Applied General Topology

Appl. Gen. Topol. 21, no. 2 (2020), 215-233
doi:10.4995/agt.2020.12929

The class of simple dynamical systems

K. Ali Akbar

Department of Mathematics, Central University of Kerala, Kasaragod - 671320, Kerala, India. (aliakbar.pkd@gmail.com, aliakbar@cukerala.ac.in)

Communicated by F. Balibrea

Abstract

In this paper, we study the class of simple dynamical systems on $\mathbb{R}$ induced by continuous maps having finitely many non-ordinary points. We characterize this class using labeled digraphs and dynamically independent sets. In fact, we classify dynamical systems up to their number of non-ordinary points. In particular, we discuss about the class of continuous maps having unique non-ordinary point, and the class of continuous maps having exactly two non-ordinary points.

2010 MSC: *54H20; 26A21; 26A48.*

Keywords: *special points; non-ordinary points; critical points; order conjugacy; order isomorphism; labeled digraph; dynamically independent set.*

1. Introduction

A dynamical system is a pair (X, f), where X is a metric space and f is a continuous self map on X. Two dynamical systems (X, f) and (Y, g) are said to be *topologically conjugate* (or simply *conjugate*) if there exists a homeomorphism $h : X \to Y$ (called topological conjugacy) such that $h \circ f = g \circ h$. We simply say that f is conjugate to g, and we write it as $f \sim g$. In the case when h happens to be an increasing homeomorphism (for example, when $X = \mathbb{R}$ or an interval) we say that f and g are *increasingly conjugate* or *order conjugate*. When we are working with a single system, any self conjugacy can utmost shuffle points with same dynamical behavior. Therefore a point which is unique up to its behavior must be fixed by every self conjugacy. On the

Received 02 January 2020 – Accepted 17 May 2020

other hand if a point is fixed by all self conjugacies then it must have a special property (some times it may not be known explicitly). These things motivated to call the set of all points fixed by all self conjugacies as set of *special points*. For $x, y \in \mathbb{R}$, we write $x \sim y$ if x and y have the same dynamical properties in the dynamical system $(\mathbb{R}, f)$. Said precisely, $x \sim y$ if there exists an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $h \circ f = f \circ h$ and $h(x) = y$. It is easy to see that $\sim$ is an equivalence relation. Since the equivalence relation is coming from self conjugacy it is important in the field of topological dynamics. Let $[x]$ denote the equivalence class of $x \in \mathbb{R}$. Let I, J be two subintervals of $\mathbb{R}$. We say that $I < J$ if $x < y$ for all $x \in I$ and $y \in J$.

The properties of dynamical systems which are preserved by topological conjugacies are called dynamical properties. The points which are unique up to some dynamical property are called *dynamically special points*. Said differently, a special point has a dynamical property which no other point has. We say that a point x is ordinary if, its "like" points near it. That is, an element $x \in \mathbb{R}$ is *ordinary* in $(\mathbb{R}, f)$ if its equivalence class $[x]$ is a neighbourhood of it. i.e, the equivalence class of x contains an open interval around x. A point which is not ordinary is called *non-ordinary*. The idea of special points and non-ordinary points are relatively new to the literature (see [1], [2], [8]). Recently, we studied the class of simple dynamical systems induced by homeomorphisms. The reader may refer [2] to get an idea of simple systems induced by homeomorphisms having finitely many non-ordinary points. Throughout this paper, we will be working with continuous self maps of the real line. Since $\mathbb{R}$ has order structure, we would like to consider the conjugacies preserving the order. Hence the conjugacies which we consider in this paper are order preserving conjugacies (increasing conjugacies). For any continuous map $f : \mathbb{R} \to \mathbb{R}$, we denote $G_{f\uparrow}$ for the set of all order conjugacies of f.

In a dynamical system (X, f), let $N(f)$ denote the set of all non-ordinary points of f. Observe that a point to be *special* if $[x] = \{x\}$. Let $S(f)$ denote the set of all special points of f. A point x in a topological space X is said to be *rigid* if it is fixed by every self homeomorphism of X. For example, the point 1 is rigid in $(0, 1]$. According to the above definition all rigid points are special, even though there is no role for the map f. By definition, the points of $[x]$ are dynamically the same. We consider the systems for which there are only finitely many equivalence classes. This means there are only finitely many kinds of orbits up to conjugacy. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\mathrm{Per}(f)$ is properly contained in $\{1, 2, 2^2, ...\}$, then f is not Li-Yorke chaotic (see [7]). Also note that, if $f : \mathbb{R} \to \mathbb{R}$ is Devaney chaotic then $6 \in \mathrm{Per}(f)$ (see [5]). Therefore, if $f : \mathbb{R} \to \mathbb{R}$ is a continuous map having finitely many non-ordinary points then neither it is Li-Yorke chaotic nor it is Devaney chaotic because of Sharkovskii's theorem. For these reasons we call such systems as *simple systems*. These are the system in which the phase portrait can be drawn. Phase portraits (see [4]) are frequently used to graphically represent the dynamics of a system. A phase portrait consists of a diagram representing possible beginning positions in the system and arrows that indicate the change in these positions under iteration of the function. The drawable systems are interesting to physicist and for this reason the study of the class of simple dynamical systems can be useful.

Our main results (see Section 3) prove that

(i) There are exactly 26 continuous maps on $\mathbb{R}$ with exactly one non-ordinary point and 90 continuous maps on $\mathbb{R}$ with exactly two non-ordinary points up to order conjugacy.

(ii) Let $\mathcal{C}$ be the class of all continuous self maps of $\mathbb{R}$, having finitely many non-ordinary points. Then

- (a) for every member of $\mathcal{C}$, there exists a maximal dynamically independent set that is a finite union of intervals.
- (b) two members of $\mathcal{C}$ are order conjugate if and only if they have order isomorphic maximal dynamically independent set as in (a), and with isomorphic labeled digraphs.
- (c) every order isomorphism between such maximal dynamically independent set extends uniquely to an order conjugacy.

2. Basic Results

In this Section, we will be discussing some basic results that will be used to prove our main theorems. Most of the results are available in [2].

Let (X, f) be a dynamical system. By the *full orbit* of a point $x \in X$ we mean the set $O(x) = \{y \in X : f^n(x) = f^m(y) \text{ for some } m, n \in \mathbb{N}\}$.

For any subset $A \subset \mathbb{R}$, let

$$O(A) = \bigcup_{x \in A} O(x) = \bigcup_{x \in A} \{y \in \mathbb{R} : f^n(y) = f^m(x) \ for\ some\ m, n \in \mathbb{N}\}.$$

A point x in a dynamical system (X, f) is said to be a *critical point* if f fails to be one-to-one in every neighbourhood of x. The set of all critical points of f is denoted by $C(f)$.

