MIXED BRUCE-ROBERTS NUMBERS

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Abstract. We extend the notion of μ^*-sequence and Tjurina number of functions to the framework of Bruce-Roberts numbers, that is, to pairs formed by the germ at 0 of a complex analytic variety $X \subseteq \mathbb{C}^n$ and a finitely $\mathcal{R}(X)$-determined analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. We analyze some fundamental properties of these numbers.

1. Introduction

Let $\mathcal{O}_n$ denote the ring of analytic function germs $(\mathbb{C}^n, 0) \to \mathbb{C}$ and let $\mathbf{m}_n$ be the maximal ideal of $\mathcal{O}_n$. If $f \in \mathcal{O}_n$ has an isolated singularity, then we denote by $\mu(f)$ the Minor number of f. That is, $\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_n/J(f)$, where $J(f) = \langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \rangle$ is the Jacobian ideal of f. If H is a general hyperplane through the origin in $\mathbb{C}^n$, then we may speak of the Milnor number of the restriction of f to H, denoted by $\mu^{(n-1)}(f)$. More generally, in [21, p. 300] Teissier introduced the sequence $\mu^*(f) = (\mu^{(n)}(f), \dots, \mu^{(1)}(f), \mu^{(0)}(f))$, where $\mu^{(i)}(f)$ denotes the Milnor number of the restriction of f to a generic linear subspace of dimension i of $\mathbb{C}^n$, for $i = 1, \dots, n$. If $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ defines a family of hypersurfaces with isolated singularities, $f_t^{-1}(0)$, where $f_t(x) = F(t, x)$, then Teissier proves that the constancy of the sequence $\mu^*(f_t)$ implies the Whitney equisingularity of the pair $(F^{-1}(0)\setminus D, D)$, where $D \subset \mathbb{C}$ is a small disc around 0 in the parameter space. In [4] Bruce and Roberts extended the notion of Milnor number to pairs formed by an analytic function $f \in \mathcal{O}_n$ and an analytic subvariety X of $\mathbb{C}^n$ (see Definition 2.2). This number, denoted $\mu_X(f)$, is called the *multiplicity of f on X* in [4]. In some references, $\mu_X(f)$ is called the *Bruce-Roberts' Milnor number of f with respect to X*. We refer to [1, 9, 18] for recent results on the relations of $\mu_X(f)$ with other classical invariants and partial results on its role on equisingularity problems in the relative case.

Let $f \in \mathcal{O}_n$ and let X denote the germ at 0 of an analytic subvariety of $\mathbb{C}^n$. This article has several purposes. We derive some consequences of the formula for $\mu_X(f)$ obtained in [18] in the case where X is a weighted homogeneous hypersurface with an isolated singularity at the origin (see Theorem 2.13). In particular, in the case where X is a linear hyperplane in $\mathbb{C}^n$, there appears a relation (see Proposition 2.18) that reminds the formula of Teissier

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saying that if $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then $\mu(f) + \mu^{(n-1)}(f)$ is equal to the Samuel multiplicity of $J(f)$ in the quotient ring $\frac{\mathcal{O}_n}{\langle f\rangle}$ (see [21, p. 322]).

Let us observe that this multiplicity is greater than or equal to $\tau(f)$, where $\tau(f)$ is the Tjurina number of f, which is defined as the colengh of the ideal $\langle f\rangle + J(f)$. By analogy with the definition of $\mu_X(f)$, in Section 3 we introduce the Bruce-Roberts' Tjurina number of f with respect to X, which we will denote by $\tau_X(f)$. We obtain an upper bound for the quotient $\frac{\mu_X(f)}{\tau_X(f)}$ and characterize the corresponding equality.

We also extend the notion of μ^*-sequence of functions to the framework of Bruce-Roberts numbers, that is, to pairs formed by the germ at 0 of a complex analytic variety $X \subseteq \mathbb{C}^n$ and a finitely $\mathcal{R}(X)$-determined analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. We analyze some of the fundamental algebraic and geometric properties of these numbers. The analogue of Teissier's result in this setting, namely, whether or not the constancy of $\mu^*_X(f_t)$ implies the Whitney equisingularity of the family of function germs with isolated singularity f_t with respect to a singular variety X, remains an open question.

2. The Bruce-Roberts' Milnor number

Let X be the germ at 0 of an analytic subvariety of $\mathbb{C}^n$ (for short, we will say that X is an analytic subvariety of $(\mathbb{C}^n, 0)$). Let $I(X)$ denote the ideal of $\mathcal{O}_n$ generated by the germs of $\mathcal{O}_n$ vanishing on X. We denote by Θ_X the $\mathcal{O}_n$-module of germs of vector fields of $\mathbb{C}^n$ at 0 which are tangent to X. That is

$$\Theta_X = \big\{\delta \in \mathcal{O}_n^n : \delta(I(X)) \subseteq I(X)\big\}.$$

This module is also usually denoted by $\mathrm{Derlog}(X)$ (see for instance [5, 6]). The elements of $\mathrm{Derlog}(X)$ are also known as *logarithmic vector fields of* X. We recall that Θ_X defines a coherent sheaf of modules in a small enough neighbourhood U of $0 \in \mathbb{C}^n$. If $x \in U$, then we denote by $\Theta_{X,x}$ the corresponding stalk at x. We also define the vector space $\Theta_X(x) = \{\delta(x) : \delta \in \Theta_X\} \subseteq \mathbb{C}^n$. We identify any given element $\delta = (\delta_1, \dots, \delta_n) \in \mathcal{O}_n^n$ with the derivation $\delta_1 \frac{\partial}{\partial x_1} + \cdots + \delta_n \frac{\partial}{\partial x_n} \in \mathrm{Der}(\mathcal{O}_n)$.

Let R denote an arbitrary ring and let M be an R-module. Given elements $u_1, \dots, u_s \in M$, we denote by $\mathrm{syz}(u_1, \dots, u_s)$ the module of syzygies of $\{u_1, \dots, u_s\}$. That is, $\mathrm{syz}(u_1, \dots, u_s)$ is the R-submodule of R^s formed by those $(g_1, \dots, g_s) \in R^s$ satisfying that $g_1u_1 + \cdots + g_su_s = 0$. Let I be an ideal of R, then we say that I is *reduced* when I is equal to its own radical.

The computation of Θ_X for general classes of varieties X is a hard problem (see Theorem 2.5). However, we will apply the following fact in order to compute Θ_X with *Singular* [7].

Lemma 2.1. *Let* $h = (h_1, \dots, h_m) : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ *be an analytic map such that the ideal* $\langle h_1, \dots, h_m\rangle$ *is reduced. Let* $X = h^{-1}(0)$. *Let* D_h *be the set of elements of* $\mathcal{O}_n^m$ *given by* *the columns of the matrix*

$$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} & h_1 & \cdots & h_m & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \frac{\partial h_2}{\partial x_1} & \cdots & \frac{\partial h_2}{\partial x_n} & 0 & \cdots & 0 & h_1 & \cdots & h_m & & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \frac{\partial h_m}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_n} & 0 & \cdots & 0 & 0 & \cdots & 0 & & h_1 & \cdots & h_m \end{bmatrix}$$

Then $\Theta_X = \pi_n(\mathrm{syz}(D_h))$*, where* $\pi_n : \mathcal{O}_n^{n+m^2} \to \mathcal{O}_n^n$ *is the projection onto the first n components.*

Proof. Let $I = \langle h_1, \dots, h_m \rangle$. Since I is reduced, given an element $\delta \in \mathcal{O}_n^n$, we have that $\delta = (\delta_1, \dots, \delta_n)$ belongs to Θ_X if and only if $\delta(h_i) \in I$, for all $i = 1, \dots, m$, which is to say that there exist $a_1^i, \dots, a_m^i \in \mathcal{O}_n$ such that $\delta_1 \frac{\partial h_i}{\partial x_1} + \cdots + \delta_n \frac{\partial h_i}{\partial x_n} = a_1^i h_1 + \cdots + a_m^i h_m$, for all $i = 1, \dots, m$. This latter condition is equivalent to saying that the element of $\mathcal{O}_n^{n+m^2}$ given by $(\delta_1, \dots, \delta_n, -a_1^1, \dots, -a_m^1, \dots, -a_1^m, \dots, -a_m^m)$ belongs to D_h. Hence the result follows. □

If $f \in \mathcal{O}_n$, then we denote by $J_X(f)$ the ideal of $\mathcal{O}_n$ generated by $\{\delta(f) : \delta \in \Theta_X\}$. In particular, we have the inclusion $J_X(f) \subseteq J(f)$.

