ldots \ , J^{\alpha_m - 1}u_m, \ldots, u_m\}.$$

It is known (see [5, 6]) that the application

$$\begin{array}{rccc} D: & \mathrm{Inv}(J) & \longrightarrow & \mathrm{Inv}(J) \\ & X & \longrightarrow & S^{-1}\,\mathrm{Ann}(X) \end{array} \tag{7}$$

is an anti-isomorphism.

We prove next that $\mathrm{Chinv}(J)$ is self-dual.

Theorem 3.8. *Let $J \in M_n(GF(2))$ be a nilpotent Jordan matrix and $\alpha = (\alpha_1, \ldots, \alpha_m)$ its Segre characteristic. Then, the lattice* $\mathrm{Chinv}(J)$ *is self-dual.*

Proof. It is enough to prove that

$$X \in \mathrm{Chinv}(J) \Rightarrow D(X) \in \mathrm{Chinv}(J).$$

In fact, the application D in (7) transforms subspaces of $\mathrm{Hinv}(J)$ into subspaces of $\mathrm{Hinv}(J)$, and subspaces of $\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$ into subspaces of $\mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$ as we show next.

1. Let $V(k_1, \ldots, k_m) \in \mathrm{Hinv}(J)$. Then,

   $$\begin{array}{l} \mathrm{Ann}(V(k_1, \ldots, k_m)) = \\ \mathrm{Ann}(\mathrm{span}\{J^{\alpha_1 - k_1}u_1, \ldots, J^{\alpha_1 - 1}u_1; \ldots; J^{\alpha_m - k_m}u_m, \ldots, J^{\alpha_m - 1}u_m\}) = \\ = \mathrm{span}\{u_1, \ldots, J^{\alpha_1 - k_1 - 1}u_1; \ldots; u_m, \ldots, J^{\alpha_m - k_m - 1}u_m\}. \end{array}$$

   Therefore,

   $$\begin{array}{l} D(V(k_1, \ldots, k_m)) = S^{-1}(\mathrm{Ann}(V(k_1, \ldots, k_m))) = \\ S^{-1}\,\mathrm{span}\{u_1, \ldots, J^{\alpha_1 - k_1 - 1}u_1; \ldots; u_m, \ldots, J^{\alpha_m - k_m - 1}u_m\} = \\ \mathrm{span}\{J^{k_1}u_1, \ldots, J^{\alpha_1 - 1}u_1; \ldots; J^{k_m}u_m, \ldots, J^{\alpha_m - 1}u_m\} = \\ = V(\alpha_1 - k_1, \ldots, \alpha_m - k_m) \in \mathrm{Hinv}(J). \end{array}$$

2. Let $X \in \mathrm{Chinv}(J) \setminus \mathrm{Hinv}(J)$. Assume that $X = Z \oplus Y$ with

$$Y = V(k_1, \dots, k_{i_1-1}, b_{i_1} - 1, k_{i_1+1}, \dots, k_{i_t-1}, b_{i_t} - 1, k_{i_t+1}, \dots, k_m),$$

where $b = (b_{i_1}, \dots, b_{i_t})$ is the char-tuple associated to X, and Z a minext subspace associated to b. Let us find $\mathrm{Ann}(X)$.
Taking into account Proposition 3.6,

$$\begin{aligned}\mathrm{Ann}(X) = \mathrm{Ann}(Z) \cap \mathrm{Ann}(Y) = \\ (\mathrm{Ann}(Z, \mathcal{Z}_t) \oplus V(\alpha_1, \dots, \tilde{\alpha}_{i_1}, \dots, \tilde{\alpha}_{i_t}, \dots, \alpha_m)) \cap \mathrm{Ann}(Y) = \\ = \mathrm{Ann}(Z, \mathcal{Z}_t) \oplus (V(\alpha_1, \dots, \tilde{\alpha}_{i_1}, \dots, \tilde{\alpha}_{i_t}, \dots, \alpha_m) \cap \mathrm{Ann}(Y)).\end{aligned}$$