See [2] and [8], for the following characterization theorem for the set $N(f)$ (and hence for $S(f)$).

Theorem 2.1. *For continuous self maps of the real line $\mathbb{R}$, the set of all non-ordinary points is contained in the closure of the union of full orbits of critical points, periodic points and the limits at infinity (if they exist and are finite).*

Remark 2.2. In Theorem 2.1, the inclusion can be strict.

Proof. Consider the map $f(x) = x + sinx$ for all $x \in \mathbb{R}$. All integral multiples of π are fixed points for this map but the increasing bijection $x \mapsto x + 2\pi$ commutes with f and fixes none of them. □

Remark 2.3 ([8]). For polynomials of even degree the equality is true in Theorem 2.1.

For a dynamical system (X, f), let $D = O(C(f) \cup P(f) \cup \{f(\infty), f(-\infty)\})$ where $f(\infty)$ and $f(-\infty)$ are the limits of f at ∞ and $-\infty$ respectively, provided they are finite. For any subset A of X, we denote $\bar{A}$ for the closure of A in X.

Now we consider the following theorem:

Theorem 2.4 ([8]). *For polynomial maps f of $\mathbb{R}$, $S(f)$ has to be either empty or a singleton or the whole $\overline{D}$.*

From the definition, it is clear that the set of special points $S(f)$ is always closed. The following theorem is about the converse and it is proved in [8].

Theorem 2.5. *Given any closed subset F of $\mathbb{R}$, there exists a continuous map $f : \mathbb{R} \to \mathbb{R}$ such that $S(f) = F$.*

Remark 2.6. For every closed subset F of $\mathbb{R}$ there exist continuous maps $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that $S(f) = Fix(g) = F$. Conversely, for every closed subset F of $\mathbb{R}$ there exist continuous maps $f, g : \mathbb{R} \to \mathbb{R}$ such that $S(f) = Fix(g) = F$.

Proof. For every closed subset F of $\mathbb{R}$ there exists a strictly increasing continuous bijection $f : \mathbb{R} \to \mathbb{R}$ such that $Fix(f) = F$. This is because, we can define $f(x) = x + \frac{1}{2}d(x, F)$. Hence the remark follows from Theorem 2.5 since $Fix(f)$ is closed. ☐

The following total order on $\mathbb{N}$ is called the Sharkovskii's ordering:
$3 \succ 5 \succ 7 \succ 9 \succ ... \succ 2 \times 3 \succ 2 \times 5 \succ 2 \times 7 \succ ...$
$\succ 2^n \times 3 \succ 2^n \times 5 \succ 2^n \times 7 \succ ...$
$...2^n \succ .... \succ 2^2 \succ 2 \succ 1$
We write $m \succ n$ if m precedes n (not necessarily immediately) in this order. An n-cycle means a cycle of length n.

Theorem 2.7 (Sharkovskii's Theorem [6]). *Let $m \succ n$ in the Sharkovskii's ordering. For every continuous self map of $\mathbb{R}$, if there is an m-cycle, then there is an n-cycle.*

For any $f : \mathbb{R} \to \mathbb{R}$, let G_f denote the set of all topological conjugacies of f and let $G_{f\uparrow}$ denote the set of all order conjugacies of f.

Proposition 2.8 ([2]). *If x is an ordinary point of f and if h is a self topological conjugacy of f, then $h(x)$ is ordinary.*

Proposition 2.9 ([2]). *If x is a non-ordinary point of f and if h is a self topological conjugacy of f, then $h(x)$ is non-ordinary.*

Now we ask: For a continuous map $f : \mathbb{R} \to \mathbb{R}$, how the equivalence classes looks like?

The following known lemma answer this question.

Lemma 2.10 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $a < b$ and $(a, b) \cap N(f) = \varnothing$. Then $x \sim y$ for all $x, y \in (a, b)$.*

Theorem 2.11 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If $|N(f)| = n$ then $|\{[x] : x \in \mathbb{R}\}| = 2n + 1$.*

Remark 2.12 ([2]). Note that, being a point in a particular equivalence class $[x]$ is a dynamical property. There are continuous maps $f : \mathbb{R} \to \mathbb{R}$ having finitely many equivalence classes, but infinitely many non-ordinary points.

Remark 2.13.

(1) If $f : \mathbb{R} \to \mathbb{R}$ has a unique fixed point then it is non-ordinary and vice versa.
(2) If $f : \mathbb{R} \to \mathbb{R}$ has finitely many fixed points (critical points) then all fixed (critical) points are special and hence non-ordinary.
(3) If there are only finitely many periodic cycles then all periodic points are special.
(4) Every special point is non-ordinary. But every non-ordinary point may not be special.

Proof. (1) Since the topological conjugacies carry fixed points to fixed points, the unique fixed point must be fixed by every self conjugacy and hence special.

Suppose $x_0 \in \mathbb{R}$ is the unique non-ordinary point of f. Then $h(x_0) = x_0$ for all $h \in G_{f\uparrow}$. Now, for any $h \in G_{f\uparrow}$ we have $h(f(x_0)) = f(h(x_0)) = f(x_0)$. That is, the point $f(x_0)$ is special. Since x_0 is the only special point, we have $f(x_0) = x_0$.

(2) It follows from the fact that under a topological conjugacy fixed points will be mapped to fixed points and critical points will be mapped to critical points and the fact that it takes the finite set of fixed points (critical points) to itself bijectively, preserving the order.

Proof of (3) is easy.

(4) It is immediate from the definition that every special point is non-ordinary. For the converse, consider the map $x \mapsto x + \sin x$ on $\mathbb{R}$ which has countably many fixed points. Note that all the fixed points are non-ordinary and they form two distinct equivalence classes, hence they are not special. □

The following proposition says that the class of maps with finitely many non-ordinary points the idea of special points and the idea of non-ordinary point, coincide.

Proposition 2.14. *If $f : \mathbb{R} \to \mathbb{R}$ has only finitely many non-ordinary points then every non-ordinary point is special.*

Proof. Since $N(f)$ is finite, it follows from proposition 2.9 that $h(N(f)) = N(f)$ for all $h \in G_{f\uparrow}$. Then we must have $h(x) = x$ for all $x \in N(f)$, because of the order preserving nature of h. Hence all points of $N(f)$ are special. □

Proposition 2.15 ([2])**.** *For maps $f : \mathbb{R} \to \mathbb{R}$ with finitely many non-ordinary points, $f(x)$ is non-ordinary whenever $x \in \mathbb{R}$ is non-ordinary.*

Definition 2.16. For any subset A of $\mathbb{R}$, we write $\partial A = \overline{A} \cap (\overline{X - A})$ and call it the boundary of A boundary of a set.

