A New Approach to the Maximum Benefit Chinese Postman Problem

Ángel Corberán[1,*], Isaac Plana[2], Antonio M. Rodríguez-Chía[3], José M. Sanchis[4]

[1] Dept. de Estadística e Investigación Operativa, Universitat de València (Spain)
[2] Dept. Matemáticas para la Economía y la Empresa, Universitat de València (Spain)
[3] Dept. de Estadística e Investigación Operativa, Universidad de Cádiz (Spain)
[4] Dept. de Matemática Aplicada, Universidad Politécnica de Valencia (Spain)

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[*] corresponding author: angel.corberan@uv.es

Abstract

The Maximum Benefit Chinese Postman Problem (MBCPP) is an interesting and practical generalization of the classical Chinese Postman Problem (CPP). Associated with each edge, the MBCPP considers a cost of traversal if the edge is serviced, a deadhead cost associated with its traversal when it is not serviced, and several benefits (one per each time the edge is traversed with service). The objective is to find a closed walk starting and ending at the depot with maximum net benefit. Unlike the classical CPP, the MBCPP is NP-hard and generalizes other arc routing problems as the Rural Postman Problem (RPP) and the Prize-Collecting RPP. Although the MBCPP was introduced in 1993, we propose here for the first time an IP formulation for the undirected case. We also study its associated polyhedron and introduce several families of valid inequalities inducing facets of it. Based on this polyhedron description, we propose a branch-and-cut algorithm for the MBCPP and present computational results on different sets of instances with up to 1000 vertices and 3000 edges.

Keywords: Chinese Postman Problem, Maximum Benefit Chinese Postman Problem, Rural Postman Problem, facets, branch-and-cut.

1 Introduction

The Chinese Postman Problem (CPP) consists of finding a minimum cost closed walk traversing each edge of a graph at least once [13]. It is well known that the CPP can be solved in polynomial time when the graph is directed or undirected (see [9]). However, if the CPP is defined on a mixed ([18]) or a 'windy' graph ([6], [12]) the problem becomes NP-hard.

The Maximum Benefit Chinese Postman Problem (MBCPP) is a generalization of the CPP in which not all the edges have to be traversed and a benefit is realized each time an edge of the graph is serviced. Each edge of the graph has an associated cost for its traversal with service, a deadhead cost for its traversal with no service, and a set of gross benefits. The objective is to find a closed walk (tour) starting and ending at the depot with maximum net benefit. Applications of the MBCPP include the routing of street cleaners and the construction of street snow-plowing and snow-salting tours. An additional benefit is derived when a street is plowed multiple times and the benefit may depend upon whether the link represents an arterial or a low-traffic neighborhood street. Unlike the classical CPP, this problem allows us to obtain solutions that do not traverse some edges while other edges can be traversed multiple times.

More precisely, the MBCPP can be defined as follows. Let $G = (V, E)$ be an undirected connected graph, where vertex $1 \in V$ represents the depot. Each edge $e \in E$ has two different costs associated, c_e^s and c_e^d. The first one represents the cost of traversing and servicing edge e, while the second one corresponds to the cost of just traversing that edge without servicing it (deadhead cost). Moreover, each edge $e \in E$ has n_e benefits, $b_e^1, b_e^2, \ldots, b_e^{n_e}$, giving the gross benefit of servicing the edge for the first, second,..., n_e-*th* time. Therefore, the net benefit of the t-th traversal of edge e is given by $b_e^t - c_e^s$ for $t = 1, \ldots, n_e$, while the net benefit of deadheading an edge is $-c_e^d$. Then, the MBCPP consists of finding a tour, starting from the depot, traversing a certain number of times some of the edges in E, and returning to the depot, with maximum total net benefit. This problem is NP-hard, since the Rural Postman Problem ([17]), which was proved to be NP-hard in [14], can be considered a special case of the MBCPP ([20]).

In the literature of routing problems we can find several attempts to study this problem. Malandraki and Daskin [15] introduced the MBCPP and studied its directed version. They modeled it as a minimum cost flow problem with subtour elimination constraints. Based on this approach, they proposed a branch-and-bound procedure and solved instances on a 25 vertices network. In [20], an approximate algorithm to solve the MBCPP on undirected graphs is devised. The procedure is illustrated on an example with 15 vertices and 26 edges. The algorithm expands the original graph by replacing each edge with a set of edges of positive net benefit. Minimal spanning tree and matching algorithms are then applied to generate a postman tour. In [19], several heuristic algorithms for solving the MBCPP on directed graphs are proposed. Authors report computational results on graphs with up to 30 vertices and 780 arcs. In all these papers it is assumed that $c_e^s \geq c_e^d$ and $b_e^1 \geq b_e^2 \geq \cdots \geq b_e^{n_e} \geq 0$. Although these assumptions seem to appropriately reflect real world situations, they are not needed for the study we present in this paper and (we think) restrict the problem unnecessarily.

Some related problems have also been subject of study. Among them, the Prize-collecting Arc Routing Problem (PCARP), also called Privatized Rural Postman Problem. In the PCARP only the edges in a given subset of edges $D \subseteq E$ have an associated benefit, and it is assumed that this benefit can be collected only once, independently of the number of times the edge is traversed. Note that this problem is a special case of the MBCPP in which $n_e = 1$ for all the edges in D, while $n_e = 0$ for the rest. The PCARP was introduced in [3], where an ILP formulation with binary variables was provided. In particular, the authors used a new family of inequalities, called the set-parity inequalities, which are an adaptation to this problem of the so-called cocircuit inequalities introduced in [5]. In [2], an LP-based algorithm to solve the problem on undirected graphs was proposed. A related problem, the Clustered Prize-collecting Arc Routing Problem, has been recently studied in [1] and [11]. In this last problem, the connected components defined by the edges with net benefit are considered, and for each component either all or none of its edges have to be serviced. The same problem defined on a 'windy' graph has been studied in [7]. Other arc routing problems with benefits have been studied in [10] and [4]. In the first paper, benefits are associated with a subset of arcs and can be collected a given number of times. The objective is to find a set of tours in the graph that maximizes the net benefit without exceeding a maximal length for each tour. Authors proposed a branch-and-price algorithm to solve this problem. In the second paper, the authors presented a branch-and-price algorithm and several heuristics for the Capacitated Arc Routing Problem with benefits.

In this paper we propose a new approach to the MBCPP that provides a more general and useful framework, present a formulation for the undirected MBCPP, study its associated polyhedron and propose a branch-and-cut algorithm for its exact resolution. More precisely, in Section 2 ...

2 Problem formulation

Consider an undirected and connected graph $G = (V, E)$. Associated with each edge $e \in E$, there are $n_e + 1$ net benefits. The n_e first ones, $\overline{b}_e^t = b_e^t - c_e^s$, $t = 1, \ldots, n_e$, correspond to the traversals of the edge servicing it, while the last one, $\overline{b}_e^{n_e+1} = -c_e^d$, is associated with the deadheading of e.

We now define

$$b_e^{odd} = \max\left\{\sum_{\ell=1}^{k} \overline{b}_e^{\ell} \;:\; k \text{ odd with } k \le n_e + 1\right\}$$

$$b_e^{even} = \max\left\{\sum_{\ell=1}^{k} \overline{b}_e^{\ell} \;:\; k \text{ even with } k \le n_e + 1\right\} - b_e^{odd}.$$

If $n_e = 0$, we define $b_e^{even} := b_e^{odd}(= -c_e^d)$. It is not difficult to see that solving the MBCPP on the graph with $n_e + 1$ parallel edges for each original edge e is equivalent to solve it on a smaller graph having only one parallel edge to each edge in the original graph. Or, equivalently, to solve the problem on the original graph G, but considering that the first traversal of edge e has net benefit b_e^{odd}, while b_e^{even} is the net benefit associated with its second traversal. Note that to get the net benefit b_e^{even} traversing edge e, we need first to traverse it with net benefit b_e^{odd}. In this way, the MBCPP can be formulated as follows.

For each $e = (i, j) \in E$ we define two binary variables x_e and y_e. Variable x_e takes value 1 if e is traversed and 0 if e is not traversed, while variable y_e takes value 1 if e is traversed twice and 0 otherwise. In other words, variables x_e and y_e represent the first and second traversal of edge e, respectively. We have the following formulation for the MBCPP:

$$\text{Maximize} \qquad \sum_{e\in E} \left(b_e^{odd} x_e + b_e^{even} y_e\right)$$

s.t.:

$$\sum_{e\in\delta(i)} \left(x_e + y_e\right) \equiv 0 \qquad (\text{mod } 2), \quad \forall i \in V \tag{1}$$

$$\sum_{e\in\delta(S)} \left(x_e + y_e\right) \ge 2x_f, \qquad \forall S \subset V \setminus \{1\}, \quad \forall f \in E(S) \tag{2}$$

$$x_e \ge y_e \qquad \forall e \in E \tag{3}$$

$$x_e, y_e \in \{0, 1\} \qquad \forall e \in E \tag{4}$$

Constraints (1) force the vertices to be of even degree in the solution, its connectivity is assured with conditions (2), and constraints (3) guarantee that a second traversal of an edge can occur only when it has been previously traversed. Note that although constraints (1) are not linear, they can be easily linearize by introducing new integer variables. However, this will be not necessary since we are proposing a polyhedral approach to solve the problem. Note also that $(x, y) = (0, 0)$ satisfies the above constraints and is, therefore, a feasible solution to the MBCPP.

3 MBCPP Polyhedron

Let us call MBCPP tour to each vector $(x, y) \in \{0, 1\}^{2|E|}$ satisfying (1) to (4) and let MBCPP(G) be the convex hull of all MBCPP tours. Obviously, it is a polytope.

Remember that a graph G is called 3-edge connected if every proper cut-set $\delta(S)$, $S \subset V$, contains, at least, 3 edges. It is well known that G is 3-edge connected if, and only if, for every pair of nodes $i, j \in V$, there are at least three edge-disjoint paths in G connecting i and j .

Theorem 1 *MBCPP(G) is a full-dimensional polyhedron (dim(MBCPP(G))= $2|E|$) if, and only if, G is 3-edge connected.*

Proof: If G is not 3-edge connected there is a cut-set $\delta(S)$ with at most 2 edges. If $\delta(S)$ contains exactly two edges, namely e and f, it can be seen that all MBCPP tours satisfy the equation $x_e - y_e = x_f - y_f$. Moreover, if $\delta(S) = \{e\}$, then all MBCPP tours satisfy $x_e = y_e$. Therefore, in both cases, the polyhedron is not full-dimensional.

On the other hand, let us suppose now that graph G is 3-edge connected. We will prove that the polyhedron is full-dimensional. Let $ax + by = c$ (that is, $\sum_{e\in E} a_e x_e + \sum_{e\in E} b_e y_e = c$) be an equation satisfied by all the MBCPP tours. We have to prove that $a = b = c = 0$.

Given that $(x, y) = (0, 0)$ is a solution to the MBCPP, $a \cdot 0 + b \cdot 0 = c$ holds and, then, $c = 0$.

