

Structure of symmetry group of some composite links and some applications

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ABSTRACT

In this paper, we study the symmetry group of a type of composite topological links, such as $2_1^2 m_1^2 \sharp 2_1^2$. We have done a complete analysis on the elements of the symmetric group of this link and show the structure of the group. The results can be generalized to the study of the symmetry group of any composite topological link, and therefore it can be used for the classification of composite topological links, which can also be potentially used to identify synthetics molecules.

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KEYWORDS: knot; link; geometric topology; symmetry group; classification of links.

1. INTRODUCTION

In topology, geometry, and physics, various knots, which are mathematically various embeddings of a circle in the 3-dimensional Euclidean space, have been interesting objects, which have been studied in recent decades (see for instance, [12], [3], and [10]). In particular, knots have been used to construct examples for the study of low-dimensional topology (see for instance, [9]). Two or more knots can make up of a link, which have appeared to be somewhat more interesting as a single knot, because of the combinatorial structure involved. The theory of knots and links has applications in many areas such as physics, biochemistry,

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and biology, in particular, DNA and enzyme action (see for instance, [19], [14], and [7]).

The structure of the symmetry group has become important information to understand the geometrical, physical properties of knots and links, as well as the enumeration of knots and links (see, for instance, [5] and [6]). As shown in [1], the knots are algebraic, and the symmetry of knots has been one of the interesting topics presented in [18]. In this article, we show heuristically that all composite topological links are actually also algebraic, in particular, the composite link, which is the knot sum of the Hopf link and its mirror image, denoted as $2_1^2 m \# 2_1^2$, has a symmetry group.

The main contribution of this paper is that we show that the symmetry group of the composite link $2_1^2 m \# 2_1^2$ is

$$(1.1) \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \alpha\} \times \{1, \beta\} \times \{1, \gamma\},$$

where

$$(1.2) \quad \alpha = (1, 1, -1, -1, (2, 3)),$$

$$(1.3) \quad \beta = (1, -1, -1, -1, e),$$

and

$$(1.4) \quad \gamma = (-1, -1, 1, 1, e).$$

The results can be generalized in the study of the symmetry group of any composite topological link, and so it can be used for the classification of composite topological links.

This paper is structured as follows: in Section 2, we show the classification of symmetries of the link $2_1^2 m \# 2_1^2$; and in Section 3, we analyze the structure of the symmetry group from the perspective of algebraic group and prove our main theorem.

2. CLASSIFICATIONS

Let us compute the compatible (p, r) permutations first. Since the compatible permutations just determine which component of each link is connected, the compatible permutations are the same with the case $2_1^2 \# 2_1^2$. So the compatible permutations are $p = (2, 3), r = (1, 2)$ and $p = e, r = e$.

Now, let's take the next step, by which we can find \bar{p}_1 and \bar{p}_2 , and indeed, $\bar{p}_1 = \bar{p}_2 = e$.

Knowing the fact that $2_1^2 m$ and 2_1^2 are in different cosets and that 2_1^2 has the symmetry group

$$(2.1) \quad \langle (1, -1, -1, e), (-1, 1, -1, e), (1, 1, 1, (1, 2)) \rangle,$$

we can see that there are 16 cases, as follows, to be considered:

$$(1) \quad \gamma = (1, 1, 1, 1, (2, 3)); \gamma_1 = (1, 1, 1, e); \gamma_2 = (1, 1, 1, e).$$

Since L_2^3 is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.