Definition 2.2. Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define

$$\mu_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J_X(f)}. \tag{1}$$

When the colength on the right of (1) is finite, the number $\mu_X(f)$ is called the *multiplicity of f on X* in [4]. In some references, $\mu_X(f)$ is called the *Bruce-Roberts' Milnor number of f with respect to X* (see for instance [1, 9, 18]).

Let $f \in \mathcal{O}_n$. Let us remark that, if $J_X(f)$ has finite colength, then $J(f)$ has also finite colength and $\mu_X(f) \geqslant \mu(f)$, since $J_X(f) \subseteq J(f)$. We also point out that when $X = \mathbb{C}^n$, then $\Theta_X = \mathcal{O}_n^n$ and consequently $\mu_X(f) = \mu(f)$. When $X = \{0\} \subseteq \mathbb{C}^n$, then $\Theta_X = \mathbf{m}_n \oplus \cdots \oplus \mathbf{m}_n$ and hence $J_X(f) = \mathbf{m}_n J(f)$.

If $X \subseteq (\mathbb{C}^n, 0)$ is the germ at 0 of an analytic subvariety and U is a sufficiently small neighbourhood of $0 \in \mathbb{C}^n$, then in [4] Bruce and Roberts introduced the notion of logarithmic stratification of U with respect to X (see [4, Definition 1.6]), based on the analogous notion for analytic hypersurfaces of $\mathbb{C}^n$ defined by Saito in [20]. If $\{X_\alpha\}_{\alpha \in A}$ denotes this stratification, then we shall refer to $\{X \cap X_\alpha\}_{\alpha \in A}$ as the *logarithmic stratification of X*. Some of the fundamental properties of $\{X_\alpha\}_{\alpha \in A}$ is that each stratum X_α is a smooth connected immersed submanifold of U and if $x \in U$ lies in a stratum X_α, then the tangent space $T_x X_\alpha$ to X_α at x coincides with $\Theta_X(x)$. The germ X is said to be *holonomic* if for some neighbourhood U of 0 in $\mathbb{C}^n$ the logarithmic stratification of U with respect to X has only finitely many strata.

Here we recall the following result from [4, p. 64].

Theorem 2.3. [4, p. 64] *Let X be an analytic subvariety of* $(\mathbb{C}^n, 0)$ *and let* $f \in \mathcal{O}_n$*. Then the following conditions are equivalent:*

(1) *$\mu_X(f)$ is finite.*
(2) $V(J_X(f)) \subseteq \{0\}$.
(3) *f has an $\mathcal{R}(X)$-versal unfolding.*
(4) *f is finitely $\mathcal{R}(X)$-determined.*
(5) *The restriction of f to each logarithmic stratum of X is a submersion except, possibly, at 0.*

Example 2.4. Let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$ and let $f \in \mathcal{O}_3$ be given by $f(x, y, z) = xy + xz + yz$, for all $(x, y, z) \in \mathbb{C}^3$. We observe that $\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z) \rangle$. Therefore $J_X(f) = \langle xy + xz, xy + yz, xz + yz \rangle$. In particular $\mu_X(f)$ is not finite, whereas f has an isolated singularity at the origin.

If X is an analytic subvariety of $(\mathbb{C}^n, 0)$, then we say that X *supports a germ with an isolated critical point* when there exist a germ $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. In this case we also say that f *has an isolated singularity on X at* 0. As shown in [4, Theorem 3.3], if U is a sufficiently small neighbourhood of $0 \in \mathbb{C}^n$, then the germ (X, x) supports a germ with an isolated critical point for each $x \in X \cap U$ if and only if $(X, 0)$ is holonomic.

We recall that a germ of hypersurface $X \subseteq \mathbb{C}^n$ is said to be a *free divisor* when Θ_X is a free $\mathcal{O}_n$-submodule of $\mathcal{O}_n^n$ (see [5, 20]). In this case, necessarily Θ_X is generated by n elements.

Let $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ. If $p \leqslant n$, then we denote by $\mathbf{J}(g_1, \dots, g_p)$ the ideal of $\mathcal{O}_n$ generated by the minors of order p of the Jacobian matrix of g. We recall that the map g, or the set $g^{-1}(0)$, is said to be an *isolated complete intersection singularity* (or an ICIS, for short) when $p \leqslant n$, $\dim \mathbf{V}(g_1, \dots, g_p) = n - p$ and the ideal $\langle g_1, \dots, g_p \rangle + \mathbf{J}(g_1, \dots, g_p)$ has finite colength in $\mathcal{O}_n$. As recalled in Theorem 2.5, an explicit generating system for Θ_X is known when $X = g^{-1}(0)$, being $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ a weighted homogeneous ICIS.

If $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an ICIS, then we denote by $\mu(g)$ the Milnor number of g (see [11, 15, 17]). We recall that, when $p = n$, then

$$\mu(g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle g_1, \dots, g_n \rangle} - 1 \tag{2}$$

(see for instance [17, p. 78]).

Given a vector of weights $w = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\geqslant 1}$, if coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$ are fixed, then we define the *Euler vector field associated to w* as $\theta_w = w_1 x_1 \frac{\partial}{\partial x_1} + \cdots + w_n x_n \frac{\partial}{\partial x_n}$.

As pointed out in [13, p. 316], the following result is due to Aleksandrov and Kersken (see also [2, p. 467], [4, p. 79, Proposition 7.2], [22, p. 617]).

Theorem 2.5. *Let $w \in \mathbb{Z}^n_{\geqslant 1}$ and let $h = (h_1, \dots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a weighted homogeneous* ICIS *with respect to w, $n - p \geqslant 1$. Let $X = h^{-1}(0)$. Then Θ_X is generated by $\{\theta_w, h_i \frac{\partial}{\partial x_j} : i = 1, \dots, p,\ j = 1, \dots, n\}$ and the derivations given by the minors of size $p + 1$*

of the matrix

$$\begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_p}{\partial x_1} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}. \tag{3}$$

In particular, given any function $f \in \mathcal{O}_n$, we have

$$J_X(f) = \langle \theta_w(f) \rangle + \langle h_1, \dots, h_p \rangle J(f) + \mathbf{J}(f, h_1, \dots, h_p). \tag{4}$$

We recall that, whenever $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an ICIS with $n - p \geqslant 1$, then the ideal $\langle h_1, \dots, h_p \rangle$ is reduced (see [17, p. 7]).

The case $p = 1$ of Theorem 2.5 leads to a substantial simplification of Θ_X, as can be seen in [22, Proposition 1.2]. We recall this case in the following theorem (see also [12, p. 249] or [18, Theorem 2.3]).

Theorem 2.6. *Let $w \in \mathbb{Z}^n_{\geqslant 1}$ and let $h \in \mathcal{O}_n$ such that h is weighted homogeneous with respect to w and h has an isolated singularity at the origin, $n \geqslant 2$. Let $X = h^{-1}(0)$. Then Θ_X is generated by θ_w and the derivations $\theta_{ij} = \frac{\partial h}{\partial x_j}\frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i}\frac{\partial}{\partial x_j}$, for $1 \leqslant i < j \leqslant n$. Hence, for all $f \in \mathcal{O}_n$, we have*

$$J_X(f) = \langle \theta_w(f) \rangle + \mathbf{J}(f, h),$$

for all $f \in \mathcal{O}_n$.

Remark 2.7. Let us observe that, even if X is a homogeneous ICIS, a simplification of Θ_X as in Theorem 2.6 is not possible in general. For instance, let $h : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ be the map given by $h(x, y, z) = (x^2 + y^2 + z^2, xyz)$, for all $(x, y, z) \in \mathbb{C}^3$, and let $X = h^{-1}(0)$. Then, using *Singular* [7] and Lemma 2.1, it is easy to check that the eight generators of Θ_X given by Theorem 2.5 consitute a minimal generating set of Θ_X.

Given an analytic map germ $h = (h_1, \dots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and a function $f \in \mathcal{O}_n$, let us define

$$c(f, h) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle h_1, \dots, h_p \rangle + \mathbf{J}(f, h_1, \dots, h_p)}.$$

Let us recall that, by [15, Theorem 3.7.1], if the maps $(h_1, \dots, h_p)$ and $(h_1, \dots, h_p, f)$ are ICIS, then $c(f, h) < \infty$ and $\mu(h_1, \dots, h_p) + \mu(h_1, \dots, h_p, f) = c(f, h)$.

Proposition 2.8. *Let $h = (h_1, \dots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an ICIS, where $p \leqslant n - 1$, and let $f \in \mathcal{O}_n$. Let $X = h^{-1}(0)$. If $\mu_X(f) < \infty$, then $c(f, h) < \infty$.*

Proof. Let $I = \langle h_1, \dots, h_p \rangle + \mathbf{J}(f, h_1, \dots, h_p)$ and let us suppose that $\dim V(I) \geqslant 1$. Let us fix a point $x \in V(I)$, $x \neq 0$. In particular $x \in V(h_1, \dots, h_p)$. Since h is an ICIS, we can assume that not all the $p \times p$ minors of the differential matrix Dh vanish at x. Moreover, the condition $x \in V(I)$ also implies that all $(p + 1) \times (p + 1)$ minors of $D(f, h)$ vanish at x. In particular $\nabla f(x)$ is a linear combination of $\nabla h_1(x), \dots, \nabla h_p(x)$.