Let $(i, j) \in E$ be an arbitrary edge. Given that G is connected, there is a path $\mathcal{P}$ joining nodes 1 and i. The solution that traverses the path $\mathcal{P}$ twice (that is, $x_e = y_e = 1 \ \ \forall e \in \mathcal{P}$) is a MBCPP tour and then $\sum_{e\in\mathcal{P}} a_e + \sum_{e\in\mathcal{P}} b_e = 0$ holds. On the other hand, the solution that traverses the path $\mathcal{P}$ and the edge (i, j) twice is also a MBCPP tour and then $\sum_{e\in\mathcal{P}} a_e + \sum_{e\in\mathcal{P}} b_e + a_{ij} + b_{ij} = 0$. By subtracting both expressions we obtain that $a_{ij} + b_{ij} = 0$ for all $(i, j) \in E$.

Let $\mathcal{C}$ be any cycle in graph G. There is a path $\mathcal{P}$ joining node 1 and a node i in the cycle. The solution that traverses the path $\mathcal{P}$ twice ($x_e = y_e = 1$) and the cycle $\mathcal{C}$ once ($x_e = 1, y_e = 0$) is a MBCPP tour and then $\sum_{e\in\mathcal{P}} a_e + \sum_{e\in\mathcal{P}} b_e + \sum_{e\in\mathcal{C}} a_e = 0$. Given that $a_{ij} + b_{ij} = 0$ we obtain that

$$\sum_{e\in\mathcal{C}} a_e = 0 \qquad \text{for any cycle } \ \mathcal{C} \ \text{ in } \ G.$$

Let $(i, j) \in E$ be an arbitrary edge. Since G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain edge (i, j). Let us suppose, w.l.o.g., that either the depot 1 is in path $\mathcal{P}_1$ or it can be joined to some node in path $\mathcal{P}_1$ without using edges in $\mathcal{P}_2$. Then, the solution (x^1, y^1) that traverses path $\mathcal{P}_1$ and edge (i, j) once and the path to the depot twice is a MBCPP tour. Let now (x^2, y^2) be the MBCPP tour obtained from (x^1, y^1) after replacing the edge (i, j) by the edges in path $\mathcal{P}_2$. By subtracting the two expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ we obtain $a_{ij} - \sum_{e\in\mathcal{P}_2} a_e = 0$, and, since $(i, j) \cup \mathcal{P}_2$ is a cycle in G, we obtain $a_{ij} + \sum_{e\in\mathcal{P}_2} a_e = 0$, and, thus, $a_{ij} = 0$ for each edge $(i, j) \in E$. Given that $a_{ij} + b_{ij} = 0$, we obtain $b_{ij} = 0$ for each edge $(i, j) \in E$. Hence, $a = b = c = 0$ and the polyhedron MBCPP(G) is full-dimensional. ♦

In the following, we will assume that graph G is 3-edge connected and thus MBCPP(G) is full dimensional. Therefore every facet of the polyhedron is induced by a unique inequality (except scalar multiples).

Theorem 2 *Inequality $y_{uv} \geq 0$, for each edge $(u,v) \in E$, is facet-inducing of MBCPP(G) (if graph G is 3-edge connected).*

Proof: Let us suppose there is another valid inequality $ax + by \geq c$ such that

$$\{(x,y) \in \text{MBCPP(G)}: \quad y_{uv} = 0\} \subseteq \{(x,y) \in \text{MBCPP(G)}: \quad ax + by = c\}$$

We will prove that inequality $ax + by \geq c$ is a scalar multiple of $y_{uv} \geq 0$. Given that $(x,y) = (0,0)$ is a MBCPP tour that satisfies $y_{uv} = 0$, we can assume that $c = 0$.

A similar argument to that used in the proof of Theorem 1 leads to $a_{ij} + b_{ij} = 0$ for all $(i,j) \in E$, $(i,j) \neq (u,v)$, and $\sum_{e \in \mathcal{C}} a_e = 0$ for any cycle $\mathcal{C}$ in G. Then, $\sum_{e \in \mathcal{C}} b_e = 0$ also holds for every cycle $\mathcal{C}$ in G that does not contain edge (u,v).

Let $(i,j) \in E$ be an arbitrary edge. Given that G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain edge (i,j). Let us suppose, w.l.o.g., that either the depot 1 is in path $\mathcal{P}_1$ or it can be connected to some node in path $\mathcal{P}_1$ without using edges in $\mathcal{P}_2$ nor edge (u,v). Then the solution (x^1, y^1) that traverses path $\mathcal{P}_1$ and edge (i,j) once and the path to the depot twice (if needed), is a MBCPP tour that satisfies $y_{uv} = 0$. Let now (x^2, y^2) be the MBCPP tour obtained from (x^1, y^1) replacing the edge (i,j) by the edges in path $\mathcal{P}_2$. This MBCPP tour also satisfies $y_{uv} = 0$.

By subtracting the two expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ and considering that $(i,j) \cup \mathcal{P}_2$ is a cycle in G, we obtain that $a_{ij} = 0$ for each edge $(i,j) \in E$. Since $a_{ij} + b_{ij} = 0$, also $b_{ij} = 0$ for each edge $(i,j) \in E \setminus \{(u,v)\}$. Then, inequality $ax + by \geq c$ turns out to be $b_{uv} y_{uv} \geq 0$ and $y_{uv} \geq 0$ is facet-inducing for MBCPP(G). ♦

Theorem 3 *Inequality $x_{uv} \leq 1$, for each edge $(u,v) \in E$, is facet-inducing for MBCPP(G) (if graph G is 3-edge connected).*

Proof: Let us suppose there is another valid inequality $ax + by \leq c$ such that

$$\{(x,y) \in \text{MBCPP(G)}: \quad x_{uv} = 1\} \subseteq \{(x,y) \in \text{MBCPP(G)}: \quad ax + by = c\}$$

Let $(i,j) \in E \setminus \{(u,v)\}$. Given that G is 3-edge connected, graph $G \setminus \{(i,j)\}$ is connected, and there is a MBCPP tour (x^1, y^1) that traverses edge (u,v) at least once and visits node i. This tour satisfies $x^1_{uv} = 1$. The MBCPP tour (x^2, y^2) obtained from (x^1, y^1) by adding edge (i,j) twice, also satisfies $x^2_{uv} = 1$. After subtracting the expressions $ax^1 + by^1 = c$ and $ax^2 + by^2 = c$, we obtain that $a_{ij} + b_{ij} = 0$ for all $(i,j) \in E \setminus \{(u,v)\}$.

Let $\mathcal{P}$ be a path joining nodes 1 and u that does not use edge (u,v). The MBCPP tour (x^1, y^1) that traverses the path $\mathcal{P}$ and the edge (u,v) twice satisfies $x^1_{uv} = 1$ and then $ax^1 + by^1 = c$, i.e., $\sum_{e \in \mathcal{P}} (a_e + b_e) + a_{uv} + b_{uv} = c$. Since $a_{ij} + b_{ij} = 0$ for all $(i,j) \in E \setminus \{(u,v)\}$ we obtain that $a_{uv} + b_{uv} = c$. The same argument can be applied to deduce that $\sum_{e \in \mathcal{C}} a_e = c$ for all cycle $\mathcal{C}$ containing edge (u,v).

Let $\mathcal{C}$ now be any cycle in graph G that does not contain edge (u,v). There is a MBCPP tour that traverses the edges in $\mathcal{C}$ once and a subset F of other edges in E including (u,v)

twice. This tour satisfies $x^1_{uv} = 1$ and then $ax^1+by^1 = c$, that is, $\sum_{e\in F}(a_e+b_e)+\sum_{e\in\mathcal{C}} a_e = c$. Since $a_{ij} + b_{ij} = 0$ for all $(i, j) \in E \setminus \{(u, v)\}$ and $a_{uv} + b_{uv} = c$ we obtain that $\sum_{e\in\mathcal{C}} a_e = 0$ for all cycle $\mathcal{C}$ not containing edge (u, v).

By combining the previous results we can deduce that $\sum_{e\in\mathcal{C}} b_e = 0$ for all cycle $\mathcal{C}$ in G.

Let $(i, j) \in E$ be an arbitrary edge. Given that G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain the edge (i, j). As $G\setminus\{(i, j)\}$ is a connected graph, there is a tour traversing paths $\mathcal{P}_1$, $\mathcal{P}_2$ exactly once, traversing the edge (u, v) and connected to the depot. Let (x^1, y^1) be the MBCPP tour consisting of the previous tour plus edge (i, j) twice, which satisfies $x^1_{uv} = 1$. Let (x^2, y^2) be the MBCPP tour obtained from (x^1, y^1) by replacing the second traversal of (i, j) by a second traversal of the edges in path $\mathcal{P}_1$, which also satisfies $x^2_{uv} = 1$. By subtracting the two expressions $ax^1 + by^1 = c$ and $ax^2 + by^2 = c$ we obtain $b_{ij} - \sum_{e\in\mathcal{P}_1} b_e = 0$ and considering that $(i, j) \cup \mathcal{P}_1$ is a cycle in G, we obtain $b_{ij} + \sum_{e\in\mathcal{P}_1} b_e = 0$ and thus, $b_{ij} = 0$ for each edge $(i, j) \in E$. Since $a_{ij} + b_{ij} = 0$ for all $(i, j) \in E\setminus\{(u, v)\}$ and $a_{uv} + b_{uv} = c$ we obtain that $a_{ij} = 0$ and $a_{uv} = c$ holds. Then, the inequality $ax+by \leq c$ turns out to be $cx_{uv} \leq c$ and $x_{uv} \leq 1$ is facet-inducing for MBCPP(G). ♦

Theorem 4 *Inequalities (3), $x_{uv} \geq y_{uv}$ for every edge $(u, v) \in E$ are facet-inducing for MBCPP(G) if, and only if, graph $G \setminus \{(u, v)\}$ is 3-edge connected.*

Proof: If graph G is 3-edge connected but graph $G \setminus \{(u, v)\}$ is not 3-edge connected, there is at least one cut-set $\delta(S)$ containing exactly the edge (u, v) and two more edges, say f, g. In this case, it can be seen that the inequality $x_{uv} \geq y_{uv}$ is not facet-inducing because it is the sum of two *parity inequalities* (6), which will be presented in Section 4.1, associated with $\delta(S)$ and with $F = \{f\}$ and $F = \{g\}$, respectively.

On the other hand, let us suppose that there is another valid inequality $ax + by \geq c$ such that

$$\{(x, y) \in \text{MBCPP(G)} : \quad x_{uv} - y_{uv} = 0\} \ \subseteq\ \{(x, y) \in \text{MBCPP(G)} : \quad ax + by = c\}$$

Again, since $(0, 0) \in G$, $c = 0$ holds. Let $(i, j) \in E$ be an arbitrary edge. Given that G is (3-edge) connected, there is a path $\mathcal{P}$ joining nodes 1 and i that does not use edge (i, j). Let (x^1, y^1) be the MBCPP tour that traverses the path $\mathcal{P}$ twice and let (x^2, y^2) be the MBCPP tour that traverses the path $\mathcal{P}$ and the edge (i, j) twice. Both tours x^1 and x^2 satisfy $x_{uv} - y_{uv} = 0$ and then they also satisfy $ax^1 + by^1 = 0$, $ax^2 + by^2 = 0$. By subtracting these expressions we obtain $a_{ij} + b_{ij} = 0$ for all $(i, j) \in E$.

