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Closure formula for ideals in intermediate rings

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Abstract

In this paper, we prove that the closure formula for ideals in $C(X)$ under m topology holds in intermediate rings also. i.e. for any ideal I in an intermediate ring with m topology, its closure is the intersection of all the maximal ideals containing I.

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1. Introduction

The m topology on $C(X)$ was defined by Hewitt in [9]. Let $C_m(X)$ denote the ring $C(X)$ equipped with m topology. $C_m(X)$ was shown to be a topological ring. In any topological ring, the closure of a proper ideal is either a proper ideal or the whole ring [8, 2M1]. Amongst other results, Hewitt in [9] showed that every maximal ideal in $C(X)$ under m topology is closed. He conjectured that every m closed ideal of $C(X)$ is an intersection of maximal ideals of $C(X)$. This conjecture was settled by Gillman, Henriksen and Jerison [7]. It was also settled independently by T.Shirota [12]. In [7](also [8, 7Q.3]), it was further shown that the closed ideals in $C^*(X)$ (under subspace m topology) coincide with the intersections of maximal ideals in $C^*(X)$ if and only if X is pseudocompact.

Intermediate rings denoted by $A(X)$, are rings of continuous functions which lie in between $C^*(X)$ and $C(X)$. These rings were studied by Donald Plank as β- subalgebras in [10]. Subsequently, a number of researchers generated renewed interests in these intermediate rings as can be seen in [11], [5], [2], [4], [3] and [1].

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Given a real number $\epsilon > 0$ and $g \in A(X)$, let $E_\epsilon(g)$ [8, 2L] denote the set $\{x \in X : |g(x)| \leq \epsilon\}$. Given $\epsilon > 0, f \in A(X)$, it is not difficult to construct a function t satisfying $ft = 1$ on the complement of $E_\epsilon(f)$. i.e. $E_\epsilon(f) \in \mathscr{Z}_A(f)\ \forall\ \epsilon > 0$. Given an ideal I in $A(X)$, let I' denote the intersection of all the maximal ideals of $A_m(X)$ that contain I. Evidently I' is closed. Let $f \in A(X)$ and $E \in Z(X)$. Then, f is said to be E^c-regular, if $\exists\ g \in A(X)$ such that $fg_{|_{E^c}} = 1$. For each $f \in A(X)$, let $\mathscr{Z}_A(f)$ denote the set $\{E \in Z(X) : f$ is $E^c - regular\}$. For an ideal I of $A(X)$, $\mathscr{Z}_A[I]$ denote the set $\bigcup_{f \in I} \mathscr{Z}_A(f)$. The set of cluster points of a z-filter $\mathscr{F}$ is denoted by $S[\mathscr{F}]$. An ideal I in $A(X)$ is said to be a β-ideal if $\mathscr{Z}_A(f) \subset \mathscr{Z}_A[I] \implies f \in I$. We shall denote intermediate rings $A(X)$ with m topology by $A_m(X)$. For undefined terms and references, we refer the reader to [8].

In this paper, we ask if Hewitt's formula for closure of an ideal holds for the case of $A_m(X)$ also. We answer this question in the affirmative, and as an outcome we obtain the result that an ideal in an intermediate ring is closed *iff* the ideal is a β-ideal.

Theorem 1.1 ([5, Theorem 3.3])**.** *Let M_A^p be the maximal ideal of $A(X)$ corresponding to the point p of βX. Then*

$$M_A^p = \{f \in A(X) : p \in S[\mathscr{Z}_A(f)]\}.$$

2. Closure formula in intermediate rings

Let $U_A(X)$ denote the set of positive units of $A(X)$. For each $f \in A(X)$ and each $u \in U_A(X)$, let $B_A(f, u)$ denote the collection $\{g \in A(X) : |f - g| < u\}$. For each $f \in A(X)$, the set $\mathscr{B}_f = \{B_A(f, u) : u \in U_A(X)\}$ forms a base for the neighborhood system at f and the topology so formed is the m topology in $A(X)$.