Recall that the properties which are preserved under topological conjugacies are called dynamical properties. Hence, if two points x, y in the dynamical system (X, f), differ by a dynamical property, then no conjugacy can map one to the other, from which it follows that,

Proposition 2.17. *For any dynamical property P, the points of ∂S_P are non-ordinary where S_P denotes the set of all points in (X, f) having the dynamical property P.*

Corollary 2.18. *Let $f : \mathbb{R} \to \mathbb{R}$ be constant in a neighbourhood of a point x_0. Then the end points of the maximal interval around x_0 on which f is constant, are non-ordinary.*

Recall that, if $f : \mathbb{R} \to \mathbb{R}$ has a unique non-ordinary point then it is a fixed point. Next we consider the following proposition. See [2] for a proof.

Proposition 2.19 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Then,*
(i) If $x \in \mathbb{R}$ is both critical and ordinary then f is locally constant at x.
(ii) If x is ordinary and f is not locally constant at x then $f(x)$ is ordinary.

Remark 2.20 ([2]). Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $\sup f(\mathbb{R})$, $\inf f(\mathbb{R})$, $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ are special (in particular, non-ordinary) provided they are finite. (Note that, for maps with finitely many non-ordinary points both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ always exists in $\mathbb{R} \cup \{-\infty, \infty\}$).

Proposition 2.21 ([2]). *The maps $x + 1$ and $x - 1$ on $\mathbb{R}$ are topologically conjugate; but not order conjugate.*

Proof. The maps $x+1$ and $x-1$ are conjugate to each other through $-x+\frac{1}{2}$.
If possible, let h be an order conjugacy from $f(x) = x + 1$ to $g(x) = x - 1$. Then $h(x+1) = h(f(x)) = g(h(x)) = h(x) - 1$. i.e, $h(x+1) - h(x) = -1 < 0$. Which is a contradiction to the assumption that h is increasing. □

Remark 2.22. Note that for the map $x + 1$ on $\mathbb{R}$, all points are ordinary. For, if $a, b \in \mathbb{R}$, then the map $x + b - a$ is the order conjugacy of $x + 1$ which maps a to b.

Recall that for any subset A of a metric space X,

$$(\partial A)^c = int(A) \cup int(A^c).$$

That is, the complement of ∂A is the union of the interior of A and the interior of 'A complement'.

The following proposition gives a characterization for the non-ordinary points of increasing homeomorphisms.

Proposition 2.23 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing bijection and let $x \in \mathbb{R}$. Then x is non-ordinary if and only if x is in the boundary of Fix(f).*

Next we consider:

Proposition 2.24 ([2]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism without fixed points.*
(i) If $f(0) > 0$ then f is order conjugate with $x + 1$.
(ii) If $f(0) < 0$ then f is order conjugate with $x - 1$.

We denote graph(f) for the range of the map f.

Corollary 2.25 ([2]). *Let $f, g : (a,b) \to (a,b)$ be homeomorphisms without fixed points. Then f is order conjugate to g if and only if both graph(f) and graph(g) are on the same side of the diagonal.*

In particular,

(i) If $f(x) > x$ for all $x \in (a,b)$ then f is order conjugate to $x+1$.

(ii) If $f(x) < x$ for all $x \in (a,b)$ then f is order conjugate to $x-1$.

Remark 2.26. In fact, in the previous corollary, the interval (a,b) can be replaced by any open ray in $\mathbb{R}$. For an increasing bijection $f : \mathbb{R} \to \mathbb{R}$ with finitely many non-ordinary points, all non-ordinary points are fixed points.

3. Class of continuous maps

We now consider the systems for which there are only finitely many equivalence classes. Recall that, if S_P denote the set of all points having the dynamical property P then the points of ∂S_P (the boundary of S_P) are non-ordinary. As in Remark 2.12, being a point in a particular equivalence class is a dynamical property of the point. Hence by the very nature of the order conjugacies, it follows that when there are finitely many non-ordinary points (therefore special points) there are only finitely many equivalence classes. Hence if $f : \mathbb{R} \to \mathbb{R}$ is an increasing bijections with finitely many non-ordinary points $x_1 < x_2 < ... < x_n$ for some $n \in \mathbb{N}$, then these points gives rise to an ordered partition $\{(-\infty, x_1), (x_1, x_2), ..., (x_n, \infty)\}$ of $\mathbb{R} \setminus \{x_1, x_2, ..., x_n\}$. This partition gives rise to a word $w(f)$ over $\{A, B, O\}$ of length $n+1$ by associating A to $f(t) > t \, \forall \, t$, B to $f(t) < t \, \forall \, t$ and O to $f(t) = t \, \forall \, t$. We now study, the class of simple systems induced by continuous maps having finitely many non-ordinary points.

We first state the following results without proof. For a proof refer [2]. These results are easy to prove and it will be used to prove our main theorems.

Proposition 3.1. *Let f, g be two increasing bijections on $\mathbb{R}$ with finitely many (same number of) non-ordinary points. Then f and g are order conjugate if and only if $w(f) = w(g)$.*

Proposition 3.2. *There is a one to one correspondence between the set of all increasing continuous bijections (up to order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points and the set of all words of length $n+1$ on three symbols **A,B,O** such that **OO** is forbidden.*

Theorem 3.3. *The number of all increasing continuous bijections (up to order conjugacy) on $\mathbb{R}$ with exactly n non-ordinary points is equal to a_n where $a_n = C_1(1+\sqrt{3})^n + C_2(1-\sqrt{3})^n$, $C_1 = \frac{(3\sqrt{3}+5)}{2\sqrt{3}}$ and $C_2 = \frac{(3\sqrt{3}-5)}{2\sqrt{3}}$.*

Proposition 3.4. *Two decreasing bijections f and g are order conjugate (respectively topologically conjugate) if and only if $f^2|_{[a,\infty)}$ and $g^2|_{[b,\infty)}$ are order conjugate (respectively topologically conjugate) where a and b are the fixed points of f and g respectively.*

Proposition 3.5. *Two decreasing bijections f and g are order conjugate (respectively topologically conjugate) if and only if $f^2|_{(-\infty,a]}$ and $g^2|_{(-\infty,b]}$ are order conjugate (respectively topologically conjugate) where a and b are the fixed points of f and g respectively.*

Proposition 3.6. *If f is a decreasing bijection from $\mathbb{R}$ to $\mathbb{R}$ with fixed point a. Then f has $2n+1$ non-ordinary points if and only if $(f\circ f)|_{(a,\infty)} : (a,\infty) \to (a,\infty)$ has n non-ordinary points.*

Theorem 3.7. *If s_n denotes the number of decreasing homeomorphisms up to order conjugacy, then*

$$s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ a_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

for all n.