With a similar reasoning to that in the proof of Theorem 3 we obtain that $\sum_{e\in\mathcal{C}} a_e = \sum_{e\in\mathcal{C}} b_e = 0$ for each cycle $\mathcal{C}$ not containing edge (u, v).

Let $(i, j) \in E \setminus \{(u, v)\}$. Given that graph $G \setminus \{(u, v)\}$ is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain edge (i, j) nor (u, v). Let us suppose, w.l.o.g., that either the depot 1 is in path $\mathcal{P}_1$ or it can be joined to some node in path $\mathcal{P}_1$ without using edges in $\mathcal{P}_2$. Then the solution (x^1, y^1) that traverses path $\mathcal{P}_1$ and edge (i, j) once and the path to the depot twice (if needed), is a MBCPP tour satisfying $x^1_{uv} - y^1_{uv} = 0$. Let now (x^2, y^2) be the MBCPP tour obtained from (x^1, y^1) after replacing edge (i, j) by path $\mathcal{P}_2$, which also satisfies $x^2_{uv} - y^2_{uv} = 0$. By subtracting the two expressions and considering that $(i,j) \cup \mathcal{P}_2$ is a cycle in G that does not contain the edge (u,v), we obtain $a_{ij} + \sum_{e \in \mathcal{P}_2} a_e = 0$ and thus, $a_{ij} = b_{ij} = 0$ for each edge $(i,j) \in E \setminus \{(u,v)\}$. The inequality $ax + by \geq c$ turns out to be $a_{uv}x_{uv} + b_{uv}y_{uv} \geq 0$ and given that $a_{uv} + b_{uv} = 0$, we obtain that $x_{uv} - y_{uv} \geq 0$ is facet-inducing for MBCPP(G). ♦

Theorem 5 *Connectivity inequalities (2) are facet-inducing for MBCPP(G) if graph G is 3-edge connected and subgraphs $G(S)$ and $G(V \setminus S)$ are 2-edge connected.*

Note: `sería faceta si G(`$V \setminus S$`) fuese solo un nodo (o dos nodos y una arista)? Y si G(`S`) fuese solo la arista `f`?`

Proof: Let us suppose there is another valid inequality $ax + by \geq c$ such that

$$\{(x,y) \in \text{MBCPP(G)} : \sum_{e \in \delta(S)} (x_e + y_e) - 2x_f = 0\} \subseteq \{(x,y) \in \text{MBCPP(G)} : \quad ax + by = c\}$$

Again, we can assume that $c = 0$. As in previous theorems, it can be seen that $a_{ij} + b_{ij} = 0$ for all $(i,j) \in E(V \setminus S) \cup E(S) \setminus \{f\}$. For each edge $(l,m) \in \delta(S)$, we can build a MBCPP tour that uses twice each edge in a path in G from node 1 to edge f traversing (l,m). This tour satisfies (2) with equality and, therefore, $a_{lm} + b_{lm} + a_f + b_f = 0$.

Let $\mathcal{C}$ be any cycle in graph $G(V \setminus S)$. It is easy to obtain that $\sum_{e \in \mathcal{C}} a_e = \sum_{e \in \mathcal{C}} b_e = 0$. Let $\mathcal{C}$ be now any cycle in graph $G(S)$ that does not contain edge f. Let $\mathcal{P}$ be a path joining node 1 with a node i belonging to the cycle and contains the edge f. If we traverse all the edges in $\mathcal{C} \cup \mathcal{P}$ twice, we obtain a MBCPP tour (x^1, y^1) satisfying $\sum_{e \in \delta(S)} (x_e^1 + y_e^1) = 2x_f^1$. If we remove one copy of each edge in $\mathcal{C}$ from (x^1, y^1) we obtain another MBCPP tour (x^2, y^2) also satisfying $\sum_{e \in \delta(S)} (x_e^2 + y_e^2) = 2x_f^2$. By subtracting the expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ we obtain that $\sum_{e \in \mathcal{C}} b_e = 0$ and also $\sum_{e \in \mathcal{C}} a_e = 0$ for all the cycles in graph $G(S)$ that do not contain edge f. Then,

$$\sum_{e \in \mathcal{C}} a_e = 0, \quad \sum_{e \in \mathcal{C}} b_e = 0 \quad \forall \text{ cycle } \mathcal{C} \text{ in graphs } \quad G(V \setminus S) \text{ or } G(S) \setminus \{f\}$$

Let $(i,j) \in E(V \setminus S)$. Given that graph G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain the edge (i,j). If $\mathcal{P}_1$, $\mathcal{P}_2$ are both in $G(V \setminus S)$, the usual process leads us to $a_{ij} = b_{ij} = 0$. Otherwise, given that graph $G(V \setminus S)$ is 2-edge connected, at least one of these paths does not traverse the cut-set $\delta(S)$. Let us suppose that path $\mathcal{P}_2$ does traverse $\delta(S)$ and notice that we can assume that $\mathcal{P}_2$ traverses the cut-set $\delta(S)$ exactly twice. Let (x^1, y^1) be the MBCPP tour that uses once the edge (i,j) and the path $\mathcal{P}_2$ (plus the edges needed to connect it with f if f is not in $\mathcal{P}_2$) and let (x^2, y^2) the MBCPP tour obtained after replacing in (x^1, y^1) the edge (i,j) by the edges in the path $\mathcal{P}_1$. Both tours satisfy inequality (2) with equality and, by comparing them, we obtain $a_{ij} = b_{ij} = 0$.

Let $(i,j) \in E(S) \setminus \{f\}$. Given that graph G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain edge (i,j). If $G(S)$ is 2-edge connected, we can assume that at most one of these paths traverses the cut-set $\delta(S)$. Let us suppose that this path is $\mathcal{P}_2$ and notice that we can assume that $\mathcal{P}_2$ traverses the cut-set

$\delta(S)$ exactly twice (once in each direction). Let (x^1, y^1) be the MBCPP that uses once the edge (i,j) and the path $\mathcal{P}_2$ (plus the edges in $E(V \setminus S)$ needed to connect it with the depot and those in $E(S)$ needed to connect the solution with f if f is not in $\mathcal{P}_2$). Let (x^2, y^2) be the MBCPP tour obtained after replacing in (x^1, y^1) the edge (i,j) by the edges in the path $\mathcal{P}_1$. Both tours satisfy $\sum_{e\in\delta(S)} \left(x^1_e + y^1_e\right) = 2x^1_f$ and, after subtracting the expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ and applying $(**)$ we obtain $a_{ij} = b_{ij} = 0$. Then, $a_{ij} = b_{ij} = 0$ for each edge $e \in E(V \setminus S) \cup E(S) \setminus \{f\}$.

Let us denote $e_1, e_2, \dots, e_p$ the edges in $\delta(S)$, where $p \geq 3$ since graph G is 3-edge connected. Consider now two edges $e_1, e_2 \in \delta(S)$. Given that graphs $\mathrm{G}(S)$ and $\mathrm{G}(V \setminus S)$ are connected, there is a MBCPP tour (x^1, y^1) using twice the edges in a path that starts at the depot, traverses e_1 and ends at edge f. Let (x^2, y^2) be the MBCPP tour obtained from (x^1, y^1) after replacing the second traversing of e_1 by the first traversals of the edges in a path joining the endpoints of e_1 and using e_2. After subtracting the two expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ we obtain that $b_{e_1} = a_{e_2}$. If we interchange the roles of the edges e_1 and e_2 we obtain that $b_{e_2} = a_{e_1}$. Proceeding in this way with all the pairs of edges in $\delta(S)$ we obtain that $a_{e_i} = b_{e_j}$ for all $i \neq j \in \{1, 2, \dots, p\}$ and then $a_{e_i} = a_{e_j} = b_{e_i} = b_{e_j}$ for all i, j (because $p \geq 3$ holds).

Finally, given that graph $\mathrm{G}(S)$ is 2-edge connected, there is a cycle $\mathcal{C}$ in graph $G(S)$ that contains the edge f. Let (x^1, y^1) be a MBCPP tour using the edges in $\mathcal{C}$ once and the edges in a path $\mathcal{P}$ joining the depot 1 to a node i belonging to the cycle $\mathcal{C}$ twice. Let (x^2, y^2) be the MBCPP tour that uses all the edges in $\mathcal{P}$, the edges in the path formed with the edges in the cycle $\mathcal{C}$ from node i to an end-point of edge f plus the edge f twice. Both MBCPP tours satisfy $\sum_{e\in\delta(S)} \left(x_e + y_e\right) = 2x_f$ and after subtracting the expressions $ax^1 + by^1 = 0$ and $ax^2 + by^2 = 0$ we obtain that $b_f = 0$. Substituting in $a_{lm} + b_{lm} + a_f + b_f = 0$ for any edge $(l,m) \in \delta(S)$ we obtain that $a_f = -2a_{lm}$ and, hence, the connectivity inequality (2) is facet-inducing for MBCPP(G). ♦

4 Other inequalities

In this section we present several new families of valid inequalities for the MBCPP, parity, K-C and p-connectivity inequalities, and we study conditions under which they are facet-inducing for MBCPP(G).

4.1 Parity inequalities

Constraints (1) are not linear inequalities. In order to force the solution to satisfy these parity constraints, we can use other linear inequalities as the set-parity inequalities proposed in [2] for the Prize-collecting Rural Postman Problem (PRPP), which as has been said is a special case of the MBCPP. A first version of these inequalities was proposed in [3], and later corrected in [2]. They are based on the so called co-circuit inequalities proposed by Barahona and Grötschel ([5]) for the binary matroid problem, and are as follows. Given a vertex set $S \subset V \setminus \{1\}$ and two edge sets $F \subseteq \delta(S)$ and $L \subseteq F$, such that $|F| + |L|$ is odd, then the set-parity inequality is

$$x(\delta(S) \setminus F) + y(F \setminus L) \geq x(F) + y(L) - (|F| + |L|) + 1. \tag{5}$$

It is easy to see that inequalities (5) are valid for the MBCPP, but we failed in proving that they induce facets of MBCPP(G). However, we found the following ones, which we will call parity inequalities, that dominate inequalities (5) and are facet inducing for MBCPP(G):

$$x(\delta(S)\setminus F) - y(\delta(S)\setminus F) \geq x(F) - y(F) - |F| + 1, \quad \forall S \subset V, \quad \forall F \subset \delta(S) \ \text{ with } |F| \text{ odd} \tag{6}$$

Theorem 6 *Parity inequalities (6) are valid for MBCPP(G).*

Proof: Let (x^*, y^*) be a MBCPP tour. We have to prove that $x^*(\delta(S)\setminus F) - y^*(\delta(S)\setminus F) \geq x^*(F) - y^*(F) - |F| + 1$. If $x^*(F) - y^*(F) \leq |F| - 1$, this inequality reduces to $x^*(\delta(S) \setminus F) - y^*(\delta(S) \setminus F) \geq 0$, which is obviously satisfied. Let us suppose then that $x^*(F) - y^*(F) = |F|$. This is only satisfied when $x^*(F) = |F|$ and $y^*(F) = 0$. In this case, the inequality becomes $x^*(\delta(S)\setminus F) - y^*(\delta(S)\setminus F) \geq 1$. Since cutset $(S : V \setminus S)$ must be traversed an even number of times, $|F|$ is odd and $x^*_e \geq y^*_e$ for any edge e, the inequality has to be satisfied. ♦

Theorem 7 *Parity inequalities (6) are stronger than set-parity inequalities* (5)*.*

Proof: Let $S \subset V \setminus \{1\}$, $F \subseteq \delta(S)$, $L \subseteq F$, $|F| + |L|$ odd. Define $F' = F \setminus L$. Since $|F| + |L|$ is an odd number, $|F'|$ is also odd. We will prove that the parity inequality (6) associated with S and F' dominates the cocircuit inequality (5) associated with sets S, F and L.