As indicated in Theorem 2.3, the condition $\mu_X(f) < \infty$ implies that the restriction of f to each logarithmic stratum of X is a submersion except possibly at 0. Let Y denote the logarithmic stratum of X such that $x \in Y$. Hence, there exists some non-zero $\xi \in \Theta_{X,x}$ such that $\xi(x)$ belongs to T_xY and $D(f|_Y)_x(\xi(x)) = (Df)_x(\xi(x)) \neq 0$. However, since $\nabla f(x)$ is a linear combination of $\nabla h_1(x), \dots, \nabla h_p(x)$ and $Y \subseteq V(h_1, \dots, h_p)$, it follows that $(Df)_x(\xi(x)) = \nabla f(x) \cdot \xi(x) = 0$, which is a contradiction. Therefore $\dim V(I) = 0$, that is, $c(f,h) < \infty$. □

Under the conditions of the previous result, the map $(h_1, \dots, h_p, f)$ is also an ICIS and $\mu(h_1, \dots, h_p) + \mu(h_1, \dots, h_p, f) = c(f,h)$, by the Lê-Greuel- formula.

Theorem 2.9. [4, Proposition 7.7, p. 82] *Let $w \in \mathbb{Z}^n_{\geqslant 1}$ and let $h = (h_1, \dots, h_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a weighted homogeneous ICIS with respect to w, $n - p \geqslant 1$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then the map $(f, h_1, \dots, h_p)$ is also an ICIS and its Milnor number is given by*

$$\mu(f, h_1, \dots, h_p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \theta_w(f), h_1, \dots, h_p \rangle + \mathbf{J}(f, h_1, \dots, h_p)}. \tag{5}$$

Remark 2.10. Let us observe that in the proof of the above result (see [4, p. 83]), the application of [4, Corollary 7.9] plays a fundamental role. In this proof it is essential to assume that $c(f,h) < \infty$. The original statement of [4, Proposition 7.7, p. 82] only requires the germ f to have an isolated critical point, but actually the correct hypothesis is to assume that $\mu_X(f) < \infty$, which in turn implies the condition $c(f,h) < \infty$, by Proposition 2.8.

As a direct application of Theorem 2.9 we have the following result, which maybe is already known for the specialists by means of other type of techniques.

Corollary 2.11. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin, $n \geqslant 2$. Let $i \in \{1, \dots, n-1\}$. If $h_1, \dots, h_{n-i}$ denotes a family of generic linear forms of $\mathbb{C}[x_1, \dots, x_n]$, then*

$$\mu^{(i)}(f) = \mu(f, h_1, \dots, h_{n-i}) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \theta(f), h_1, \dots, h_{n-i} \rangle + \mathbf{J}(f, h_1, \dots, h_{n-i})}$$

where $\theta(f) = x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n}$.

Proof. It is known, by the definition of Milnor number of an ICIS, that for generic linear forms $h_1, \dots, h_{n-i} \in \mathbb{C}[x_1, \dots, x_n]$, we have $\mu^{(i)}(f) = \mu(f, h_1, \dots, h_{n-i})$. Let us fix such a family of linear forms $h_1, \dots, h_{n-i}$ and let $H = \mathbf{V}(h_1, \dots, h_{n-i})$. Let us remark that $(h_1, \dots, h_{n-i}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-i}, 0)$ is smooth map germ, and hence it is a homogeneous ICIS of dimension i.

By Proposition 2.3, $\mu_H(f) < \infty$ if and only if the restriction $f|_H$ has an isolated singularity at the origin, which is the case by taking the forms $h_1, \dots, h_{n-i}$ accordingly. Thus the result follows as a direct application of Theorem 2.9. □

Because of its similitude with (5), it is worth to recall the following result of Briançon-Maynadier [3].

Theorem 2.12. [3] *Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be semi-weighted homogeneous* ICIS *with respect to w. Then $\mu(h)$ only depends on w and $d_w(h)$. Moreover $\mu(h)$ is expressed as*

$$\mu(h) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle \theta_w(h_1), \dots, \theta_w(h_p)\rangle + \mathbf{J}(h_1, \dots, h_p)}.$$

We remark that the previous result was proven by Greuel in [11, Korollar 5.8] (see also [17, (5.11.a)]) when the map h is assumed to be weighted homogeneous (in this case we have $\langle \theta_w(h_1), \dots, \theta_w(h_p)\rangle = \langle h_1, \dots, h_p\rangle$).

The following theorem follows as an application of Theorem 2.6, Theorem 2.9 and [4, Corollary 7.9], where this last result from [4] provides a formula expressing the colength of an ideal of maximal minors of a matrix as a sum of colengths of suitable ideals.

Theorem 2.13. [18, Theorem 3.1] *Let $w \in \mathbb{Z}^n_{\geqslant 1}$, $n \geqslant 2$. Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be weighted homogeneous with respect to w with isolated singularity at the origin and let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then $(f, h) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ is an* ICIS *whose Milnor number satisfies the relation*

$$\mu_X(f) = \mu(f) + \mu(f, h). \tag{6}$$

Remark 2.14. We observe that in Theorem 2.13 the condition that h has an isolated singularity at the origin can not be removed, as Example 2.15 shows. Obviously, if $X = h^{-1}(0)$, where $h : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is weighted homogeneous with respect to a given $w \in \mathbb{Z}^n_{\geqslant 1}$, and $f \in \mathcal{O}_n$ verifies that $\langle \theta_w(f)\rangle + \mathbf{J}(f, h)$ has finite colength, then this colength is an upper bound for $\mu_X(f)$ (this bound is not tight, as is also reflected in Example 2.15).

Example 2.15. Let f and h be the functions of $\mathcal{O}_3$ defined by $f(x, y, z) = x^3 + y^3 + z^3$ and $h(x, y, z) = xyz$, for all $(x, y, z) \in \mathbb{C}^3$. Let $X = h^{-1}(0)$. We have that $\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z)\rangle$. Thus $J_X(f) = \langle x^3, y^3, z^3\rangle$, which implies that $\mu_X(f) = 27$. It is straightforward to check that the ideal $\langle f, h\rangle + \mathbf{J}(f, h)$ has finite colength. Hence, $(f, h) : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ is an ICIS. By the Lê-Greuel formula we have the relation $\mu(f) + \mu(f, h) = \dim_{\mathbb{C}} \mathcal{O}_n/(\langle f\rangle + \mathbf{J}(f, h)) = 57$, which is different from $\mu_X(f)$ in this case.

As a direct application of Theorem 2.13, the following result follows.

Corollary 2.16. *Let $f, h \in \mathbb{C}[x_1, \dots, x_n]$ be weighted homogeneous polynomials, not necessarily with respect to the same vector of weights, $n \geqslant 2$. Let $X = h^{-1}(0)$ and $Y = f^{-1}(0)$. Let us suppose that $\mu_X(f)$ and $\mu_Y(h)$ are finite. Then*

$$\mu_X(f) - \mu_Y(h) = \mu(f) - \mu(h).$$

Proof. The condition $\mu_X(f) < \infty$ implies that $J(f)$ has finite colength. Analogously, $J(h)$ has finite colength. Therefore, by Theorem 2.13, (f, h) is an ICIS and $\mu_X(f) - \mu(f) = \mu(f, h) = \mu_Y(h) - \mu(h)$. □

Corollary 2.17. *Let $w \in \mathbb{Z}^n_{\geqslant 1}$, $n \geqslant 2$. Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be weighted homogeneous with respect to w with isolated singularity at the origin. Let $f \in \mathcal{O}_n$. Let us suppose that the ideal*

$\langle\theta_w(f)\rangle + \mathbf{J}(f,h)$ has finite colength. Then $\langle f\rangle + \mathbf{J}(f,h)$ has also finite colength and

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f\rangle + \mathbf{J}(f,h)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle\theta_w(f)\rangle + \mathbf{J}(f,h)}. \tag{7}$$

Proof. Let $X = h^{-1}(0)$. Hence $J_X(f) = \langle\theta_w(f)\rangle + \mathbf{J}(f,h)$. By Proposition 2.8, we have $c(f,h) < \infty$, which implies that (f,h) is an ICIS. Since f has an isolated singularity at the origin, we have that

$$\mu(f) + \mu(f,h) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f\rangle + \mathbf{J}(f,h)},$$

by [15, Theorem 3.7.1]. Then (7) follows as a direct consequence of Theorem 2.6. □

Let $f \in \mathcal{O}_n$ and let $i \in \{1,\dots,n\}$. By virtue of Theorem 2.5 and the upper semicontinuity of the colength of ideals, we can consider the minimum value of $\mu_H(f)$ when H varies in the set of linear subspaces of $\mathbb{C}^n$ of dimension i. Let us denote this number by $\mu_{H^{(i)}}(f)$. We will also write $f|_{H^{(i)}}$ to refer to the restriction of f to a generic linear subspace of $\mathbb{C}^n$ of dimension i.

