Master's Thesis - 2019/ 2020

Notes on differentiability, norm-preserving extensions, and renormings in Banach spaces

Author: **Christian Cobollo**

Tutor: Antonio José Guirao
Tutor: Vicente Montesinos

DEPARTAMENTO DE MATEMÁTICA APLICADA

Acknowledgements

I cannot put in words how grateful I am to my master's thesis supervisors Professor Antonio José Guirao and Professor Vicente Montesinos, for their encouragement and invaluable help. Without their commitment and effort, this Memoir would not have been possible.

Also, I would like to express my gratitude to the Universitat de València and the Universitat Politècnica de València, for providing very good working conditions during my master's degree academic year, specially to all professors and staff involved on INVESTMAT, that even in this atypical year, they have done their best to complete the course as normally as possible.

Last, but not least, I would like to thank my family, friends and colleagues for their emotional support and encouragement.

Contents

Introduction

This Memoir is a work on the geometry of Banach spaces. The central subject is differentiability and renorming.

0.1 The results

There are essentially **three aspects where we succeeded in saying something new**. Besides this, many of our minor contribution along this Memoir are also new. We shall mention them in due term.

1.- A recent paper by E. Jordá and A. M. Zarco elaborates on the classical Šmulyan Lemma for Gâteaux and Fréchet differentiability of the norm at a nonzero point x by checking the way x exposes a selected subset of the dual unit ball —Šmulyan Lemma does it by taking the whole dual unit ball. Our contribution is the following: (a) **We prove that the "new" Gâteaux test is nothing else but the classical Šmulyan test**, and (b) that **the Fréchet test can still be significantly improved**. Jordá–Zarco's approach rely on the concept of James boundary. Ours, on the concept of a 1-norming subset of B_{X^*}. Of course, every James boundary is a 1-norming set —but not conversely.

2.-Very recently, the latest E. Oja, together with T. Viil and D. Werner, proved a substantial extension of a theorem of F. Sullivan that showed that if the norm of a separable Banach space X is Hahn–Banach smooth (a property that ensures the uniqueness of the norm-preserving extension of any continuous linear functional on X to its bidual), then it can be renormed to have a stronger property, the so called totally smoothness (the possibility of a unique norm-preserving extension to the bidual of X of any continuous linear functional on any closed subspace of X). Oja, Viil, and Werner, achieved this in the context of all weakly compactly generated Banach spaces (WCG) —a class substantially larger that the class of all separable Banach spaces. They did it by using a very useful renorming of WCG spaces through a transfer method. **We prove that, surprisingly, the result holds *for any Banach space***, and that **our renorming result applies to a strictly larger class of spaces that the WCG spaces having a HBS equivalent**

norm. Maybe our result has the only merit of observing how a very powerful renorming result of M. Raja gives the sought renorming. Anyhow, we think that our result deserves to be known. **It has been published in RACSAM**. The reference is in the bibliography ([CoGuiMon20]).

3.-An important result of A. Moltó, J. Orihuela, S. Troyanski, and M. Valdivia, ensures that, under a special property of the duality mapping, the space has a LUR renorming. One of the essential applications is that Banach spaces whose norm is Fréchet differentiable and its dual norm is Gâteaux differentiable have a LUR renorming. **We slightly enlarge the setting of the application**: A property of the norm introduced by F. Sullivan is the very rotundness. This property is stronger than the strict convexity, and yet it does not imply the Gâteaux differentiability of the dual norm —even more, the norm can be very rotund and the dual may not be Gâteaux renormable. **Under the Fréchet differentiability of the norm and, simultaneously, its very rotundness, the space has a LUR renorming.**

0.2 The techniques and auxiliary results

Chapter 1 provides a number of definitions and results needed along the Memoir. We collect, for the reader's convenience, some about differentiability, the Minkowski's functional, and duality. Of course, those can be found in any book devoted to Banach space theory. Here we select some for later reference.
The main idea that pervades the three aforementioned directions of research, and that in a sense unifies the whole Memoir, is that some special subsets of the dual unit ball of a Banach space are enough for getting important properties of the norm (differentiability, renorming, etc.). Those subsets are James boundaries and, more generally, 1-norming subsets and extreme points. Šmulyan characterization of Gâteaux and Fréchet differentiability at a point x on the unit sphere computes the intersection of the value-1 hyperplane (of a half-space, respectively) defined by x with the dual unit ball. Instead, we consider their intersection with a 1-norming subset of this ball. In the second case, we hope to get a set of small diameter. For this, **we need an enhancement of the classical Namioka–Bourgain Superlemma**. This is presented in Section 2.1 in Chapter 2, and **we believe that our version may have, besides the application that follows it, a certain interest by itself**. Since some more or less well-known results on James boundaries and 1-norming subsets are needed, we comment on those in Section 2.2. We give examples and counterexamples for the connections between extreme points, James boundaries and 1-norming sets. Not all of our remarks are explicit in the literature. The end —and the most important part— of this first chapter concerns the comments and extensions on the Jordá–Zarco result. We distinguish between

the Gâteaux —treated later— and the Fréchet case, considered first. Lemma 2.12 is usually overlooked in the regular literature. The main result is Theorem 2.13. **The set of examples and remarks following the Gâteaux and Fréchet versions should be considered also as our genuine contribution**. For example, we could not find in the literature an example of a Banach space $(X, \|\cdot\|)$ such that the set of all extreme points of B_{X^*} should not be w^*-closed in S_{X^*} —a condition used by Jordá and Zarco. We provide an example of this situation. Proposition 2.19 is also our contribution.

R. R. Phelps has been one of the most influential mathematician in the area of the geometry of Banach spaces. He throughfully studied norm-attaining functionals, differentiability, extensions, extremality, dentability, integral representation and much more. A seminal paper of him [Ph60] was devoted to the so-called Hahn–Banach extensions —extensions of continuous linear functionals that preserve the norm. In order to properly present our theorem on totally smooth renorming, we need to inform the reader about Phelps' U property. This is what Chapter 3 is mainly devoted to. **Many of the remarks and observations about the U and other properties are ours. In the spirit of the Memoir, we introduce property wU and explore its features in Theorem 3.22. This should be consider as our contribution**.

Then we pass to Chapter 4, where we introduce the concept of Hahn–Banach smoothness, very smoothness and very rotundness of the norm of a Banach space, and explore their connections. The first part relies on the work of F. Sullivan, although **many of the proofs are different from the original Sullivan's presentation**. For example, we show that **it is not necessary to use the Principle of Local Reflexivity** to get some of Sullivan's results. The proof of Proposition 4.3 is ours, and Proposition 4.4 does not appear in Sullivan's paper —although we show that it can be improved, as we do in Proposition 4.10, that probably is known. In order to complete some information, we list some equivalent properties to the Asplundness of a Banach space. The concepts of very smoothness and very rotundness are introduced for properly formulate our extension of a renorming result of Moltó, Orihuela, Troyanski and Valdivia (Theorem 5.5). We prove a result of Dixmier (Theorem 4.18) mentioned by Sullivan without a proof. The last part develops techniques regarding the coincidence of several topologies on the unit sphere of the dual unit ball, a question that turns out to be central in the Memoir. **Those results are new**.
In order to properly present Raja's result, we need to consider the fundamental results on non-linear transfer developed by Moltó, Orihuela, Troyanski, and Valdivia, in their relevant Memoir [MOTV09] quoted in the reference list. Here, we just mention the results, and provide, as it was mentioned above, **an improvement of one of their fundamental contributions. Our formulation covers**

some cases that are not included in their theorem.
The next goal is to offer a self-contained approach to Raja's result. This is done in Chapter 7. The proof is very technical, and to understand its position in the whole renorming theory of Banach spaces we need first to consider projectional resolutions of the identity (PRI's) , V. Zizler result on LUR renorming for spaces having a PRI, then PRI's in duals to Asplund spaces, all of them the subject of a preliminary Chapter 6. This is just a technical one containing the most important results about PRI's. As we mentioned, Chapter 7 contains Raja's fundamental result.
All this allows for a precise formulation of our renorming theorem by a totally smooth norm that **extends the aforementioned partial results and completes the understanding of the connection between both properties**. This is done in Chapter 8. It uses all the techniques presented above. We end by discussing the similarities and differences between the HBS and wHBS properties used in our main theorem in this chapter.
A final chapter presents briefly **the conclusions and some suggestions for future work**.

Chapter 1

Preliminaries

1.1 Differentiability

A norm on a Banach space X is a real-valued function defined on X. Thus, we may consider the directional differentiability of this function at points in X (a function of the directions), asking for its continuity and linearity.
The directional derivative $D_v f(x)$ of $f : U \to Y$, where U is a nonempty open subset of X, and X and Y are normed spaces, at a point $x \in X$ in the direction $v \in X$, is the limit (in case that it exists)

$$\lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

In the case of the norm, it certainly appears as

$$\lim_{t\to 0} \frac{\|x+tv\|-\|x\|}{t}.$$

When the function $v \to D_v\|\cdot\|(x)$ is linear and continuous, i.e., an element in X^*, then we get the concept of Gâteaux differentiability. When uniformity in directions is requested, we get the concept of Fréchet differentiability. It turns out that the Gâteaux differentiabilty at a point $x \in X$ is equivalent to the fact that x exposes B_{X^*} (at a single point $x^* \in S_{X^*}$). Thus, an alternative definition of Gâteaux differentiability is the following:

Definition 1.1

*Let $(X, \|\cdot\|)$ be a Banach space. The norm $\|\cdot\|$ is said to be **Gâteaux differentiable** at $x \in X$ if there exists a unique $x^* \in S_{X^*}$ such that*

$$\langle x, x^*\rangle = \|x\|.$$

A given point $x \in X$ will be called *a Gâteaux differentiability point* whenever the norm is Gâteaux differentiable at x. Trivially, 0 cannot be a differentiability point for any norm, and by homogeneity, it is clear that if $0 \neq x$ is a Gâteaux differentiability point, then λx with $\lambda \neq 0$ is also a Gâteaux differentiability point. Thus, it is said that the norm $\|\cdot\|$ is Gâteaux differentiable if it is Gâteaux differentiable at every point in $S_{(X,\|\cdot\|)}$.
Another key concept related to differentiability is the duality mapping.

Definition 1.2

Let X be a Banach space. Define the multivalued mapping $\partial\|\cdot\| : X \longrightarrow 2^{X^*}$ by

$$\partial\|\cdot\|(x) := \{x^* \in B_{X^*} : \langle x, x^*\rangle = \|x\|\}.$$

This mapping is called the **duality mapping**

By the Hahn–Banach Theorem, the set $\partial\|\cdot\|(x)$ is always nonempty. Another way to get the same conclusion is to observe that x is a linear w^*-continuous function defined on the dual X^*. By the Alaoglu–Bourbaki Theorem, B_{X^*} is a w^*-compact set, so x attains its supremum (i.e., its norm) on (B_{X^*}, w^*) by the Weierstrass Theorem.

This map is always **$\|\cdot\|$-w^*-upper semicontinuous** at every point $x_0 \in X$, i.e., for any w^*-open subset W of X^* that contains $\partial\|\cdot\|(x_0)$, there exists a $\|\cdot\|$-open subset V of x_0 such that $\partial\|\cdot\|(x) \subset W$ for all $x \in V$. Also, observe that $\partial\|\cdot\|(x)$ is by definition a convex w^*-closed set on the dual unit ball, so by the Alaoglu-Bourbaki Theorem, a w^*-compact set. Thus, the duality map is also **w^*-compactly valuated**. Those multivalued maps that are upper semicontinuous and compactly valuated are called **usco** for short.

Hence, Gâteaux differentiability of the norm $\|\cdot\|$ on a point $x \in X$ can be expressed in terms of the duality map as follows: a point $x \in X$ is a Gâteaux differentiability point of $\|\cdot\|$ if and only if $\partial\|\cdot\|(x)$ is a single point. By the previous paragraph, in this case we have that $\partial\|\cdot\|$ is $\|\cdot\|$-w^*-upper semicontinuous and single-valued, so $\partial\|\cdot\|$ is $\|\cdot\|$-w^*-continuous.

Similarly to the Gâteaux differentiability case, Šmulyan's Lemma provides a geometric characterization of Fréchet differentiability of the norm, which we take as an alternative definition.

