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Characterizing compromise solutions for investors with uncertain risk preferences

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Abstract The optimum portfolio selection for an investor with particular preferences was proven to lie on the normalized efficient frontier between two bounds defined by the Ballestero (1998) bounding theorem. A deeper understanding is possible if the decision-maker is provided with visual and quantitative techniques. Here, we derive useful insights as a way to support investor's decision-making through: (i) a new theorem to assess balance of solutions; (ii) a procedure and a new plot to deal with discrete efficient frontiers and uncertain risk preferences; and (iii) two quality metrics useful to predict long-run performance of investors.

Keywords: Finance; portfolio selection; compromise programming; discrete efficient-frontiers; performance prediction.

1 Introduction

Investment decision makers not only care about profit but also about the amount of risk accepted to achieve that profit. In portfolio selection, profit and risk can be measured by the mean and the variance (or the standard

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deviation) of portfolio returns. In the search for the best portfolio, Compromise Programming proposed that utility optimization was considered as a distance minimization problem (Zeleny, 1982; Yu, 1985) that has been widely used in many recent works (Bilbao-Terol et al., 2006; Amiri et al., 2011; Pla-Santamaria and Bravo, 2013; Ballestero et al., 2015)

However, individual or collective investors may have particular risk preferences different to an average investor, i.e., one unbiased towards profitability or risk. This bias is summarized by Ballestero (1998) through the use of a parameter reflecting the investor's preference. From that, a bounding theorem was proposed by the author to determine the bounds for approximating the optimum portfolio on the efficient frontier for an investor with particular risk preferences. Here, efficient frontier refers to the set of non-dominated portfolios in terms of profitability and risk. In an attempt to reduce the number of available solutions, Ballestero's theorem introduces risk preferences in the optimization problem to provide a bounded solution. However, this theorem does not provide an optimum, but an approximation to the optimum characterized by two points in the efficient frontier delimiting an efficient set of portfolios.

Subsequent works have followed two separated ways. On the one hand, some of them have successfully applied this theorem for selecting portfolios for mutual funds (Ballestero and Pla-Santamaria, 2004) and for eliciting balanced solutions by using an additive linear-quadratic composite metric (Ballestero, 2007; Bravo et al., 2012). On the other hand, a number of related works have recently mentioned Ballestero's theorem as an important contribution to portfolio selection (Ehrgott et al., 2004; Steuer et al., 2007; Xidonas et al., 2012). However, little attention has been paid to the geometry of risk preferences, its relationship with the efficient frontier and its utility to enhance the understanding of the multi-objective decision-making process.

The main goal of this paper is to characterize Ballestero's bounding theorem to enhance its understanding and implications for investment decision makers through:

- a new theorem to assess balance of solutions for different metrics.
- a procedure and a new stairs plot to deal with discrete efficient frontiers and risk preference uncertainty.
- two quality metrics to predict long-run performance of investors.

The structure of this paper is as follows. Section 2 presents the Ballestero's bounding theorem that we characterize using two metrics to obtain the best compromise solutions in Section 3. Next, Section 4 compares linear and quadratic metrics which leads to a new theorem, namely, the balance theorem. An algorithm and a new plot is proposed in Section 6 to obtain solutions for discrete efficient frontiers and uncertain risk preferences. Both the algorithm an the new plot will be illustrated with an example. In Section 7, two quality measures are proposed to predict long-run investor performance. Finally, we elaborate some conclusions in Section 8.

2 Background : Ballestero's bounding theorem

Since Markowitz (1952), portfolio selection is usually approached through mean-variance (E-V) efficient frontier techniques. Each point in this efficient frontier represents an alternative portfolio that produces higher profitability and lower risk than any alternative portfolio in the E-V space. It is said, then, that this portfolio dominates the alternative one. In order to avoid meaningless comparison, Ballestero proposed the efficient frontier to be built through the use of the following indexes: (i) index of profitability, θ_1; (ii) index of safety, θ_2, which are defined as follows:

$$\theta_1 = \frac{E - E_{min}}{E_{max} - E_{min}} \tag{1}$$

$$\theta_2 = \frac{V_{min} - V}{V_{min} - V_{max}} \tag{2}$$

where (E, V) is a point of the mean-variance efficient frontier, (E_{max}, V_{min}) is the ideal point and (E_{min}, V_{max}) is the anti-ideal point. Note that the index of safety, θ_2, is computed as the reciprocal of the index of risk or, equivalently, *index of safety* = 1 - *index of risk*. These indexes, θ_1 and θ_2, transform the E-V space in a normalized profitability-safety (E-S) space where $0 \leq \theta_1 \leq 1$ and $0 \leq \theta_2 \leq 1$.