Definition 2.1. Let $A(X)$ be an intermediate subring. For an ideal I in $A(X)$, let $\Delta_A(I) = \{p \in \beta X : M_A^p \supset I\}$.

Theorem 2.2. *Let I be an ideal in $A(X)$ and $p \in \beta X - \Delta_A(I)$. Then, $\exists\ f \in I \cap C^*(X)$ such that $f^\beta(p) = 1$.*

Proof. Since $p \notin \Delta_A(I)$, so $M_A^p \not\supset I$. Therefore, $\exists\ g \in I$, such that $g \notin M_A^p$. So, $\exists$ a neighborhood U of p (in βX) which does not meet E, for some $E \in \mathscr{Z}_A(g)$. Now $E \in \mathscr{Z}_A(g) \implies gl_{|_{E^c}} = 1$ for some $l \in A(X)$. Let $f \in C^*(X)$ be such that $0 \leq f \leq 1$, $f^\beta(p) = 1$ and

$$f^\beta(U^c) = 0. \tag{2.1}$$

We define $h : X \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{f(x)}{(|f(x)|+1)l(x)g(x)}, \text{ if } x \in \mathrm{cl}_{\beta X} U \cap X \\ 0, \text{ if } x \in (\beta X - U) \cap X. \end{cases}$$

Then, h is well-defined and continuous. In fact $h \in A(X)$ since $h \in C^*(X)$. Moreover the definition of h shows that f is a multiple of g so that $f \in I$, which completes the proof. $\square$

Theorem 2.3. *Let Ω be an open subset of βX such that $\Omega \supset \Delta_A(I)$ for some ideal I in $A(X)$. Then, given ϵ with $0 < \epsilon < 1$, $\exists\ g \in I$ with $0 \leq g \leq 1$ such that $\Omega \cap X \supset E_\epsilon(g)$.*

Proof. Let $p \in \beta X - \Omega$. Then, $p \notin \Delta_A(I)$. By theorem 2.2 we see that $\exists\ g_p \in I \cap C^*(X)$ such that $g_p^\beta(p) = 1$. We choose an $\epsilon \in \mathbb{R}$ with $0 < \epsilon < 1$. Let

$$\Sigma_p = \{q \in \beta X : g_p^\beta(q) > \sqrt{\epsilon_0}\}.$$

Then, Σ_p is open in βX and non-empty as $p \in \Sigma_p$. Now, the collection $\{\Sigma_p : p \in \beta X - \Omega\}$ forms an open cover for the compact set $\beta X - \Omega$. Let $\{\Sigma_{p_1}, \Sigma_{p_2}, \ldots, \Sigma_{p_n}\}$ be a finite subcover of this open cover. Let $g = g_{p_1}^2 + g_{p_2}^2 + \ldots + g_{p_n}^2$. For any $p \in \beta X - \Omega$, we then have $g^\beta(p) = (g_{p_1}^\beta(p))^2 + (g_{p_2}^\beta(p))^2 + \ldots + (g_{p_n}^\beta(p))^2 > \epsilon$. Therefore, if $|g^\beta(p)| \leq \epsilon$, then $p \notin \beta X - \Omega$. i.e. $p \in \Omega$. Hence, $E_\epsilon(g) \subset \Omega \cap X$. $\square$

Definition 2.4. Let $f \in A(X)$. We say that f is ZC-related to I, if $\exists\ \epsilon > 0$, such that $Z(f) \supset C \supset E_\epsilon(g)$ for some cozero-set C and some $g \in I$.