Definition 1.3

Let $(X, \|\cdot\|)$ be a Banach space, and let $S(B_{X^*}, x, \delta) := \{x^* \in B_{X^*} : \langle x, x^*\rangle \geq 1-\delta\}$ be the **slice** *of B_{X^*} defined by x and δ. The norm $\|\cdot\|$ is said to be* ***Fréchet***

differentiable *at* $x \in X$ *if*

$$\operatorname{diam} S(B_{X^*}, x, \delta) \xrightarrow{\delta \to 0} 0.$$

Diestel also provides in [Di75] a characterization in terms of the continuity for the duality map: the norm $\|\cdot\|$ is Fréchet differentiable if and only if its duality map $\partial\|\cdot\|$ is $\|\cdot\|$-$\|\cdot\|$-continuous.

1.2 Duality

Along this Memoir we shall need some basic facta on the structure of normed and Banach spaces. First, for a subspace M of a normed space $(X, \|\cdot\|)$, we shall consider M always endowed with the restriction of the norm $\|\cdot\|$, denoted again by $\|\cdot\|$, if there is no risk of misunderstanding. When M is, moreover, closed, then X/M is endowed with a norm (its "canonical quotient norm") defined by $\|\widehat{x}\| := \inf\{\|x\| : x \in \widehat{x}\}$, where $\widehat{x} = x + M$ is the class containing x. It is easy to see that $\|\widehat{x}\| = \operatorname{dist}(x, M)$ for all $x \in X$.
The following simple facta about duality of subspaces and quotients of normed spaces are certainly well known: Let $(X, \|\cdot\|)$ be a normed space, and let $M \subset X$ be a subspace. Then M^* is isometrically isomorphic to $X^*/M^\perp$. The isometry associates to each $m^* \in M^*$ the class of all continuous linear extensions of m^* to X. This is a consequence of the fundamental Hahn–Banach extension theorem. When M is, moreover, closed, then $(X/M)^*$ is isometrically isomorphic to $M^\perp$. The isometry associates to each $f \in (X/M)^*$ the (well-defined) function $x^* \in X^*$ such that $x^* = f \circ q$, where $q : X \to X/M$ is the canonical quotient mapping.

1.3 Minkowski's functional

Now, for renorming purposes, we introduce a fundamental tool for obtaining equivalent norms on a Banach space.

Definition 1.4

Let X *be a normed space and* $C \subset X$*. We define the* **Minkowski functional of** C *as* $\mu_C : X \longrightarrow [0, +\infty]$*, by*

$$\mu_C(x) = \begin{cases} \inf\{\lambda > 0 : \ x \in \lambda C\} & \text{if } \ \{\lambda > 0 : \ x \in \lambda C\} \neq \emptyset, \\ +\infty & \text{if } \ \{\lambda > 0 : \ x \in \lambda C\} = \emptyset. \end{cases}$$

Observe that by the definition, this kind of function will be finite if the set C is **absorbing**, i.e., for every $x \in X$ there exists $\lambda > 0$ such that $x \in \lambda C$. This is the case when C is a neighbourhood of the origin. Indeed, if C is a convex neighbourhood of the origin, then its Minkowski functional μ_C is a finite, non-negative, positively homogeneous, subadditive and continuous function from X to $\mathbb{R}$, that also satisfies

$$\text{Int}(C) = \{x \in X : \ \mu_C(x) < 1\},$$
$$\overline{C} = \{x \in X : \ \mu_C(x) \leq 1\}$$

(see [FHHMZ11, Lemma 2.11]). As said before, we are interested in Minkowski functionals as a way of creating new equivalent norms.

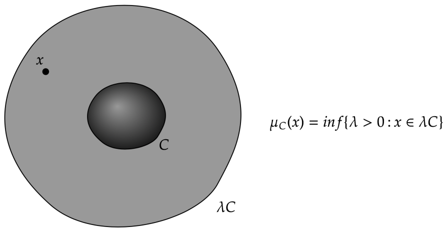

Figure 1.1: The Minkowski functional of $C \subset X$.

Proposition 1.5

Let X be a Banach space and $C \subset X$ a nonempty closed absolutely convex subset which is bounded and has the origin as an interior point. Then, its Minkowski functional μ_C defines an equivalent norm on X.

This proposition provides a way for obtaining equivalent norms on Banach spaces, so this works also for dual spaces X^*. However, most of the time the interest on renorming dual spaces lies in the fact of inducing an equivalent norm in its predual X, which may inherit some good properties. This is a powerful tool, due to the fact that many properties of norms in X can be characterized by some behaviour in the dual space, and even, in some circumstances it is easier to renorm a dual space, given the extra properties these have (for example, the unit ball of a dual space is always w^*-compact, by the Alaoglu–Bourbaki Theorem). Those norms in dual spaces that induce equivalent norms in its predual are called **equivalent dual norms**.

The following result (sometimes not excesively explicit in the literature) characterizes thoses norms on a dual that are dual norms.

Proposition 1.6

Let X be a Banach space, and let $|||\cdot|||$ be an equivalent norm on X^*. The following statements are equivalent:
(i) $|||\cdot|||$ *is a dual norm;*
(ii) $B_{(X^*,|||\cdot|||)}$ *is w^*-closed;*
(iii) $\overline{S_{(X^*,|||\cdot|||)}}^{w^*} \subset B_{(X^*,|||\cdot|||)}$*;*
(iv) $|||\cdot||| : X^* \longrightarrow \mathbb{R}$ *is w^*-lower semicontinuous (i.e., all sets $\{x^* \in X^* : |||x^*||| \leq r\}$ with $r \in \mathbb{R}$ are w^*-closed).*

1.4 Higher duals, reflexivity

Let X be a normed space. As usual, we identify X with a linear subspace of its bidual X^{**}. This identification is done via the canonical embedding $J : X \longrightarrow X^{**}$, defined by $Jx(x^*) := x^*(x)$, for every $x^* \in X^*$. It is clear that J is a linear isometry from X into X^{**}. If X is Banach, then JX is a closed subspace of X^{**}. The space X is said to be **reflexive** whenever $JX = X^{**}$. Although in general X is not complemented into its bidual, Dixmier proved that X^* is always complemented in X^{***}. In fact, $X^{***} = X^* \oplus X^{\perp}$, and the associated projection from X^{***} onto X^* has norm 1.

Chapter 2

On the Superlemma

The main purpose of this chapter is to present a very recent result of E. Jordá and A. M. Zarco on differentiability of the norm of a Banach space, to comment on their Gâteaux version (in Subsection 2.3.2), and to improve their Fréchet version (in our Theorem 2.13). All this will be accomplished in Section 2.3. The Gâteaux version is seen to be equivalent to the classical Smulyan's Lemma —and so independent of any norm-attaining considerations. Our theorem for the Fréchet version does not need the Gâteaux differentiability hypothesis. Moreover, our approach is more "geometric" than the original one. The proof we provide here for the Fréchet version is based on a little precision on the classical Namioka–Bourgain "Superlemma" (Lemma 2.2) that is worth to present in full. So we devote to it Section 2.1. We do not claim that our "improved" version of this last famous lemma is absent from the literature. However, we may say that it does not appear this way in the most conspicuous available references —in fact in none we had the opportunity to check—, so we may honestly claim that the observation that enhances the lemma and the way the addition is proved is genuinely our contribution. Since we shall use not only James boundaries, but also norming and 1-norming sets, as well as the set of norm-attaining functionals, we devote the entire Section 2.2 to those concepts and their connections. Again, we claim that those are many times either disperse or not clearly presented in the available literature, so it should be considered as our contribution, too. In particular, we provide Example 2.3.2, that seems absent from any reference.

2.1 The Namioka–Bourgain Superlemma

The following plays an important role in arguments related to dentability and the Radon–Nikodým property. In the literature it is known as the "Superlemma". A good account can be found in [Di84, page 157]. Despite the wonderful exposition of the result there, we think that it is possible to give an alternative —equivalent— more natural and simple formulation. For the sake of completeness, we shall formulate the two versions; we shall give below the (simple) proof (in Proposition 2.3) of the fact that the two statements are equivalent.

Lemma 2.1 (Namioka–Bourgain (see, e.g., [Di84, page 157]))

Let C, C_0, and C_1 be closed convex bounded subsets of the Banach space X, and let $\varepsilon > 0$. Suppose that
(i) *C_0 is a subset of C having diameter less than ε.*
(ii) *C is not a subset of C_1.*
(iii) *C is a subset of $\overline{\text{conv}}\,(C_0 \cup C_1)$.*
Then there is a slice of C having diameter less than ε that intersects C_0.

Here is our streamlined (equivalent) version of this result (see Figure 2.1):

Lemma 2.2 (Namioka–Bourgain)

Let X be a Banach space. Let C_0 and C_1 be two closed convex subsets of X. Assume that C_1 is bounded and that for some $\varepsilon > 0$, $\text{diam}\,(C_0) < \varepsilon$. Assume too that $C_0 \not\subset C_1$. Then there exists a slice of $\overline{\text{conv}}\,(C_0 \cup C_1)$ that cuts C_0 and has diameter less than ε.

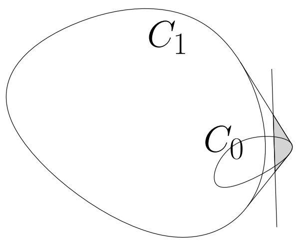

Figure 2.1: A small section of the set $\overline{\text{conv}}\,(C_0 \cup C_1)$ when C_0 is small

Proposition 2.3

Lemmata 2.1 and 2.2 are equivalent.

Proof: Indeed, assume that Lemma 2.1 holds, and let C_0 and C_1 be two sets as in Lemma 2.2. Put $C := \overline{\text{conv}}\,(C_0 \cup C_1)$. Then C is a closed, convex, and bounded subset of X. Certainly, C is not a subset of C_1 (due to the fact that $C_0 \not\subset C_1$). For sure, C is a subset of $\overline{\text{conv}}\,(C_0 \cup C_1)$ (they are equal). Thus, Lemma 2.1 gives a slice S of C that cuts C_0 and whose diameter is less than ε, and this is the conclusion of Lemma 2.2.
Assume now that Lemma 2.2 holds, and let C_0, C_1, and C, be three subsets of X as in Lemma 2.1. Lemma 2.2 applied to C_0 and C_1 provides a slice S of $\overline{\text{conv}}\,(C_0 \cup C_1)$ having diameter less than ε and cutting C_0. Then it cuts C. Since $C \subset \overline{\text{conv}}\,(C_0 \cup C_1)$, the diameter of $S \cap C$ is less than ε, too. ■

Figure 2.1 hints at the slight improvement of the Superlemma that we mention as Lemma 2.4 below. It is not explicit in the original version —collected by Diestel—, and we think that it may help in some situations. For example, we shall apply this to a result on norming subsets that improves a result in [JoZa19]. We believe that without our version of the Superlemma, it will be more difficult to achieve this extension. To be sure, the proof of our slight improvement of the Namioka–Bourgain Superlemma is not too far from the original proof of Namioka and its modification by Bourgain. The last part of the statement turns out to be essential for applying the previous results to differentiability arguments (in particular, see the proof of Theorem 2.13 below).

In the proof, if $x^* \in X^*$ and $S \subset X$, we shall use the notation $\sup\langle S, x^*\rangle$ instead of $\sup\{\langle s, x^*\rangle : s \in S\}$.

Lemma 2.4 (Superlemma (modified))

Let X be a Banach space. Let C_0 and C_1 be two closed convex subsets of X. Assume that C_1 is bounded and that for some $\varepsilon > 0$, $\text{diam}\,(C_0) < \varepsilon$. Assume too that $C_0 \not\subset C_1$. Then there exists a slice S of $\overline{\text{conv}}\,(C_0 \cup C_1)$ such that $S \cap C_0 \neq \emptyset$, $S \cap C_1 = \emptyset$, $\text{diam}\,(S) < \varepsilon$, and if $x_0^* \in S_{X^*}$ satisfies $\sup\langle C_1, x_0^*\rangle < \sup\langle C_0, x_0^*\rangle$, then the slice S can be defined by x_0^*.

Proof: For $0 \le r \le 1$, let (see Figure 2.2)

$$D_r := \{(1-\ell_\infty)c_0 + \ell_\infty c_1 : \ r \le \ell_\infty \le 1,\ c_0 \in C_0,\ c_1 \in C_1\}. \tag{2.1}$$

Put $C := \overline{\text{conv}}\,(C_0 \cup C_1)$. Let us list some properties of the sets D_r.
(i) Clearly, D_r is convex for all $0 \le r \le 1$.
(ii) $D_0 = \text{conv}\,(C_0 \cup C_1)$ (so $C = \overline{D_0}$), and $D_1 = C_1$.