In subsections 3.1 and 3.2, we discuss the class of continuous maps with unique (respectively two) non-ordinary points. In the Subsection 3.3, we consider the class of continuous maps with finitely many non-ordinary points.

3.1. Class of continuous maps having unique non-ordinary point.

We consider the following propositions that will help us to prove our main theorem.

Proposition 3.8. *Let f, $g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.
(2) $f|_{(0,\infty)}$, $g|_{(0,\infty)} : (0,\infty) \to (0,\infty)$ *are increasing bijections.*
(3) $f|_{(-\infty,0)}$, $g|_{(-\infty,0)} : (-\infty,0) \to (0,\infty)$ *are decreasing bijections.*

Then f is order conjugate to g if and only if $f|_{(0,\infty)}$ is order conjugate to $g|_{(0,\infty)}$.

Proof. Suppose $h : (0,\infty) \to (0,\infty)$ is an order conjugacy from $f|_{(0,\infty)}$ to $g|_{(0,\infty)}$. For $x < 0$, define $h(x) = (g|_{(-\infty,0)})^{-1}hf(x)$, and $h(0) = 0$. □

Proposition 3.9. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that*

(1) $f(0) = g(0) = 0$.
(2) $f|_{(-\infty,0)}$, $g|_{(-\infty,0)} : (-\infty,0) \to (-\infty,0)$ *are increasing bijections.*
(3) $f|_{(0,\infty)}$, $g|_{(0,\infty)} : (0,\infty) \to (-\infty,0)$ *are decreasing bijections.*

Then f is order conjugate to g if and only if $f|_{(-\infty,0)}$ is order conjugate to $g|_{(-\infty,0)}$.

Proof. Suppose $h : (-\infty,0) \to (-\infty,0)$ is an order conjugacy from $f|_{(-\infty,0)}$ to $g|_{(-\infty,0)}$. For $x > 0$, define $h(x) = (g|_{(0,\infty)})^{-1}hf(x)$, and $h(0) = 0$. □

We denote the symbol $\biguplus$ for disjoint union.

Proposition 3.10.

(1) *Let* $f : (-\infty, 0] \to (-\infty, 0]$ *be an increasing bijection (it follows that* $f(0) = 0$*).*
   (a) *If* $f(x) > x$ *for all* $x \in (-\infty, 0)$ *then* f *is order conjugate to* $\frac{x}{2}$.
   (b) *If* $f(x) < x$ *for all* $x \in (-\infty, 0)$ *then* f *is order conjugate to* $2x$.

(2) *Let* $f : [0, \infty) \to [0, \infty)$ *be an increasing bijection (it follows that* $f(0) = 0$*).*
   (a) *If* $f(x) > x$ *for all* $x \in (0, \infty)$ *then* f *is order conjugate to* $2x$.
   (b) *If* $f(x) < x$ *for all* $x \in (0, \infty)$ *then* f *is order conjugate to* $\frac{x}{2}$.

(3) *Let* $f : [1, \infty) \to [1, \infty)$ *be an increasing bijection (it follows that* $f(1) = 1$*).*
   (a) *If* $f(x) > x$ *for all* $x \in (1, \infty)$ *then* f *is order conjugate to* $2x - 1$.
   (b) *If* $f(x) < x$ *for all* $x \in (1, \infty)$ *then* f *is order conjugate to* $\frac{x+1}{2}$.

Proof. (1). **Proof of (a):**

Let $f : (-\infty, 0] \to (-\infty, 0]$ be an increasing bijection satisfying $f(x) > x$ for all $x < 0$. It follows that $f(0) = 0$. Note that for any such map $\biguplus_{n\in\mathbb{Z}}[f^n(x), f^{n+1}(x)) = (-\infty, 0)$ for all points $x \in (-\infty, 0)$. Then f is topologically conjugate to the map $x/2$. We construct a topological conjugacy $h : (-\infty, 0] \to (-\infty, 0]$. For this, take any point other than 0, say -1, in the domain. We take an arbitrary increasing homeomorphism h from $[-1, f(-1))$ to $[-1, -1/2)$. Then as noted above, $\biguplus_{n\in\mathbb{Z}}[f^n(-1), f^{n+1}(-1)) = (-\infty, 0)$. That is, for every $x \in (-\infty, 0)$, there exists a unique $n_0 \in \mathbb{Z}$ such that $f^{n_0}(x) \in [-1, f(-1))$. We define $h(x) = 2^{n_0} h(f^{n_0}(x))$. This is well defined. It is an increasing homeomorphism from $(-\infty, 0)$ to $(-\infty, 0)$. This h commutes with f. This h is a conjugacy from f to the map $x/2$.

Similarly, we can prove (1).b, (2) and (3). □

Theorem 3.11 ([1])**.** *There are exactly* 26 *maps on* $\mathbb{R}$ *with a unique non-ordinary point, up to order conjugacy.*

Proof. By Corollary 3.15, Propositions 3.8, 3.9, 3.10 (1), and 3.10 (2), the theorem follows. □

Definition 3.12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. Let $I \subset \mathbb{R}$ be an interval such that $f^k(I) = I$, and $f^m(I) \neq I$ for all $1 \leq m < k$. Then we say that $I \to f(I) \to f^2(I) \to ... \to f^{k-1}(I) \to I$ is a k-cycle through I. Cycles through I, J are said to be distinct if $f^m(I) \neq f^n(J)$ for all $m, n \in \mathbb{N}$.

We can represent each member of this class as a graph as follows:

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with a unique non-ordinary point. Now, let I_1, I_2 be non singleton equivalence classes such that $I_1 < I_2$. Define a labeled digraph (G, V_f) with vertex set $V_f = \{I_1, I_2\}$, an edge from I_j to I_m if $f(I_j) = I_m$, and a symbol I (respectively D) on this edge whenever f is increasing (respectively decreasing) on I_j. Label the symbols A, B, O on the least vertex of each k-cycle depends on the graph of f^k, $k = 1, 2$ which is above the diagonal or below the diagonal or on the diagonal respectively.

Note: If f is a decreasing homeomorphism then we can consider the labeled digraph of f^2 instead of the labeled digraph f (see Proposition 3.4).

3.2. Class of continuous maps having exactly two non-ordinary points.

Next we consider the following proposition.