Given that $F' = F \setminus L$, $\delta(S) \setminus F' = (\delta(S) \setminus F) \cup L$ and the inequality (6) associated with sets S and F' can be written as:

$$x(\delta(S) \setminus F) + x(L) - y(\delta(S) \setminus F) - y(L) \geq x(F) - x(L) - y(F) + y(L) - (|F| - |L|) + 1,$$

or equivalently

$$x(\delta(S) \setminus F) + y(F) - y(L) \geq x(F) + y(L) - 2x(L) + y(\delta(S) \setminus F) - (|F| - |L|) + 1,$$

which can be written as

$$x(\delta(S) \setminus F) + y(F \setminus L) \geq x(F) + y(L) - (|F| + |L|) + 1 + 2|L| - 2x(L) + y(\delta(S) \setminus F) =$$
$$x(F) + y(L) - (|F| + |L|) + 1 + 2(|L| - x(L)) + y(\delta(S) \setminus F),$$

whose RHS is, obviously, greater than or equal to the RHS in the set-parity inequality. ♦

Note 1 Before proving that parity inequalities induce facets of MBCPP(G), we will describe in what follows two types of MBCPP tours satisfying (6) with equality. Given a graph $G = (V, E)$ and $T \subset V$, with $|T|$ even, recall that a subset of edges $E' \subset E$ is a T-join if, in the subgraph $G' = (V, E')$, the degree of v is odd if and only if $v \in T$ (see [16]).

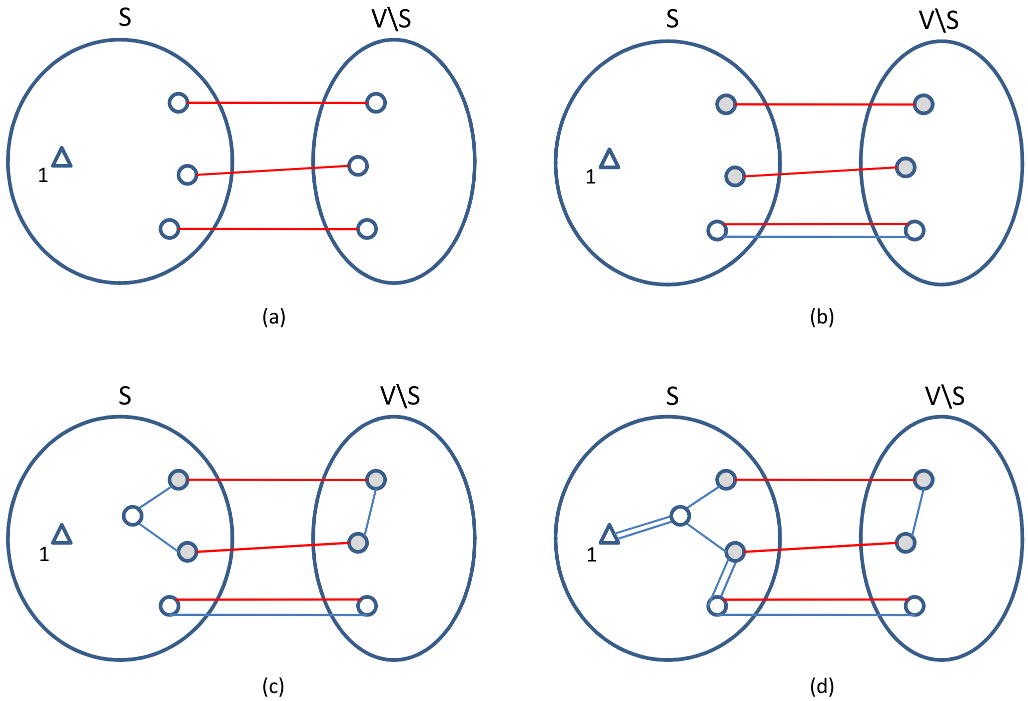

Figure 1: Construction of MBCPP tours of type 1 satisfying (6) with equality.

Type 1: Let us consider the cut-set depicted in Figure 1(a), with $|F| = 3$. Each MBCPP solution traversing all the edges in F has to traverse the cut-set $(S, V \setminus S)$ at least once more time. Let e' be either an edge in $\delta(S) \setminus F$ or a copy of an edge in F. Figure 1(b) shows this second case. Let $T \subset S$ be the set of vertices incident with an odd number of edges in $F \cup \{e'\}$. Given that $|F \cup \{e'\}|$ is even, $|T|$ is also even and there is a T-join E' in G(S) (see Figure 1(c)). The subgraph in G(S), G^*, induced by the edges in E', the vertices incident with $F \cup \{e'\}$, and the depot can be disconnected. If graph G(S) is connected, G^* can be converted into a connected graph (see Figure 1(d)) by adding two copies of some edges connecting its components (and any other vertex i as needed in the proof of Theorem 8). This same process is done in G($V \setminus S$). Then, the two subgraphs build in G(S) and G($V \setminus S$) plus the edges in $F \cup \{e'\}$ define a MBCPP tour that satisfies (6) with equality.

Note that the above procedure can be used also if in the cut-set $(S, V \setminus S)$, besides the edges in $F \cup \{e'\}$, we consider also two copies of any q edges in $\delta(S) \setminus F$. In this case we would obtain:

$x(F) = |F|$, $y(F) = 0$, $x(\delta(S) \setminus F) = 1 + q$ and $y(\delta(S) \setminus F) = 0 + q$ if $e' \in \delta(S) \setminus F$, or

$x(F) = |F|$, $y(F) = 1$, $x(\delta(S) \setminus F) = 0 + q$ and $y(\delta(S) \setminus F) = 0 + q$ if $e' \in F$.

In both cases, the MBCPP tours satisfy (6) with equality.

Type 2: Consider now all the edges in F, except one of them, and two copies of any q edges in $\delta(S) \setminus F$. From these $|F| - 1 + 2q$ edges we can apply the above procedure to obtain a MBCPP tour satisfying:

$x(F) = |F| - 1$, $y(F) = 0$, $x(\delta(S) \setminus F) = 0 + q$ and $y(\delta(S) \setminus F) = 0 + q$, and, therefore, satisfies (6) with equality.

Theorem 8 *Parity inequalities (6) are facet-inducing for MBCPP(G) if graph G is 3-edge connected and graphs $G(S)$ and $G(V \setminus S)$ are 2-edge connected.*

Proof: Inequalities (6) can be written in the following form:

$$x(F) - y(F) - x(\delta(S)\backslash F) + y(\delta(S)\backslash F) \leq |F| - 1 \tag{7}$$

Let us suppose there is another valid inequality $ax + by \leq c$ such that

$$\{(x, y) \in \text{MBCPP(G)} : x(F) - y(F) - x(\delta(S)\backslash F) + y(\delta(S)\backslash F) = |F| - 1\} \subseteq$$
$$\subseteq \{(x, y) \in \text{MBCPP(G)} : \quad ax + by = c\},$$

where, we can assume that $c = |F| - 1$.

Let $(i, j) \in E(S) \cup E(V \setminus S)$. Given that G is 3-edge connected, graph $G \setminus \{(i, j)\}$ is connected, and there is a MBCPP tour (x^1, y^1) that satisfies (7) with equality and visits node i (see Note 1). The MBCPP tour (x^2, y^2) obtained from (x^1, y^1) by adding the traversal of edge (i, j) twice, also satisfies (7) with equality. Then $ax^1 + by^1 = ax^2 + by^2 = c$ and, after subtracting both expressions, we obtain

$$a_{ij} + b_{ij} = 0 \quad \forall (i, j) \in E(S) \cup E(V \setminus S) \qquad (*)$$

Let $(i, j) \in E(S)$. Given that G is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes i and j that do not contain the edge (i, j). Note that given that $\text{G}(S)$ is 2-edge connected, at least one of this paths is in $\text{G}(S)$. Let us suppose first that both paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are in $\text{G}(S)$. Given that graphs $G \setminus \{(i, j)\}$, $\text{G}(S)\backslash\{(i, j)\}$ and $\text{G}(V \setminus S)$ are connected, proceeding as in Note 1, we can build a MBCPP tour (x^1, y^1) in G satisfying (7) with equality such that it uses edge (i, j) exactly once. To do that, the "parity" label of vertices i and j is switched before the T-join is computed and then the edge (i, j) is added.

We define three more MBCPP tours in the following way. Consider (x^1, y^1) and suppose we add one copy of each edge in paths $\mathcal{P}_1$, $\mathcal{P}_2$. The resulting tour is even and connected, but it is not necessarily a MBCPP tour because some edges can appear three times. If we remove two copies of each one of these edges used three times, we obtain an even and connected tour (x^2, y^2) that is a feasible solution for the MBCPP. Let (x^3, y^3) be the tour obtained from (x^1, y^1) after removing the edge (i, j), adding one copy of each edge in path $\mathcal{P}_1$, and then removing two copies of each one of these edges used three times. Finally, let (x^4, y^4) be the tour obtained from (x^1, y^1) after removing the edge (i, j), adding one copy of each edge in path $\mathcal{P}_2$, and then removing two copies of each one of these edges used three times. All these four MBCPP tours satisfy (7) with equality and then also satisfy $ax + by = c$.

Let us call $\alpha(P_i) = \sum_{e \in P_i^1} a_e + \sum_{e \in P_i^2} b_e$, where P_i^1 is the set of edges in path P_i that are traversed only once in (x^1, y^1) and P_i^2 the set of edges in path P_i that are not traversed or traversed twice in (x^1, y^1). If we subtract the expressions $ax^1 + by^1 = c$ and $ax^2 + by^2 = c$ we obtain $\alpha(\mathcal{P}_1) + \alpha(\mathcal{P}_2) = 0$. In the same way, by comparing tours 3 and 4 above we obtain that $\alpha(\mathcal{P}_1) = \alpha(\mathcal{P}_2)$ and then $\alpha(\mathcal{P}_1) = \alpha(\mathcal{P}_2) = 0$. Finally, if we compare the tours 1 and 3, we obtain that $a_{ij} = \alpha(\mathcal{P}_1)$ and then $a_{ij} = 0$. From $(*)$, also $b_{ij} = 0$. For each edge $(i, j) \in E(V \setminus S)$, a similar process leads to $a_{ij} = b_{ij} = 0$.