The last identity is a consequence of the fact that $\mathrm{Chinv}(J)$ is modular and $\mathrm{Ann}(Z, \mathcal{Z}_t) \subset \mathrm{Ann}(Y)$ (see Lemma 3.4, condition (6) and Corollary 3.7). We have that

$$\begin{aligned}\mathrm{Ann}(Y) = \mathrm{span}\{u_1, \dots, J^{\alpha_1-k_1-1}u_1; \dots; u_{i_1}, \dots, J^{\alpha_{i_1}-b_{i_1}-1}u_{i_1}; \dots; \\ u_{i_t}, \dots, J^{\alpha_{i_t}-b_{i_t}-1}u_{i_t}; \dots; u_m, \dots, J^{\alpha_m-k_m-1}u_m\},\end{aligned}$$

then,

$$\begin{aligned}V(\alpha_1, \dots, \alpha_{i_1-1}, \tilde{\alpha}_{i_1}, \alpha_{i_1+1}, \dots, \alpha_{i_t-1}, \tilde{\alpha}_{i_t}, \alpha_{i_t+1}, \dots, \alpha_m) \cap \mathrm{Ann}(Y) = \\ \mathrm{span}\{u_1, \dots, J^{\alpha_1-1}u_1; \dots; u_{i_1}, \dots, J^{\alpha_{i_1}-b_{i_1}-1}u_{i_1}, J^{\alpha_{i_1}-b_{i_1}+1}u_{i_1}, \dots, \\ J^{\alpha_{i_1}-1}u_{i_1}; \dots; u_{i_t}, \dots, J^{\alpha_{i_t}-b_{i_t}-1}u_{i_t}, J^{\alpha_{i_t}-b_{i_t}+1}u_{i_t}, \dots, J^{\alpha_{i_t}-1}u_{i_t}; \dots; \\ u_m, \dots, J^{\alpha_m-1}u_m\} \ \cap \ \mathrm{span}\{u_1, \dots, J^{\alpha_1-k_1-1}u_1; \dots; u_{i_1}, \dots, J^{\alpha_{i_1}-b_{i_1}}u_{i_1}; \\ \dots; u_{i_t}, \dots, J^{\alpha_{i_t}-b_{i_t}}u_{i_t}; \dots; u_m, \dots, J^{\alpha_m-k_m-1}u_m\} = \\ = \mathrm{span}\{u_1, \dots, J^{\alpha_1-k_1-1}u_1; \dots; u_{i_1}, \dots, J^{\alpha_{i_1}-b_{i_1}-1}u_{i_1}; \dots; \\ u_{i_t}, \dots, J^{\alpha_{i_t}-b_{i_t}-1}u_{i_t}; \dots; u_m, \dots, J^{\alpha_m-k_m-1}u_m\}.\end{aligned}$$

Applying the inverse of the change of basis S to this set, we obtain

$$\begin{aligned}S^{-1}(V(\alpha_1, \dots, \tilde{\alpha}_{i_1}, \dots, \tilde{\alpha}_{i_t}, \dots, \alpha_m) \cap \mathrm{Ann}(Y)) = \\ \mathrm{span}\{J^{\alpha_1-1}u_1, \dots, J^{k_1+1}u_1; \dots; J^{\alpha_{i_1}-1}u_{i_1}, \dots, J^{b_{i_1}+1}u_{i_1}; \dots; \\ J^{\alpha_{i_t}-1}u_{i_t}, \dots, J^{b_{i_t}+1}u_{i_t}; \dots; J^{\alpha_m-1}u_m, \dots, J^{k_m+1}u_m\} = \\ = V(\alpha_1 - k_1, \dots, \alpha_{i_1} - b_{i_1}, \dots, \alpha_{i_t} - b_{i_t}, \dots \alpha_m - k_m).\end{aligned}$$