Proposition 3.13. *Let $f : \mathbb{R} \to \mathbb{R}, g : \mathbb{R} \to \mathbb{R}$ be two continuous maps having finitely many non-ordinary points such that $N(f) = N(g)$ and let $N(f)^c = \biguplus I_k$. For $m, n \in \mathbb{N}$, there exist increasing bijections $h_n : \bar{I}_n \to \bar{I}_n$ and $h_m : \bar{I}_m \to \bar{I}_m$ such that $g \circ h_n(x) = h_m \circ f(x)$ for all $x \in \bar{I}_n$ whenever $f(\bar{I}_n) = \bar{I}_m$ and $g(\bar{I}_n) = \bar{I}_m$ then $f \sim g$.*

Proof. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = h_n(x)$$

for all $x \in \bar{I}_n$ and $n \in \mathbb{N}$. Then h is an increasing bijection such that $h \circ f = g \circ h$. Hence the proposition. □

Note that if a continuous bijection has finitely many non-ordinary points then we can assume that these points are arbitrary. Because, let $a_1 < a_2 < ... < a_n$ be the non-ordinary points of a continuous map $f : \mathbb{R} \to \mathbb{R}$. Let $b_1 < b_2 < ... < b_n$ be arbitrary points in $\mathbb{R}$. Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing bijection such that $h([a_i, a_{i+1}]) = [b_i, b_{i+1}]$ for $i = 1, 2, ..., n-1$. Then define $g = h \circ f \circ h^{-1}$. We can easily verify $b_1, b_2, ..., b_n$ are the only non-ordinary points of g since h is an order conjugacy from f to g.

The following theorem help us to classify the class of continuous maps having finitely many non-ordinary points.

Theorem 3.14. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be two continuous maps having finitely many non-ordinary points such that $N(f) = N(g)$. Let $N(f)^c = \biguplus_n I_n$.*

1. *If f and g have the same type of monotonicity (i.e., either both f and g are increasing or both are decreasing) in the closure of each equivalence class and contains exactly n distinct cycles of length k_i through $I_{j_{k_i}}$ for $i = 1, 2, ..., n$ and for some j_{k_i}. Then $f \sim g$ whenever graph $(g^{k_i}|_{\bar{I}_{j_{k_i}}})$ and graph $(g^{k_i}|_{\bar{I}_{j_{k_i}}})$ are in the same side of the diagonal.*
2. *If f and g have the same type of monotonicity in the closure of each equivalence class and does not contain any cycle then $f \sim g$.*

Proof. For simplicity we consider the case whenever f and g have exactly one cycle through its equivalence classes.

Let f and g have the same type of monotonicity in the closure of each equivalence class and contain exactly one cycle through its equivalence classes (say of length k and through their equivalence class I_j for some j).

Claim: $f \sim g$ whenever graph $(f^k|_{\bar{I}_j})$ and graph $(g^k|_{\bar{I}_j})$ are same side of the diagonal.

Let $I_j = J_1 \to J_2 = f(J_1) \to ... \to J_k = f^{k-1}(J_1) \to J_1$ be the k-cycle. Given that graph $(f^k|\bar{I}_j)$ and graph $(g^k|\bar{I}_j)$ are same side of the diagonal. Then there exists an increasing bijection $h : \bar{J}_1 \to \bar{J}_1$ such that

$$f^k \circ h = h \circ g^k \tag{3.1}$$

Choose $h_1 = h$. Find $h_i : \bar{J}_i \to \bar{J}_i$ for $i = 2, 3, ..., k$ and $h_1' : \bar{J}_1 \to \bar{J}_1$ such that $f|_{\bar{J}_i} \circ h_i = h_{i+1} \circ g|_{\bar{J}_i}$ and $f \circ h_k = h_1' \circ g$. Recursively, we can prove that $h_1' = f^k \circ h_1 \circ g^{-k}$. This implies $h_1' = h$ by equation (3.1). Define $s_m(x) = x$ for all $x \in I_m$ whenever f, g are constant on I_m. In all the other equivalence classes choose h arbitrary and define h' (or vice versa) such that $f \circ h = h' \circ g$. This gives a well defined $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = s_n(x)$ where $s_n : I_n \to I_n$ is a continuous increasing bijection obtained as above such that $f \circ s_n = s_m \circ g$. This implies $f \circ h = h \circ g$. Similarly we can prove the general case of (1) and the case (2). □

Corollary 3.15. *Let f, g be two increasing bijections having finitely many fixed points such that $Fix(f) = Fix(g)$ and let $Fix(f)^c = \biguplus I_n$. If $f|_{\bar{I}_n} \sim g|_{\bar{I}_n}$ for every n then $f \sim g$.*

Remark 3.16. Note that the complement of $Fix(f)$ is a countable union of open intervals (including rays) whose end points are fixed points. Since f is increasing and the end points are fixed, no point in a component interval can be mapped to a point in any other component interval by f.

Definition 3.17. Let $A, B \subset \mathbb{R}$, $A \neq B$, and $f, g : A \to B$ be continuous maps. We say that f is order conjugate to g if there exist increasing bijections $h_f : A \to A$ and $h_g : B \to B$ such that $f \circ h_f = h_g \circ g$.

Now we consider:

Proposition 3.18.

(1) *Let $f : (-\infty, 0] \to [0, 1)$ be a decreasing bijection (it follows that $f(0) = 0$). Then f is order conjugate to $\frac{-x}{-x+1}$.*
(2) *Let $f : [1, \infty) \to [0, 1)$ be an increasing bijection (it follows that $f(1) = 0$). Then f is order conjugate to $\frac{x-1}{x}$.*
(3) *Let $f : [1, \infty) \to (0, 1]$ be a decreasing bijection (it follows that $f(1) = 1$). Then f is order conjugate to $\frac{1}{x}$.*
(4) *Let $f : [1, \infty) \to (-\infty, 0]$ be a decreasing bijection (it follows that $f(1) = 0$). Then f is order conjugate to $1 - x$.*
(5) *Let $f : (-\infty, 0] \to (0, 1]$ be an increasing bijection (it follows that $f(0) = 1$). Then f is order conjugate to $\frac{-1}{x-1}$.*

Proof. This Proposition easily follows from the following fact.