Let us suppose now that path $\mathcal{P}_2$ is not in G(S), i.e., it leaves the graph G(S) and traverses the cut-set $\delta(S)$. Given that graph G($V \setminus S$) is connected, we can assume that path $\mathcal{P}_2$ traverses the cut-set $\delta(S)$ exactly once in each direction through two edges, say e_1 and e_2. We consider three cases:

(1) $e_1, e_2 \in F$. Let (x^1, y^1) be the MBCPP tour of type 2, satisfying (7) with equality, which traverses all the edges in $F \setminus \{e_2\}$ once and does not traverse e_2. It can be seen that three MBCPP tours (x^2, y^2), (x^3, y^3) and (x^4, y^4) defined as above satisfy also (7) with equality. Note that when we add one copy of each edge in path $\mathcal{P}_2$ we obtain a MBCPP tour of type 1 that uses each edge in $F \setminus \{e_1\}$ exactly once and edge e_1 twice.

(2) $e_1, e_2 \notin F$. Let (x^1, y^1) be the MBCPP tour of type 1, satisfying (7) with equality, which traverses all the edges in $F \cup \{e_1\}$ once and does not traverse e_2. Again, three MBCPP tours (x^2, y^2), (x^3, y^3) and (x^4, y^4) defined as above satisfy also (7) with equality. Note that when we add one copy of each edge in path $\mathcal{P}_2$ we obtain a MBCPP tour of type 1 that uses each edge in $F \setminus \{e_1\}$ exactly once and the edge e_1 twice.

(3) $e_1 \in F, e_2 \notin F$. Let (x^1, y^1) be the MBCPP tour of type 2 satisfying (7) with equality, traversing all the edges in $F \setminus \{e_1\}$ once and not traversing e_2. Again, the three MBCPP tours (x^2, y^2), (x^3, y^3) and (x^4, y^4) defined as above satisfy also (7) with equality. Note that when we add one copy of each edge in path $\mathcal{P}_2$ we obtain a MBCPP tour of type 1 that uses each edge in $F \cup \{e_2\}$ exactly once.

In any of the 3 cases above, following a similar reasoning to that of the case in which path $\mathcal{P}_2$ is in G(S), we obtain that $a_{ij} = b_{ij} = 0$ for all edges $(i, j) \in E(S) \cup E(V \setminus S)$.

Consider now an edge $(i, j) \in \delta(S)$. As we have seen in Note 1, there is a MBCPP tour (x^1, y^1) that satisfies (7) with equality and does not use (i, j), while visiting nodes i and j. Let (x^2, y^2) be the tour obtained after adding edge (i, j) twice to (x^1, y^1). Since $ax^1 + by^1 = ax^2 + by^2 = c$, subtracting these expressions we obtain that $a_{ij} + b_{ij} = 0$ for all $(i, j) \in \delta(S)$.

Let $e_1, e_2 \in F$. Let (x^1, y^1) be the MBCPP tour that uses all the edges in $F \setminus \{e_1\}$ exactly once and edge e_1 twice and let (x^2, y^2) be the tour that uses all the edges in $F \setminus \{e_2\}$ exactly once and edge e_2 twice. Both tours can be constructed satisfying (7) with equality. By comparing them, and considering that $a_{ij} = b_{ij} = 0$ for all edges $(i, j) \in E(S) \cup E(V \setminus S)$, we obtain that $b_{e_1} = b_{e_2}$ and, therefore, $a_{e_1} = a_{e_2}$. By iterating this argument we obtain that $a_{ij} = \lambda$ and $b_{ij} = -\lambda$ for all $(i, j) \in F$.

For each $e_1 \notin F$, consider any $e_2 \in F$. Let (x^1, y^1) be the MBCPP tour that uses all the edges in $F \cup e_1$ exactly once and let (x^2, y^2) be the tour that uses all the edges in $F \setminus \{e_2\}$ exactly once and edge e_2 twice. By comparing them, we obtain that $a_{e_1} = b_{e_2} = -\lambda$ and, therefore, $b_{e_1} = \lambda$.

Then we have that inequality $ax + by \leq c$ reduces to $\lambda x(F) - \lambda y(F) - \lambda x(\delta(S) \setminus F) + \lambda y(\delta(S) \setminus F) \leq |F| - 1$. Given that the MBCPP tour (x^1, y^1) above, for example, satisfies this inequality with equality, we obtain that $\lambda|F| - \lambda = |F| - 1$ and, therefore, $\lambda = 1$, which completes the proof. ♦

4.2 K-C inequalities

K-C inequalities ([8]) are a well-known family of facet-inducing inequalities for the RPP and many other arc routing problems. In this section we show that these inequalities can be transformed in order to obtain new valid and facet-inducing inequalities for MBCPP(G) that we will continue calling K-C inequalities for the sake of simplicity.

Let $M_0, M_1, \dots, M_K$, with $K \geq 3$, be a partition of V, where $1 \in M_0 \cup M_K$. Given an edge $e_i \in E(M_i)$ for each $i = 1, \dots, K-1$, and a subset of edges $F \subseteq (M_0 : M_K)$ with $|F|$ even, the K-C inequalities for the MBCPP are defined as:

$$(K-2)\big(x((M_0 : M_K) \setminus F) - y((M_0 : M_K) \setminus F)\big) - (K-2)(x(F) - y(F)) +$$
$$+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(M_i : M_j) + (2-j+i)y(M_i : M_j)\Big) - 2\sum_{i=1}^{K-1} x_{e_i} \geq -(K-2)|F| \tag{8}$$

The coefficients and structure of the K-C inequalities are shown in Figure 2, where for each pair (a, b) associated with an edge e, a and b represent the coefficients of x_e and y_e, respectively. Note that if we add $2(K-2)(x(F) - y(F))$ to both sides of inequality (8), it can be written as

$$(K-2)\big(x((M_0 : M_K)) - y((M_0 : M_K))\big) + \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(M_i : M_j) + (2-j+i)y(M_i : M_j)\Big)$$
$$\geq (K-2)\big(2(x(F) - y(F)) - |F|\big) + 2\sum_{i=1}^{K-1} x_{e_i},$$

and, if a given solution traverses exactly once each edge in F and each edge e_i, it has to satisfy

$$(K-2)\big(x((M_0 : M_K)) - y((M_0 : M_K))\big) + \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(M_i : M_j) + (2-j+i)y(M_i : M_j)\Big)$$
$$\geq (K-2)|F| + 2(K-1),$$

which, regarding the x variables, resembles the version of the K-C inequality for the undirected RPP ([8]).

If the depot $1 \notin M_0 \cup M_K$ but $1 \in M_d$, for some $d \in \{1, 2, \dots, K-1\}$, and $|F| \geq 2$, the corresponding K-C inequality is

$$(K-2)\big(x((M_0 : M_K) \setminus F) - y((M_0 : M_K) \setminus F)\big) - (K-2)(x(F) - y(F)) +$$
$$+ \sum_{\substack{0 \leq i < j \leq K \\ (i,j) \neq (0,K)}} \Big((j-i)x(M_i : M_j) + (2-j+i)y(M_i : M_j)\Big) - 2\sum_{\substack{i=1 \\ i \neq d}}^{K-1} x_{e_i} \geq -(K-2)|F| + 2. \tag{9}$$

Note that when $|F| = 0$ or $K = 2$ the K-C inequality (9) is not valid. Moreover, when $K = 2$, the K-C inequality (8) reduces to a connectivity inequality (2).

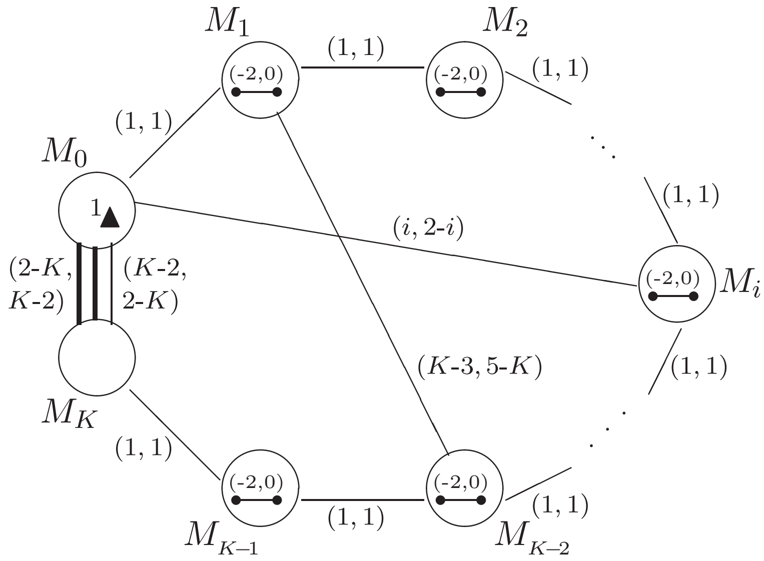

Figure 2: Coefficients of the K-C inequality.

Theorem 9 *K-C inequalities* (8) *and* (9) *are valid for MBCPP(G).*

Proof: We will prove only the validity of inequalities (8) since the proof for inequalities (9) is analogous. Let (x^*, y^*) be a MBCPP tour. We have to prove that (x^*, y^*) satisfies (8).

Note that if $x^*_{e_j} = 0$ for some $j \in \{1, \dots, K-1\}$, then we could consider a new K-C configuration with $K-1$ nodes where nodes M_j and M_{j+1} have been merged into a single node $M_j \cup M_{j+1}$, and it can be proved that, if its associated K-C inequality is satisfied by (x^*, y^*), then the original K-C inequality is also satisfied by (x^*, y^*). Then, in what follows, we can assume that $x^*_{e_1} = x^*_{e_2} = \dots x^*_{e_{K-1}} = 1$ and we have to prove that

$$\begin{aligned}&(K-2)\big(x^*((M_0 : M_K) \setminus F) - y^*((M_0 : M_K) \setminus F)\big) - (K-2)(x^*(F) - y^*(F)) + \\ &+ \sum_{\substack{0 \le i < j \le K \\ (i,j) \ne (0,K)}} \Big((j-i)x^*(M_i : M_j) + (2-j+i)y^*(M_i : M_j)\Big) \ge 2(K-1) - (K-2)|F|. \qquad (10)\end{aligned}$$

Let us suppose first that $x^*(F) - y^*(F) = |F|$. This means that (x^*, y^*) traverses once each edge in F. In this case, inequality (10) reduces to

$$\begin{aligned}&(K-2)\big(x^*((M_0 : M_K) \setminus F) - y^*((M_0 : M_K) \setminus F)\big) + \\ &+ \sum_{\substack{0 \le i < j \le K \\ (i,j) \ne (0,K)}} \Big((j-i)x^*(M_i : M_j) + (2-j+i)y^*(M_i : M_j)\Big) \ge 2(K-1). \qquad (11)\end{aligned}$$

Since (x^*, y^*) starts at M_0, visits all the nodes $M_1, \dots, M_{K-1}$ and returns to M_0 (see Figures 3a and 3b, where edges in F used by the tour are depicted in bold), it is easy to see that inequality (11) holds.