Let I be an ideal of finite colength in a Noetherian local ring $(R,\mathbf{m})$ of dimension d and let $i \in \{0,1,\dots,d\}$. Then $e_i(I)$ will denote the mixed multiplicity $e(I,\dots,I,\mathbf{m},\dots,\mathbf{m})$, where I is repeated i times and $\mathbf{m}$ is repeated $n-i$ times (we refer to [14] and [21] for the definition and basic properties of mixed multiplicities). We recall that $e_n(I) = e(I)$, where $e(I)$ denotes the Samuel multiplicity of I.

Proposition 2.18. *Let $f \in \mathcal{O}_n$ and let $i \in \{0,1,\dots,n-1\}$. If f has an isolated singularity at the origin, then*

$$\mu_{H^{(i)}}(f|_{H^{(i+1)}}) = \mu^{(i+1)}(f) + \mu^{(i)}(f) = e_i\left(J(f)\frac{\mathcal{O}_n}{\langle f\rangle}\right). \tag{8}$$

Proof. The second equality in (8), for all $i \in \{0,1,\dots,n-1\}$, is a result of Teissier in [21, p. 322]. Let H be a linear subspace of $\mathbb{C}^n$ of dimension $n-1$ and let $h \in \mathbb{C}[x_1,\dots,x_n]$ be a linear form such that $H = h^{-1}(0)$. Since the logarithmic stratification of H is given by H itself, Theorem 2.3 shows that $\mu_H(f) < \infty$ if and only if the restriction of f to H is a submersion except, possibly, at the origin, which is to say that the restriction $f|_H$ has, at most, an isolated singularity at the origin. The latter condition holds for a generic choice of H in the Grassmannian variety of linear subspaces of $\mathbb{C}^n$ of dimension $n-1$ (see for instance [21, p. 299]). Therefore, we can apply Theorem 2.13 to say that for a generic linear subspace H of $\mathbb{C}^n$ of dimension $n-1$ we have that $\mu_H(f) = \mu(f) + \mu(f,h)$. We recall that $\mu(f,h) = \mu(f|_H) = \mu^{(n-1)}(f)$. Hence

$$\mu_{H^{(n-1)}}(f) = \mu(f) + \mu^{(n-1)}(f). \tag{9}$$

Let us fix an index $i \in \{1,\dots,n-1\}$. If we apply (9) to $f|_{H^{(i+1)}}$, then we obtain that $\mu_{H^{(i)}}(f|_{H^{(i+1)}}) = \mu(f|_{H^{(i+1)}}) + \mu^{(i)}(f|_{H^{(i+1)}}) = \mu^{(i+1)}(f) + \mu^{(i)}(f)$. □

In the next example we see that the numbers $\mu_{H^{(i)}}(f|_{H^{(i+1)}})$ and $\mu_{H^{(i)}}(f)$ are different in general. Let us remark that in the first case the subscript $H^{(i)}$ makes reference to a linear subspace of codimension 1 in $\mathbb{C}^{i+1}$.

Example 2.19. Let $f \in \mathcal{O}_4$ be the function given by $f(x,y,z,t) = x^3 + xy^4 + y^3z + t^3 + yz^5$. We have that $\mu^*(f) = (60, 12, 4, 2, 1)$. Therefore relation (8) shows that $\mu_{H^{(0)}}(f|_{H^{(1)}}) = 3$, $\mu_{H^{(1)}}(f|_{H^{(2)}}) = 6$, $\mu_{H^{(2)}}(f|_{H^{(3)}}) = 16$. Moreover $\mu_{H^{(3)}}(f) = 72$, $\mu_{H^{(2)}}(f) = 68$ and $\mu_{H^{(1)}}(f) = 66$, $\mu_{H^0}(f) = 64$.

The following result shows another aspect of Bruce-Roberts' Milnor numbers.

Corollary 2.20. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a weighted homogeneous function with an isolated singularity at the origin. Let $Y = f^{-1}(0)$. Then*

$$\mu^{(n-1)}(f) = \mu_Y(h)$$

for a generic choice of a linear form $h \in \mathbb{C}[x_1, \dots, x_n]$.

Proof. Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be a generic linear form. Let $X = h^{-1}(0)$. Obviously, the restriction of h to any logarithmic stratum of Y is a submersion except possibly at 0. Therefore $\mu_Y(h) < \infty$. By Corollary 2.16 we have that $\mu_X(f) = \mu_Y(h) + \mu(f) - \mu(h) = \mu_Y(h) + \mu(f)$, since $\mu(h) = 0$. Moreover, by (9) we obtain that $\mu_X(f) = \mu_{H^{(n-1)}}(f) = \mu(f) + \mu^{(n-1)}(f)$. Joining both relation, the result follows. $\square$

3. The Bruce-Roberts' Tjurina number

In this section we introduce the notion of Tjurina number in the context described in the previous section. We will compare this number with Bruce-Roberts' Milnor numbers in Theorem 3.2.

Definition 3.1. Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ and let $f \in \mathcal{O}_n$. We define

$$\tau_X(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f \rangle + J_X(f)}. \tag{10}$$

When the colength on the right of (10) is finite, we refer to $\tau_X(f)$ as the *Bruce-Roberts' Tjurina number of f with respect to X*.

Let R be a ring and let I be an ideal of R. Let $f \in R$. We denote by $r_f(I)$ the minimum of those $r \in \mathbb{Z}_{\geqslant 1}$ such that $f^r \in I$. If no such r exist, then we set $r_f(I) = \infty$. Let us also denote by $\varphi_{f,I}$ the morphism $R/I \to R/I$ defined by $g + I \mapsto fg + I$, for all $g \in R$. If M is an R-module, then we denote by $\ell(M)$ the length of M. As usual, we refer to $\ell(R/I)$ as the colength of I. With aim of comparing Bruce-Robert's Milnor and Tjurina numbers, we show the following result, which is inspired by the main result of Liu in [16].

Theorem 3.2. *Let $(R, \mathbf{m})$ be a Noetherian local ring. Let I be an ideal of R of finite colength and let $f \in R$ such that $r_f(I) < \infty$. Then*

$$\frac{\ell\left(\frac{R}{I}\right)}{\ell\left(\frac{R}{\langle f\rangle+I}\right)} \leqslant r_f(I) \tag{11}$$

and equality holds if and only if $\ker(\varphi_{f,I}) = \frac{\langle f^{r-1}\rangle+I}{I}$, *where* $r = r_f(I)$.

Proof. Let $A = R/I$ and $B = R/(\langle f\rangle + I)$. Let $r = r_f(I)$. Let us consider the following chain of ideals

$$0 = \frac{\langle f^r\rangle + I}{I} \subseteq \frac{\langle f^{r-1}\rangle + I}{I} \subseteq \cdots \subseteq \frac{\langle f^2\rangle + I}{I} \subseteq \frac{\langle f\rangle + I}{I} \subseteq A. \tag{12}$$

From (12) it follows that

$$\ell\left(\frac{R}{I}\right) = \sum_{i=0}^{r-1} \ell\left(\frac{\langle f^i\rangle + I}{\langle f^{i+1}\rangle + I}\right). \tag{13}$$

Let $\varphi = \varphi_{f,I}$. It is immediate to see that the sequence

$$0 \longrightarrow \ker(\varphi) \xrightarrow{j} \frac{R}{I} \xrightarrow{\varphi} \frac{R}{I} \longrightarrow \frac{R}{\langle f\rangle + I} \longrightarrow 0. \tag{14}$$

is exact. So

$$\ell\big(\ker(\varphi)\big) = \ell\left(\frac{R}{\langle f\rangle + I}\right). \tag{15}$$

Let us fix any $i \in \{1, \dots, r-1\}$. The sequence (14) induces the exact sequence

$$0 \longrightarrow \ker(\varphi) \cap \frac{\langle f^i\rangle + I}{I} \xrightarrow{j} \frac{\langle f^i\rangle + I}{I} \xrightarrow{\varphi} \frac{\langle f^i\rangle + I}{I} \longrightarrow \frac{\langle f^i\rangle + I}{\langle f^{i+1}\rangle + I} \longrightarrow 0. \tag{16}$$