- (2) $\gamma = (1, 1, 1, -1, (2, 3)); \gamma_1 = (1, 1, 1, e); \gamma_2 = (1, 1, -1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (3) $\gamma = (1, 1, -1, 1, (2, 3)); \gamma_1 = (1, 1, -1, e); \gamma_2 = (1, 1, 1, e)$.
 Since $L_2^{\gamma_1}$ is in the coset of L_1 , but $L_1^{\gamma_2}$ is not in the coset of L_2 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (4) $\gamma = (1, 1, -1, -1, (2, 3)); \gamma_1 = (1, 1, -1, e); \gamma_2 = (1, 1, -1, e)$.
 Since $L_2^{\gamma_1}$ is in the coset of L_1 , and $L_1^{\gamma_2}$ is in the coset of L_2 , this case is in the symmetry group of $2_1^2 m \# 2_1^2$.
- (5) $\gamma = (1, -1, 1, 1, (2, 3)); \gamma_1 = (1, -1, 1, e); \gamma_2 = (1, -1, 1, e)$.
 Since $L_2^{\gamma_1}$ is in the coset of L_1 , and $L_1^{\gamma_2}$ is in the coset of L_2 , this case is in the symmetry group of $2_1^2 m \# 2_1^2$.
- (6) $\gamma = (1, -1, 1, -1, (2, 3)); \gamma_1 = (1, -1, 1, e); \gamma_2 = (1, -1, -1, e)$.
 Since $L_1^{\gamma_2}$ is not in the coset of L_2 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (7) $\gamma = (1, -1, -1, 1, (2, 3)); \gamma_1 = (1, -1, -1, e); \gamma_2 = (1, -1, 1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (8) $\gamma = (1, -1, -1, -1, (2, 3)); \gamma_1 = (1, -1, -1, e); \gamma_2 = (1, -1, -1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
 The next 8 cases of mirror image are the followings:
- (9) $\gamma = (-1, 1, 1, 1, (2, 3)); \gamma_1 = (-1, 1, 1, e); \gamma_2 = (-1, 1, 1, e)$.
 Since $L_2^{\gamma_1}$ is in the coset of L_1 , and $L_1^{\gamma_2}$ is in the coset of L_2 , this case is in the symmetry group of $2_1^2 m \# 2_1^2$.
- (10) $\gamma = (-1, 1, 1, -1, (2, 3)); \gamma_1 = (-1, 1, 1, e); \gamma_2 = (-1, 1, -1, e)$.
 Since $L_1^{\gamma_2}$ is not in the coset of L_2 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (11) $\gamma = (-1, 1, -1, 1, (2, 3)); \gamma_1 = (-1, 1, -1, e); \gamma_2 = (-1, 1, 1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (12) $\gamma = (-1, 1, -1, -1, (2, 3)); \gamma_1 = (-1, 1, -1, e); \gamma_2 = (-1, 1, -1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (13) $\gamma = (-1, -1, 1, 1, (2, 3)); \gamma_1 = (-1, -1, 1, e); \gamma_2 = (-1, -1, 1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (14) $\gamma = (-1, -1, 1, -1, (2, 3)); \gamma_1 = (-1, -1, 1, e); \gamma_2 = (-1, -1, -1, e)$.
 Since $L_2^{\gamma_1}$ is not in the coset of L_1 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.
- (15) $\gamma = (-1, -1, -1, 1, (2, 3)); \gamma_1 = (-1, -1, -1, e); \gamma_2 = (-1, -1, 1, e)$.
 Since $L_1^{\gamma_2}$ is not in the coset of L_2 , this case is not in the symmetry group of $2_1^2 m \# 2_1^2$.

- (16) $\gamma = (-1, -1, -1, -1, (2, 3)); \gamma_1 = (-1, -1, -1, e); \gamma_2 = (-1, -1, -1, e)$.
 Since $L_2^{\gamma_1}$ is in the coset of L_1 , and $L_1^{\gamma_2}$ is in the coset of L_2 ,
 this case is in the symmetry group of $2_1^2 m \# 2_1^2$.

In summary, the set of elements involving $(2, 3)$ as the permutation in the symmetry group is

$$(2.2) \quad S_1 = \{(1, 1, -1, -1, (2, 3)), (1, -1, 1, 1, (2, 3)), (-1, 1, 1, 1, (2, 3)), (-1, -1, -1, -1, (2, 3))\}.$$

For the other compatible permutation $p = r = e$, we can compare $L_1^{\gamma_1}$ and L_1 , as well as $L_2^{\gamma_2}$ and L_2 , then we obtain the following set of elements in the symmetry group,

$$(2.3) \quad S_2 = \{(1, 1, 1, 1, e), (1, -1, -1, -1, e), (-1, -1, 1, 1, e), (-1, 1, -1, -1, e)\}.$$

3. STRUCTURE ANALYSIS AND THEOREM

In this section, we analyze the structure of the symmetry group with the multiplication operation of the group and have the following theorem.