Let $A, B \subset \mathbb{R}$ be intervals such that either both $f, g : A \to B$ are decreasing bijections or both are increasing bijections. Then there exist increasing bijections $h_f : A \to A$ and $h_g : B \to B$ such that $f \circ h_f = h_g \circ g$. This is because we can always take an arbitrary h_f and define $h_g = f \circ h_f \circ g^{-1}$. □

Next we consider:

Proposition 3.19. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with exactly two non-ordinary points* $0, 1$ *such that*

1. $f(0) = 0$.
2. *Either* $f|_{[1,\infty)} : [1, \infty) \to [1, \infty)$ *and it is an increasing bijection or* $f|_{[1,\infty)} : [1, \infty) \to (0, 1]$ *and it is a decreasing bijection.*
3. *Either* $f|_{(-\infty,0]} : (-\infty, 0] \to (-\infty, 0]$ *and it is an increasing bijection or* $f|_{(-\infty,0]} : (-\infty, 0] \to [0, 1)$ *and it is a decreasing bijection.*

Then there are only 48 *such maps up to order conjugacy.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with exactly two non-ordinary points $0, 1$ such that it satisfies the three conditions mentioned in the statement of Proposition 3.19. Then these non-ordinary points provides an ordered partition $\{(-\infty, 0), (0, 1), (1, \infty)\}$ of $\mathbb{R} \setminus \{0, 1\}$. Now by Proposition 2.19, Remark 2.20, Proposition 3.10, Theorem 3.14, and Proposition 3.18, the map $f : \mathbb{R} \to \mathbb{R}$ has exactly four choices in the interval $(-\infty, 0)$, exactly three choices in the interval $(0, 1)$, and exactly five choices in the interval $(1, \infty)$ up to order conjugacy. But because of the assumed conditions there are only 48 maps up to order conjugacy. See figure 1 (a). □

Remark 3.20. There are sixty eight maps (up to order conjugacy) having exactly two non-ordinary points $0, 1$ such that both are fixed.

Proof. By Corollary 2.18, if $f : \mathbb{R} \to \mathbb{R}$ be constant in a neighbourhood of a point x_0 then the end points of the maximal interval around x_0 on which f is constant, are non-ordinary. Hence by the arguments involved in the proof of Proposition 3.19, this remark follows. See figure 1 (a). □

Now we have:

Proposition 3.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with exactly two non-ordinary points* $0, 1$ *such that*

1. $f(1) = 1$.
2. *Either* $f|_{[1,\infty)} : [1, \infty) \to [1, \infty)$ *and is an increasing bijection or* $f|_{[1,\infty)} : [1, \infty) \to (0, 1]$ *and is a decreasing bijection.*
3. *Either* $f|_{(-\infty,0]} : (-\infty, 0] \to (-\infty, 0]$ *and is a decreasing bijection or* $f|_{(-\infty,0]} : (-\infty, 0] \to (0, 1]$ *and is a increasing bijection.*

Then there are only 8 *such maps up to order conjugacy.*

Proof. Proof follows from Theorem 3.14, Remark 2.20 and Propositions 3.10, 3.18 and 2.19 (see figure 1(b)). □

Remark 3.22. There are eleven maps on $\mathbb{R}$ (up to order conjugacy) having exactly two non-ordinary points $0, 1$ such that 1 is a fixed point and the image of 0 is 1.

Proof. This remark follows from Corollary 2.18 together with the results used for the proof of the Proposition 3.21 (see figure 1(b)). □

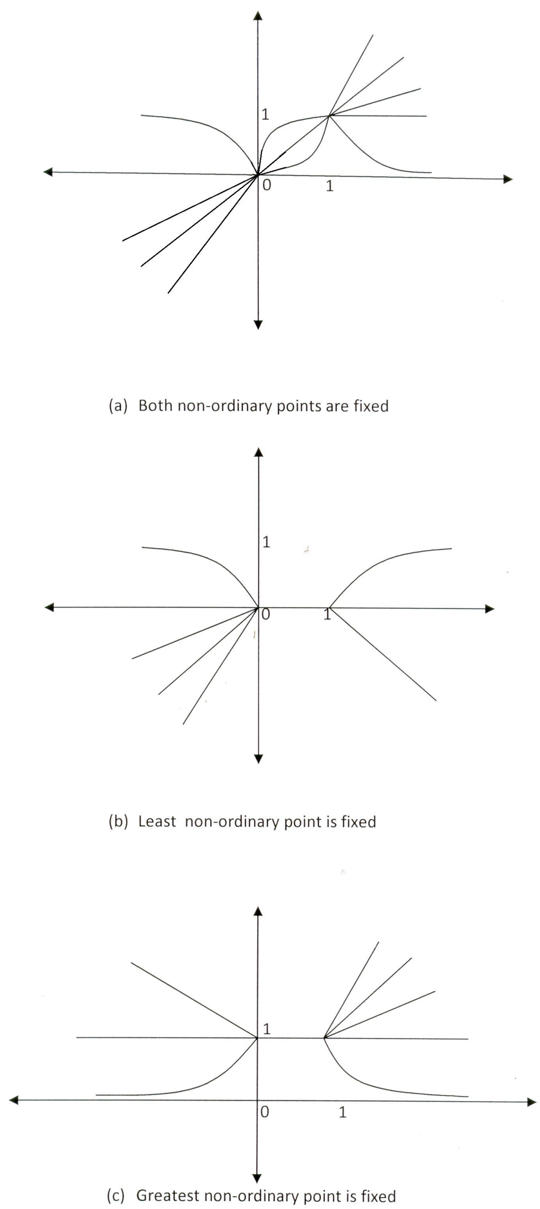

FIGURE 1. Maps with exactly two non-ordinary points

Next we have:

Proposition 3.23. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map with exactly two non-ordinary points* $0, 1$ *such that*

(1) $f(0) = g(0) = 0$.
(2) *Either* $f|_{(-\infty,0]} : (-\infty, 0] \to (0, 1]$ *and is a decreasing bijection or* $f|_{(-\infty,0]} : (-\infty, 0] \to (-\infty, 0]$ *and is an increasing bijection.*
(3) *Either* $f|_{[1,\infty)} : [1, \infty) \to (-\infty, 0]$ *and is a decreasing bijection or* $f|_{[1,\infty)} : [1, \infty) \to [0, 1)$ *and is an increasing bijection.*

Then there are only 8 *such maps up to order conjugacy.*

Proof. Proof follows from Theorem 3.14, Remark 2.20 and Propositions 3.10, 3.18 and 2.19 (see figure 1 (c)). □

Remark 3.24. There are eleven continuous maps on $\mathbb{R}$ (up to order conjugacy) having exactly two non-ordinary points $0, 1$ such that 0 is a fixed point and the image of 1 is 0.

Proof. This remark follows from Corollary 2.18 together with results used for the proof of the Proposition 3.23 (see figure 1 (c)). □

Now we are ready to consider our first main theorem:

Theorem 3.25 (Main Theorem 1)**.** *There are exactly* 90 *continuous maps on* $\mathbb{R}$ *with exactly two non-ordinary points, up to order conjugacy.*

Proof. Let a and b be the two non-ordinary points such that $a < b$. Then $\{a, b\}$ is invariant under f by Proposition 2.15. Hence at least one of these two points is a fixed point; the other is either a fixed point or goes to a fixed point.

Case 1: Both a and b are fixed points.