The exactness of (16) implies that

$$\ell\left(\ker(\varphi) \cap \frac{\langle f^i\rangle + I}{I}\right) = \ell\left(\frac{\langle f^i\rangle + I}{\langle f^{i+1}\rangle + I}\right). \tag{17}$$

Relations (15) and (17) imply that

$$\ell\left(\frac{\langle f^i\rangle + I}{\langle f^{i+1}\rangle + I}\right) \leqslant \ell\left(\frac{R}{\langle f\rangle + I}\right) \tag{18}$$

for all $i = 1, \dots, r-1$. Hence, by (13), we have that

$$\ell\left(\frac{R}{I}\right) = \sum_{i=1}^{r-1} \ell\left(\frac{\langle f^i\rangle + I}{\langle f^{i+1}\rangle + I}\right) + \ell\left(\frac{R}{\langle f\rangle + I}\right) \leqslant r\ell\left(\frac{R}{\langle f\rangle + I}\right)$$

and thus (11) follows. The above relation shows that

$$\ell\left(\frac{R}{I}\right)=r\ell\left(\frac{R}{\langle f\rangle+I}\right)\Longleftrightarrow \ell\big(\ker(\varphi)\big)=\ell\left(\ker(\varphi)\cap\frac{\langle f^i\rangle+I}{I}\right),\text{ for all } i=1,\dots,r-1$$

$$\Longleftrightarrow \ker(\varphi)=\ker(\varphi)\cap\frac{\langle f^i\rangle+I}{I},\text{ for all } i=1,\dots,r-1$$

$$\Longleftrightarrow \ker(\varphi)\subseteq\frac{\langle f^{r-1}\rangle+I}{I}\Longleftrightarrow \ker(\varphi)=\frac{\langle f^{r-1}\rangle+I}{I},$$

where the last equivalence follows as a consequence of the definition of r. □

We remark that it is easy to find examples where the analogous inequality to (11) obtained when replacing the ideal $\langle f\rangle$ by an arbitrary ideal does not hold in general. As an immediate application of the previous theorem we have the following result.

Corollary 3.3. *Let X be an analytic subvariety of $(\mathbb{C}^n,0)$. Let $f\in\mathcal{O}_n$ such that $\mu_X(f)<\infty$. Then*

$$\frac{\mu_X(f)}{\tau_X(f)}\leqslant r_f(J_X(f)) \tag{19}$$

and equality holds if and only if $\ker(\varphi_{f,J_X(f)})=\frac{\langle f^{r-1}\rangle+J_X(f)}{J_X(f)}$, *where $r=r_f(J_X(f))$.*

Corollary 3.4. *Let $w\in\mathbb{Z}^n_{\geqslant 1}$ and let $h=(h_1,\dots,h_p):(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ be a weighted homogeneous* ICIS *with respect to w, $p\leqslant n-1$. Let $f\in\mathcal{O}_n$ such that $\mu_X(f)<\infty$. Then the map $(f,h_1,\dots,h_p)$ is also an* ICIS *and*

$$\mu(h)\leqslant(r-1)\mu(f,h) \tag{20}$$

where $r=r_{\pi(\theta_w(f))}(\pi(J(f,h_1,\dots,h_p)))$ and π denotes the natural projection $\mathcal{O}_n\to\frac{\mathcal{O}_n}{\langle h_1,\dots,h_p\rangle}$. Moreover, if R denotes the ring $\mathcal{O}_n/\big(\langle h_1,\dots,h_p\rangle+\mathbf{J}(f,h_1,\dots,h_p)\big)$, then equality holds in (20) if and only if the kernel of the automorphism of R defined by multiplication by $\theta_w(f)$ is equal to the ideal generated by the image of $\theta_w(f)^{r-1}$ in R.

Proof. By Theorem 2.9 we know that $(f,h_1,\dots,h_p)$ is an ICIS whose Milnor number is equal to the colength of the ideal $\pi(\langle\theta_w(f)\rangle+\mathbf{J}(f,h_1,\dots,h_p))$ in $\frac{\mathcal{O}_n}{\langle h_1,\dots,h_p\rangle}$. By Proposition 2.8 we also know that the number $c(f,h)$ is finite. Let us recall that $c(f,h)$ is equal to the colength of $\pi(\mathbf{J}(f,h_1,\dots,h_p))$. Therefore, by Theorem 3.2 and the Lê-Greuel formula, we obtain that

$$\frac{\mu(h)+\mu(f,h)}{\mu(f,h)}=\frac{c(f,h)}{\mu(f,h)}\leqslant r,$$

which is equivalent to saying that $\mu(h)\leqslant(r-1)\mu(f,h)$. The characterization of equality in (20) is a direct application of Theorem (3.2) . □

The bound given in (3.3) is sharp, as the following example shows.

Example 3.5. Let $h\in\mathcal{O}_2$ be the polynomial given by $h(x,y)=xy^6+x^4y^4+x^{10}$ and let $X=h^{-1}(0)$. Hence $\Theta_X=\langle(-2x^4y^3,5y^6+2x^3y^4+5x^9),(2x,3y)\rangle$. Let us consider the function $f(x,y)=x+y$. We have $\mu_X(f)=6$ and $\tau_X(f)=1$. Moreover $r_f(J_X(f))=6$. This shows that in this example equality holds in (19).

4. Derlog and lowerable vector fields

Given an integer $i \in \{1, \dots, n\}$, we denote by π_i the projection $\mathbb{C}^n \to \mathbb{C}^i$ onto the first i coordinates. Let $\mathrm{id}_{\mathbb{C}^n}$ be the identity map $\mathbb{C}^n \to \mathbb{C}^n$. We denote by $L_{i,n}$ the set of linear maps $p : \mathbb{C}^i \to \mathbb{C}^n$ such that $\pi_i \circ p = \mathrm{id}_{\mathbb{C}^i}$, that is, of the form

$$p(x_1, \dots, x_i) = \big(x_1, \dots, x_i, \ell_{i+1}(x_1, \dots, x_i), \dots, \ell_n(x_1, \dots, x_i)\big),$$

where $\ell_{i+1}, \dots, \ell_n$ denote linear forms of $\mathbb{C}[x_1, \dots, x_n]$. If $1 \leqslant i \leqslant n-1$, then the set $L_{i,n}$ can be identified with the set of matrices of size $(n-i) \times i$ with entries in $\mathbb{C}$.

Let $X \subseteq (\mathbb{C}^n, 0)$ be an analytic subvariety, $n \geqslant 2$, and let $p \in L_{i,n}$, where $i \in \{1, \dots, n-1\}$. The aim of this section is to obtain information about $\Theta_{p^{-1}(X)}$ in terms of p and Θ_X.

Definition 4.1. Let $p : \mathbb{C}^i \to \mathbb{C}^n$ be a linear map, where $i \in \{1, \dots, n\}$, and let X be an analytic subvariety of $(\mathbb{C}^n, 0)$. We define

$$\mathrm{Low}_X(p) = \big\{\theta \in \mathcal{O}_i^i : Dp \circ \theta = \eta \circ p, \text{ for some } \eta \in \Theta_X\big\},$$

where Dp denotes the differential of p. The elements of $\mathrm{Low}_X(p)$ are also known as *lowerable vector fields with respect to* p *and* X.

If $\eta \in \Theta_X$ verifies that there exists some $\theta \in \mathcal{O}_i^i$ such that $Dp \circ \theta = \eta \circ p$, then we say that η is *liftable with respect to* p. Let us denote by $\mathrm{Lif}_X(p)$ the set of such vector fields. Let us remark that $\mathrm{Low}_X(p)$ is an $\mathcal{O}_i$-submodule of $\mathcal{O}_i^i$ and $\mathrm{Lif}_X(p)$ is an $\mathcal{O}_n$-submodule of $\mathcal{O}_n^n$.