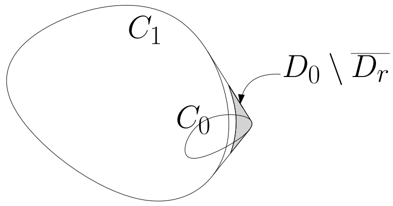

Figure 2.2: In gray, the sets $D_0 \setminus \overline{D_r}$

(iii) $C_0 \not\subset \overline{D_r}$ for $0 < r \leq 1$. In order to prove this, and having in mind that C_1 is closed and convex, and that $C_0 \not\subset C_1$, we can find $x^* \in S_{X^*}$ such that

$$s_1 := \sup\langle C_1, x^*\rangle < \sup\langle C_0, x^*\rangle =: s_0.$$

If $x \in D_r$, then $x = (1-\ell_\infty)x_0 + \ell_\infty x_1$ for some $x_0 \in C_0$, $x_1 \in C_1$, and $r \leq \ell_\infty \leq 1$. Thus,

$$\begin{aligned}\langle x, x^*\rangle &= (1-\ell_\infty)\langle x_0, x^*\rangle + \ell_\infty\langle x_1, x^*\rangle \\ &\leq (1-\ell_\infty)s_0 + \ell_\infty s_1 \in [s_1, s_0), \text{ for } x \in D_r,\ 0 < r \leq 1. \end{aligned} \tag{2.2}$$

We can then find $c_0 \in C_0$ such that $\langle c_0, x^*\rangle > (1-\ell_\infty)s_0 + \ell_\infty s_1$. Thus, noticing that (2.2) also holds for any $x \in \overline{D_r}$, we get $c_0 \notin \overline{D_r}$. This proves (iii).
(iv) $D_0 \setminus \overline{D_r}$ is dense in $\overline{D_0} \setminus \overline{D_r}$ ($= C \setminus \overline{D_r}$). To show this, notice first that this is trivial for $r = 0$. Assume $r > 0$. Observe that $C \setminus \overline{D_r} \neq \emptyset$, thanks to (iii) above. Let $x \in C \setminus \overline{D_r}$. Find a sequence $\{x_n\}$ in $D_0 = \operatorname{conv}(C_0 \cup C_1)$ such that $x_n \to x$. Since $x \notin \overline{D_r}$, there is $n_0 \in \mathbb{N}$ such that $x_n \notin \overline{D_r}$ for $n \geq n_0$. Note that $x_n \in D_0 \setminus \overline{D_r}$ for $n \geq n_0$. This proves (iv).
Let $x \in D_0 \setminus \overline{D_r}$ for some $r > 0$. Then $x = (1-\ell_\infty)c_0 + \ell_\infty c_1$, for some $c_0 \in C_0$, $c_1 \in C_1$, and $0 \leq \ell_\infty < r$. Note that

$$\|x - c_0\| = \|\ell_\infty(c_0 - c_1)\| < r.\sup\{\|c_0 - c_1\| :\ c_o \in C_0\ \ c_1 \in C_1\} = r\delta,$$

where $\delta := \sup\{\|c_0 - c_1\| :\ c_0 \in C_0,\ c_1 \in C_1\}$ (> 0). Since $D_0 \setminus \overline{D_r}$ is dense in $C \setminus \overline{D_r}$, we have $\operatorname{diam}(C \setminus \overline{D_r}) \leq \operatorname{diam} C_0 + 2r\delta$. Find $0 < r < 1$ such that $\operatorname{diam}(C_0) + 2r\delta < \varepsilon$.
Take $c_0 \in C_0 \setminus \overline{D_r}$ (for the previous r). Find $x^* \in S_{X^*}$ that separates c_0 and $\overline{D_r}$. This defines a section of C, say S, that does not intersect $\overline{D_r}$ (thus $\operatorname{diam}(S) < \varepsilon$), and $x_0 \in S$. Note that $S \cap C_1 = \emptyset$, since $C_1 \subset \overline{D_r}$.
Let us show now the last part of the statement: Assume that for some $x_0^* \in S_{X^*}$ we have

$$\sup\langle C_1, x_0^*\rangle < \sup\langle C_0, x_0^*\rangle.$$

Then x_0^* can be used for defining the slice S in the conclusion of the lemma. In other words, x_0^* defines a slice S of $\overline{\text{conv}}\,(C_0 \cup C_1)$ with the following properties:

(i) $\text{diam}\,(S) < \varepsilon$.

(ii) $S \cap C_0 \neq \emptyset$.

(iii) $S \cap C_1 = \emptyset$.

Indeed, notice that in the proof above we got $r > 0$ with $\text{diam}\,(C \setminus \overline{D_r}) < \varepsilon$. Then we found x^* that separated some previously chosen $c_0 \in C_0 \setminus \overline{D_r}$ (we know that $C_0 \setminus \overline{D_r} \neq \emptyset$) from $\overline{D_r}$. If $x = (1-\ell_\infty)c_0 + \ell_\infty c_1 \in D_r$, where $r \leq \ell_\infty \leq 1$, $c_0 \in C_0$, and $c_1 \in C_1$, we have $\langle x, x_0^* \rangle = (1-\ell_\infty)\langle c_1, x_0^* \rangle + \ell_\infty \langle c_1, x_0^* \rangle \leq (1-\ell_\infty)s_0 + \ell_\infty s_1$, where $s_0 := \sup\langle C_0, x_0^* \rangle$ and $s_1 := \sup\langle C_1, x_0^* \rangle$. Since $0 < r \leq \ell_\infty \leq 1$ and $s_1 < s_0$, we get $\langle x, x_0^* \rangle \leq (1-r)s_0 + rs_1 < s_0$, and so there exists $c_0 \in C_0$ such that x_0^* separates $\overline{D_r}$ and c_0. This shows that x_0^* does the job from the very beginning. ■

We need also a w^*-version of Lemma 2.4:

Lemma 2.5 (Superlemma (modified), w^*-version)

Let $(X, \|\cdot\|)$ be a Banach space, and let C_0 and C_1 be convex w^*-compact subsets of X^*. Assume that $C_0 \not\subset C_1$, and that for some $\varepsilon > 0$ we have $\text{diam}\,(C_0) < \varepsilon$. Put $C := \overline{\text{conv}}^{\,w^*}(C_0 \cup C_1)$. Then there exists a w-slice S of C such that $\text{diam}\,(S) < \varepsilon$, $S \cap C_0 \neq \emptyset$, and $S \cap C_1 = \emptyset$. Moreover, if $x_0 \in S_X$ satisfies $\sup\langle x_0, C_1 \rangle < \sup\langle x_0, C_0 \rangle$, then the slice S can be defined by x_0.

Proof: (Sketch) We follow the notation in the proof of Lemma 2.4. Given $0 \leq r \leq 1$, put $D_r := \{(1-\ell_\infty)c_0^* + \ell_\infty c_1^* : \ c_0^* \in C_0, \ c_1^* \in C_1, \ r \leq \ell_\infty \leq 1\}$. We proceed as in the proof of Lemma 2.4, by taking now w^*-closures instead, and considering w^*-density. The Separation Theorem gives $x \in X$. The computations needed are fairly similar. ■

In Section 2.3 we shall give some applications of the previous results. Section 2.2 introduce and discuss some relevant definitions needed.

2.2 On James boundaries and 1-norming sets

Let us motivate Theorem 2.13 below. The basic Šmulyan Lemma for Fréchet differentiability says that *the norm $\|\cdot\|$ of a Banach space X is Fréchet differentiable at a given point $x_0 \in S_X$ if, and only if, x_0 "cuts" slices*

$$S(x_0, B_{X^*}, \delta) = \{x^* \in B_{X^*} : \ \langle x_0, x^* \rangle > 1 - \delta\}$$

from B_{X^*} *whose* $\|\cdot\|$*-diameter goes to* 0 *with* δ. It will be a neat improvement for testing Fréchet differentiability of the norm to check the diameter of slices *cut from some distinguished subset of* B_{X^*} —instead of slices of the full ball. Theorem 2.13 is a modified version of a result of E. Jordá and A. M. Zarco (and in fact it is equivalent to their theorem, as it is explained below, see Remark 2.14). What is new is the proof and the formulation. The statement and the argument we propose is independent of the fact that a James boundary realizes the supremum on the dual unit ball of elements in X —and stresses just the 1-norming character of the set involved; moreover, it exhibits a geometric feature common to all versions of the classical Šmulyan's Lemma. Let us state here there result (only in the Fréchet differentiable case; we shall discuss the Gâteaux differentiable one later on).

To rightly formulate the statement, we need some concepts. If $(X, \|\cdot\|)$ is a Banach space, a subset J of B_{X^*} is said to be a **James boundary for** X if every $x \in X$ attains its norm at some point in J. This is a very important concept, whose origin is, undoubtedly, the one of an extreme point. Indeed, on a compact convex set of a locally convex space, every continuous linear functional attains its supremum at an extreme point —a result preceded by the essential Krein–Milman theorem (see below), showing that every such a set has at least an extreme point, what amounts to say that it is the closed convex hull of the set of its extreme points. Thus, the set $\mathrm{Ext}(B_{X^*})$ of all extreme points of B_{X^*} is the prototype of a James boundary. However, there are James boundaries for X *disjoint* from $\mathrm{Ext}(B_{X^*})$. An example is the set $J := \{x^* \in \ell_\infty(\Gamma) : \ \|x^*\|_\infty = 1, \ |\mathrm{supp}\ x^*| \leq \aleph_0\}$. This is a James boundary for $\ell_1(\Gamma)$, where Γ is an uncountable index set. Since $\mathrm{Ext}(B_{\ell_\infty(\Gamma)})$ is the set of all $x^* \in \ell_\infty(\Gamma)$ with all coordinates ± 1, the two sets $\mathrm{Ext}(B_{\ell_\infty(\Gamma)})$ and J are disjoint. The reason for J being a James boundary is that every element in $\ell_1(\Gamma)$ has a countable support, as it is easy to see.

Remark 2.6

Despite the fact that the term "James boundary" applies to a subset of B_{X^*}, it is clear that a point $x \in X$, $x \neq 0$, attains its supremum on B_{X^*} at a point of S_{X^*}. Thus, most of the time we shall restrict ourselves to James boundaries contained in S_{X^*}. This is in contrast with the concept of a 1-norming subset of B_{X^*}, to be discussed below. In that case, we shall prefer to work with subsets of B_{X^*}. ®

A related concept is that of a **norming** subset of B_{X^*}. A set $N \subset B_{X^*}$ is said to be α-norming, for some $\alpha \geq 1$, whenever $\|x\|_N \leq \|x\| \leq \alpha \|x\|_N$ for every $x \in X$, where $\|x\|_N := \sup\langle x, N\rangle$. When N is α-norming for some $\alpha \geq 1$, we say that N **is norming**. Notice that, in particular, N is 1-norming whenever $\|x\| = \sup\langle x, N\rangle$ for all $x \in X$. Quite often, and when speaking about a subspace N of X^*, we say that the subspace N is $\boldsymbol{\alpha}$**-norming** whenever $B_N := \{x^* \in N : \ \|x\| \leq 1\}$ is

α-norming.

The following is an easy characterization of 1-norming sets:

Proposition 2.7

Let $(X, \|\cdot\|)$ be a Banach space. A subset N of B_{X^*} is 1-norming for X if, and only if, $\overline{\text{conv}}^{w^*}(N) = B_{X^*}$.

Proof: Let N be 1-norming. Assume that $\overline{\text{conv}}^{w^*}(N) \neq B_{X^*}$ and find $x_0^* \in B_{X^*} \setminus \overline{\text{conv}}^{w^*}(N)$. The Separation Theorem gives $x_0 \in S_X$ such that

$$\sup\langle x_0, \overline{\text{conv}}^{w^*}(N)\rangle < \langle x_0, x_0^*\rangle \ (\leq \|x_0\| = 1).$$

This violates the 1-norming character of N.
Assume now that $\overline{\text{conv}}^{w^*}(N) = B_{X^*}$. By continuity and convexity, it is clear that for any $x \in X$, $\sup\langle x, N\rangle = \sup\langle x, \overline{\text{conv}}^{w^*}(N)\rangle = \sup\langle x, B_{X^*}\rangle = \|x\|$, and the conclusion follows. ■

The following corollary gives another characterization of 1-norming subsets of B_{X^*}, this time in terms of the set of extreme points of B_{X^*}. It is a straightforward consequence of the Krein–Milman theorem and its "converse", the Milman's theorem. The first one ensures that *every nonempty compact convex subset of a locally convex space is the closed convex hull of the set of all its extreme points*—in particular, that this last set is nonempty. The second says that *if K is a nonempty compact convex subset of a locally convex space, and $K = \overline{\text{conv}}(S)$ for some $S \subset K$, then* $\text{Ext}(K) \subset \overline{S}$. For details, see, e.g., [FHHMZ11, Theorems 3.65 and 3.66].