Definition 2.5. For an ideal I of $A(X)$, we define

$$K_A(I) = \{f \in A(X) : f \text{ is } ZC\text{-related to } I\}.$$

Theorem 2.6. *For every ideal I of an intermediate subring $A_m(X)$, we have $K_A(I) \subset I$ and $cl_m(K_A(I)) = cl_m(I)$.*

Proof. Let $f \in K_A(I)$. Then, $\exists\ \epsilon > 0$ such that $Z(f) \supset C \supset E_\epsilon(g)$ for some cozero-set C and some $g \in I$. Let us denote $E_\epsilon(g)$ by E. Since $E \in \mathscr{Z}_A(g)$, $\exists\ l \in A_m(X)$ such that $(gl)_{|_{E^c}} = 1$. Now, we define h by

$$h(x) = \begin{cases} 0, \text{ if } x \in \text{cl}_X C \\ \frac{f}{(|f|+1)lg} \text{ if } x \notin C. \end{cases}$$

Then, h is a well-defined bounded function. Moreover, h is continuous. i.e. $h \in C^*(X) \subset A(X)$. Also, we get $f = h(|f| + 1)lg$, which shows that $f \in I$. Thus $K_A(I) \subset I$ and hence $\text{cl}_m(K_A(I)) \subset \text{cl}_m(I)$. To prove that $\text{cl}_m(I) \subset \text{cl}_m(K_A(I))$, it is enough to prove that $I \subset \text{cl}_m(K_A(I))$. So, we take a $g \in I$. Let $\pi \in U_A(X)$. We define f by

$$f(x) = \begin{cases} 0, \text{ if } -\frac{\pi(x)}{2} \leq g(x) \leq \frac{\pi(x)}{2} \\ g(x) - \frac{\pi(x)}{2}, \text{ if } g(x) > \frac{\pi(x)}{2} \\ g(x) + \frac{\pi(x)}{2}, \text{ if } g(x) < -\frac{\pi(x)}{2}. \end{cases}$$

Then, f lies in the π neighborhood of g. We also notice that $f \in A_m(X)$ since f may be rewritten as follows :

$$f(x) = \big[\big(g(x) - \frac{\pi(x)}{2}\big) \vee 0\big] + \big[\big(g(x) + \frac{\pi(x)}{2}\big) \wedge 0\big].$$

We shall now show that $f \in K_A(I)$. Let $C = \{x \in X\colon -\frac{\pi(x)}{2} < g(x) < \frac{\pi(x)}{2}\}$. Then $Z(f) \supset C$. Moreover, C is the cozero-set of the function $h \in A(X)$ defined by:

$$h(x) = \big(|g(x)| - \frac{\pi(x)}{2}\big) \wedge 0.$$

We choose any real number $\epsilon > 0$ and define a function θ by $\theta(x) = \frac{4\epsilon g(x)}{\pi(x)}$. Clearly, $\theta \in I$. Moreover $|\theta(x)| \leq \epsilon \iff |g(x)| \leq \frac{\pi(x)}{4}$. In otherwords, $x \in E_\epsilon(\theta) \iff |g(x)| \leq \frac{\pi(x)}{4}$. But, $|g(x)| \leq \frac{\pi(x)}{4} \implies x \in Z(f)$. Hence $Z(f) \supset C \supset E_\epsilon(\theta)$ which completes the proof. □

Example 2.7. Now, we will give an example of an ideal I such that $K_A(I) \subsetneq I$. Let $X = \mathbb{R}$ and $A(X) = C(X)$. Let $I = M_0$. We will show that $K_A(I) = O_0$. Firstly, if $f \in O_0$, then $\exists$ an open set C such that $0 \in C \subset Z(f)$. Now, $\exists\ \epsilon > 0$ such that $E = [-\epsilon, \epsilon] \subset C$. Then $E = E_\epsilon(g)$, where g is the identity map on $\mathbb{R}$. Moreover, C is a cozero-set as X is a metric space. Hence we have $f \in K_A(I)$. Secondly, if $f \in K_A(I)$, then $\exists\ g \in I$, $\epsilon > 0$ such that $Z(f) \supset C \supset E_\epsilon(g)$ for some cozero-set C. Since $0 \in E_\epsilon(g)$, this gives that $Z(f)$ is a neighborhood of 0 i.e. $f \in O_0$.