Let us fix a map $p \in L_{i,n}$, for some $i \in \{1, \dots, n\}$, and let $J(p)$ denote the Jacobian module of p, that is, $J(p) = \langle \frac{\partial p}{\partial x_1}, \dots, \frac{\partial p}{\partial x_i} \rangle \subseteq \mathcal{O}_i^n$. By abuse of notation, let us also denote by π_i the projection $\mathcal{O}_n^n \to \mathcal{O}_n^i$ onto the first i components. Let $p^*(\Theta_X) = \{\eta \circ p : \eta \in \Theta_X\} \subseteq \mathcal{O}_i^n$. An elementary computation shows that

$$\mathrm{Lif}_X(p) = \big\{\eta \in \Theta_X : p\big(\pi_i(\eta \circ p)\big) = \eta \circ p\big\} \tag{21}$$

$$\mathrm{Low}_X(p) = \big\{\pi_i(\eta \circ p) : \eta \in \mathrm{Lif}_X(p)\big\} = \pi_i\big(p^*(\Theta_X) \cap J(p)\big). \tag{22}$$

Given a map $p : \mathbb{C}^i \to \mathbb{C}^n$ and an analytic subvariety $X \subseteq (\mathbb{C}^n, 0)$, then p is said to be *algebraically transverse to* X *off* 0 when there exists an open neighbourhood U of 0 in $\mathbb{C}^n$ such that

$$Dp(T_x\mathbb{C}^i) + \Theta_X(p(x)) = T_{p(x)}\mathbb{C}^n \tag{23}$$

for all $x \in U \setminus \{0\}$. We will denote this condition by $p \,\overset{\circ}{\pitchfork}_{\mathrm{alg}}\, X$. We recall that $p \,\overset{\circ}{\pitchfork}_{\mathrm{alg}}\, X$ if and only if p is finitely $\mathcal{K}_X$-determined (see [5, p. 9]). Let us remark that if p is an immersion, then relation (23) holds only if $\dim_{\mathbb{C}} \Theta_X(p(x)) \geqslant n - i$. Here we recall a result from [5] relating the modules $\Theta_{p^{-1}(X)}$ and $\mathrm{Low}_X(p)$.

Theorem 4.2. [5, p. 17] *Let* X *be an analytic subvariety of* $(\mathbb{C}^n, 0)$ *and let* $p : \mathbb{C}^i \to \mathbb{C}^n$ *be a map such that* $p \,\overset{\circ}{\pitchfork}_{\mathrm{alg}}\, X$. *Then there exists some* $k \geqslant 1$ *such that*

$$\mathbf{m}_i^k \Theta_{p^{-1}(X)} \subseteq \mathrm{Low}_X(p) \subseteq \Theta_{p^{-1}(X)}. \tag{24}$$

The following example shows that the second inclusion of (24) can be strict. In Proposition 4.4 we give a sufficient condition for the inclusion $\mathrm{Low}_X(p) \subseteq \Theta_{p^{-1}(X)}$ to hold without imposing the condition $p \,\overline{\pitchfork}^{\circ}_{\mathrm{alg}}\, X$.

Example 4.3. Let us consider the function $h \in \mathcal{O}_2$ given by $h(x,y) = x^3y^2 + x^2y^3 + x^6 + y^6$ and let $X = h^{-1}(0)$. We observe that X is a plane curve with an isolated singularity at the origin. Let us consider the immersive linear map $p : \mathbb{C} \to \mathbb{C}^2$ given by $p(x) = (x,x)$, for all $x \in \mathbb{C}$. Hence $h(p(x)) = 2x^5(1+x)$, for all $x \in \mathbb{C}$, which implies that $p^{-1}(X) = \{0\}$, as germs at 0. In particular, there exists an open neighbourhood U of $0 \in \mathbb{C}$ such that $h(p(x)) \neq 0$, for all $x \in U \setminus \{0\}$. Therefore, the dimension of $\Theta_X(p(x))$ as a complex vector space is 2, for all $x \in U \setminus \{0\}$. This shows that $p \,\overline{\pitchfork}^{\circ}_{\mathrm{alg}}\, X$. A basic computation with *Singular* [7] shows that $\mathrm{Low}_X(p) = \pi_1(p^*(\Theta_X) \cap J(p)) = \langle x^3 \rangle$, whereas $\Theta_{p^{-1}(X)} = \langle x \rangle$. That is, $\mathrm{Low}_X(p) \subsetneq \Theta_{p^{-1}(X)}$.

Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be an analytic map. We say that h is *reduced* when the ideal of $\mathcal{O}_n$ generated by the components of h is reduced.

Proposition 4.4. *Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$, $n \geqslant 2$, and let $i \in \{1, \ldots, n-1\}$. Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a reduced analytic map such that $X = h^{-1}(0)$ and let $p \in L_{i,n}$ such that the map $h \circ p : (\mathbb{C}^i, 0) \to (\mathbb{C}^n, 0)$ is also reduced. Then*

$$\mathrm{Low}_X(p) \subseteq \Theta_{p^{-1}(X)}.$$

Proof. Let J be the ideal of $\mathcal{O}_n$ generated by the components of h and let $\theta \in \mathrm{Low}_X(p)$, $\theta = (\theta_1, \ldots, \theta_i)$. By relations (21) and (22) it follows that there exists some $\eta \in \Theta_X$ such that $p(\pi_i(\eta \circ p)) = \eta \circ p$ and $\theta = \pi_i(\eta \circ p)$.

Let $p^* : \mathcal{O}_n \to \mathcal{O}_i$ be the morphism given by $p^*(f) = f \circ p$, for all $f \in \mathcal{O}_n$. We have that $I(p^{-1}(X)) = I((h \circ p)^{-1}(0)) = \mathrm{rad}(p^*(J)) = p^*(J)$. We will see that $\theta(h_k \circ p) \in p^*(J)$, for all $k = 1, \ldots, m$, where $h = (h_1, \ldots, h_m)$.

Let us write p as $p(x_1, \ldots, x_i) = (x_1, \ldots, x_i, \sum_{j=1}^{i} a_{i+1,j}x_j, \ldots, \sum_{j=1}^{i} a_{n,j}x_j)$, for some coefficients $a_{\ell,j} \in \mathbb{C}$. Let us fix an index $k \in \{1, \ldots, m\}$. Computing $\theta(h_k \circ p)$ we obtain the following:

$$\begin{aligned}
\theta(h_k \circ p) &= \sum_{j=1}^{i} \theta_j \frac{\partial(h_k \circ p)}{\partial x_j} = \sum_{j=1}^{i} \theta_j \left(\frac{\partial h_k}{\partial x_j} \circ p + \sum_{\ell=i+1}^{n} a_{\ell,j} \frac{\partial h_k}{\partial x_\ell} \circ p \right) \\
&= \sum_{j=1}^{i} \theta_j \left(\frac{\partial h_k}{\partial x_j} \circ p \right) + \sum_{j=1}^{i} \theta_j \left(\sum_{\ell=i+1}^{n} a_{\ell,j} \frac{\partial h_k}{\partial x_\ell} \circ p \right) \\
&= \sum_{j=1}^{i} (\eta_j \circ p) \left(\frac{\partial h_k}{\partial x_j} \circ p \right) + \sum_{\ell=i+1}^{n} \left(\sum_{j=1}^{i} \theta_j a_{\ell,j} \right) \frac{\partial h_k}{\partial x_\ell} \circ p \\
&= \sum_{j=1}^{n} (\eta_j \circ p) \left(\frac{\partial h_k}{\partial x_j} \circ p \right) = \eta(h_k) \circ p \in p^*(J).
\end{aligned}$$

Therefore the inclusion $\mathrm{Low}_X(p) \subseteq \Theta_{p^{-1}(X)}$ holds. □

Let $n \in \mathbb{Z}_{\geqslant 1}$ and let us fix coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$. Then we denote by $\theta^{(n)}$ the Euler derivation $x_1 \frac{\partial}{\partial x_1} + \cdots + x_n \frac{\partial}{\partial x_n}$. In the next result we show a case where the equality $\mathrm{Low}_X(p) = \Theta_{p^{-1}(X)}$ holds.

Proposition 4.5. *Let $h : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a homogeneous* ICIS *such that $n - m \geqslant 1$ and let $X = h^{-1}(0)$. Let $i \in \{m+1, \dots, n\}$ and let $p : \mathbb{C}^i \to \mathbb{C}^n$ be an immersive linear map such that $h \circ p : (\mathbb{C}^i, 0) \to (\mathbb{C}^m, 0)$ is an* ICIS *of positive dimension. Then*

$$\mathrm{Low}_X(p) = \Theta_{p^{-1}(X)}. \tag{25}$$

Proof. Let H denote the image of p. Let $R : \mathbb{C}^n \to \mathbb{C}^n$ be a rotation such that $R(H)$ is given by the equations $x_{i+1} = \cdots = x_n = 0$. Let $q = R \circ p : \mathbb{C}^i \to \mathbb{C}^n$. Therefore $q(x_1, \dots, x_i) = (x_1, \dots, x_i, 0, \dots, 0)$, for all $(x_1, \dots, x_i) \in \mathbb{C}^i$. Let $Z = R(X)$.

Let $Y = q^{-1}(Z) = p^{-1}(X)$. By hypothesis, Y is a homogeneous ICIS. Let $f = h \circ R^{-1}$. Therefore $Z = f^{-1}(0)$ and $Y = (f \circ q)^{-1}(0)$. Let us write $f = (f_1, \dots, f_m) : (\mathbb{C}^i, 0) \to (\mathbb{C}^m, 0)$.