Without loss of generality we can assume that $a = 0$ and $b = 1$. This is because, let f be a continuous map having only two non-ordinary points a and b. Let $h : \mathbb{R} \to \mathbb{R}$ be an increasing bijection such that $h([a, b]) = [0, 1]$. Then consider $g = hfh^{-1}$. Then $g(0) = 0$ and $g(1) = 1$. From Remark 3.20, it follows that that there are only 68 continuous maps of this type up to order conjugacy.

Case 2: $f(a) = a = f(b)$.

Without loss of generality we can assume that $a = 0,\ b = 1$ (a similar proof as in Case 1 will work). From Remark 3.24, it follows that there are only 11 maps of this type up to order conjugacy.

Case 3. $f(b) = b = f(a)$.

Without loss of generality we can assume that $a = 0$ and $b = 1$ (a similar proof as in Case 1 will work). From Remark 3.22, it follows that there are only 11 maps of this type up to order conjugacy.

Hence the proof. □

Remark 3.26. From Corollary 2.18, Remark 2.20, Theorem 3.14 and Propositions 3.10, 3.18 and 2.19, there are 16 somewhere constant continuous maps up to order conjugacy such that interval of constancy is bounded, 31 somewhere constant continuous maps up to order conjugacy such that the interval of constancy is unbounded, 18 nowhere constant continuous maps up to order conjugacy with unique critical point (among them 9 maps having unique critical point as a local maximum, remaining 9 maps having unique critical point as a local minimum), 3 continuous maps up to order conjugacy with exactly two critical points, and 22 continuous maps up to order conjugacy with no critical points. Hence there are exactly 90 continuous maps (up to order conjugacy) on $\mathbb{R}$ with having exactly two non-ordinary points. It gives another way of counting involved in Theorem 3.25.

We can represent each member of this class as a digraph as follows.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous map having exactly two non-ordinary points. Let I_1, I_2, I_3 be non-singleton equivalence classes such that $I_1 < I_2 < I_3$, and define a graph (G, V_f) with vertex set $V_f = \{I_1, I_2, I_3\}$, and an edge from I_j to I_m if $f(I_j) = I_m$, and a symbol I (respectively D) on this edge whenever f is increasing (respectively decreasing) on I_j for $j = 1, 2, 3$. If there is a k-cycle, label one of the symbols A, B, O in the least vertex (say J_1) of the cycle depends on the $graph(f^k|_{J_1})$ is above the diagonal or below the diagonal or on the diagonal respectively.

Note: If f is a decreasing homeomorphism then we can consider the labeled digraph of f^2 instead of the labeled digraph of f because of Proposition 3.4.

Example 3.27. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 2x & \text{if } x \leq 0 \\ x^2 & \text{if } 0 < x \leq 1 \\ 2x - 1 & \text{if } x > 1 \end{cases}$$

The map $f : \mathbb{R} \to \mathbb{R}$ is continuous with exactly two non-ordinary points 0 and 1. The labeled digraph associated to the map f has three vertices I_1, I_2, I_3 with one loop at each vertex with an edge labeling I on each loop. The additional symbol labeled to the vertices I_1, I_2 and I_3 are B, A and A respectively. The associated graph does not have any other edge.

Example 3.28.

$$f(x) = \begin{cases} -2x + 1 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x \leq 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$$

The map $f : \mathbb{R} \to \mathbb{R}$ is continuous with exactly two non-ordinary points 0 and 1. The labeled digraph associated to the map $f : \mathbb{R} \to \mathbb{R}$ has three vertices I_1, I_2, I_3 with a directed edge from I_3 to I_2 with edge labeling D. The vertex I_1 has an additional symbol labeled as A and the remaining vertices does not have any other additional label. The associated graph does not have any other edge.

3.3. Class of continuous maps having finitely many non-ordinary points.

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous map from this class. Then the set of all non-ordinary points is invariant by Proposition 2.15. By Corollary 2.18, the end points of the maximal interval around every point on which f is constant are non-ordinary. By Proposition 2.19, if $x\in\mathbb{R}$ is both critical and ordinary then f is locally constant at x, and if x is ordinary then so is $f(x)$ whenever f is not locally constant in a neighbourhood of x. Also note that $lim_{x\to-\infty}f(x)$ and $lim_{x\to+\infty}f(x)$ are non-ordinary whenever f has only finitely many non-ordinary points. Hence the following informations will help us to characterizes the set of all continuous maps from $\mathbb{R}$ to $\mathbb{R}$ having finitely many non-ordinary points (see Theorem 3.32).

(1) graph(f) is above the diagonal or below the diagonal or on the diagonal on each equivalence class (a,b) if $f|_{(a,b)}$ is increasing, and $f(a)=a$, $f(b)=b$.
Related to this we will assign the symbols A,B,O on the vertex (a,b) of the labeled digraph of $(\mathbb{R},f)$ depends on the graph$(f|_{(a,b)})$ is above the diagonal or below the diagonal or on the diagonal respectively.

(2) Increasing or decreasing or constant on each equivalence class.
Related to this we will assign symbols I,D on the edge from the equivalence class to itself of the labeled digraph of $(\mathbb{R},f)$ depends on the graph (f) on corresponding equivalence class is increasing or decreasing respectively.

(3) If $f^k(I)=I$, $f^m(I)\neq I$, $m<k$ then consider $f^k|_J$ and ask (1) for least $J\in\{I,f(I),...,f^{k-1}(I)\}$
Related to this we will assign the symbols A,B,O on the vertex J of the labeled digraph of $(\mathbb{R},f)$ depends on the graph $(f^k|_J)$ is above the diagonal or below the diagonal or on the diagonal respectively.

Now we introduce some labeled digraph for each $(\mathbb{R},f)$ as follows:

Let $I_1,I_2,...,I_n$ be non-singleton equivalence classes such that $I_1<I_2<...<I_n$. Define a graph (G,V_f) with vertex set $V_f=\{I_1,I_2,...,I_n\}$, and define an edge from I_j to I_k if $f(I_j)=I_k$. But this graph would not give full information of the dynamical system $(\mathbb{R},f)$. To achieve this, we give more labels on each edge on the graph of the map, and on the least vertex of each cycle(see (1), (2) and (3) for details). Observe that, if f,g are order conjugate then the associated labeled digraph should be isomorphic.

Note: If f is a decreasing homeomorphism having odd number of non-ordinary points then we can consider the graph of f^2 instead of f because of Proposition 3.4.

Image of each non-trivial equivalence Class

Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous map having $n-1$ non-ordinary points, and $I_1<I_2<...<I_n$ be the n non-singleton equivalence classes of f. Then $f(I_1)=I_m$ for some $m\in\{1,2,...,n\}$ or a constant; and $f(I_2)=I_{m+1}$ or a constant if $m=1$, and I_m or I_{m+1} or a constant if $m>1$. In general for

$3 \leq k \leq n-2$, $f(I_k) = I_l$ or I_{l-1} or I_{l+1} or a constant, l depends on m; and $f(I_n) = I_j$ or a constant; and $f(I_{n-1}) = I_{j-1}$ or a constant if $j = n$, and I_{j-1} or I_j or a constant if $j < n$. Note that j depends on m.