Theorem 2.8. *$k \in I' \iff S[\mathscr{Z}_A(k)] \supset \Delta_A(I)$.*

Proof. ($\Rightarrow$) We assume that $k \in I'$. Let $p \in \Delta_A[I]$. Then, $M^p_A \supset I$ and so $k \in M^p_A$. By definition of M^p_A, $p \in S[\mathscr{Z}_A(k)]$.

($\Leftarrow$) Let M^p_A be a maximal ideal which contains I. So, $p \in \Delta_A(I)$ and thus, $p \in S[\mathscr{Z}_A(k)]$. Therefore, $k \in M^p_A$ and hence $k \in I'$. □

We now prove the main result.

Theorem 2.9. *The m closure of any ideal I in $A_m(X)$ is the intersection of all the maximal ideals containing I.*

Proof. We have $\mathrm{cl}_m(I) \subset I'$ as I' is closed. To prove $I' \subset \mathrm{cl}_m(I)$, it is sufficient to prove that $K_A(I') \subset K_A(I)$. Then, by theorem 2.6, we will get $I' \subset \mathrm{cl}_m I$.

Let $f \in K_A(I')$. Then, $\exists$ a cozero-set C, a real number $\epsilon > 0$ and $\theta \in I'$ such that

$$Z(f) \supset C \supset E_\epsilon(\theta) = E(\text{say}). \tag{2.2}$$

Let $Z = X - C$. Then, Z and E are completely separated being disjoint zero-sets. Therefore, $\exists\ h \in C^*(X),\ 0 \leq h \leq 1$ such that $h(E) = 0$ and $h(Z) = 1$.

Let $\Omega = \{p \in \beta X\colon h^\beta(p) < 1\}$. We observe that $X = C \cup Z$, so $\beta X = \mathrm{cl}_{\beta X} C \cup \mathrm{cl}_{\beta X} Z$. If $p \in \Omega$, i.e. $h^\beta(p) < 1$, then $p \notin \mathrm{cl}_{\beta X} Z$ as $h^\beta(\mathrm{cl}_{\beta X} Z) = 1$. So $p \in \mathrm{cl}_{\beta X} C$.

$$\text{i.e. } \mathrm{cl}_{\beta X} C \supset \Omega. \tag{2.3}$$

Since $E \in \mathscr{Z}_A(\theta)$, therefore $\Omega \supset S[\mathscr{Z}_A(\theta)]$ because $p \in S[\mathscr{Z}_A(\theta)]$ gives $h^\beta(p) = 0$. Hence by theorem 2.8, we see that $\Omega \supset \Delta_A(I)$. Theorem 2.3 now gives a $g \in I$ with $0 \leq g \leq 1$ and some ϵ with $0 < \epsilon < 1$ such that

$$\Omega \cap X \supset E_\epsilon(g). \tag{2.4}$$

From (2.2) and (2.3), we get,

$$\mathrm{cl}_{\beta X} Z(f) \supset \mathrm{cl}_{\beta X} C \supset \Omega.$$

Then $\mathrm{cl}_{\beta X} Z(f) \cap X \supset \Omega \cap X$. Thus $Z(f) \supset \Omega \cap X$. Therefore, by (2.4) $Z(f) \supset \Omega \cap X \supset E_\epsilon(g)$. Finally, we have $\Omega \cap X$ is a co-zero-set as $\Omega \cap X = \{p \in X : h(p) < 1\}$. □

Corollary 2.10. *Every closed ideal is a β-ideal.*

Proof. First we claim that an arbitrary intersection of β-ideals is also a β-ideal. Let $\{I_\alpha : \alpha \in \Lambda\}$ be a collection of β-ideals. Let $\mathscr{Z}_A(f) \subset \mathscr{Z}_A[\bigcap_{\alpha\in\Lambda} I_\alpha]$. Since each I_α is a β-ideal, it is enough to prove that $\mathscr{Z}_A(f) \subset \mathscr{Z}_A[I_\alpha]\ \forall\ \alpha \in \Lambda$, for this would imply that $f \in I_\alpha\ \forall\ \alpha \in \Lambda$. So take $E \in \mathscr{Z}_A(f)$. Therefore $E \in \mathscr{Z}_A(g)$ for some $g \in \bigcap_{\alpha\in\Lambda} I_\alpha$. This then gives $E \in \mathscr{Z}_A[I_\alpha]\ \forall\ \alpha \in \Lambda$. Now, let I be a closed ideal in $A_m(X)$. Therefore, I is an intersection of maximal ideals. But, as every maximal ideal is a β-ideal, therefore I is an intersection of β-ideals and hence a β-ideal. □