The efficient frontier takes the form $T(\theta_1, \theta_2) = 0$ and the investor's utility function is expressed as $U(\theta_1, \theta_2)$. Then, if U_{θ_1} and U_{θ_2} are the partial derivatives of U with respect to θ_1 and θ_2, the marginal rate of substitution (MRS) is defined as:

$$MRS(\theta_1, \theta_2) = \frac{-d\theta_2}{d\theta_1} = \frac{U_{\theta_1}}{U_{\theta_2}} \tag{3}$$

As argued in Ballestero (1998), a number of common assumptions are usually accepted in economics which are enumerated as follows:

- A1: More is better, in the sense of increasing profitability and safety.
- A2: Differentiability of U.
- A3: MRS monotonically decreases as θ_1 increases.
- A4: Convexity, regarding the efficient set of portfolios.
- A5: Differentiability of T.
- A6: θ_2 is a decresing function of θ_1 along T, that is, $d\theta_2/d\theta_1 < 0$.
- A7: Strict concavity, that is, $d^2\theta_2/d\theta_1^2 < 0$.
- A8: The Lagrangean maximum of the investors utility function U is given by the intersection of T with $U_{\theta_1}/U_{\theta_2} = T_{\theta_1}/T_{\theta_2}$ where T_{θ_1} and T_{θ_2} are the partial derivatives of T with respect to θ_1 and θ_2 and it exists on the efficient frontier.

Moreover, it is assumed that, in the investor's utility function, the MRS has a stable and constant value r_0 along the path $\theta_1 = \theta_2$, which is called the investor's particular risk preference. For instance, an investor willing to lose 1.2 marginal units of safety for increasing her profitability in one marginal

unit it is said to have $r_0 = 1.2$ and to be a risky investor. On the other hand, an investor willing to lose 0.5 marginal units of safety for increasing her profitability in one marginal unit it is said to have $r_0 = 0.5$ and to be a conservative investor. From the previous set of assumptions and definitions, the Ballestero's bounding theorem is expressed as follows:

Theorem 1 *The Langrangean maximum M of the investor's utility U on the efficient frontier $T(\theta_1, \theta_2) = 0$ with $\theta_1 \geq 0$ and $\theta_2 \geq 0$ is bounded between the points L_∞ and L defined as:*

- *L_∞ is the intersection of $\theta_1 = \theta_2$ with $T(\theta_1, \theta_2)$.*
- *L is the maximum of the linear utility $V = r_0\theta_1 + \theta_2$ subject to $T(\theta_1, \theta_2)$.*

It is interesting to point out that L_∞ does not depend on the particular investor's preferences, expressed here as r_0, but L does. The reader interested in the proof of this theorem is referred to Ballestero (1998). The underlying idea behind the Ballestero's bounding theorem is approximating solutions (optimum portfolio selections) for an investor with particular risk preferences. In this sense, it generalizes a previous model by Ballestero and Romero (1996), for the case of an average investor without bias for profits or risk. To this end, risk preferences are expressed through risk parameter r_0, resulting in the transformation of Yu's bounds (Yu, 1973) into the Ballestero's bounds L_∞ and L, the latter depending on r_0.

As a result, the utility of the Ballestero's bounding theorem roots in the possibility to precisely delimit the efficient set of portfolios according to the particular bias for risk of an investor. This theorem has been recently used in portfolio selection research (Ballestero and Pla-Santamaria, 2004; Ballestero, 2007; Bravo et al., 2012). Next we characterize this theorem to the selection of the best portfolio according to particular preferences.

3 Characterizing Ballestero's bounding theorem for different metrics

The main goal of this section is to provide a visual and quantitative characterization of Ballestero's bounding theorem for different metrics. This characterization consists of: (i) providing an optimal solution within the bounds L and L_∞; and (ii) improving understanding of the selection of best portfolios through graphical tools.

The concept of ideal point is at the core of compromise programming (Zeleny, 1982; Yu, 1985; Ballestero and Romero, 1998). In portfolio selection, the ideal point in the normalized E-S space is $(1, 1)$, because the goal is to maximize profit and safety. Since this ideal point is usually unfeasible, it is necessary to look for compromise solutions by minimizing the distance to this ideal point. A general distance function between two bidimensional points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, is the Minkowski distance of order h defined as:

$$\left(|x_1 - x_2|^h + |y_1 - y_2|^h\right)^{1/h} \tag{4}$$

Then, by computing the distance between the ideal point $(1, 1)$ and a particular point in the efficient frontier (θ_1, θ_2), we are in a position to determine whether a given portfolio is better than another. Moreover, when considering particular preferences defined by r_0, distances must be computed between $(r_0, 1)$, as the new ideal point, and $(r_0\theta_1, \theta_2)$, as the new particular point in the efficient frontier. From that, a family of normalized distance functions that includes parameter r_0 determining the investor's risk preferences (Ballestero and Romero, 1998) can be expressed as:

$$\mathcal{L}_h = \left[r_0^h \cdot (1-\theta_1)^h + (1-\theta_2)^h\right]^{1/h} \tag{5}$$

Note that $\mathcal{L}_1$ is the Manhattan distance, $\mathcal{L}_2$ is the Euclidean distance, and $\mathcal{L}_\infty$ is the Chebyshev distance. They are the most used distances in practice for interpretation and computational reasons (Ringuest, 1992; Ballestero, 2007). In order to characterize Theorem 1, we rely on the concept of isometric curves in the normalized $E-S$ space (Zeleny, 1982; Yu, 1985; Ballestero and Romero, 1998), defined by varying a parameter over the previous family of distance functions to the ideal point. The CP objective function is given by the family of distances defined in (5) to be minimized subject to an efficient frontier, which is equivalent to maximize the following Yu-Zeleny utility (Ballestero, 2007):

$$U = K - \mathcal{L}_h \tag{6}$$

where K is a constant sufficiently large to assure non-negativity. From that, we are in a position to obtain the best solutions derived from Ballestero's theorem for both Manhattan and Euclidean distances.