Let us consider the matrices

$$A_Y = \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_i} \\ \frac{\partial (f_1 \circ q)}{\partial x_1} & \cdots & \frac{\partial (f_1 \circ q)}{\partial x_i} \\ \vdots & & \vdots \\ \frac{\partial (f_m \circ q)}{\partial x_1} & \cdots & \frac{\partial (f_m \circ q)}{\partial x_i} \end{bmatrix}, \qquad A_Z = \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

By Theorem 2.5, we have that Θ_Y is generated by $\{\theta^{(i)}, (f_\ell \circ q) \frac{\partial}{\partial x_j} : \ell = 1, \dots, m,\ j = 1, \dots, i\}$ and the minors of size $m+1$ of A_Y. Let us denote this generating system by W_Y. Also by Theorem 2.5, a generating system of Θ_Z is given by $\{\theta^{(n)}, f_\ell \frac{\partial}{\partial x_j} : \ell = 1, \dots, m,\ j = 1, \dots, n\}$ and the minors of size $m+1$ of A_Z. Let us denote this generating system of Θ_Z by W_Z. Given indices $1 \leqslant j_1 < \cdots < j_{m+1} \leqslant i$, let $\theta_{j, \dots, j_{m+1}}$ denote the minor of A_Y formed by the columns $j_1, \dots, j_{m+1}$ of A_Y and let $\theta'_{j_1, \dots, j_{m+1}}$ denote the analogous minor of A_Z. Then, it is immediate to check that the following relations hold:

$$\begin{aligned} \theta^{(i)} &= \pi_i(\theta^{(n)} \circ q) \\ (f_\ell \circ q) \frac{\partial}{\partial x_j} &= \pi_i\big((f_\ell \frac{\partial}{\partial x_j}) \circ q\big), \text{ for all } \ell = 1, \dots, m,\ j = 1, \dots, i \\ \theta_{j, \dots, j_{m+1}} &= \pi_i(\theta'_{j, \dots, j_{m+1}} \circ q), \text{ for all } 1 \leqslant j_1 < \cdots < j_{m+1} \leqslant i. \end{aligned}$$

Therefore we found that for any $\theta \in W_Y$, there exists some $\eta = (\eta_1, \dots, \eta_n) \in W_Z$ such that $\theta = \pi_i(\eta \circ q)$ and $\eta_{i+1} = \cdots = \eta_n = 0$. In particular $\eta_{i+1} \circ q = \cdots = \eta_n \circ q = 0$, which means that η is liftable with respect to q. Therefore

$$\Theta_Y \subseteq \mathrm{Low}_Z(q). \tag{26}$$

An elementary computation shows that $\Theta_Z = (R^{-1})^*(R(\Theta_X))$, where $R(\Theta_X) = \{R(\eta) : \eta \in \Theta_X\}$. Hence $\mathrm{Low}_Z(q) = \mathrm{Low}_X(p)$ and thus (26) implies that $\Theta_Y \subseteq \mathrm{Low}_X(p)$.

By hypothesis, the map $h \circ p : (\mathbb{C}^i, 0) \to (\mathbb{C}^m, 0)$ is an ICIS with $(h \circ p)^{-1}(0)$ of dimension $i - m \geqslant 1$. Then $h \circ p$ is reduced (see [17, p. 7]). Thus, as a direct application of Proposition 4.4, the reverse inclusion $\Theta_Y \supseteq \mathrm{Low}_X(p)$ follows. Therefore $\Theta_Y = \mathrm{Low}_X(p)$. □

Remark 4.6. We have found that equality (25) holds in a wide variety of examples where X has not an isolated singularity at the origin. We conjecture that Proposition 4.5 holds at least when X is homogeneous, not necessarily an ICIS with isolated singularity at the origin. In particular, when X is a generic determinantal variety.

5. Bruce-Roberts numbers and linear sections

Let us fix a function $f \in \mathcal{O}_n$ and a complex analytic subvariety $X \subseteq (\mathbb{C}^n, 0)$. If $i \in \{1, \dots, n\}$, then we denote by $L_{i,n}(f, X)$ the set of those $p \in L_{i,n}$ such that $\mu_{p^{-1}(X)}(f \circ p)$ is finite. As is already known in the case $X = \mathbb{C}^n$, the set $L_{i,n}(f, X)$ can be strictly contained in $L_{i,n}$ even if $\mu_X(f)$ is finite.

Let us suppose that f has an isolated singularity at the origin and let $i \in \{1, \dots, n\}$. In [21, p. 299] Teissier showed that there exists a dense Zariski open set $U_{i,n}$ of the Grassmannian variety of linear subspaces of dimension i of $\mathbb{C}^n$ such that the topological type of $f^{-1}(0) \cap H$ does not depend on H whenever $H \in U_{i,n}$. This leads to the definition of $\mu^{(i)}(f)$ as the Milnor number of the restriction $f|_H$, where H varies in $U_{i,n}$. Moreover, due to the semicontinuity of the colength of ideals, the minimium possible value of the colength of the ideal $J(f \circ p) = \langle \frac{\partial (f\circ p)}{\partial x_1}, \dots, \frac{\partial (f\circ p)}{\partial x_i} \rangle$, where p varies in $L_{i,n}(f, \mathbb{C}^n)$, is actually equal to $\mu^{(i)}(f)$. Motivated by this version of $\mu^{(i)}(f)$ we introduce in Definition 5.2 the analogous concept in the context of Bruce-Roberts' Milnor numbers.

Lemma 5.1. *Let $f \in \mathcal{O}_n$ and let X be an analytic subvariety of $(\mathbb{C}^n, 0)$. Let $i \in \{1, \dots, n\}$ and let $p \in L_{i,n}(f, X)$. Then*

$$\mu_{p^{-1}(X)}(f \circ p) \geqslant \mu^{(i)}(f).$$

Proof. The inclusion $J_{p^{-1}(X)}(f \circ p) \subseteq J(f \circ p)$ is obvious, by the definition of $J_{p^{-1}(X)}(f \circ p)$. The condition $p \in L_{i,n}(f, X)$ means that $J_{p^{-1}(X)}(f)$ has finite colength. Therefore $\mu(f \circ p)$ is finite and thus $\mu_{p^{-1}(X)}(f \circ p) \geqslant \mu(f \circ p) \geqslant \mu^{(i)}(f)$. □

Definition 5.2. Let $f \in \mathcal{O}_n$ and let X be an analytic subvariety of $(\mathbb{C}^n, 0)$. For any $i \in \{1, \dots, n\}$ such that $L_{i,n}(f, X) \neq \emptyset$, we define the number

$$\mu_X^{(i)}(f) = \min_{p \in L_{i,n}(f,X)} \mu_{p^{-1}(X)}(f \circ p).$$

If $L_{i,n}(f, X) = \emptyset$, then we set $\mu_X^{(i)}(f) = \infty$. We denote the vector $(\mu_X^{(n)}(f), \dots, \mu_X^{(1)}(f))$ by $\mu_X^*(f)$. We refer to $\mu_X^*(f)$ as the vector of *mixed Bruce-Roberts numbers of f with respect to X*.

If $f \in \mathcal{O}_n$, $f \neq 0$, then the *order* of f is defined as $\operatorname{ord}(f) = \max\{r \in \mathbb{Z}_{\geqslant 1} : f \in \mathbf{m}_n^r\}$. The order $\operatorname{ord}(I)$ of a non-zero ideal I of $\mathcal{O}_n$ is defined analogously.

Proposition 5.3. *Let X be an analytic subvariety of $(\mathbb{C}^n, 0)$ with $\dim(X) < n$, $n \geqslant 2$. Let $f \in \mathcal{O}_n$, $f \neq 0$. Then $\mu_X^{(1)}(f) = \operatorname{ord}(f)$. Consequently, if $\mu_X(f) < \infty$ and $\operatorname{ord}(f) \geqslant 3$, then*

$$\mu_X(f) \geqslant \mu_X^{(1)}(f).$$

Proof. Since $\dim(X) < n$, the intersection of X with a generic line passing through the origin is equal to $\{0\}$. Let $p \in L_{1,n}$ such that $p^{-1}(X) = \{0\}$. Let $Y = \{0\} \subseteq (\mathbb{C}, 0)$.