Remark 2.11. In [6, Theorem 3.13], it was shown that the β-ideals of an intermediate ring are just the intersections of maximal ideals of the ring. This says that β-ideals are closed, since maximal ideals are closed. Hence the class of β-ideals and the class of closed ideals in intermediate rings coincide. This coincidence also occurs in the case of the subring $C^*(X)$ with m topology. Here, the class of e-ideals is the same as the class of closed ideals [8, 2M]. However, this coincidence does not extend to z-ideals in $C_m(X)$ since the ideal O^p is a z-ideal which is not closed.

Remark 2.12. In [1], it was proven that if an intermediate ring $A(X)$ is different from $C(X)$, then there exists at least one non-maximal prime ideal P in $A(X)$. Thus, P is not closed in $A_m(X)$. On the other hand if $A(X) = C(X)$ and X is a P space then each ideal in $A_m(X)$ is closed [8, 7Q4]. Thus within the class of P spaces X, for an intermediate ring $A(X)$, each ideal in $A_m(X)$ is closed $\iff A(X) = C(X)$ - this is a special property of $C(X)$ which distinguishes $C(X)$ amongst all the intermediate rings (in the category of P spaces X).

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References

[1] S. K. Acharyya and B. Bose, A correspondence between ideals and z-filters for certain rings of continuous functions - some remarks, Topology and its Applications 160, no. 13 (2013), 1603–1605.

[2] S. K. Acharyya, K. C. Chattopadhyay and D. P. Ghosh, A class of subalgebras of $C(X)$ and the associated compactness, Kyungpook Math. J. 41, no. 2 (2001), 323–324.

[3] S. K. Acharyya and D. De, An interesting class of ideals in subalgebras of $C(X)$ containing $C^*(X)$, Comment. Math. Univ. Carolin. 48, no. 2 (2007), 273–280.

[4] S. K. Acharyya and D. De, Characterization of function rings between $C^*(X)$ and $C(X)$, Kyungpook Math. J. 46 (2006), 503–507.

[5] H. L. Byun and S. Watson, Prime and maximal ideals in subrings of $C(X)$, Topology and its Applications 40 (1991), 45–62.

[6] J. M. Domínguez and J.-Gómez Pérez, Intersections of maximal ideals in algebras between $C^*(X)$ and $C(X)$, Topology and its Applications 98 (1999), 149–165.

[7] L. Gillman, M. Henriksen and M. Jerison, On a theorem of Gelfand and Kolmogoroff concerning maximal ideals in rings of continuous functions, Proc. Amer. Math. Soc. 5 (1954), 447–455.

[8] L. Gillman and M. Jerison, Rings of continuous functions, Univ. Ser. Higher Math, D. Van Nostrand Company, Inc., Princeton, N. J., 1960.

[9] E. Hewitt, Rings of real-valued continuous functions I, Trans. Amer. Math. Soc. 64, no. 1 (1948), 45–99.

[10] D. Plank, On a class of subalgebras of $C(X)$ with applications to $\beta X \setminus X$, Fund. Math. 64 (1969), 41–54.

[11] L. Redlin and S. Watson, Maximal ideals in subalgebras of $C(X)$, Proc. Amer. Math. Soc. 100, no. 4 (1987), 763–766.

[12] T. Shirota, On ideals in rings of continuous functions, Proc. Japan Acad. 30, no. 2 (1954), 85–89.