3.1 Characterization of the solution for Manhattan distances

The Manhattan distance between two points in a bidimensional space is the sum of the absolute values of the differences between the first and second components of each point. Given two points (x_1, y_1) and (x_2, y_2), their Manhattan distance is calculated as:

$$|x_1 - x_2| + |y_1 - y_2|. \tag{7}$$

It can be graphically represented by the strictly horizontal and vertical path between two points located in an imaginary grid. The Manhattan distance is linear and this fact facilitates computations within an optimization problem. In practice, isometric points in terms of Manhattan distances can be represented by lines in a E-S space. Indeed, practitioners can graphically derive the optimal portfolio by plotting the line with slope r_0 that is tangent to the efficient frontier as the following theorem formally states.

Theorem 2 *The Lagrangean maximum M of the investor's utility:*

$$U_1 = K - \mathcal{L}_1 = K - r_0(1-\theta_1) - (1-\theta_2) \tag{8}$$

is the solution to $r_0\theta_1 + \theta_2 - b_M = 0$*, with* $b_M > 0$*, being the intercept of the tangent to the efficient frontier* $T(\theta_1, \theta_2) = 0$ *between the* L *and* L_∞ *bounds, with* $\theta_1 \geq 0$ *and* $\theta_2 \geq 0$*.*

Proof A family of parallel isometric lines with slope r_0, that is, with the same particular preferences, can be obtained by varying a parameter b over:

$$r_0\theta_1 + \theta_2 = b. \tag{9}$$

Hence, there is only a value $b_M > 0$ such that the line defined by r_0 and b_M is tangent to the efficient frontier $T(\theta_1, \theta_2)$ between L and L_∞. In this tangential point, $r_0\theta_1 + \theta_2 - b_M = T(\theta_1, \theta_2)$ and the condition for the Lagrangean maximum, $U_{\theta_1}/U_{\theta_2} = T_{\theta_1}/T_{\theta_2}$, holds. Then, the maximum utility portfolio is given by r_0 and b_M as shown in Figure 1.

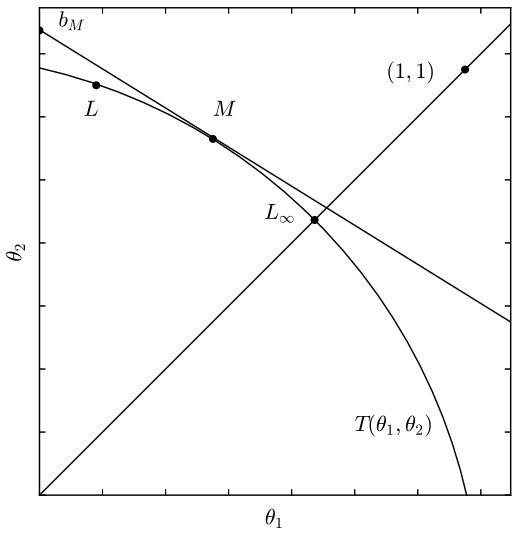

Fig. 1 Normalized efficient frontier $T(\theta_1, \theta_2)$ and position of relevant points in the characterization for Manhattan distances

3.2 Characterization of the solution for Euclidean distances

The Euclidean distance between two points in a bidimensional space is the square root of the sum of the squared differences between the first and second components of each point. Given two points (x_1, y_1) and (x_2, y_2), their

Euclidean distance is calculated as:

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{10}$$

It can be graphically represented by the straight line between two points. The Euclidean distance is non-linear and this fact makes computations within an optimization problem more difficult than in the linear case. In practice, isometric points in terms of Euclidean distances can be represented by ellipses in a E-S space. Indeed, practitioners can graphically derive the optimal portfolio by plotting the ellipse with horizontal semiaxis depending on r_0 that is tangent to the efficient frontier as the following theorem formally states.

Theorem 3 *The Lagrangean maximum M of the investor's utility*

$$U_2 = K - \mathcal{L}_2 = K - \left[r_0^2(1-\theta_1)^2 + (1-\theta_2)^2\right]^{1/2} \tag{11}$$

is the solution to $r_0^2(\theta_1 - 1) + (\theta_2 - 1)^2 - d_M^2 = 0$, with $d_M > 0$, being d_M the semiaxis of the ellipse centered in the ideal point $(1,1)$ and tangent to $T(\theta_1, \theta_2)$, between L and L_∞, with $\theta_1 \geq 0$ and $\theta_2 \geq 0$.

Proof A family of isometric ellipses centered in $(1,1)$, with particular preferences defined by r_0, can be obtained by varying a parameter d over:

$$r_0^2(\theta_1 - 1)^2 + (\theta_2 - 1)^2 = d^2. \tag{12}$$

This family of ellipses is characterized by having the same ratio a/b, between the horizontal semiaxis (a) and the vertical semiaxis (b), which depends only on r_0, since equation (12) can be rewritten as:

$$\frac{(\theta_1 - 1)^2}{d^2/r_0^2} + \frac{(\theta_2 - 1)^2}{d^2} = 1 \tag{13}$$

where $a = d/r_0$ is the horizontal semiaxis and $b = d$ is the vertical semiaxis. Then, $a/b = 1/r_0$ or $b/a = r_0$.