Consider now that $x^*(F) - y^*(F) = |F| - 1$, that is, (x^*, y^*) traverses the edges in F an odd number of times. In this case, inequality (10) reduces to

$$\begin{aligned}&(K-2)\big(x^*((M_0 : M_K) \setminus F) - y^*((M_0 : M_K) \setminus F)\big) + \\ &+ \sum_{\substack{0 \le i < j \le K \\ (i,j) \ne (0,K)}} \Big((j-i)x^*(M_i : M_j) + (2-j+i)y^*(M_i : M_j)\Big) \ge K. \qquad (12)\end{aligned}$$

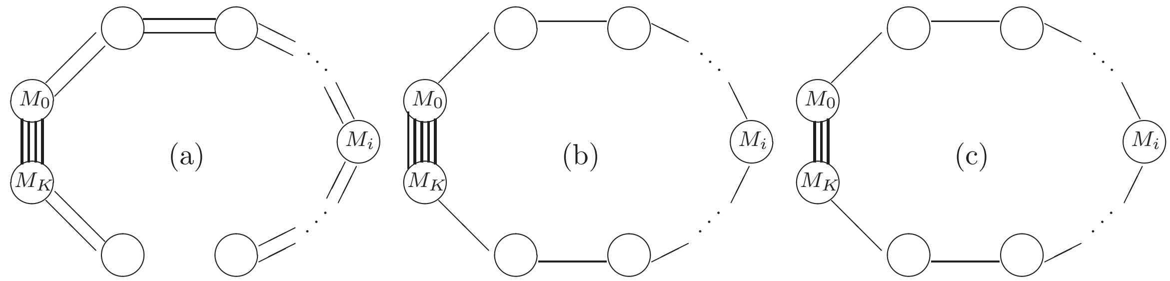

Figure 3: MBCPP solutions in the proof of Theorem 9.

The best way of starting at M_0, visit $M_1, \ldots, M_{K-1}$ and ending at M_K (remember that the edges in F are traversed an odd number of times) has an F-cost cambiarlo por a left-hand side cost greater than or equal to K (see Figure 3c), and inequality (12) holds.

All the other cases when $x^*(F) - y^*(F) < |F| - 1$ can be proved easily. Note that, when $x^*(F) - y^*(F) = |F| - 2$, the right-hand side of the inequality expressed as in (12) is 2, while, when $x^*(F) - y^*(F) \leq |F| - 3$, the right-hand side is non-positive. ♦

Note 2 There are two types of MBCPP solutions that satisfy a K-C inequality (8) with equality that will be used in the proof of Theorem 10:

Type 1: MBCPP tours where $x_{e_i} = 1 \quad \forall i = 1, \ldots, K-1$. There are 3 possibilities:

(a) solutions traversing exactly once each edge in F, twice each edge e_i ($x_{e_i} = y_{e_i} = 1$), $i = 1, 2, \ldots, K-1$, and connecting sets M_j, $j = 0, 1, 2, \ldots, K-1$, with either two different edges in $(M_j : M_{j+1})$ used once or an edge used twice as in Figure 3(a).

(b) solutions traversing each edge in F and one more edge in $(M_0 : M_K)$ (this could be a second traversal of an edge in F), once each edge e_i, $i = 1, 2, \ldots, K-1$, and connecting sets M_j, $j = 0, 1, 2, \ldots, K-1$, with exactly an edge in each set $(M_j : M_{j+1}), j = 0, \ldots, K-1$ (see Figure 3(b)).

(c) solutions traversing exactly once each edge in F except one of them, once each edge e_i, $i = 1, 2, \ldots, K-1$, and connecting sets M_j, $j = 0, 1, 2, \ldots, K-1$, with exactly an edge in each set $(M_j : M_{j+1}), j = 0, \ldots, K-1$ (see Figure 3(c)).

Type 2: MBCPP tours traversing exactly once each edge in F but having some variables x_{e_i} equal to zero. There are 3 possibilities:

(d) solutions with $x_{e_i} = 0 \quad \forall i = 1, \ldots, K-1$ (see Figure 4(d)).

(e) solutions with $x_{e_i} = y_{e_i} = 1 \quad \forall i = 1, \ldots, l$ and $x_{e_i} = y_{e_i} = 0 \quad \forall i = l+1, l+2, \ldots, K-1$, and connecting sets M_j, $j = 1, 2, \ldots, l$, as in Figure 4(e).

(f) solutions traversing twice each edge e_i except one of them, say e_p, and connecting sets $M_j, j \neq p$ as in Figure 4(f).

Note that in the previous MBCPP solutions we have not described how the edges in each set $E(M_i)$ are traversed. It can be seen that all these solutions can be completed by using T-joins as described in Note 1 for the parity inequalities.

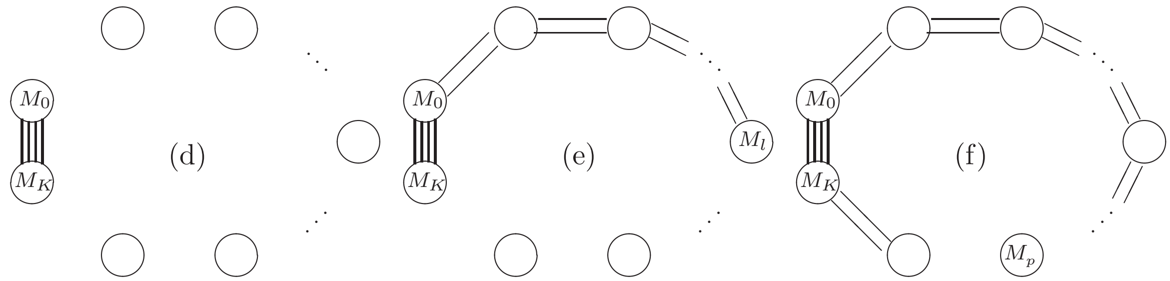

Figure 4: MBCPP solutions satisfying (8) with equality

Theorem 10 *K-C inequalities* (8) *and* (9) *are facet-inducing for MBCPP(G) if graph G is 3-edge connected, graph $G(M_i)$ is 3-edge connected for $i = 0, 1, \dots, K$, $|(M_i : M_{i+1})| \geq 2$ for $i = 0, \dots, K-1$, and $|F| \geq 2$.*

Proof: Let us prove that inequalities (8) are facet inducing. The proof for inequalities (9) is similar and is omitted here. Assume, w.l.o.g., that $1 \in M_0$. Let us suppose there is another valid inequality $ax + by \leq c$ such that

$$\Big\{(x,y) \in \text{MBCPP(G)} : (x,y) \text{ satisfies (8) with equality}\Big\} \subseteq \Big\{(x,y) \in \text{MBCPP(G)} : \ ax+by = c\Big\},$$

where $c = (2-K)|F|$.

Let $e = (u,v) \in E(M_i) \setminus \{e_i\}$, $i \in \{0,1,\dots,K\}$. Given that G is 3-edge connected, graph $G \setminus \{(u,v)\}$ is connected, and there is a MBCPP tour (x^1, y^1) of type (b) in Note 2 that satisfies (8) with equality and visits node u. The MBCPP tour (x^2, y^2) obtained from (x^1, y^1) by adding the traversal of edge (u,v) twice, also satisfies (8) with equality. Then $ax^1 + by^1 = ax^2 + by^2 = c$ and, after subtracting these expressions, we obtain

$$a_{uv} + b_{uv} = 0 \quad \forall (u,v) \in E(M_i) \setminus \{e_i\}, \quad i \in \{0,1,\dots,K\} \quad (*)$$

Let $e = (u,v) \in (M_0 : M_K)$ and let (x^1, y^1) be a MBCPP tour of type (c) in Note 2 that does not traverse edge (u,v) and visits node u. The MBCPP tour (x^2, y^2) obtained from (x^1, y^1) by adding the traversal of edge (u,v) twice, also satisfies (8) with equality, since $x_e = y_e = 1$ and the sum of the coefficients of both variables in (8) is zero. Then $ax^1 + by^1 = ax^2 + by^2 = c$ and, after subtracting these expressions, we obtain

$$a_{uv} + b_{uv} = 0 \quad \forall (u,v) \in (M_0 : M_K) \quad (**)$$

Let $(u,v) \in E(M_i) \setminus \{e_i\}$, $i \in \{0,1,\dots,K\}$. Given that $G(M_i)$ is 3-edge connected, there are two edge-disjoint paths $\mathcal{P}_1$, $\mathcal{P}_2$ joining nodes u and v that do not contain the edge (u,v) and both paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are in G(M_i). Given that the graph G(M_i)$\setminus\{(u,v)\}$ is connected, we can build a MBCPP tour (x^1, y^1) in G of type (b) satisfying (8) with equality such that it uses edge (u,v) exactly once. We define three more MBCPP tours in the following way:

Consider (x^1, y^1) and suppose we add one copy of each edge in paths $\mathcal{P}_1, \mathcal{P}_2$. The resulting tour is even and connected, but it is not necessarily a MBCPP tour because some edges could appear three times. If we remove two copies of each one of these edges used three times, we obtain an even and connected tour (x^2, y^2) that is a feasible solution for the MBCPP. Let (x^3, y^3) be the tour obtained from (x^1, y^1) after removing the edge (u,v), adding one copy of each edge in path $\mathcal{P}_1$ and then removing two copies of each one of these edges used three times. Finally, let (x^4, y^4) be the tour obtained from (x^1, y^1) after removing the edge (u,v), adding one copy of each edge in path $\mathcal{P}_2$ and then removing two copies of each one of these edges used three times. All these four MBCPP tours satisfy (8) with equality and then also satisfy $ax + by = c$.

Let us call $\alpha(P_r) = \sum_{e \in P_r^1} a_e + \sum_{e \in P_r^2} b_e$, where P_r^1 is the set of edges in path P_r that are traversed only once in (x^1, y^1) and P_r^2 the set of edges in path P_r that are not traversed or traversed twice in (x^1, y^1). If we subtract the expressions $ax^1 + by^1 = c$ and $ax^2 + by^2 = c$ we obtain $\alpha(\mathcal{P}_1) + \alpha(\mathcal{P}_2) = 0$. In the same way, by comparing tours 3 and 4 above we obtain that $\alpha(\mathcal{P}_1) = \alpha(\mathcal{P}_2)$ and then $\alpha(\mathcal{P}_1) = \alpha(\mathcal{P}_2) = 0$. Finally, if we compare the tours 1 and 3 we obtain that $a_{uv} = \alpha(\mathcal{P}_1)$ and then $a_{uv} = 0$. From $(*)$, also $b_{uv} = 0$.

For each $i \in \{1, 2, \ldots, K-1\}$, let (x^1, y^1) be the MBCPP tour of type (a) in note 2 traversing twice an edge in each set $(M_j : M_{j+1})$, $j \neq i$ and let (x^2, y^2) be the MBCPP tour of type (f) traversing twice the same edge in each set $(M_j : M_{j+1})$, $j \neq i-1, i$. Both tours satisfy (8) with equality and then also satisfy $ax + by = c$. Hence, $a_{uv} + b_{uv} = -a_{e_i} - b_{e_i}$ for all $(u, v) \in (M_{i-1} : M_i)$. If we consider the MBCPP tour (x^3, y^3) of type (a) traversing twice an edge in each set $(M_j : M_{j+1})$, $j \neq i-1$, we can conclude $a_{uv} + b_{uv} = -a_{e_i} - b_{e_i}$ for all $(u, v) \in (M_i : M_{i+1})$. Iterating this argument, we obtain: $a_{uv} + b_{uv} = 2\lambda$ for all $(u, v) \in (M_i : M_{i+1})$ and $a_{e_i} + b_{e_i} = -2\lambda$ for all $i = 1, \ldots, K-1$, where λ is a certain constant value.