Corollary 2.8

Let $(X, \|\cdot\|)$ be a Banach space. A subset N of B_{X^*} is 1-norming if, and only if, $\text{Ext}(B_{X^*}) \subset \overline{N}^{w^*}$, where $\text{Ext}(B_{X^*})$ denotes the set of all extreme points of B_{X^*}.

Proof: If N is 1-norming then, by Proposition 2.7, $\overline{\text{conv}}(N)^{w^*} = B_{X^*}$, and Milman's theorem concludes that $\text{Ext}(B_{X^*}) \subset \overline{N}^{w^*}$. On the other hand, if $\text{Ext}(B_{X^*}) \subset \overline{N}^{w^*}$, then, by the Krein–Milman theorem,

$$B_{X^*} = \overline{\text{conv}}^{w^*}(\text{Ext}(B_{X^*})) \subset \overline{\text{conv}}^{w^*}(\overline{N}^{w^*}).$$

It is obvious that $\overline{\text{conv}}^{w^*}(\overline{N}^{w^*}) = \overline{\text{conv}}^{w^*}(N)$, and the conclusion follows from Proposition 2.7. ■

Remark 2.9

1. Of course, if $J \subset B_{X^*}$ is a James boundary for X, then $J \cap S_{X^*}$ is 1-norming (it is clearly nonempty).

2. Certainly, there are 1-norming sets that are not James boundaries. An almost trivial example is given by a w^*-dense subset N of the open dual unit ball. Another, this time in the unit sphere, is the following: Let $(X, \|\cdot\|)$ be a Banach space whose dual norm is strictly convex. Let $x_0 \in S_X$, and let $\partial\|\cdot\|(x_0) = \{x_0^*\}$. Then, the set $N := B_{X^*} \setminus \{x_0^*\}$ is 1-norming, and certainly it is not a James boundary for X, since x_0 does not attain its norm on N. A third one is the following: If $(X, \|\cdot\|)$ is a nonreflexive Banach space, we consider the Banach space $(X^*, \|\cdot\|^*)$. Naturally, $B_X \subset B_{X^{**}}$ is a 1-norming set. It cannot be a James boundary, due to James' compactness theorem: *if every $x^* \in X^*$ attains its norm on B_X, then X is reflexive.*

3. On the other hand, a w^*-closed 1-norming set N of B_{X^*} is a James boundary. Indeed, given $x \in S_X$ and $n \in \mathbb{N}$, find $x_n^* \in N$ such that $\langle x, x_n^*\rangle > 1 - 1/n$. The sequence $\{x_n^*\}$ so obtained has a w^*-cluster point x^*. Certainly, $\langle x, x^*\rangle = 1$, and $x^* \in N$ due to the fact that N is w^*-closed. This proves that N is a James boundary. Three examples of 1-norming sets which are not w^*-closed were given in Item 2 above.

4. As a consequence, if $N \subset B_{X^*}$ is 1-norming for X, then $\overline{N}^{w^*}$ is a James boundary for X. Indeed, if $N \subset M \subset B_{X^*}$ and N is 1-norming, then obviously M is 1-norming, too, and Item 3 applies.

5. As we mentioned above, there is a connection —a subtle one— between the set $\mathrm{Ext}(B_{X^*})$ of all the extreme points of B_{X^*} and any James boundary, and between this two sets and any 1-norming subset of B_{X^*}. Of course, $\mathrm{Ext}(B_{X^*})$ is a James boundary, and then a 1-norming subset. As a consequence of the Milman's theorem mentioned above, if $A \subset B_{X^*}$ is a James boundary, or if A is a 1-norming set for X, then $\mathrm{Ext}(B_{X^*}) \subset \overline{A}^{w^*}$. Above, we gave an example of a James boundary in S_{X^*} that is disjoint from $\mathrm{Ext}(B_{X^*})$. ®

2.3 Some applications

In Subsections 2.3.1 and 2.3.2 below we shall check Fréchet and Gâteaux differentiability of the norm of a Banach space at a point, respectively, by using Šmulyan's approach. Of course, the relevant part is that as soon as $\partial\|\cdot\|(x_0)$ (or, more generally, the slices cut by some $x_0 \in S_X$ from B_{X^*}), has some specific property, then $\|\cdot\|$ is Gâteaux or Fréchet differentiable at x_0. A "more efficient" (or "less expensive", in a sense) method is to observe the properties of $\partial\|\cdot\|(x_0) \cap J$ or the slices that x_0 cuts from J, where J is some subset of B_{X^*}. The smaller the set J, the more efficient will be the method. This is the purpose of the next two subsections. However, it must be pointed out from the beginning that the Gâteaux version we give in Theorem 2.17 in Subsection 2.3.2 is equivalent to the classical Šmulyan's Lemma, reducing Proposition 2.5 in [JoZa19] (for its Gâteaux version) to a particular case of this lemma.

2.3.1 Fréchet differentiability of the norm

A result of E. Jordá and A. M. Zarco on Fréchet differentiability

The following result appears in [JoZa19] as an essential tool in dealing with differentiability of vector-valued mappings. It contains only the Fréchet version of their theorem. The Gâteaux version will be treated in Subsection 2.3.2 below.

Theorem 2.10 (Jordá and Zarco (Fréchet version), [JoZa19]**)**

Let $J \subset B_{X^*}$ be a James boundary for a Banach space $(X, \|\cdot\|)$. Assume that $\|\cdot\|$ is Gâteaux differentiable at $x \in S_X$. Let x^* be the differential of $\|\cdot\|$ at x. Assume that every sequence $\{x_n^*\}$ in $J \cap S_{X^*}$ which w^*-converges to x^* also converges in norm. Then $\|\cdot\|$ is Fréchet differentiable at x.

Improving the Jordá–Zarco Fréchet differentiability result

The first thing to do will be to change the "sequential" condition in Theorem 2.10 by a more geometric one concerning diameters of slices. This formulation is closer in spirit to the classical Šmulyan test for differentiability. However, this change will turn out to be not only cosmetic, but will allow us to get rid off the Gâteaux differentiability assumption in the statement of the result, leading to a neat improvement of it (Theorem 2.13 below). Let us formulate the equivalence in the form of a simple proposition (observe that for $\delta > 0$, $x^* \in J \cap S_{X^*} \cap S(x, B_{X^*}, \delta)$; in particular, the set $J \cap S_{X^*} \cap S(x, B_{X^*}, \delta)$ is nonempty):

Proposition 2.11

Let $(X, \|\cdot\|)$ be a Banach space, $x \in S_X$, and $\emptyset \neq A \subset X^*$. Assume that x is bounded above on A and that x attains its supremum on A at a single point $x^* \in A$. Then the two following statements are equivalent:

(i) *Every sequence $\{x_n^*\}$ in A which w^*-converges to x^* also converges in norm.*
(ii) $\operatorname{diam}(S(x, A, \delta)) \to 0$ *as $\delta \to 0$, where $S(x, A, \delta) := \{x^* \in A : \langle x, x^* \rangle > s - \delta\}$ and $s := \sup\langle x, A\rangle$.*

Proof: Assume that (ii) holds. Then, given $\varepsilon > 0$ we can find $\delta > 0$ such that $\operatorname{diam}(S(x, A, \delta)) < \varepsilon$. Let $\{x_n^*\}$ be a sequence in A that w^*-converges to x^*. Thus, there exists $n_0 \in \mathbb{N}$ such that $x_n^* \in S(x, A, \delta)$ for all $n \geq n_0$. Since $x^* \in S(x, A, \delta)$, we get $\|x_n^* - x^*\| < \varepsilon$ for all $n \geq n_0$. This proves $x_n^* \to x^*$ in norm.
Assume now that $\operatorname{diam}(S(x, A, \delta)) \not\to 0$ as $\delta \to 0$. Then, there exists $\varepsilon > 0$ such that $\operatorname{diam}(S(x, A, \delta)) > \varepsilon$ for all $\delta > 0$. Obviously, $x^* \in S(x, A, \delta)$ for all $\delta > 0$. By letting $\delta = 1/n$ for $n \in \mathbb{N}$, we may find $x_n \in S(x, A, 1/n)$ such that $\|x_n^* - x^*\| > \varepsilon/2$. This provides a sequence $\{x_n^*\}$ that does not $\|\cdot\|$-converges to x^*. ■

The following lemma is almost trivial. We state it and provide its proof here because it is frequently overlooked in the literature.

Lemma 2.12

Let $\varepsilon > 0$ and let $(X, \|\cdot\|)$ be a Banach space.
(i) *If $A \subset X$ satisfies* $\operatorname{diam}(A) < \varepsilon$, *then* $\operatorname{diam}(\overline{A}^{w^*}) < 2\varepsilon$, *where $\overline{A}^{w^*}$ denotes the closure of A in (X^{**}, w^*) and the diameter is computed in $\|\cdot\|^{**}$.*
(ii) *If $A \subset X^*$ satisfies* $\operatorname{diam}(A) < \varepsilon$, *then* $\operatorname{diam}(\overline{A}^{w^*}) < 2\varepsilon$, *where w^* denotes the topology $w(X^*, X)$ on X^*.*

Proof: Fix $a_0 \in A$. Note then that, in both cases, $A \subset B(a_0, \varepsilon)$.
(i) Certainly, $\overline{A}^{w^*} \subset \overline{B(a_0, \varepsilon)}^{w^*} = BB(a_0, \varepsilon)$, where $BB(a_0, \varepsilon)$ denotes here the closed unit ball in X^{**} centered at a_0 and having radius ε. The result follows.
(ii) In this case is even simpler: $B(a_0, \varepsilon)$ is already w^*-closed, so $\overline{A}^{w^*} \subset B(a_0, \varepsilon)$, and the result follows. ■

Notice that, due to Proposition 2.7, if $N \subset B_{X^*}$ is a 1-norming set for X, and $S := S(x, B_{X^*}, \delta)$ is a slice of B_{X^*} defined by $x \in X$, then $S \cap N \neq \emptyset$.

The following is the main result of this Subsection.

Theorem 2.13

Let $(X, \|\cdot\|)$ be a Banach space. Let $N \subset B_{X^*}$ be a 1-norming set. If $\operatorname{diam}(N \cap S(x, B_{X^*}, \delta)) \to 0$ *as* $\delta \to 0$, *then* $\|\cdot\|$ *is Fréchet differentiable at* x.

Proof: Without loss of generality, we may assume $x \in S_X$.
Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $S := S(x, B_{X^*}, \delta)$, then $\operatorname{diam}(N \cap S) < \varepsilon$ (notice that $N \cap S \neq \emptyset$). Put $C_1 := \overline{\operatorname{conv}}^{w^*}(N \setminus S)$, and $C_0 := \overline{\operatorname{conv}}^{w^*}(N \cap S)$. These two sets C_0 and C_1 are w^*-closed, convex subsets of B_{X^*}, and if $\delta > 0$ is small enough, they are both nonempty. Lemma 2.12 shows that $\operatorname{diam}(C_0) < 2\varepsilon$.