Hence, there is only a value $d_M > 0$ such that the isometric ellipse is tangent to the efficient frontier $T(\theta_1, \theta_2)$. In the tangential point, given by $r_0^2(\theta_1 - 1)^2 + (\theta_2 - 1)^2 - d_M^2 = T(\theta_1, \theta_2)$, the condition for the Lagrangean maximum, defined as $U_{\theta_1}/U_{\theta_2} = T_{\theta_1}/T_{\theta_2}$, holds. Then, the maximum utility portfolio is given by d_M and r_0 as shown in Figure 2.

4 Comparing linear and quadratic metrics

The bounding theorem is characterized by L and L_∞ in the normalized E-S space. Then, the vector $\boldsymbol{e_L}$ starting at L_∞ and ending at L characterizes the bounding theorem. By translating the origin of coordinates from $(0,0)$ to point $L_\infty = (\theta_{1\infty}, \theta_{2\infty})$, this vector $\boldsymbol{e_L}$ presents the following properties:

1. A negative first coordinate of $\boldsymbol{e_L}$ denotes a conservative bias or $r_0 < 1$.

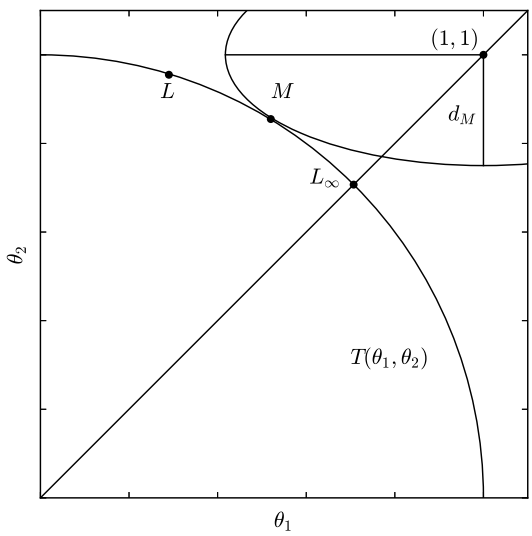

Fig. 2 Normalized efficient frontier $T(\theta_1, \theta_2)$ and position of relevant points in the characterization for Euclidean distances

2. A positive first coordinate of $\boldsymbol{e_L}$ denotes a risky bias or $r_0 > 1$.
3. The norm $||\boldsymbol{e_L}||$ is proportional to r_0.
4. The angle α of $\boldsymbol{e_L}$ with respect to the x-axis is determined by L.

This vector $\boldsymbol{e_L}$ does not only depend on an investor's particular risk preferences, but also on the efficient frontier and its particular form in the E-S space. Next, we use this vector $\boldsymbol{e_L}$ when comparing balance of solutions.

Choosing the appropriate metric $\mathcal{L}_h$ in equation (5) is an ongoing issue in the compromise programming setting (Ballestero, 2007). Manhattan distances tend to outcome imbalanced solutions, i.e., close to the corner solutions $(1, 0)$ and $(0, 1)$ given by the intersection of $T(\theta_1, \theta_2)$ with the x-axis and the y-axis. On the other hand, Euclidean distances produce more balanced solutions.

Definition 1 We say that a solution $R = (\theta_{1R}, \theta_{2R})$ is more balanced than $P = (\theta_{1P}, \theta_{2P})$ if R is closer than P to L_∞, in terms of Euclidean distance, i.e., if $||\boldsymbol{e_R}|| < ||\boldsymbol{e_P}||$, being $\boldsymbol{e_R}$ and $\boldsymbol{e_P}$ vectors from L_∞ to R and L_∞ to P, respectively.

Within the limits set by Ballestero's bounding theorem, we next provide a necessary condition to obtain balanced solutions using metrics $\mathcal{L}_1$ and $\mathcal{L}_2$.

Theorem 4 *Given an efficient frontier $T(\theta_1, \theta_2)$, a point R within L and L_∞ maximizing utility U_2 is more balanced than point $P = (\theta_{1P}, \theta_{2P})$ derived by maximizing U_1, when $\theta_1 > d_M/\sqrt{2}r_0$.*

Proof From Theorem 2, we know that the slope of the tangent determining P is r_0. Then, if S is a more balanced solution than P, then the absolute value of the slope of the tangent line to $T(\theta_1, \theta_2)$ at R must be greater than r_0. The slope of the tangent line to $T(\theta_1, \theta_2)$ at R is given by the first derivative with respect to θ_1 of the iso-ellipse equation:

$$\theta_2' = \frac{d\theta_2}{d\theta_1} = \frac{-r_0^2\theta_1}{[d_M^2 - r_0^2\theta_1^2]^{1/2}} \tag{14}$$

with $0 < d_M < 1$. If $|\theta_2'| > r_0$ then:

$$r_0\theta_1 > [d_M^2 - r_0^2\theta_1^2]^{1/2} \tag{15}$$

which holds when:

$$\theta_1 > \frac{d_M}{\sqrt{2}r_0} \tag{16}$$

5 The balance theorem

From the previous section, we know that solutions maximizing U_2 are more balanced than solutions maximizing U_1 when $\theta_1 > d_M/\sqrt{2}r_0$ holds. This property can be generalized to U_h through the following theorem:

Theorem 5 *The Lagrangean maximum M of the investors utility for metric $\mathcal{L}_h$:*

$$U_h = K - \mathcal{L}_h = K - \left[r_0^h(1-\theta_1)^h + (1-\theta_2)^h\right]^{1/h} \tag{17}$$

between L and L_∞ for investors with risk preferences determined by $r_0 > 0$ is more balanced that the maximum for metric $\mathcal{L}_{h-1}$.

Proof From A8, we know that the condition for a minimum distance to the optimum is $U_{\theta_1}/U_{\theta_2} = T_{\theta_1}/T_{\theta_2}$, which is given by:

$$\frac{dU_h/d\theta_1}{dU_h/d\theta_2} = \frac{r_0 r_0^{h-1}\theta_1^{h-1}}{\theta_2^{h-1}} = \frac{T_{\theta_1}}{T_{\theta_2}} \quad or \quad \frac{\theta_2}{\theta_1} = \left[\frac{r_0 T_{\theta_2}}{T_{\theta_1}}\right]^{\frac{1}{h-1}} \tag{18}$$

which tends to one if h tends to ∞ with $r_0 > 0$, and which is an increasing function of h when $r_0 T_{\theta_2}/T_{\theta_1} < 0$, and a decreasing function of h when $r_0 T_{\theta_2}/T_{\theta_1} > 0$. Then, as long as h increases, the maximum moves closer to L_∞, producing more balanced solutions according to Definition 2.

From that, balance of solution R defined by vector $||\boldsymbol{e_R}(\mathcal{L}_h)||$ depending on metric $\mathcal{L}_h$ is proportional to h, i.e., vector norms are monotonically increasing with h or $||\boldsymbol{e}(\mathcal{L}_{h-1})|| \leq ||\boldsymbol{e}(\mathcal{L}_h)|| \leq ||\boldsymbol{e}(\mathcal{L}_{h+1})||$.

Corollary 1 *The most balanced solutions are produced using the Chebyshev distance $\mathcal{L}_\infty$.*

6 Dealing with discrete efficient frontiers and uncertain risk preferences

In practice, most of the efficient frontiers are discrete and most of the risk preferences are uncertain. Discrete efficient frontiers are the set of E-S points representing each of the available portfolios. On the other hand, risk preferences are uncertain in the sense that they are likely to be expressed in a loosely manner rather than as a single value r_0. For instance, my r_0 is around 1.5, neither higher than 2 nor lower than 1. Here, we translate these expressions into a interval that is used as the main input to Algorithm 1 that selects the best set of compromise solutions. This algorithm outputs the best set of compromise solutions when given: (i) a discrete efficient frontier as a set of n points, namely, $F = \{(\theta_{11}, \theta_{21}), \ldots, (\theta_{1n}, \theta_{2n})\}$; (ii) an interval $[r_{min}, r_{max}]$; (iii) a metric $\mathcal{L}_h$.

Algorithm 1: Approximation algorithm to obtain compromise solutions based on interval risk preferences.

Inputs: Discrete efficient frontier F; risk preference interval $[r_{min}, r_{max}]$; metric $\mathcal{L}_h$;
Output: Interval compromise solutions $M = [M_{min}, M_{max}]$;
Obtain $L_{min} \in F$ for r_{min};
Obtain $L_{max} \in F$ for r_{max};
$D_{min} = h$-distances for points in F ranging in $[L_{min}, L_\infty]$;
$D_{max} = h$-distances for points in F ranging in $[L_{max}, L_\infty]$;
Obtain M_{min} as the point minimizing D_{min};
Obtain M_{max} as the point minimizing D_{max};
$M = [M_{min}, M_{max}]$;

For illustrative purposes, let us consider the efficient frontier from the example given in Ballestero (1998) summarized in Table 1. Assume now that three different investors express their risk preferences using common words. Chances are that risk preferences are expressed as: (i) a declaration of being risky or conservative (recall that the efficient frontier has a risky zone when $\theta_1 > \theta_2$, and a conservative zone when $\theta_1 < \theta_2$); (ii) a threshold value, i.e., greater/less than a particular value; or (iii) as an interval, between a minimum and a maximum value. These expressions have an equivalent representation in the form of an interval $[r_{min}, r_{max}]$, that can be used as an input to Algorithm 1 to obtain the set of compromise solutions that fits their risk preferences. For instance, consider that the following hypothetical investors are willing to minimize distances to the ideal with metric $\mathcal{L}_1$ by means of Algorithm 1:

1. Investor 1: I am a conservative investor but I am unable to specify any particular risk preference, equivalent to $[r_{min}, r_{max}] = [0, 1]$, since the conservative zone ranges in $0 \leq r_0 \leq 1$.