On the other hand,

$$\mathrm{Ann}(Z, \mathcal{Z}_t) = \{w \in \mathcal{Z}_t = \mathrm{span}\{z_1, \dots, z_t\} | \ w \cdot z = 0, \forall z \in Z\},$$

which by Proposition 3.6 is a minext subspace associated to the char-tuple b. Applying the inverse of the change of basis S to this subspace, we obtain that $S^{-1}(\mathrm{Ann}(Z, \mathcal{Z}_t))$ is a minext subspace generated by the elements

$$z_i' = J^{\alpha_{i_j}-b_{i_j}}u_{i_j}, \ j = 1, \dots, t.$$

As a consequence, $D(X) = S^{-1}\operatorname{Ann}(X)$ is the subspace

$$\operatorname{Ann}(Z, \mathcal{Z}_t) \oplus V(\alpha_1 - k_1, \dots, \alpha_{i_1} - b_{i_1}, \dots, \alpha_{i_t} - b_{i_t}, \dots \alpha_m - k_m),$$

and, by Theorem 2.3, $D(X) \in \operatorname{Chinv}(J) \setminus \operatorname{Hinv}(J)$ associated to the char-tuple

$$b' = (\alpha_{i_1} - b_{i_1} + 1, \dots, \alpha_{i_t} - b_{i_t} + 1).$$

□

Example 3.9. Let $\alpha = (12, 7, 4, 2, 1)$ be the Segre partition of a Jordan matrix J and $\Omega_t = \{2, 4\}$. Let $b = (6, 2)$ be a char-tuple. $Y = V(9, 5, 3, 1, 1)$ is a hyperinvariant subspace associated to b ($V(9, 6, 3, 2, 1)$ is also hyperinvariant). Define $z_1 = Ju_2, z_2 = u_4$ and $Z = \operatorname{span}\{z_1 + z_2\}$. Then $X = Z \oplus Y \in \operatorname{Chinv}(J) \setminus \operatorname{Hinv}(J)$. Let S be the change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{B}'$ mentioned above. We find $D(X)$:

$$V(9, 5, 3, 1, 1) = \operatorname{span}\{J^3u_1, \dots, J^{11}u_1; J^2u_2, \dots, J^6u_2; Ju_3, \dots, J^3u_3; Ju_4; u_5\},$$

$$\operatorname{Ann}(V(9, 5, 3, 1, 1)) = \operatorname{span}\{u_1, Ju_1, J^2u_1; u_2, Ju_2; u_3; u_4\},$$
$$V(12, \tilde{7}, 4, \tilde{2}, 1) = \operatorname{span}\{u_1, \dots, J^{11}u_1; u_2, J^2u_2, \dots, J^6u_2; u_3 \dots J^3u_3; Ju_4; u_5\},$$
$$V(12, \tilde{7}, 4, \tilde{2}, 1) \cap \operatorname{Ann}(V(9, 5, 3, 1, 1)) = \operatorname{span}\{u_1, Ju_1, J^2u_1; u_2; u_3\},$$

$$\begin{aligned} D(X) = S^{-1}\operatorname{span}\{u_1, Ju_1, J^2u_1; u_2; u_3\} = \\ \operatorname{span}\{J^{11}u_1, J^{10}u_1, J^9u_1; J^6u_2; J^3u_3\} = \\ = V(12-9, 7-6, 4-3, 2-2, 1-1) = V(3, 1, 1, 1, 0) \in \operatorname{Hinv}(J), \end{aligned}$$

associated t the char-tuple $b' = (7 - 6 + 1, 2 - 2 + 1) = (2, 1)$.

$$\operatorname{Ann}(Z, \mathcal{Z}_t) = \{w \in \operatorname{span}\{z_1, z_2\} | w \cdot (z_1 + z_2) = 0\} = \operatorname{span}\{z_1 + z_2\},$$

$$S^{-1}\operatorname{Ann}(Z, \mathcal{Z}_t) = \operatorname{span}\{J^5u_2 + Ju_4\},$$

therefore, $S^{-1}\operatorname{Ann}(Z, \mathcal{Z}_t)$ is a minext subspace associated to the char-tuple $b' = (2, 1)$.

Finally,

$$D(X) = S^{-1}\operatorname{Ann}(X) = \operatorname{span}\{J^5u_2 + Ju_4\} \oplus V(3, 1, 1, 1, 0) \in \operatorname{Chinv}(J) \setminus \operatorname{Hinv}(J).$$

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