Claim $B_{X^*} = \overline{\operatorname{conv}}^{w^*}(C_1 \cup C_0)$. To prove the Claim recall (see Proposition 2.7 above) that $\overline{\operatorname{conv}}^{w^*}(N) = B_{X^*}$. Then, given $x^* \in B_{X^*}$ there exists a net $\{c_i^*\}$ in $\operatorname{conv}(N)$ that w^*-converges to x^*. Let $c^* \in \operatorname{conv}(N)$, say $c^* = \sum_{k=1}^p \ell_{\infty k} n_k^*$, where $\ell_{\infty k} \in [0,1]$, $n_k^* \in N$ for all $k = 1, 2, \ldots, p$, and $\sum_{k=1}^p \ell_{\infty k} = 1$. Assume that $n_k^* \in N \setminus S$ for $k = 1, 2, \ldots, q$, and $n_k^* \in S$ for all $k = q+1, \ldots, p$. Then

$$c^* = \left(\sum_{j=1}^q \ell_{\infty j}\right) \sum_{k=1}^q \frac{\ell_{\infty k}}{\sum_{j=1}^q \ell_{\infty j}} n_k^* + \left(\sum_{j=q+1}^p \ell_{\infty j}\right) \sum_{k=q+1}^p \frac{\ell_{\infty k}}{\sum_{j=q+1}^p \ell_{\infty j}} n_k^* = \left(\sum_{j=1}^q \ell_{\infty j}\right) c_1^* + \left(\sum_{j=q+1}^p \ell_{\infty j}\right) c_0^* \quad (2.3)$$

where $c_1^* \in C_1$ and $c_0^* \in C_0$. If this is applied to each c_i^*, we get $c_i^* = \ell_{\infty i,1} c_{i,1}^* + \ell_{\infty i,0} c_{i,0}^*$ for all $i \in I$, where $c_{i,0}^* \in C_0$ and $c_{i,1}^* \in C_1$, $\ell_{\infty i,1} \in [0,1]$, and $\ell_{\infty i,0} \in [0,1]$, and moreover $\ell_{\infty i,1} + \ell_{\infty i,0} = 1$ (thus, $c_i^* \in \operatorname{conv}(C_1 \cup C_0)$), for each $i \in I$. Passing to a subnet if necessary, we may assume that $\ell_{\infty i,1} \to \ell_{\infty 1}$, $\ell_{\infty i,0} \to \ell_{\infty 0}$, $c_{i,1}^* \to c_1^*$ $(\in C_1)$ and $c_{i,0}^* \to c_0^*$ $(\in C_0)$. This shows that $x^* \in \overline{\operatorname{conv}}^{w^*}(C_1 \cup C_0)$, as we wanted to prove.
Finally, Lemma 2.5 above shows then that there exists $x_0 \in S_X$ and a slice $S_0 := S(x_0, B_{X^*}, \delta_0)$ of B_{X^*} such that $\operatorname{diam}(S_0) < 2\varepsilon$. Due to the fact that $\sup\langle x, C_1\rangle < \sup\langle x, C_0\rangle$, it is possible to take $x_0 = x$. The Šmulyan Lemma concludes that $\|\cdot\|$ is Fréchet differentiable at x. ■

Remark 2.14

Theorem 2.13 obviously implies Theorem 2.10. This is due to Proposition 2.11 and the fact that every James boundary J satisfies that $J \cap S_{X^*}$ is a 1-norming set (see Item 1 in Remark 2.9). ®

2.3.2 Gâteaux differentiability of the norm

The Gâteaux differentiability of the norm on a Banach space $(X, \|\cdot\|)$ at a given point $x \in S_X$ is checked by looking at the points of B_{X^*} exposed by x (this is, essentially, the Gâteaux differentiability version of the Šmulyan Lemma, see, e.g., [FHHMZ11, Corollary 7.22]).
The following result appears in [JoZa19]. It gives a sufficient condition for Gâteaux differentiability of the norm at a given point $x \in X$ by checking exposure of some special subset by x, in the same spirit as in Subsection 2.3.1, where the Fréchet differentiability case was presented.

Theorem 2.15 (Jordá and Zarco (Gâteaux version), [JoZa19])
Let $(X, \|\cdot\|)$ be a Banach space. Assume that there is a James boundary $J \subset B_{X^*}$ such that $\overline{J}^{w^*} \cap S_{X^*} \subset J$ and for a certain $x \in X$, the set $\partial\|\cdot\|(x) \cap J$ is a single point. Then $\|\cdot\|$ is Gâteaux differentiable at x.

We claim that Theorem 2.15 is, in fact, nothing else than the classical Šmulyan's Lemma for Gâteaux differentiability. In any case, let us discuss the kind of James boundary mentioned in its statement.

Remark 2.16

1. For a James boundary $J \subset S_{X^*}$, notice that the condition used on J in Theorem 2.15 clearly amounts to request that J should be closed in the topological space (S_{X^*}, w^*). Of course, not all James boundaries in S_{X^*} satisfy this condition. There are James boudaries contained in S_{X^*} that are not w^*-closed. An example is given at the beginning of Section 2.2: The set $J := \{x = (x_\gamma)_{\gamma\in\Gamma} \in \ell_\infty(\Gamma) : \ |x_\gamma| = 1 \text{ for all } \gamma \in \Gamma;\ \text{supp}\, x \leq \aleph_0\}$ is a James boundary for the space $X := \ell_1(\Gamma)$, where Γ is an uncountable set. It is a subset of $S_{X^*} = S_{\ell_\infty(\Gamma)}$, and it is not w^*-closed in S_{X^*}. The reason is that the net $\{\sum_{\gamma\in F} e^*_\gamma :\ F \subset \Gamma,\ F \text{ finite},\ <\}$, where $e^*_\gamma = \xi_{\{\gamma\}}$, indexed by the set $\mathcal{PF}(\Gamma)$ of all the finite subsets of Γ partially ordered by inclusion, w^*-converges to $e \in S_{X^*}$, where $e(\gamma) = 1$ for all $\gamma \in \Gamma$.

   Notice that the Banach space $X := (c_0, \|\cdot\|_\infty)$ has the property that $E := \text{Ext}B_{X^*} = \{\pm e_n :\ n \in \mathbb{N}\}$ is not w^*-closed. In fact, $\overline{E}^{w^*} = E \cup \{0\}$. In particular, E is w^*-closed in S_{X^*}. On the other hand, if K is a compact topological space and $X := (C(K), \|\cdot\|_\infty)$, then $\text{Ext}B_{X^*} = \{\pm\delta_k :\ k \in K\}$, where δ_k is the Dirac delta of the element k. This set is w^*-compact.

2. Notice that Theorem 2.15 may fail if no extra condition is required on the James boundary J —and this makes a difference with respect to Theorem 2.10. Indeed, let $J \subset S_{X^*}$ has the property that $J \cap \partial\|\cdot\|(x)$ is a singleton for every $x \neq 0$ in X. This can always be done by just a reduction argument. Then, by the very definition $\partial\|\cdot\|(x) \cap J$ is a singleton for every $x \neq 0$ in X, although maybe the norm was not Gâteaux differentiable at some $x \in X$. Ⓡ

Example: Inspired by the example of c_0 in Remark 2.16 above we are able to provide an example of a Banach space $(X, \|\cdot\|)$ where the set of all extreme points of B_{X^*} *is not closed in* (S_{X^*}, w^*). The setting is quite general. Let $(X, \|\cdot\|)$ be any infinite-dimensional Banach space such that its dual is strictly convex (any infinite-dimensional separable space will do the job after renorming, see, e.g., [DGZ93, Theorem II.2.6]). Let $x_0 \in 2S_X$. The set $\operatorname{conv}(B_{(X,\|\cdot\|)} \cup \{x_0\} \cup \{-x_0\})$ is closed, bounded, absolutely convex, and it contains B_X, so it is the closed unit ball of an equivalent norm $|||\cdot|||$ in X. Notice that $|||\cdot|||$ is neither Gâteaux differentiable at x_0 nor at $-x_0$. Put $H_\alpha := \{x^* \in X^* : \langle x_0, x^*\rangle = \alpha\}$, where $\alpha \in \mathbb{R}$. This is a translate of the w^*-closed hyperplane H_0. It is simple to see that the dual closed unit ball is $B_{(X^*,|||\cdot|||^*)} = \{x^* \in B_{(X^*,\|\cdot\|^*)} : |\langle x_0, x^*\rangle| \leq 1\}$ (see Figure 3.1 for the closed unit ball of $|||\cdot|||$ and its dual unit ball).

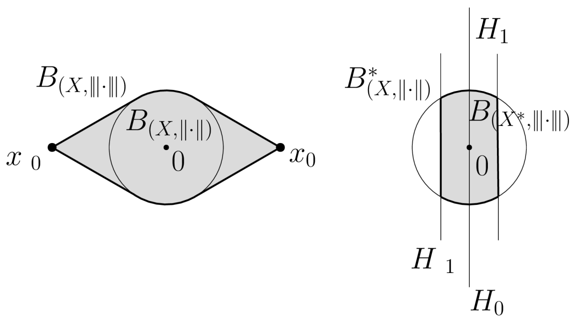

Figure 2.3: The closed unit ball and its dual of an equivalent norm

The idea of the following construction is that $F_1 := B_{(X^*,|||\cdot|||^*)} \cap H_1$ is a "face" of $B_{(X^*,|||\cdot|||^*)}$, a set that is homeomorphic to a bounded neighborhood of 0 in H_0. We have two alternatives:

The first one consists in applying the Josefson–Nissenzweig theorem to the dual space H_0 (it is the dual space of $X/\operatorname{span}(x_0)$) in order to see that we can find a sequence $\{x_n^*\}$ in $S_{(X^*,|||\cdot|||^*)}$ (in fact, in $B_{(X^*,\|\cdot\|^*)} \cap H_1$) that w^*-converges to a point that is in $S_{(X^*,|||\cdot|||^*)}$ and is not an extreme point of $(B_{X^*}, |||\cdot|||^*)$. By scaling, it is possible to select the sequence in $\operatorname{Ext}B_{(X^*,|||\cdot|||^*)}$.

The second one is simpler: Let us provide the details: First, notice that $G_1 := H_1 \cap \operatorname{Int}B_{(X^*,\|\cdot\|^*)} \neq \emptyset$. This is easy: Since $\sup\langle x_0, \operatorname{Int}B_{(X^*,\|\cdot\|^*)}\rangle = 2$, we certainly are able to find $h_1^* \in G_1$. Note that the boundary ∂F_1 of F_1 in the space $(H_1, \|\cdot\|^*)$ is $F_1 \setminus G_1$. Let us work now in the topological space (H_1, w^*), where w^* denotes the restriction of the w^*-topology on X^* to H_1. A simple argument about the unboundedness of the w^*-neighbourhood of 0 shows that h_1^* is in the closure in (H_1, w^*) of the boundary ∂F_1. It is also trivial (due to the fact that $\|\cdot\|^*$ in X^* is strictly convex), that all points in ∂F_1 are extreme points of $B_{(X^*,\|\cdot\|^*)}$ (and so also of $B_{(X^*,|||\cdot|||^*)}$). However, h_1^* is a point in $S_{(X^*,|||\cdot|||^*)}$ that is not an extreme point of $B_{(X^*,|||\cdot|||^*)}$. In the particular case that X was a separable Banach space, the point h_1^* is even the w^*-limit of a sequence in $\mathrm{Ext}B_{(X^*,|||\cdot|||^*)}$.

In order to enhance Theorem 2.17, notice that the same argument shows that even the set $\mathrm{Ext}B_{X^*} \cap \mathrm{NA}(X)$ can fail to be closed in (S_{X^*}, w^*). Indeed, it is enough to use our construction together with the Bishop–Phelps theorem and to notice that we were assuming that $\|\cdot\|^*$ was strictly convex, so any perturbation of points in $\mathrm{Ext}B_{X^*}$ to get norm-attaining functionals on S_{X^*} remains in the set $\mathrm{Ext}B_{X^*}$ (certainly, the argument by using Bishop–Phelps must be applied to the dual space H_0 and to its homeomorphic set H_1). ✓

Recall that for $x \in X\setminus\{0\}$, the set $\partial\|\cdot\|(x) = \{x^* \in S_{X^*} : \langle x^*, x\rangle = \|x\|\}$ is always a non-empty, w^*-closed convex subset of the unit ball (and so, a w^*-compact set) and —as the Krein–Milman theorem states— it is the w^*-closed convex hull of the set of all its extreme points. In particular, $\partial\|\cdot\|(x)$ is not reduced to a single point if, and only if, it contains more that one extreme point. By the way, a simple argument shows that any of those extreme points is an extreme point of B_{X^*}.

Example: Regarding Theorem 2.17 below, it is important to notice that $\mathrm{Ext}(B_{X^*})$ can be strictly bigger than $\mathrm{Ext}(B_{X^*}) \cap \mathrm{NA}(X)$. For instance, take any separable nonreflexive space X. Thus, X^* can be renormed with a strictly convex dual norm $|||\cdot|||$, so every point in the new dual unit sphere $S_{(X^*,|||\cdot|||)}$ would be an extreme point of $B_{(X^*,|||\cdot|||)}$. However, as X is not reflexive, James' compactness theorem ensures that there must be a point in $S_{(X^*,|||\cdot|||)}$ that does not attains its norm. ✓

The kind of James boundaries considered in Theorem 2.15 always contain the set $\mathrm{Ext}B_{X^*}$. It may be then tempting to use this smaller set in checking Gâteaux differentiability. Even better, we may think of using the (in general, smaller, see Example 2.3.2 above) set $\mathrm{Ext}B_{X^*} \cap \mathrm{NA}(X)$. However, all this attempts (including Theorem 2.15) amount to nothing better than the Šmulyan's Lemma. Indeed, observe that the following version *is* this classical result. To be convinced look at its "proof":

Theorem 2.17 (Šmulyan)

Let X be a Banach space and let $x \in X$. Then, $\|\cdot\|$ is Gâteaux differentiable at x if, and only if, x exposes $\mathrm{Ext} B_{X^*} \cap NA(X)$.