2. Investor 2: I am a risky investor but I am willing to loose no more than 1.5 units of safety to achieve an increase of one unit of profitability, equivalent to $[r_{min}, r_{max}] = [1, 1.5]$.
3. Investor 3: I am a risky investor but I am willing to loose neither more than 2 nor less than 1.5 units of safety to achieve an increase of one unit of profitability, equivalent to $[r_{min}, r_{max}] = [1.5, 2]$.

Following the steps detailed in Algorithm 1, we can determine the set of portfolios that would satisfy each investor. As an example, consider Investor 1 with $[r_{min}, r_{max}] = [0, 1]$. From Theorem 1, we know that $L_{min} = P_1$ and $L_{max} = P_5$. In this case, L_∞ is a point between P_5 and P_6, set to $(0.62, 0.62)$ in the example by Ballestero (1998). Next, we compute Manhattan distances for each point in the efficient frontier ranging in $[P_1, L_\infty]$ and $[P_5, L_\infty]$, as shown in Table 1. The rest of points are excluded from the analysis because the optimum does not lie among them according to Theorem 1. From this set of distances, we compute the minimum values to the ideal to determine the range that satisfies Investor1, namely, any portfolio between P_1 and P_5. Following the same procedure for the rest of investors, we obtain that any portfolio between P_5 to P_7 satisfies Investor 1, and that any portfolio between P_7 to P_8 satisfies Investor 2 as detailed in Table 1.

Table 1 Efficient frontier and satisfactory portfolios (marked with ✓) for different investors.

Portfolio	θ_1	θ_2	$\mathcal{L}_1(r_0 = 0)$	$\mathcal{L}_1(r_0 = 1)$	Investor 1	Investor 2	Investor 3
P_1	0	1	1.00	–	✓	–	–
P_2	0.105	0.991	1.01	–	✓	–	–
P_3	0.267	0.939	1.06	–	✓	–	–
P_4	0.406	0.851	1.15	–	✓	–	–
P_5	0.532	0.736	1.26	0.73	✓	✓	–
P_6	0.669	0.576	–	–	–	✓	–
P_7	0.803	0.389	–	–	–	✓	✓
P_8	0.893	0.232	–	–	–	–	✓
P_9	1	0	–	–	–	–	–

Further insight can be achieved through a graphical representation of the change of the best compromise solution for different values of the risk preference parameter r_0. Two consecutive points in any discrete efficient frontier present different values for θ_1 and θ_2. Then, plotting the variation of the particular values θ_1 or θ_2 of the best solution for r_0 ranging in $[0, \infty]$ shows how the best solution M changes with r_0. Since θ_2 is usually used as the vertical axis variable, here we choose θ_2 to show changes in the best compromise solution. The outcome is a kind of decreasing stairs plot as shown in Figure 3. Just as the efficient frontier representation in Figures 1 and 2 is divided in two areas by the diagonal $\theta_1 = \theta_2$, the stairs plot is divided by a vertical line $r_0 = 1$, delimiting the risky zone and the conservative zone.

The utility of such a plot is double: (i) risky/conservative investors should look for portfolios in the risky/conservative zone according to the their risk

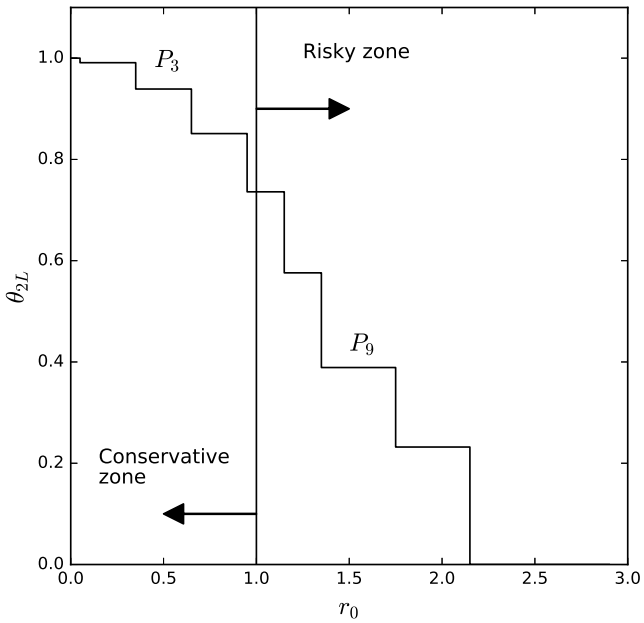

Fig. 3 Stairs plot: safety index θ_{2M} of the best compromise solution M for different risk preferences r_0 and metric $\mathcal{L}_1$

preferences, regardless of whether these preferences are defined by a single value or by an interval statement; (ii) validity of particular portfolios for different ranges of r_0 is immediately assessed by the width of the step of each portfolio, e.g., in Figure 3, portfolio P_7 is valid for a wider range of risk preferences than P_6, thus, P_7 can be considered a more stable portfolio.