Let us write p as $p(x) = (x, a_2x, \dots, a_nx)$, for some $a_2, \dots, a_n \in \mathbb{C}$, for all $x \in \mathbb{C}$. Let us take coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$. Since $\Theta_Y = \mathbf{m}_1$, we have

$$J_Y(f \circ p) = \left\langle x\frac{\partial(f \circ p)}{\partial x} \right\rangle = \left\langle x\frac{\partial f}{\partial x_1}(p(x)) + a_2x\frac{\partial f}{\partial x_2}(p(x)) + \cdots + a_nx\frac{\partial f}{\partial x_n}(p(x)) \right\rangle.$$

Let $I(f)$ denote the ideal of $\mathcal{O}_n$ generated by $x_1\frac{\partial f}{\partial x_1}, \dots, x_n\frac{\partial f}{\partial x_n}$. We have $J_Y(f \circ p) \subseteq p^*(I(f))$ and $\operatorname{ord}(J_Y(f \circ p)) = \operatorname{ord}(p^*(I(f)) = \operatorname{ord}(I(f)) = \operatorname{ord}(f)$, for a generic choice of the coefficients $a_2, \dots, a_n$. Then

$$\mu_X^{(1)}(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_1}{J_Y(f \circ p)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_1}{p^*(I(f))} = \operatorname{ord}(f).$$

If additionally we assume that $\mu_X(f) < \infty$, then $\mu_X(f) \geqslant \mu(f) \geqslant (\operatorname{ord}(f)-1)^n$. We finally have that $(\operatorname{ord}(f) - 1)^n \geqslant \operatorname{ord}(f)$, since we are assuming that $\operatorname{ord}(f) \geqslant 3$ and $n \geqslant 2$. □

The following example shows that the sequence $\mu_X^*(f)$ is not decreasing in general.

Example 5.4. Let $f \in \mathcal{O}_3$ be the function given by $f(x, y, z) = x + y + z$ and let $X = \{(x, y, z) \in \mathbb{C}^3 : xyz = 0\}$. We have $\Theta_X = \langle (x, 0, 0), (0, y, 0), (0, 0, z)\rangle$. Therefore $\mu_X(f) = 1$.

Let $p \in L_{2,3}$ be given by $p(x, y) = (x, y, ax + by)$, where $a, b \in \mathbb{C} \setminus \{-1, 0\}$. Therefore

$$p^{-1}(X) = \left\{(x, y) \in \mathbb{C}^2 : xy(ax + by) = 0\right\}.$$

By Theorem 2.6, we have that $\Theta_{p^{-1}(X)} = \langle (x, y), (ax^2 + 2bxy, -2axy - by^2)\rangle$. Thus

$$J_{p^{-1}(X)}(f \circ p) = \left\langle x(a + 1) + y(b + 1), a(a + 1)x^2 + 2(b - a)xy - b(b + 1)y^2\right\rangle \subseteq \mathcal{O}_2.$$

This implies that

$$\mu_X^{(2)}(f) = \mu_{p^{-1}(X)}(f \circ p) = \dim_{\mathbb{C}} \frac{\mathcal{O}_2}{J_{p^{-1}(X)}(f \circ p)} = 2.$$

It is immediate to check that $\mu_X^{(1)}(f) = 1$. So $\mu_X^*(f) = (1, 2, 1)$.

Example 5.5. Let us consider the function $h : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ given by $h(x, y, z, t) = x^a + y^a + z^a + t^a$, for some $a \in \mathbb{Z}_{\geqslant 2}$. Let $X = h^{-1}(0)$. Let $f \in \mathcal{O}_4$ be given by $f(x, y, z, t) = \alpha x^b + \beta y^b + \gamma z^b + \delta t^b$, where $b \in \mathbb{Z}_{\geqslant 2}$, and $\alpha, \beta, \gamma, \delta$ denote generic complex coefficients. Therefore, we can apply [18, Corollary 3.12] to deduce that

$$\begin{aligned}
\mu_X(f) &= b^4 + (a - 4)b^3 + (a^2 - 4a + 6)b^2 + (a^3 - 4a^2 + 6a - 4)b \\
\mu_X^{(3)}(f) &= b^3 + (a - 3)b^2 + (a^2 - 3a + 3)b \\
\mu_X^{(2)}(f) &= b^2 + (a - 2)b \\
\mu_X^{(1)}(f) &= b.
\end{aligned}$$

If $p \leqslant n$, given an integer $i \in \{1, \dots, n-p+1\}$, we denote by $\mu^{(i)}(g)$ the Milnor number of the ICIS given by $(g_1, \dots, g_p, h_1, \dots, h_{n-p-i+1}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n-i+1}, 0)$, where $h_1, \dots, h_{n-p-i+1}$ is a family of generic linear forms of $\mathbb{C}[x_1, \dots, x_n]$ (see [10] or [19]). Then $\mu^{(n-p+1)}(g) = \mu(g)$. Let us set $\mu^{(0)}(g) = 1$. Hence, as in the case $p = 1$ (see [21, p. 300]), we also have a decreasing sequence of integers

$$\mu^{(n-p+1)}(g) \geqslant \mu^{(n-p)}(g) \geqslant \dots \geqslant \mu^{(1)}(g) \geqslant \mu^{(0)}(g).$$

We will denote the vector $(\mu^{(n-p+1)}(g), \dots, \mu^{(1)}(g), \mu^{(0)}(g))$ by $\mu^*(g)$ and we refer to it as the *μ^*-sequence of g*. Let us remark that, by (2), we have

$$\mu^{(1)}(g) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle g_1, \dots, g_p, h_1, \dots, h_{n-p} \rangle} - 1,$$

where $h_1, \dots, h_{n-p}$ is a family of generic linear forms of $\mathbb{C}[x_1, \dots, x_n]$.

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an isolated complete intersection singularity. We recall that, if $n - p \geqslant 1$, then the ring $\mathcal{O}_n / \langle g_1, \dots, g_p \rangle$ is reduced (see [17, p. 7]). Following [8, p. 215], we denote by $JM(g)$ the submodule of $(\mathcal{O}_n / \langle g_1, \dots, g_p \rangle)^p$ generated by the partial derivatives $\frac{\partial g}{\partial x_1}, \dots, \frac{\partial g}{\partial x_n}$. Given a module of finite colength M of a free module R^p, where R denotes a given Noetherian local ring, then we denote by $e(M)$ the Buchsbaum-Rim multiplicity of M.

Proposition 5.6. *Let $h \in \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous polynomial with isolated singularity at the origin and let $X = h^{-1}(0)$, $n \geqslant 2$. Let $f \in \mathcal{O}_n$ such that $\mu_X(f) < \infty$. Then, for all $i \in \{2, \dots, n\}$:*

$$\mu_X^{(i)}(f) = \mu^{(i)}(f) + \mu^{(i-1)}(f, h). \tag{27}$$

Moreover, we have

$$\mu_X(f) + \mu_X^{(n-1)}(f) = e\left(J(f) \frac{\mathcal{O}_n}{\langle f \rangle} \right) + e\big(JM(f, h)\big). \tag{28}$$

Proof. Let us fix an index $i \in \{2, \dots, n\}$. For a general $p \in L_{i,n}$, we have that $h \circ p : (\mathbb{C}^i, 0) \to (\mathbb{C}, 0)$ is also homogeneous with an isolated singularity at the origin. By Theorem 2.13, we have

$$\mu_{p^{-1}(X)}(f \circ p) = \mu(f \circ p) + \mu(f \circ p, h \circ p). \tag{29}$$

Let $p_{i+1}, \dots, p_n$ denote the last $n - i$ components of p. The Milnor number of the map $(f \circ p, h \circ p) : (\mathbb{C}^i, 0) \to (\mathbb{C}^2, 0)$ is equal to the Milnor number of $(f, h, x_{i+1} - p_{i+1}, \dots, x_n - p_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2 \times \mathbb{C}^{n-i}, 0)$, which in turn is equal to $\mu^{(i-1)}(f, h)$, by the definition of the sequence of mixed Milnor numbers of an isolated complete intersection singularity. Then (29) shows relation (27).

By [21, Corollaire 1.5] we know that $\mu(f) + \mu^{(n-1)}(f) = e(J(f) \frac{\mathcal{O}_n}{\langle f \rangle})$. Moreover, by the Lê-Greuel formula and the definition of the sequence $\mu^*(f, h)$, for a generic choice of a linear form $\ell_1 \in \mathbb{C}[x_1, \dots, x_n]$, we have that

$$\mu^{(n-1)}(f, h) + \mu^{(n-2)}(f, h) = \mu(f, h) + \mu(f, h, \ell_1) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle f, h \rangle + \mathbf{J}(f, h, \ell_1)}.$$

This last colength is equal to $e\big(JM(f,h)\big)$, by [8, Proposition 2.6]. Then, by using (27) in the case $i=n$, we obtain that

$$\begin{aligned}\mu_X(f)+\mu_X^{(n-1)}(f)&=\mu^{(n)}(f)+\mu^{(n-1)}(f,h)+\mu^{(n-1)}(f)+\mu^{(n-2)}(f,h)\\&=e\left(J(f)\frac{\mathcal{O}_n}{\langle f\rangle}\right)+e\big(JM(f,h)\big).\end{aligned}$$

□

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