Theorem 3.1. *The symmetry group of the composite link $2_1^2 m \# 2_1^2$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. Let $\alpha = (1, 1, -1, -1, (2, 3)), \beta = (1, -1, -1, -1, e), \gamma = (-1, -1, 1, 1, e), \delta = (-1, 1, -1, -1, e)$, and the unit element $1 = (1, 1, 1, 1, e)$, we have

$$(3.1) \quad (1, -1, 1, 1, (2, 3)) = \alpha\beta,$$

$$(3.2) \quad (-1, 1, 1, 1, (2, 3)) = \alpha\delta,$$

and

$$(3.3) \quad (-1, -1, -1, -1, (2, 3)) = \alpha\gamma.$$

Noticing that $\delta = \beta\gamma$, we now have the symmetry group

$$(3.4) \quad G = \langle 1, \alpha, \beta, \gamma \rangle.$$

Since any other element in G than the identity is of order 2, then we know that G is abelian. Therefore, G is an abelian group of order 8. By the fundamental theorem of finitely generated abelian group (see for instance [8]), the structure of G is $\mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. But since any other element in G than the identity has an order 2, the structure of G must be $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence,

$$(3.5) \quad G = \{1, \alpha\} \times \{1, \beta\} \times \{1, \gamma\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

where $\alpha = (1, 1, -1, -1, (2, 3)), \beta = (1, -1, -1, -1, e)$, and $\gamma = (-1, -1, 1, 1, e)$. □

So it turns out that the structure of the symmetry group of the composite link $2_1^2 m \# 2_1^2$ is the same as the composite link $2_1^2 \# 2_1^2$, but the symmetry group of $2_1^2 m \# 2_1^2$ has different elements in its symmetry group.

Remark 3.2. Topological knots and links were studied by using integral geometry, on which one can refer to [13], but other theoretic work of integral geometry such as [2], [16], [4], [15], and [17], may also be used to study knots and links. On the other hand, the fundamental theorem of finitely generated abelian group, in the case if the group is abelian, the classification of finite simple groups (see for instance [11]), and other algebraic theories, can be used to determine the structure of the symmetry group.

Another remark about the applications of the symmetry groups of knots we would like to make is

Remark 3.3. In some physical movements or processes of DNA, the group structure of the double helix strands of DNA is invariant, and therefore, it can be used to track these movements or processes. Furthermore, the chirality of synthetics molecules (see for instance, [20]), which should be induced by the symmetry groups, can be used to identify synthetics molecules and can be potentially applied to the testing on virus infections, which might potentially help with the control on diseases, in particular, the infectious disease, COVID-19, in the recent pandemic.

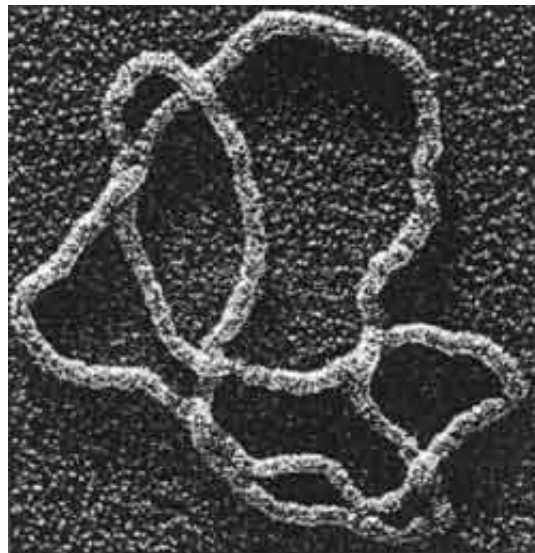


FIGURE 1. Discovered DNA Knot (c.f. [21])

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