For each $i \in \{1, 2, \ldots, K-1\}$, let (x^1, y^1) be the MBCPP tour of type (c) in note 2 traversing edge $e_i = (u, v)$ once. Given that subgraph $G(M_i)$ is 3-edge connected, we can find a path connecting u and v that does not use e_i. If we add this path plus one copy of e_i to (x^1, y^1), we obtain a MBCPP tour (x^2, y^2) satisfying (8) with equality. By comparing both tours, and given that $a_e = b_e = 0$ for all $e \in E(M_i) \setminus \{e_i\}$, we obtain $b_{e_i} = 0$ and therefore $a_{e_i} = -2\lambda$.

For each $i \in \{0, 1, 2, \ldots, K-1\}$, let $(u, v), (w, z)$ be two edges in $E(M_i : M_{i+1})$ (recall that $|(M_i, M_{i+1})| \geq 2$ for $i = 0, \ldots, K-1$, holds). Then, it is easy to see that there are two MBCPP tours (x^1, y^1) and (x^2, y^2) of type (c) in note 2 traversing edges (u, v) and (w, z) once respectively. Comparing both tours, we get $a_{uv} = a_{wz}$. Since we have proved that $a_{uv} + b_{uv} = 2\lambda = a_{wz} + b_{wz}$, we have $b_{uv} = b_{wz}$. Furthermore, let (x^3, y^3) be a tour of type (f) traversing edge (u, v) twice and (x^4, y^4) a similar tour traversing (u, v) and (w, z) once. By comparing both tours, we obtain $b_{uv} = a_{wz}$ and, since $a_{wz} = a_{uv}$, we get $a_{uv} = b_{uv}$. Therefore $a_{uv} = b_{uv} = \lambda$ for each edge $(u, v) \in E(M_i : M_{i+1})$ for all i.

From the MBCPP tour of type (d), we obtain that $\sum_{e \in F} a_e = (2-K)|F|$. Let e, f be two edges in F and let (x^1, y^1) and (x^2, y^2) be the tours of type (c) that traverse all edges in F except for e and f respectively. By comparing both tours, we get $a_e = a_f$ and, iteratively, all the edges in F have the same coefficient $a_e = 2-K$, and also $b_e = K-2$. If there is any edge e in $E(M_0 : M_K) \setminus F$, comparing a tour of type (b) traversing e and another tour of type (c), we get $a_e = K-2$ and therefore $b_e = 2-K$.

If we compare a tour of type (a) and another one of type (c), we obtain $2-K+(K-1)\lambda = \lambda$. Hence $\lambda = 1$.

Finally, for each edge $e \in E(M_i : M_j)$, $|i-j| > 1$, comparing tours of type (g) and (h) in Figure 5, we obtain $a_e + b_e = 2$. Then, comparing tours of type (h) and (i), we obtain $a_e + |i-j| = a_e + b_e + 2(|i-j|-1)$. Therefore, $b_e = 2 - |i-j|$ and $a_e = |i-j|$, which completes the proof. ♦

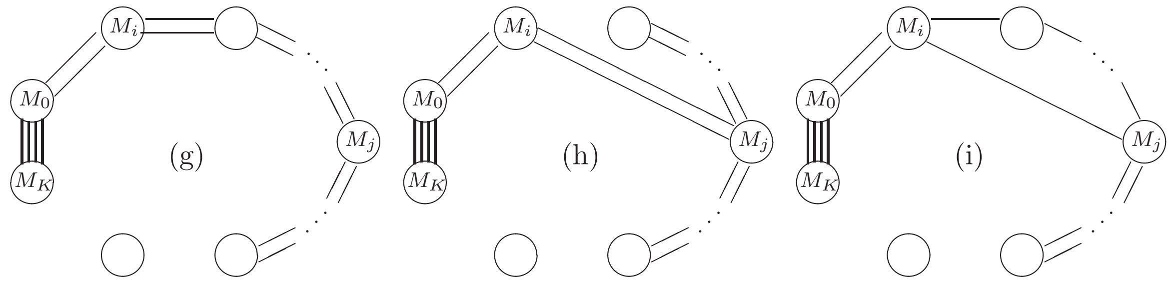

Figure 5: MBCPP solutions satisfying (8) with equality

Note 3 The 3-edge-connectivity hypothesis on graphs $G(M_i)$ may be unnecessary for a given K-C inequality to induce a facet, but we needed it in the proof above. In the same way, if $|E(M_i : M_{i+1})| = 1$ for some i, we do not know if the K-C inequality is facet-inducing or not. Only in the case $K = 3$ and $|E(M_1 : M_2)| = 1$ we have proved that the K-C inequality is dominated by a 2-connectivity inequality (see Section 4.3) defined with $S_0 = M_0 \cup M_3$.

4.3 p-connectivity inequalities

All the previous inequalities do not describe the polyhedron MBCPP(G) completely. For example, consider a MBCPP instance defined on the complete graph with 5 vertices, K_5. Consider the fractional solution shown in Figure 6(a), where a double solid (dotted) line means that both variables x and y take value 1 (0.5). It can be seen that it satisfies all the inequalities presented in previous sections. Specifically, it satisfies the connectivity inequalities (2) associated with sets $S_1 = \{2, 3\}$ and $S_2 = \{4, 5\}$ and edges $(2, 3)$ and $(4, 5)$, respectively.

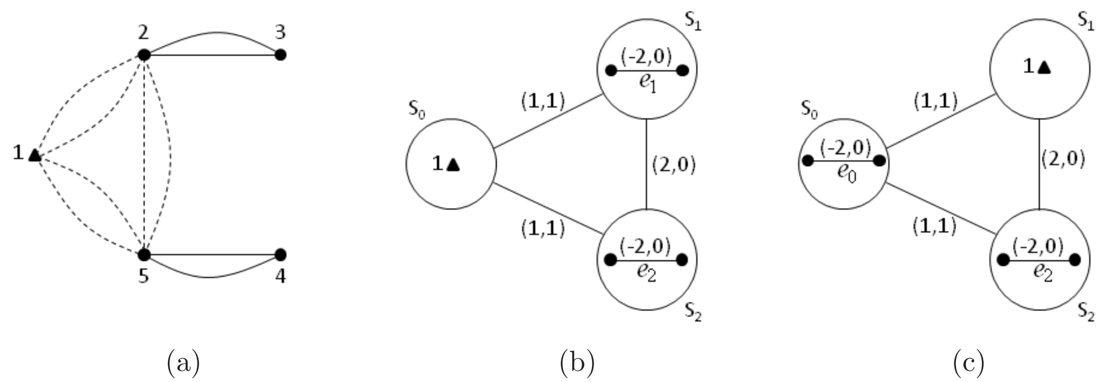

Figure 6: MBCPP fractional solution satisfying connectivity inequalities

Note, however, that some features concerning connectivity are not satisfied by this fractional solution. As we will show later, the following inequality is valid for the MBCPP and cuts fractional solutions like the one depicted in Figure 6(a):

$$x(\delta(\{1\}) + y(\delta(\{1\}) + 2x(S_1 : S_2) \geq 2x_{23} + 2x_{45}.$$

The above inequality can be extended as follows. Let S_0, S_1, S_2 be a partition of V and assume that $1 \in S_0$. Let $e_1 \in E(S_1)$ and $e_2 \in E(S_2)$. The following inequality will be referred to as 2-connectivity inequality:

$$x(\delta(S_0)) + y(\delta(S_0)) + 2x(S_1 : S_2) \geq 2x_{e_1} + 2x_{e_2}.$$

This inequality is represented in Figure 6(b), where for each pair (a, b) associated with an edge e, a and b represent the coefficients of x_e and y_e, respectively. A different version of these 2-connectivity inequalities is obtained when the depot is not in S_0. If, for example, $1 \in S_1$ (see Figure 6(c)), the inequality takes the form:

$$x(\delta(S_0)) + y(\delta(S_0)) + 2x(S_1 : S_2) \geq 2x_{e_0} + 2x_{e_2},$$

where e_0 is a given edge in S_0.

2-connectivity inequalities can be generalized by considering any number $p+1$ of sets. Let $S_0, S_1, \ldots, S_p$ be a partition of V. Assume that $1 \in S_d$, $d \in \{0, 1, \ldots, p\}$ and consider an edge $e_j \in E(S_j)$ for every $j \in \{0, 1, \ldots, p\} \setminus \{d\}$. The following inequality

$$x(\delta(S_0)) + y(\delta(S_0)) + 2 \sum_{1 \leq r < t \leq p} x(S_r : S_t) \geq 2 \sum_{i=0, i \neq d}^{p} x_{e_i} \tag{13}$$

is valid and will be referred to as p-connectivity inequality.

Theorem 11 *p-connectivity inequalities* (13) *are valid for MBCPP(G).*

Proof: For the sake of simplicity we will assume that $1 \in S_0$. Let (x^*, y^*) be a MBCPP tour. We have to prove that $x^*(\delta(S_0)) + y^*(\delta(S_0)) + 2\sum_{1 \leq r < t \leq p} x^*(S_r : S_t) \geq 2\sum_{i=1}^{p} x^*_{e_i}$.

Note that if $x^*_{e_j} = 0$ for some $j \in \{1, \ldots, p\}$, then we could consider a new p-connectivity configuration with $p-1$ nodes where nodes S_j and S_{j+1} have been merged into a single node $S_j \cup S_{j+1}$, and it can be proved that, if its associated $(p-1)$-connectivity inequality is satisfied by (x^*, y^*), then the original p-connectivity inequality is also satisfied by (x^*, y^*). Then, in what follows, we can assume that $x^*_{e_1} = x^*_{e_2} = \ldots x^*_{e_p} = 1$.

Similarly, if $x^*_e = 1$ for some $e \in (S_r, S_t)$ with $1 \leq r < t \leq p$, we can define a new partition with $p-1$ elements where S_r and S_t have been merged into $S'_r = S_r \cup S_t$ and where $e'_r = e_r$. Again, if its associated $(p-1)$-connectivity inequality is satisfied by (x^*, y^*), then the original p-connectivity inequality is also satisfied by (x^*, y^*). Hence, we can assume that $x^*(S_i, S_j) + y^*(S_i, S_j) = 0$ for any $i, j \in \{1, \ldots, p\}$.