Moreover, from the particular characteristics of the stairs plot, an interesting property of discrete efficient frontiers in contrast to continuous frontiers can be derived as follows:

Theorem 6 *Given a discrete efficient frontier* $F = \{(\theta_{11}, \theta_{21}), \ldots, (\theta_{1n}, \theta_{2n})\}$*, of* n *points and a metric* $\mathcal{L}_h$ *with finite* h*, there exists a value* r_{max}*, which depends on* h*, such that the best compromise solution* M *in a normalized space is always* $(1, 0)$ *for all* $r_0 > r_{max}$*.*

Proof For $h = 1$, let $P = (\theta_{1P}, \theta_{2P})$ be the next-to-last point and $(1, 0)$ the last point in F, if $r_{max} = \theta_{2P}/(1-\theta_{1P})$, is the slope of the line passing through P and $(1, 0)$, all isolines with slope greater than r_{max} in absolute value are necessarily tangent to F only in $(1, 0)$. A similar reasoning leads to the same result for $h > 1$.

For instance, in the example from Table 1, $r_{max} = 0.232/(1-0.893) = 2.2$, as reflected in the stairs plot of Figure 3. Recall that we are dealing with discrete efficient frontiers. In the case of continuous efficient frontiers, $r_{max} = \infty$ since the only isoline tangent to $(1, 0)$ is a vertical line with infinite slope. Summarizing, it is important to highlight that: (i) risk preference intervals allow a wider and more flexible characterization than a single value; (ii) even though risk preferences are defined by an interval for a particular investor, the best compromise set of portfolios can be formed by a single portfolio; and (iii) a further step can be taken by considering not only interval but fuzzy risk preferences in which there is no sharp boundary between interval limits.

7 Predicting long-run investor performance

The basic idea in portfolio selection is diversification over a given set of available assets to achieve a maximum return for a certain level of risk. This idea assumes the existence of only one efficient frontier for a given set of available assets. However, chances are that different sets of assets produce different efficient frontiers and, eventually, different long-run performance. For instance, investors may be interested in selecting their best portfolio from assets located either in America, Europe or Asia. Other investors may focus on selecting assets either in the banking, industrial or telecommunications sector. Even within the same sector, investors may establish criteria to select companies according to their size. Determining the location, sector or company size that is expected to perform better and quantitatively estimating such improvement are two interesting challenges that are worth tackling. To this end, we propose the use of a single quality measure, namely, the Area Under the Curve (AUC).

Recall that the closer the portfolio to the ideal point the better the portfolio. Similarly, the closer the whole efficient frontier to the ideal point the better the efficient frontier. From that, a measure of quality of a particular efficient frontier in the normalized E-S space is the AUC, a widely used metric in machine learning (Fawcett, 2006; Hernández-Orallo et al., 2013) for assessing performance. In the context of portfolio selection, we define the AUC for a given efficient frontier $T(\theta_1, \theta_2)$ as:

$$AUC(T, \theta_1, \theta_2) = \int_0^1 T(\theta_1, \theta_2) d\theta_1 \tag{19}$$

ranging in $[0, 1]$ and determining the average quality of portfolios obtained from a given set of available assets. For example, say that a group of investors consider alternative portfolios from assets located in Europe, America and Asia. After normalization, they derive three different efficient frontiers: T_1 for Europe; T_2 for America; and T_3 for Asia, as depicted in Figure 4. Clearly, portfolios in T_1 are better than those in T_2 (and portfolios in T_2 are better than those in T_3) since they are closer to the ideal point $(1, 1)$. A measure that is able to quantitatively characterize which efficient frontier (location) is better (and how much better) in the long-term is the AUC. The group of investors in

the example should invest in Europe since the expected performance should be better due to $AUC(T_1, \theta_1, \theta_2) > AUC(T_2, \theta_1, \theta_2) > AUC(T_3, \theta_1, \theta_2)$. Generalizing, investors considering alternative sets of assets should first derive the efficient frontier for each set, then compute the AUC for each set, and finally select the set of assets with the higher AUC since the expected long-term performance is higher.

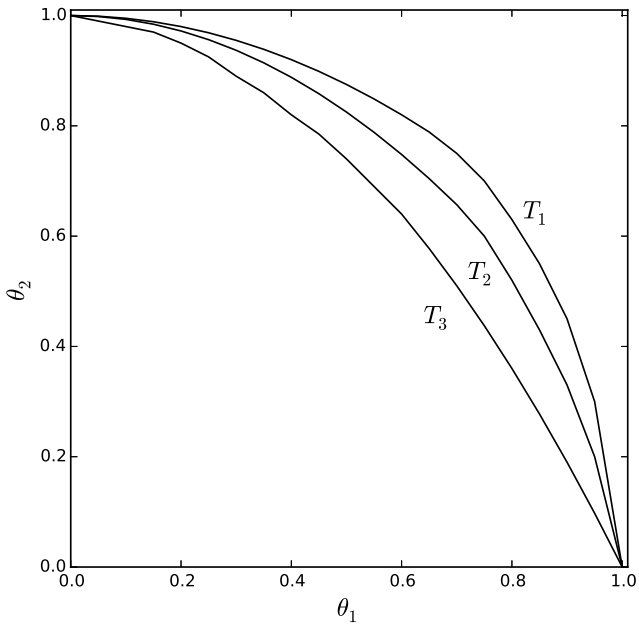

Fig. 4 Different efficient frontiers lead to different long-term performance.