Proof: If $\|\cdot\|$ is Gâteaux differentiable at x, then Šmulyan's Lemma says that $\partial\|\cdot\|(x)$ is a singleton, that certainly belongs to $\mathrm{Ext} B_{X^*} \cap \mathrm{NA}(X)$. So x exposes $\mathrm{Ext} B_{X^*} \cap \mathrm{NA}(X)$. Conversely, if x exposes $\mathrm{Ext} B_{X^*} \cap \mathrm{NA}(X)$, notice that $\partial\|\cdot\|(x)$ is the w^*-closed convex hull of its extreme points (and all of them belong to $\mathrm{NA}(X)$), so $\partial\|\cdot\|(x)$ is a singleton. Šmulyan's lemma concludes that $\|\cdot\|$ is Gâteaux differentiable at x. ■

Let us finalize this section by including Proposition 2.19 below, that complements Šmulyan's Lemma and [JoZa19, Lemma 2.2]. It will be preceded by a simple result —usually formulated for sequences, although valid for nets with the same proof, that we shall omit— followed by a remark in the same direction.

Lemma 2.18

Let $x \in X$ be a Banach space and $x \in X$ be a point where $\|\cdot\|$ is Gâteaux differentiable with $\partial\|\cdot\|(x) = \{x^*\}$. Then, a net $\{x_i^* : i \in I, \prec\}$ in B_{X^*} is w^*-convergent to x^* if, and only if, $\langle x, x_i^*\rangle \to \langle x, x^*\rangle$.

Proposition 2.19

Let X be a Banach space, and $N \subset B_{X^*}$ be a 1-norming set. Then, the following are equivalent:
(i) *The norm $\|\cdot\|$ is Gâteaux differentiable at x.*
(ii) *For every pair $\{x_n^*\}_{n=1}^\infty$ and $\{y_n^*\}_{n=1}^\infty$ of sequences in N such that $\langle x, x_n^*\rangle \to 1$ and $\langle x, y_n^*\rangle \to 1$, we have $\{x_n - y_n\}_{n=1}^\infty \xrightarrow{w^*} 0$.*
(iii) *For every sequence $\{x_n^*\}_{n=1}^\infty \subset N$ such that $\langle x, x_n^*\rangle \to 1$, then $\{x_n^*\}$ is w^*-convergent.*

Proof: That (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) is obvious from Lemma 2.18. The reverse implications can be proven by contradiction, using the same geometric argument. Assume that $\|\cdot\|$ is not Gâteaux differentiable at x. Thus, $\partial\|\cdot\|(x)$ contains at least two different extreme points, say e_1^* and e_2^*. By the Separation Theorem, there exists a hyperplane defined by $y \in X$ that separates e_1^* and e_2^* (see Figure 2.4). We may assume, without loss of generality, that $\langle y, e_1^* - e_2^*\rangle > 0$.

Corollary 2.8 ensures that $e_1^*, e_2^* \in \overline{N}^{w^*}$, so we can find two sequences $\{x_n^*\}_{n=1}^\infty$ and $\{y_n^*\}_{n=1}^\infty$ in N that converge to e_1^* and e_2^*, respectively, on x and y, i.e.,

$$\langle x, x_n^* \rangle \to \langle x, e_1^* \rangle, \text{ and } \langle x, y_n^* \rangle \to \langle x, e_2^* \rangle.$$

Thus,

$$\langle x, x_n^* \rangle \to 1 \text{ and } \langle x, y_n^* \rangle \to 1.$$

So we get a contradiction with the statement (ii), because

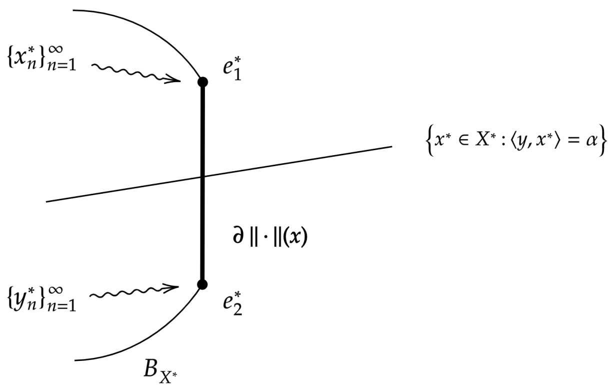

Figure 2.4: Sketch of the proof of Proposition 2.19

$$\langle y, x_n^* - y_n^* \rangle \to \langle y, e_1^* - e_2^* \rangle > 0,$$

hence the sequence $\{x_n^* - y_n^*\}_{n=1}^\infty$ does not w^*-converge to 0. The contradiction with statement (iii) is even easier, since the alternated sequence $\{x_1^*, y_1^*, x_2^*, y_2^*, ...\}$ cannot w^*-converge (indeed, the sequences $\{x_n^*\}_{n=1}^\infty$ and $\{y_n^*\}_{n=1}^\infty$ are eventually separated by y). ■

Remark 2.20

Notice that the previous result is still valid for nets with almost the same proof. In (ii) we must impose that both nets $\{x_\alpha^*\}$ and $\{y_\alpha^*\}$ must be directed by the same index set. For the construction of the alternated net required in the proof of (iii) $\Longrightarrow$ (i), it is enough to notice that a partially ordered set Δ can be *duplicated* by defining the set $\Delta \times \{0, 1\}$ and endow it with the natural lexicographic partial order. Thus, the net $\{z_\beta^*\}_{\beta \in \Delta \times \{0,1\}}$ such that $z_\beta^* = x_\alpha^*$ if $\beta = (\alpha, 0)$ and $z_\beta^* = y_\alpha^*$ if $\beta = (\alpha, 1)$ plays the rôle of the alternated sequence there.

Chapter 3

Phelps' Unique Extension Property

3.1 Introduction

The main result of a very recent paper [OVW19] provides a renorming theorem for the class of the weakly compactly generated (WCG) Banach spaces. It shows that every WCG Banach space that has property HBS can be renormed to have the stronger property TS. This is an improvement of an older result of Sullivan, that shows that the result holds for separable spaces. The relevant definitions are the following: The norm $\|\cdot\|$ of a Banach space is **Hahn–Banach smooth** (HBS) if every continuous linear functional on X has a unique Hahn–Banach (i.e., norm-preserving) extension to its bidual space X^{**} (Definition 4.1). A result that seems to be attributed to G. Godefroy (see Proposition 4.3) characterizes norms with HBS as those such that on the dual unit sphere the w and the w^* topologies coincide. The norm $\|\cdot\|$ is said to be **totally smooth** (TS) whenever for every closed subspace M of X and every continuous linear functional on M, there exists a unique Hahn–Banach extension to X^{**}. This last property is strictly more general than the first one, if only because a result of Taylor and Foguel (Theorem 3.14 below) that ensures that TS is equivalent to the addition of the properties HBS for the norm and strict convexity of its dual norm. Thus, if $(X, \|\cdot\|)$ is a reflexive Banach space, its norm has the HBS property (due to the fact that w and w^* coincide on X^*). If the dual norm is not strictly convex, then $\|\cdot\|$ has not the TS property. The existence of such a norm in every reflexive Banach space is guaranteed by using the construction in (ii) in Remark 3.1 below.

We are able to dramatically improve the Oja, Viil and Werner result: The renorming result holds, according to our result, *without any condition on the space* (Theorem 8.1). It is true that this is a consequence of a deep theorem of M. Raja

(Theorem 7.13 below), as we shall mention later.

In order to properly present these results we organize our work in the following way: In the actual chapter, we shall review Phelps' results on the property U and the "dual" Haar property, complemented with some of Sullivan's results and the Taylor–Foguel Theorem 3.14. The last part of this chapter will focus on a property that we introduce in the spirit of Phelps and Sullivan, concerning the possibility to uniquely extending the norm-attaining functionals keeping their norm. We shall prove that this is equivalent to the Gâteaux differentiability of the norm, what in a sense completes the information provided by property U. This way to link the Gâteaux smoothness of the norm with the set of norm-attaining functionals recalls the discussion in Chapter 2.

In Chapter 4 we shall discuss Sullivan's results about the so-called Hahn–Banach smoothness and weak-Hahn–Banach smoothness properties—improving some proofs and adding comments and characterizations. Since this concept is strongly related to the nice smoothness, very smoothness and very rotundness properties of the norm, as well as to the Asplund property of the space, we explore the connections with these properties. As the reader will notice, the fundamental Kadets–Klee property —the coincidence of two topologies on the unit sphere of a Banach space or of its dual— is essential in renorming theory —see, specially, Theorem 7.13. Section 4.4 develops a series of results that are completely new. They are certainly inspired in the fact —in Raja's Theorem 7.13— that the topology induced by a dual LUR norm on the dual unit sphere of a Banach space coincides with the w^*-topology. This fact is not explicit in the aforementioned reference. We provide a set of techniques that ensure this kind of behaviour.

Chapter 8 is the main objective of all Chapters 3, 4, 6, and 7. There, after some recapitulation on those chapters, we shall present a neat improvement of the aforementioned theorems of Sullivan and Oja–Viil–Werner. Some comments on the properties involved inform the reader on the scope of our result. We mention also an open problem in this direction.

3.2 Property U

In this section we shall review some of the results in Phelps paper [Ph60], giving in many cases easier alternative proofs or providing those missing, adding some extra information and results needed, filling blanks, and presenting some examples or arguments absent there. We should insist in the fact that Phelps' paper is sixty years old, and that new techniques and results —some of them provided by Phelps himself—, are now available. Of course, we shall refer properly to the results there and to the later sources consulted. If a proof or a result is new, we shall conveniently inform the reader.

It is well known that, by the classical Hahn–Banach Theorem, every linear continuous functional defined on a linear subspace of a normed space can be extended to the whole space preserving the norm of the original functional. We say then that this extension is a **Hahn–Banach extension**, also called a **norm-preserving extension**. It is by no means guaranteed that this norm-preserving extension is unique.

Remark 3.1

1. Let us give, for the sake of clarity, an example of a Banach space such that every closed subspace has the uniqueness property for the norm-preserving extension. As it is expected, any Hilbert space $(H, \|\cdot\|)$ has this property. Indeed, let M be a closed subspace. Let $i : M \to H$ be the injection mapping, and let $q := i^t$ its adjoint mapping. This is just the canonical quotient mapping $q : H^* \to H^*/M^\perp$, where $M^\perp := \{m^\perp \in H^* : \langle m, m^\perp\rangle = 0\}$ is the **annihilator of** M . Given $m^* := q(m) \in S_{M^*}$ for some $m \in H^*$, the set of all extensions of m^* to H^* is the affine hyperplane $m^* + M^\perp = \{x^* \in H^* : q(x^*) = m^*\} = q^{-1}(m^*) = \{m^* + m^\perp : m^\perp \in M^\perp\}$. Notice that $1 = \|m^*\| = \inf\{\|x^*\| : x^* \in q^{-1}(m^*)\}$. There is one and only one element in $m^* + M^\perp$ at minimum distance from 0 (i.e., at distance 1 from 0), a consequence of the Parallelogram Identity in Hilbert spaces, and this is precisely the (unique) Hahn–Banach extension of m^* to H. Notice, too, that the Parallelogram Identity can be substituted by the w^*-compactness of the dual unit ball plus the strict convexity of the dual unit ball.

   Despite the "elementary" argument behind this example, it already shows the basic connection between the uniqueness extension property and proximinality properties of the annihilator of a given subspace of a Banach space (to be more precise, see Theorem 3.7 below).