Therefore, since $x^*_{e_i} = 1$ for each $i = 1, \ldots, p$, $x^*(S_0, S_i) + y^*(S_0, S_i) \geq 2$, and in this case the inequality holds. ♦

Theorem 12 *p-connectivity inequalities* (13) *are facet-inducing for MBCPP(G) if graph G is 3-edge connected, subgraphs $G(S_i)$, $i = 1, \ldots, p$, are 3-edge connected, $|(S_0, S_i)| \geq 2$, $\forall\, i = 1, \ldots, p$, and the graph induced by $V \setminus S_0$ is connected.*

Proof: We will assume that $1 \in S_0$. The case $1 \in S_i$, $i \neq 0$, is similar and the proof is omitted here. Let us suppose there is another valid inequality $ax + by \geq c$ such that

$$\{(x, y) \in \text{MBCPP(G)} : (x+y)(\delta(S_0)) + 2\sum_{1 \leq r < t \leq p} x(S_r : S_t) - 2\sum_{i=1}^{p} x_{e_i} = 0\} \subseteq$$
$$\subseteq \{(x, y) \in \text{MBCPP(G)} : ax + by = c\},$$

where, w.l.o.g., we can assume that $c = 0$.

Similar arguments to those used in the proof of Theorem 10, lead to prove that $a_{uv} = b_{uv} = 0$ for every edge $(u, v) \in E(S_i) \setminus \{e_i\}$ for all $i = 0, 1, 2, \ldots, p$ and that $b_{e_i} = 0$ for all $i = 1, 2, \ldots, p$.

Let S_i and S_j, $i, j \neq 0$ be two sets such that there is an edge $(u, v) \in (S_i, S_j)$. Since $(S_0, S_i) \neq \emptyset$, we can construct the tour that traverses twice both an edge $f \in (S_0, S_i)$ and edge e_i, which satisfies the inequality (13) as an equality. If we compare this tour with the empty tour, we obtain $a_f + b_f + a_{e_i} = 0$. This result also holds for set S_j. We construct now three tours satisfying (13) with equality such as those depicted in Figure 7. Comparing them, we conclude $a_{0i} + b_{0i} = a_{ij} + b_{ij} = a_{0j} + b_{0j} = -a_{e_i} = -a_{e_j}$, where a_{kl} (b_{kl}) represents the coefficient of variable x (y) corresponding to any edge in (S_k, S_l). Given that the graph induced by $V \setminus S_0$ is connected, we can iterate this argument to conclude that

$$a_{uv} + b_{uv} = -a_{e_i} = 2\lambda$$

for every edge (u, v) joining any pair of sets S_u and S_v and for every edge e_i, $i = 1, \ldots, p$.

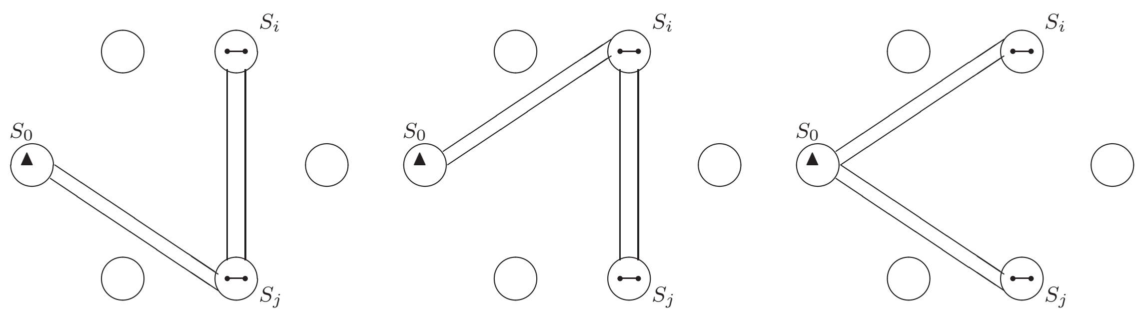

Figure 7: MBCPP solutions satisfying (13) with equality

Given that $|(S_0, S_i)| \geq 2$ for each $i = 1, \ldots, p$, using a similar argument as in the proof of Theorem 10 we obtain $a_{uv} = b_{uv} = \lambda$ for each edge $(u, v) \in (S_0, S_i)$, $i = 1, \ldots, p$.

As above, let S_i and S_j, $i, j \neq 0$ be two sets such that there is an edge $(u, v) \in (S_i, S_j)$. The tour that traverses once edge (u, v), one edge in (S_0, S_i), one edge in (S_0, S_j), and edges e_i and e_j satisfies inequality (13) with equality. Then, it satisfies $a_{uv} + a_{0i} + a_{0j} + a_{e_i} + a_{e_j} = a_{uv} + \lambda + \lambda - 2\lambda - 2\lambda = 0$, and therefore $a_{uv} = 2\lambda$, which implies $b_{uv} = 0$.

Finally, since the right-hand side c is 0, dividing the inequality by λ we get the coefficients of the p-connectivity inequality and the proof is completed. ♦

Set	# inst	Gap 0	# opt 0	Nodes	Time	Gap 0	# opt 0	Time 0
D16	9	0,000%	9	0,00	0,05	0,51%	8	0,3
D36	9	0,000%	9	0,00	0,10	0,12%	5	14,58
D64	9	0,000%	9	0,00	0,21	0,14%	7	105,34
D100	9	0,000%	9	0,00	1,75	0,40%	4	1890,71
G16	9	0,000%	9	0,00	0,03	0,00%	9	0,28
G36	9	0,000%	9	0,00	0,12	0,00%	9	18,31
G64	9	1,111%	8	0,44	0,67	1,85%	8	139,97
G100	9	0,452%	7	1,11	3,52	0,44%	6	2798,35
R20	5	0,000%	5	0,00	0,05	0,26%	4	0,4
R30	5	0,000%	5	0,00	0,11	0,00%	5	2,7
R40	5	0,000%	5	0,00	0,13	0,00%	4	3,69
R50	5	0,065%	4	0,20	0,24	0,07%	4	60,86
P	24	0,068%	20	0,21	0,07	0,44%	20	1,97
AlbaidaA	1	0,000%	1	0,00	0,38	0,19%	0	562,5
AlbaidaB	1	0,000%	1	0,00	0,22	0,00%	1	25,15
	118	0,136%	110			0,369%	94	

Table 1: Add caption

Set	Con. H	Con. H2	Con. Ex	Imp. H	Imp. 1v	Imp. Ex	CP_K	P-con.
D16	8,4	1,2	1,6	1,4	16,7	1,6	0,2	1,1
D36	17,6	2,4	4,3	5,1	47,0	0,2	0,2	2,9
D64	24,3	8,6	14,4	15,1	83,1	3,4	2,0	7,4
D100	40,6	39,6	38,7	23,4	130,1	93,3	10,6	32,4
G16	9,0	1,9	0,2	0,2	16,0	0,3	0,0	0,0
G36	22,7	10,3	3,6	3,6	47,2	8,4	0,8	3,1
G64	37,3	20,4	17,8	7,6	80,3	24,3	4,9	34,2
G100	52,7	101,8	50,1	18,2	131,2	173,8	13,4	86,0
R20	10,4	0,8	2,4	1,6	21,8	0,4	0,0	0,0
R30	23,8	1,8	7,0	10,2	41,4	3,8	0,0	1,8
R40	26,0	2,2	3,2	4,4	45,0	14,2	0,8	0,0
R50	33,2	6,6	11,6	9,8	66,2	20,6	0,0	2,2
P	7,8	1,3	2,7	1,6	25,0	2,5	0,9	1,3
AlbaidaA	43,0	6,0	14,0	8,0	93,0	44,0	0,0	0,0
AlbaidaB	33,0	11,0	6,0	4,0	70,0	10,0	0,0	2,0

Table 2: Add caption

Set	# inst	Gap 0	# opt 0	Nodes	Time
B3	3	0,18%	0	3,00	55,2
B4	3	0,08%	1	4,33	119,8
B5	3	0,00%	3	0,00	63,4
B6	3	0,00%	3	0,00	42,3
C3	3	0,03%	1	1,33	251,4
C4	3	0,00%	2	0,33	503,0
C5	3	0,01%	1	2,67	831,4
C6	3	0,00%	3	0,00	410,0
D3	3	0,01%	1	0,67	1265,6
D4	3	0,00%	2	0,33	2366,3
D5	3	0,00%	3	0,00	1252,8
D6	3	0,00%	3	0,00	380,1

Table 3: Add caption

Set	Conn. H1	Conn. H2	Conn. E.	Odd H	Odd E1v	Odd E.	CP_K	P-conn.
B3	200,7	492,0	373,3	211,0	465,0	300,7	12,7	110,3
B4	74,7	63,3	559,0	291,3	525,3	702,3	48,7	83,0
B5	50,0	22,0	154,0	269,3	490,0	84,3	33,0	9,3
B6	25,3	4,0	24,3	158,7	449,7	44,7	5,3	1,7
C3	251,3	995,3	1073,7	312,0	668,0	1186,7	1,3	179,3
C4	120,3	51,0	535,0	369,3	736,7	397,3	16,7	53,0
C5	65,3	8,7	609,3	357,0	724,3	984,3	16,3	19,3
C6	37,0	0,7	105,3	364,7	727,3	812,7	10,0	6,3
D3	447,7	2261,0	1599,0	549,7	993,0	4195,0	6,3	317,3
D4	174,7	204,3	1123,3	523,0	979,7	1604,3	72,7	119,0
D5	124,7	18,0	745,7	457,3	973,3	367,0	33,7	31,3
D6	68,0	3,7	90,3	328,3	933,3	379,3	0,0	37,7

Table 4: Add caption

Set	# inst.	Gap N0	# opt N0	# opt	Nodes	Time
B3	3	0,02%	1	3	3,33	190,9
B4	3	0,01%	2	3	1,33	100,8
B5	3	0,00%	3	3	0,00	70,3
B6	3	0,00%	3	3	0,00	58,6
C3	3	0,01%	1	2	0,67	857,9
C4	3	0,00%	3	3	0,00	420,3
C5	3	0,00%	3	3	0,00	385,2
C6	3	0,00%	3	3	0,00	317,4
D3	3	0,00%	1	2	1,00	1896,2
D4	3	0,00%	3	3	0,00	3578,8
D5	3	0,00%	3	3	0,00	350,9
D6	3	0,00%	3	3	0,00	660,4

Table 5: Add caption

Set	Conn. H1	Conn. H2	Conn. E.	Odd H	Odd E1v	Odd E.	CP_K	P-conn.
B3	176,0	220,3	1388,3	176,3	461,0	615,0	98,0	351,0
B4	53,7	9,3	145,7	297,0	518,3	620,0	45,7	33,3
B5	4,0	0,3	19,0	263,7	479,3	80,0	0,3	0,0
B6	1,7	0,0	2,0	149,0	449,0	82,0	0,0	0,7
C3	225,7	358,7	3416,7	308,0	737,7	3219,3	46,0	998,3
C4	48,7	8,7	400,0	187,7	643,3	74,0	2,7	75,0
C5	5,7	3,0	62,3	364,3	707,0	102,0	0,0	1,0
C6	0,3	0,3	7,7	381,3	759,3	38,7	0,0	0,3
D3	387,3	1280,3	8580,3	474,3	1006,0	3274,7	80,3	1322,0
D4	56,0	17,7	2617,3	496,0	958,0	174,3	5,3	139,3
D5	2,7	0,0	11,3	278,0	884,0	19,7	0,0	0,0
D6	0,7	0,0	9,0	317,0	941,0	59,0	0,0	0,3

Table 6: Add caption

5 Branch-and-cut algorithm for the MBCPP

6 Computational results

7 Conclusions

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