Note: This information gives the possible choices of edge sets for the assigned labeled digraph.

Now we consider the following definitions:

Definition 3.29. A graph isomorphism between two graphs G and H can be defined as a bijection $f : V_G \to V_H$ that such that a pair of vertices u, v is adjacent in V_G if and only if the image pair $f(u), f(v)$ is adjacent in V_H. In full generality, a graph isomorphism $f : G \to H$ is a pair of bijections $f_V : V_G \to V_H$ and $f_E : E_G \to E_H$ such that for every edge $e \in E_G$, the endpoints of e are mapped onto the endpoints of $f_E(e)$. A digraph isomorphism is an isomorphism of the underlying graphs such that the edge correspondence preserves all edge directions. A labeled digraph isomorphism is an isomorphism of the underlying digraphs such that the correspondence preserves labeling. Two graphs are isomorphic if there is an isomorphism from one to the other, or informally, if their mathematical structures are identical.

Definition 3.30. Let $(S, \leq_S)$, $(T, \leq_T)$ be two partially ordered sets. An *order isomorphism* from $(S, \leq_S)$ to $(T, \leq_T)$ is a surjective map $h : S \to T$ such that for all u and v in S, $h(u) \leq_T h(v)$ if and only if $u \leq_S v$. In this case, the posets S and T are said to be order isomorphic. All surjective order isomorphisms are bijective (see [3]).

Definition 3.31. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map. A subset of $\mathbb{R}$ is said to be a *dynamically independent* set if any two points of the set have disjoint orbits. A subset of $\mathbb{R}$ is said to be *maximal dynamically independent* if it is dynamically independent and no other super set is dynamically independent.

Now we are ready to prove our second main theorem.

Theorem 3.32 (Main Theorem 2)**.** *Let $\mathcal{C}$ be the class of all continuous self maps of $\mathbb{R}$, having finitely many non-ordinary points. Then*

(1) *for every member of $\mathcal{C}$, there exists a maximal dynamically independent set that is a finite union of intervals.*
(2) *two members of $\mathcal{C}$ are order conjugate if and only if they have order isomorphic maximal dynamically independent set as in (1), and with isomorphic labeled digraphs.*
(3) *every order isomorphism between such maximal dynamically independent set as in* (1) *extends uniquely to an order conjugacy.*

Proof. Let $\mathcal{C}$ be the class of all continuous self maps of $\mathbb{R}$, having finitely many non-ordinary points.

(1) Let $f \in \mathcal{C}$ and let $z_0 = x_1 < x_2 < ... < x_n$ be the n non-ordinary points of f. Then $I_1 = (-\infty, x_1), I_{n+1} = (x_n, \infty)$ and $I_i = (x_i, x_{i+1})$, $i = 1, ..., n-1$ be the $n+1$ non-singleton equivalence classes by Lemma

2.11. Choose $y_i \in I_i$ whenever f is not a constant on I_i for $i = 1, 2, ..., n$; and let J be the set of all i such that y_i has been chosen. Let z_1 be the least x_i not in $O(z_0)$. Define, inductively, z_{i+1} as the least x_i not in $O(z_i)$. Let Z be the the set of all elements such that $z_{i+1} \notin O(z_i)$ (it may be empty but always finite). Define $y_{i+1} = f^{k_i}(y_i)$ if k_i is least such that $f^{k_i}(I_i) = I_i$ for $i \in \mathrm{J}$. Then consider $\mathcal{M} = \bigcup_{i\in \mathrm{J}}(y_i, y_{i+1}) \cup Z$. Then $\mathcal{M}$ is a maximal dynamically independent set. Note that this $\mathcal{M}$ is always non-empty.

(2) First part is easy because having maximal dynamically independent set is invariant under order conjugacy. ie., if $\mathcal{M}_1$ is a maximal dynamically independent set of f. Then $h(\mathcal{M}_1)$ is a maximal dynamically independent set of g whenever h is an order conjugacy from f to g.

Conversely, let $f, g \in \mathcal{C}$ have order isomorphic maximal dynamically independent set as in (1) (say $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively), and with isomorphic labeled digraphs. Consider all non-empty intersection of each equivalence classes of f with $\mathcal{M}_1$ and g with $\mathcal{M}_2$. Observe that there is a one to one correspondence between these intersections. Because of maximal dynamical independency, we can extend restriction of the order isomorphism on these sets to a homeomorphism on its corresponding equivalence class. By a similar proof as in Theorem 3.14 we can extend it to a unique conjugacy from f to g since f and g have isomorphic labeled digraphs and order isomorphic maximal dynamically independent sets.

(3) Easily follows from (2).

□

4. Summary

For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the class of all continuous self maps of $\mathbb{R}$ with finitely many non-ordinary points up to order conjugacy. For each $n \in \mathbb{N}$, with the help of Theorem 3.32, we can determine the cardinality of $\mathcal{C}_n$. Note that the cardinality of $\mathcal{C}_1$ is 26 and the cardinality of $\mathcal{C}_2$ is 90. Our main theorem characterizes the class of all continuous self maps of $\mathbb{R}$ with finitely many non-ordinary points up to order conjugacy.

Acknowledgements. *The author is very thankful to the referee for giving valuable suggestions. The author acknowledges SERB-MATRICS Grant No. MTR/2018/000256 for financial support.*

References

[1] K. Ali Akbar, Some results in linear, symbolic, and general topological dynamics, Ph. D. Thesis, University of Hyderabad (2010).
[2] K. Ali Akbar, V. Kannan and I. Subramania Pillai, Simple dynamical systems, Applied General Topology 2, no. 2 (2019), 307–324.
[3] A. Brown and C. Pearcy, An introduction to analysis (Graduate Texts in Mathematics), Springer-Verlag, New York, 1995.
[4] R. A. Holmgren, A first course in discrete dynamical systems, Springer-Verlag, NewYork, 1996.
[5] S. Patinkin, Transitivity implies period 6, preprint.
[6] A. N. Sharkovskii, Coexistence of cycles of a continuous map of a line into itself, Ukr. Math. J. 16 (1964), 61–71.
[7] J. Smital, A chaotic function with some extremal properties, Proc. Amer. Math. Soc. 87 (1983), 54–56.
[8] B. Sankara Rao, I. Subramania Pillai and V. Kannan, The set of dynamically special points, Aequationes Mathematicae 82, no. 1-2 (2011), 81–90.