2. Maybe the most elementary observation in order to see the lack of uniqueness in some easily-described situations is the following: Let $(X, \|\cdot\|)$ be a Banach space such that its norm is not Gâteaux differentiable at some $x_0 \in S_X$. Let $M := \text{span}\{x_0\}$. Take x_1^* and x_2^* two distinct points in $\partial\|\cdot\|(x_0)$ (they exist thanks to the Šmulyan characterization of a Gâteaux differentiable norm). In other words, assume that $x_i^* \in S_{X^*}$ is such that $\langle x_0, x_i^*\rangle = 1$ for $i = 1, 2$. Obviously, $x_1^*|_M = x_2^*|_M$, so we have two different norm-preserving extensions to the whole of X. It is worth to mention (again an almost trivial observation) that in every Banach space $(X, \|\cdot\|)$ it is possible to define an equivalent norm $|||\cdot|||$ that has some point $x_0 \in S_{(X,|||\cdot|||)}$ where the norm $|||\cdot|||$ is not Gâteaux differentiable. For this, let $x_0 \in 2S_{(X,\|\cdot\|)}$. Let $B := \text{conv}\,(B_{(X,\|\cdot\|)} \cup \{x_0\} \cup \{-x_0\})$. It is simple to prove that B is the

closed unit ball of an equivalent norm $|||\cdot|||$, and that $|||\cdot|||$ is not Gâteaux differentiable at x_0 (see Figure 3.1 for the geometric aspect of B and of its dual unit ball).

3. It is worth to mention from the very beginning that the uniqueness of the norm-preserving extension is an isometric property. This can be observed by combining the two previous items: Take a Hilbert space $(H, \|\cdot\|)$ and define on it an equivalent norm $|||\cdot|||$ which is not Gâteaux differentiable at some point. The space $(H, \|\cdot\|)$ has the uniqueness extension property. This is not the case for $(H, |||\cdot|||)$.

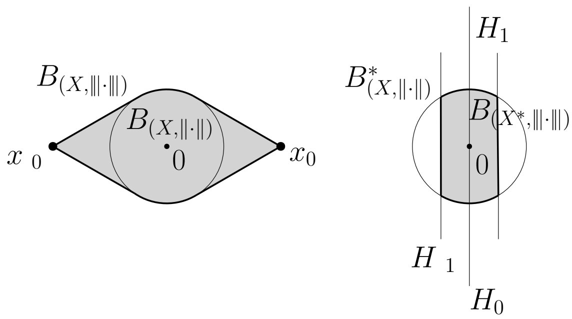

Figure 3.1: The closed unit ball of a non-Gâteaux differentiable equivalent norm and its dual unit ball

The previous discussion and examples justify the introduction of the so-called property U of a subspace of a Banach space.

Definition 3.2 (Phelps, [Ph60])

Let X be a Banach space, and M a linear (not necessarily closed) subspace of X. We will say that M has **property U in** X *if each linear continuous functional on M has a unique norm-preserving extension to X.*

The property of the uniqueness of the Hahn–Banach extension was considered already at early stages of the theory. For example, from [Tay39] and [Fo58] together it follows a characterization of those Banach spaces X for which *every* subspace has property U in X. This is reflected in Theorem 3.14 below. One of the relevant contributions to the study of the U property was due to R. R. Phelps, who in [Ph60] discussed, among some other aspects, this property of a given subspace in connection with its "dual" one, the so-called **Haar property** of its annihilator (see Definition 3.3; this was somehow advanced when we presented the straightforward

example of a Hilbert space, in Remark 3.1). Part of the discussion below comes from the aforementioned seminal Phelps' paper. It also contains a through analysis of the most interesting examples and an up-to-date list of references.
As we mentioned above, Phelps related the U property of a subspace of a Banach space X with approximation properties of subspaces in the dual X^*. Here we mention the relevant geometric property that plays a rôle in this context.

Definition 3.3

It is said that a subset M of a normed space X is ***proximinal*** *if for all $x \in X$ there exists at least one element $y \in M$ such that $\|x - y\| = dist(x, M)$. If such point y is unique, we say that M has the* ***Haar property*** *in X, or that M is a Haar set in X.*

Nonempty subsets with the Haar property are also called in optimization literature **Chebyshev sets**.

Remark 3.4

(1) Trivially, if X is a normed space, then any nonempty subset with the Haar property is also proximinal.
(2) Any nonempty proximinal subset M of X is closed. Indeed, let $x \in \overline{M}$. Due to the fact that M is proximinal, there exist $y \in M$ such that $\|x-y\| = \text{dist}(x, M)$ $(= 0)$. Thus, $x = y \in M$.
(3) The property of being proximinal is obviously invariant by translation. The same applies to the property of being a Haar set. ®

In view of the previous remark, most of the results below will concern closed proximinal sets —usually subspaces. Since trivially a subspace of a Banach space has property U if and only if its closure has property U, we will restrict our attention, without loss of generality, to closed linear subspaces.

According to what was mentioned in Chapter 1, if M is a subspace of a normed space X, then M^* is isometrically isomorphic to $X^*/M^\perp$ (note that $M^\perp$ is always a closed subspace of X^*, in fact w^*-closed). It is part of this assertion that given $f \in X^*$, its restriction to M (an element in M^*) has a norm (denoted $\|f\|_M$) that coincides with the canonical quotient norm $\|q(f)\|$, where $q : X^* \to X^*/M^\perp$ is the canonical quotient mapping. As it is well known (and was again mentioned in Chapter 1), $\|q(f)\| = \text{dist}(f, M^\perp)$. Since this construction will be used in our arguments about proximinality, we start by giving a precise argument. If M is a subspace of a normed space $(X, \|\cdot\|)$, we shall denote by B_M the set $\{m \in M : \|m\| \leq 1\}$, and by S_M the set $\{m \in M : \|m\| = 1\}$.

Lemma 3.5

Let X be a normed space, M a closed linear subspace of X, and $f \in X^*$. Then $\|f\|_M = dist(f, M^\perp)$.

Proof:
By letting $g \in M^\perp$, we have

$$\|f\|_M = \sup_{x\in S_M} \{|f(x)|\} = \sup_{x\in S_M} \{|(f-g)(x)|\} \leq \sup_{x\in S_X} \{|(f-g)(x)|\} = \|f-g\|.$$

Thus, computing the infimum on $g \in M^\perp$, we get $\|f\|_M \leq \operatorname{dist}(f, M^\perp)$.
For the other inequality, let us consider $f|_M$. The Hahn–Banach Theorem shows that there exists $h \in X^*$ such that $h|_M = f|_M$ and $\|h\| = \|f\|_M$. Then, $f - h \in M^\perp$ and we get

$$\|f\|_M = \|h\| = \|f-(f-h)\| \geq \operatorname{dist}(f, M^\perp).$$

This proves the result. ■

Remark 3.6

Of course, not every closed subspace of a Banach space is proximinal. A Banach space is said to be **reflexive** whenever the canonical embedding into the bidual $J : X \longrightarrow X^{**}$ (i.e., $\langle Jx, x^*\rangle = \langle x, x^*\rangle$ for all $x^* \in X^*$) maps X onto X^{**}. Observe, then, that in every non-reflexive Banach space there exists a closed hyperplane that is not proximinal. The proof of this depends on the deep James' compactness theorem. Indeed, if X is not reflexive, there exists, according to this result, a continuous linear functional $f \in S_{X^*}$ that does not attains its norm. Let $K := \ker f \subset X$. We claim that K is not proximinal. Assume for a moment the contrary. Let $x \in X$ such that $\operatorname{dist}(x, K) = 1$. There exists $k_0 \in K$ such that $\|x - k_0\| = 1$. We may apply Lemma 3.5 to the subspace $M := \operatorname{span}\{f\} \subset X^{**}$ and $x \in X^{**}$ to get $1 = \operatorname{dist}(x, K) = \|x\|_M = f(x)$. This shows that f attains its norm at x, a contradiction.
It is equally easy to observe that this result is, in fact, an equivalence: f attains its norm on X if and only if $\ker f$ is proximinal.
It is a useful simple observation that *every w^*-closed subspace M of the dual X^* of a Banach space X is proximinal.* Indeed, fix $x_0^* \in X^* \setminus M$. Let $d := \operatorname{dist}(x_0^*, M)$ (> 0). Thus, the set $M_0 := M \cap B(x_0^*, d+1)$ is non-empty. The distance function $m^* \to \|x_0^* - m^*\|$ defined on M_0 is w^*-lower semicontinuous. Since M_0 is w^*-compact, this distance function attains its minimum, and this guarantees the existence of a point in M_0 at minimum distance from x_0. Clearly, this point realizes the minimum distance from x_0 to M. Another argument, based on the nested intersection property characterizing compactness, will be presented in Proposition 3.13. ®

The next theorem will present the interesting basic duality between property U and Haar subspaces.

Theorem 3.7 (Phelps, [Ph60, Theorem 1.1])

Let X a normed linear space. Then a closed linear subspace M has property U if and only if its annihilator $M^{\perp}$ is a Haar subspace.

Proof: First, suppose that M do not have property U. Then there exist $h \in S_{M^*}$ and $F, G \in S_{X^*}$ such that $F_{|M} = G_{|M} = h$ with $F \neq G$. Then, $0 \neq F - G \in M^{\perp}$ and by using the Lemma 3.5,

$$1 = \|F\| = \|F - (F - G)\| \geq \operatorname{dist}(F, M^{\perp}) = \|F\|_M = \|h\|_M = 1$$

So we achieve the expression $\|F\| = \|F-(F-G)\| = \operatorname{dist}(F, M^{\perp})$, i.e., the distance between F and $M^{\perp}$ is attained at the two distinct points 0 and $F - G$, both in $M^{\perp}$. Thus, $M^{\perp}$ is not a Haar subspace.
Conversely, suppose that $M^{\perp}$ is not a Haar subspace of X^*. Then, there exist an element $f \in S_{X^*}$ and $g \in M^{\perp} \setminus \{0\}$ such that $\operatorname{dist}(f, M^{\perp}) = 1 = \|f - g\|$. By taking $h := f_{|M} = (f - g)_{|M}$ and considering that $\|h\|_M = \|f\|_M = \operatorname{dist}(f, M^{\perp}) = 1$ (due to Lemma 3.5), we get that $f - g$ and f are two distinct norm-preserving extensions of h, so M does not have property U. ■

If M is a linear subspace of the dual space X^*, then the closed linear subspace $M_{\perp} := \{x \in X : \ y(x) = 0 \ \text{ for all } y \in M\}$ is usually called the **pre-annihilator** of M. Since obviously $\overline{M}^{w^*}$ is the annihilator of the space $M_{\perp}$, the previous result has the following immediate consequence.

Corollary 3.8

Let M be a w^*-closed subspace of X^*. Then, M is a Haar subspace of X^* if and only if $M_{\perp}$ has property U.

Remark 3.9

Observe that, without the hypothesis of M being w^*-closed, the previous corollary may fail. Indeed, let X be a non-reflexive Banach space and let $F \in X^{**} \setminus X$ an element that does not attains its norm on X^*. Put $K := \ker F$ $(\subset X^*)$. This is a $\|\cdot\|$-closed subspace of X^* that is w^*-dense in X^* (we shall mention later, see [GuiLisMon19, Proposition 5] , that in fact K is a norming subspace); thus, $K_{\perp} = \{0\}$. We proved in Remark 3.6 that K is not proximinal (in particular, it is not a Haar subspace of X^*). Theorem 3.7 shows that $K_{\perp}$ has property U (due to the fact that $\{0\}^\perp$ $(= X^*)$ is certainly a Haar subspace of X^*). ®

Looking at Remark 3.9 and to Theorem 3.7, it is natural to wonder whether the duality between the two properties is complete. More precisely, we may ask whether a closed subspace M of a Banach space is a Haar subspace if, and only if, $M^\perp$ has property U or, at least, if some implication holds. In order to throw some light on this question, let us start by a simple expected result in this direction:

Theorem 3.10

Let X be a reflexive space, and let M be a closed linear subspace of X. Then M is a Haar subspace if and only if $M^\perp$ has property U.

Proof: Just apply Theorem 3.7 in X^*, and observe that $(M^\perp)^\perp = J(M) = \overline{M}^{w^*} = M$. This is due to the fact that the topologies w and w^* on X^* coincide, a consequence of the reflexivity of the space. ■

Before answering the previous questions, we shall present the following results that will show how the properties previously introduced are closely linked to the geometry of the space. Before that, we shall introduce some notation. For $x, y \in X$, we use $[x, y]$ to denote the line segment between x and y. Analogously, if the extremes x and y are excluded, then we shall write $]x, y[$ instead. If $x \neq y$, the one-dimensional affine subspace (a line) through x and y will be denoted by $l(x, y)$.