However, when particular risk preferences of investors are considered, we know from Theorem 1 that the set of compromise solutions is bounded by L and L_∞. Then, the rest of available solutions are irrelevant to investors and can be discarded. From that, we propose a more refined measure of the quality for investors with particular risk preferences, which is the Area Between Bounds (ABB), defined as:

$$ABB(T, \tilde{\theta_1}, \tilde{\theta_2}) = \int_0^1 T(\tilde{\theta_1}, \tilde{\theta_2}) d\tilde{\theta_1} \tag{20}$$

where $T(\tilde{\theta_1}, \tilde{\theta_2})$ is a re-scaled efficient frontier when the point L is now the corner solution $(0, 1)$ and L_∞ is the other corner solution $(1, 0)$. The procedure is shown in Figure 5. After deriving L from r_0, a new origin of coordinates is

determined by the intersection of the vertical line containing L and the horizontal line containing L_∞. For meaningful comparisons, the efficient frontier is re-scaled to be fitted in a new normalized E-S space by mapping L to $(0, 1)$ and L_∞ to $(1, 0)$.

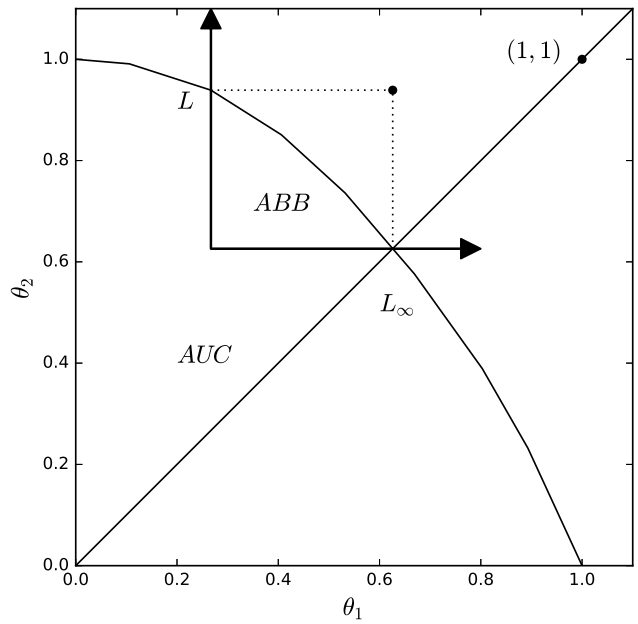

Fig. 5 Two different measures to predict long-run performance: the Area Under the Curve (AUC) and the Area Between Bounds (ABB)

The utility of the ABB metric is better understood through the following example. Assume that two different investors, I_1 and I_2, want to select portfolios from the same set of available assets that produces $T(\theta_1, \theta_2)$ with $AUC = 0.7$. Assume also that I_1 is a risky investor with risk preference given by $r_1 = 1.5$ and that I_2 is a conservative investor with risk preference given by $r_2 = 0.5$. After deriving L for both r_1 and r_2, re-scaling T and computing ABB for both investors, assume that the results yield $ABB_1 = 0.6$ and $ABB_2 = 0.55$ respectively. From that, we infer that investor I_1 would outperform I_2 since $ABB_1 > ABB_2$. This different performance denotes a certain bias of T that would act in favor of risky investors and that can be measured by the proposed metric ABB.

Summarizing, the AUC metric help us predict the long-run performance of investors selecting portfolios from different sets of available assets. On the other hand, the ABB metric help us predict the long-run performance of investors

with particular risk preferences since portfolios located out of the limits L and L_∞ are irrelevant to them.

8 Conclusions

In this paper, we characterize Ballestero's bounding theorem in portfolio selection for investors with particular risk preferences. We claim this characterization is necessary since investors are more interested in finding a particular optimal solution rather than a set between two bounds. Visual techniques allow to enhance the understanding of decision-making techniques based on compromise programming. Even though this paper particularizes on two specific loss functions, namely, $\mathcal{L}_1$ for Theorem 2 and $\mathcal{L}_2$ for Theorem 3, it is likely that no other loss function is used in practice due to computational and interpretation problems. Further insight in the characterization of the bounded compromise set is given. More precisely, we show the utility of:

1. Theorem 4 determining the necessary condition to obtain more balanced solutions using the linear and quadratic common metrics in compromise programming. This condition depends on the particular risk preferences of investors.
2. Theorem 5 showing the balance relation of solutions obtained using different loss functions $\mathcal{L}_h$. This theorem implies that the balance of solutions depending on metric $\mathcal{L}_h$ monotonically increases with h.
3. An algorithm and a stairs plot to show the variation of solutions for discrete efficient frontiers and risk preferences expressed as a single value r_0 or as an interval statement.
4. Two quality measures to predict long-run investor performance, the Area Under the Curve and the Area Between Bounds which is directly linked to Ballestero's bounding theorem.

Finally, it is important to point out that visual techniques help enhance the understanding of complex decision-making problems such as portfolio optimization. Investors with particular risk preferences can rely on this kind of techniques to make better decisions. Plots and metrics allow a more informed decision-making process. Discrete efficient frontiers and risk preference intervals for particular criteria introduces a new practical perspective to multiobjective decision problems that deserves further research.

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