A normed linear space is called **rotund** or **strictly convex** if the unit sphere does not contain non-trivial line segments. Throughout the text we will also refer to the rotundity of the space (or the norm on it) as **property R**.

The following simple characterization of the strict convexity appears in [Ph60]:

Proposition 3.11

Let X be a normed space. Then, X is strictly convex if and only if each line through the origin has the Haar property.

Proof: Assume that X is not strictly convex. Then, there must exist two distinct points $x, y \in S_X$ such that $[x, y] \subset S_X$. In view of Remark 3.4, it is enough to prove that the line $l(x, y)$ does not have the Haar property, and this is obvious, since $\operatorname{dist}(0, l(x, y)) = 1$ and the distance from 0 to x (to y) is 1.
Conversely, if there is a line l through the origin that does not have the Haar property, then there is $x \notin l$ and two distinct points y and z in l such that $d := \operatorname{dist}(x, l) = \|x - y\| = \|x - z\|$, By convexity, the segment $[y, z]$ is in the boundary of $B(x, d)$. ■

It should be noted that if we take $f \in X^* \backslash \{0\}$, then the line $\operatorname{span}(f)$ is the annihilator of the hyperplane $f^{-1}(0)$. Then the geometry of the dual space can be characterized in terms of uniqueness of the norm-preserving extensions.

Proposition 3.12

Let X be a normed space. Then X^* is strictly convex if and only if each closed hyperplane in X has property U.

Proof: By Theorem 3.7 and the previous observation, we know that every hyperplane in X has property U if and only if every line through the origin in X^* has the Haar property. By Proposition 3.11 this is equivalent to the strict convexity of X^*. ∎

Proposition 3.13

Let X be a normed space. Then, every w^*-closed linear subspace M of X^* is proximinal. If we also assume that X^* is strictly convex, then every w^*-closed linear subspace M of X^* is indeed a Haar subspace.

Proof: In Remark 3.6 we provided a proof of the first part of the assertion based on the w^*-lower semicontinuity of the dual norm. The following alternative proof is based on the nested intersection property that characterizes a compact topological space: Let $f \in X^* \backslash M$ and, for $n \in \mathbb{N}$, define $C_n := f + \big(\mathrm{dist}(f, M) + \frac{1}{n}\big) B_{X^*}$. We get a sequence $\{C_n\}_{n=1}^{\infty}$ of nonempty w^*-compact sets (thanks to the Alaoglu–Bourbaki Theorem). By compactness, $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$. Finally, note that for any $g \in \bigcap_{n=1}^{\infty} \{C_n \cap M\}$, we have $g \in M$ and $\|f - g\| = \mathrm{dist}(f, M)$.
Assume now that $(X^*, \|\cdot\|)$ is strictly convex. Let g_1 and g_2 be two elements in M such that $d := \|g_i - f\| = \mathrm{dist}(f, M)$ for $i = 1, 2$. The strict convexity of $\|\cdot\|$ shows that $\|f - (g_1 + g_2)/2\| < d$, a contradiction except in case that $g_1 = g_2$. ∎

Combining this two previous results, we get a short alternative proof of the Taylor–Foguel theorem mentioned after Definition 3.2.

Theorem 3.14 (Taylor–Foguel)

Let X be a normed space. Then X^* is strictly convex if and only if every closed linear subspace M of X has property U in X.

Proof: First, suppose that X^* is strictly convex. Then, taking an arbitrary closed linear subspace M of X, by the Proposition 3.13 we know that its annihilator $M^{\perp}$, which is a w^*-closed linear subspace, has the Haar property. Thus, by Theorem 3.7 it follows that M has property U.
Conversely, if every closed subspace has property U, then, in particular, every hyperplane in X has property U, and by Proposition 3.12 this is equivalent to the rotundity of X^*. ∎

An easy corollary of this result is that, for checking the U property on every linear subspace, it is enough to check the property only on hyperplanes:

Corollary 3.15

Let X be a normed space. Then, every closed linear subspace has property U if and only if every hyperplane in X has property U in X.

Proof: As we have observed before, every hyperplane in X having property U is equivalent to every line through the origin in X^* being a Haar subspace. Then apply 3.11 on X^* and Theorem 3.14. ■

It will be useful to compile the above results in a single statement.

Theorem 3.16

Let X be a normed space. Then, the following statements are equivalent:
(i) *X^* is strictly convex.*
(ii) *Every closed linear subspace of X has property U in X.*
(iii) *Every hyperplane in X has property U in X.*

We end this section by offering some criteria that relate the properties U and Haar with arguments of dimensions.

Definition 3.17

Let C be a convex subset of a linear space. The ***dimension*** *of C (shortly denoted by $\dim(C)$) is the dimension of the linear space $\operatorname{span}(C-x)$ for any $x \in C$.*

We will make use of the following notation. For any element $x_0 \in X\backslash\{0\}$ the set $F^{x_0} := \{x^* \in S_{X^*} : x^*(x_0) = \|x_0\|\}$ is called the **face** defined by x_0 in B_{X^*}. If $x_0^* \in X^*\backslash\{0\}$, analogously, we can set the face defined by x_0^* in B_X as $F_{x_0^*} := \{x \in S_X : x_0^*(x) = \|x_0^*\|\}$.

Proposition 3.18 ([Ph60, Theorem 1.4])

Let X be a normed space and M a closed linear subspace with $\operatorname{codim}(M) = n$. If there exists an $f \in S_{X^*}$ such that $f(M) = 0$ and $\dim(F_f) \geq n$, then M is not a Haar subspace.

Proof: Take $x_0 \in F_f$. Since $\dim(F_f) \geq n$ we know that $\operatorname{span}(F_f - x_0)$ contains an n-dimensional subspace L, so $(F_f - x_0) \cap L$ has non-empty interior relative to L. Thus, by a traslation argument, we can assume without loss of generality that

0 is one of these interior points. Then, $(F_f - x_0) \cap L$ is an absorbing set in L, that is, every element in L is a positive multiple of one elements in $(F_f - x_0) \cap L$.
Now, by contradiction, suppose that $L \cap M = \{0\}$. Since $\dim(L) = \text{codim}(M) = n$, then $X = M \oplus L$. But as f vanishes on $(F_f - x_0) \cap L$ which is an absorbing set of L, then implies $f(L) = 0$, and by hypothesis we know that $f(M) = 0$, so f vanishes on the whole space X and thus $f = 0$, a contradiction with the fact that $f \in S_{X^*}$.
This means that there must exists $y \in (M \cap L) \setminus \{0\}$. Moreover, we can assume without loss of generality that $-y \in M \cap (F_f - x_0)$. As any element of F_f has norm one, we have that $\|x_0 - y\| = 1 = \|x_0\|$. Moreover, taking $z \in M$, $\|x_0 - z\| \geq f(x_0 - z) = f(x_0) = 1$. Then $\text{dist}(x_0, M) = \|x_0 - y\| = \|x_0\| = 1$, and this shows that M is not a Haar subspace. ■

The following is a result dual to the one we just proved.

Proposition 3.19 ([Ph60, Theorem 1.5])

Let X be a normed linear space and M a closed linear subspace of X such that $\dim(M) = n$. Assume that there exists $x_0 \in S_X \cap M$ with $\dim(F_{x_0}) \leq n$. Then, M does not have property U.

Proof: Recall that $J : X \longrightarrow X^{**}$ denotes the canonical embedding of X into its bidual. We note that $F^{x_0} = F_{Jx_0}$. Also $x_0 \in S_X \cap M$ implies that $Jx_0 \in S_{X^{**}} \cap (M^\perp)^\perp$. Finally, remark that $n = \dim(M) = \text{codim}(M^\perp)$, so we can apply Proposition 3.18 above to $M^\perp$, and deduce that $M^\perp$ is not a Haar subspace. Thus, by Phelp's Theorem 3.7, M does not have property U. ■

3.3 Property wU

Definition 3.20

Let $(X, \|\cdot\|)$ be a normed space. Let $C \subset D \subset X$. We say that C is an ***extreme subset*** *of D if:*

(1) C is a convex set.
(2) For all $x, y \in \text{conv}(D)$ such that $]x, y[\cap C \neq \emptyset$, then $[x, y] \in C$.

If C is just a point, then it is called an ***extreme point*** *of D. The set of extreme points of a set D is denoted by $\text{Ext}(D)$.*

Obviously, the set of extreme points of B_X is contained in S_X.

The duality mapping was introduced in Chapter 1. Let us repeat its definition here: If $(X, \|\cdot\|)$ is a normed space, define the duality mapping as the multivalued mapping $\partial\|\cdot\|$ from X into 2^{X^*} such that $\partial\|\cdot\|(x) = \{x^* \in S_{X^*} : \ x^*(x) = \|x\|\}$ for $x \neq 0$, and $\partial\|\cdot\|(0) = B_{X^*}$.

The fundamental definition of Gâteaux differentiability of the norm was introduced in Chapter 1, Definition 1.1. We may rephrase this concept here by saying that $\|\cdot\|$ is Gâteaux differentiable at $x \in X$ if $\partial\|\cdot\|(x)$ is a singleton $\{x^*\}$ (and the element x^* is said to be the Gâteaux derivative of $\|\cdot\|$ at x).
It is said that an element $x_0^* \in X^*$ is a norm-attaining element if there exist x_0 such that $x_0^*(x_0) = \|x_0\|$. The set of norm-attaining linear continuous functions on X is denoted with $\mathrm{NA}(X)$. We will consider the following generalization of property U. That this concept is a meaningful one can be seen in Theorem 3.22 below: It characterizes Gâteaux differentiability.

Definition 3.21

Let X be a Banach space, and M a linear subspace. We will say that M has ***property weak U*** *(**wU** for short) in X if each linear continuous norm-attaining functional on M has a unique norm-preserving extension to X.*

Observe that if M is a finite-dimensional subspace, then every element in M^* attains its norm, so properties U and wU are equivalent in this case. The following theorem is the version of Theorem 3.16 for the property wU.

Theorem 3.22

Let X be a normed space. Then, the following statements are equivalent:
(1) *X is Gâteaux differentiable.*
(2) *Every closed linear subspace of X has property wU in X.*
(3) *Every closed hyperplane in X has property wU in X.*
(4) *For every $x \in X\backslash\{0\}$, $\mathrm{span}(\{x\})$ has property wU (or U) in X.*

Proof: (1) $\Longrightarrow$ (2). Let M be a closed linear subspace of X, and $f_0 \in M^*$ be a norm-attaining element. Then, there exist $x_0 \in S_M$ such that $f_0(x_0) = \|f_0\|$ (norm in M^*). Let f and g be two norm-preserving extensions of f_0 to X. Thus, $\|f\| = \|g\| = \|f_0\| = f_0(m_0) = f(m_0) = g(m_0)$. Since $\|\cdot\|$ is Gâteaux differentiable at x_0, we get $f = g$. This shows that M has property wU.
Also, it is clear that (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). So we only need to prove that (4) $\Longrightarrow$ (1). To see this, conversely, assume that there exists a point $x_0 \in S_X$ such that $\|\cdot\|$ is not Gâteaux differentiable in x_0, it is, $\dim(F^{x_0}) \geq 1$. Then, we can apply proposition 3.19 to the closed linear space $M = \mathrm{span}(\{x_0\})$, concluding that

span($\{x_0\}$) does not have property U. As span($\{x_0\}$) is finite-dimensional, then neither have property wU. ■

Chapter 4

Hahn–Banach Smoothness

4.1 HBS and wHBS properties

Property U of a subspace of a Banach space was introduced in Chapter 3. Sullivan, in [Su77], introduced a related property, nowadays called **Hahn–Banach smoothness**. This, and the close **total smoothness** property, were also mentioned in the Introduction to Chapter 3. Let us repeat the definition of the first concept here:

Definition 4.1 (Sullivan)

*A normed space $(X, \|\cdot\|)$ is said to be **Hahn–Banach Smooth** at a point $x^* \in S_{X^*}$ (**HBS** for short) if x^* has a unique norm-preserving extension to X^{**}. We say that $(X, \|\cdot\|)$ is* **Hahn–Banach smooth** *if it is Hahn–Banach smooth at every point $x^* \in S_{X^*}$ (in other words, if $(X, \|\cdot\|)$ has property U in X^{**}).*

We shall prove later on (see Remark 4.5 below) that Definition 4.1 should have been formulated by saying that "the norm $\|\cdot\|$ has property HBS", due to the fact that this is an isometric property.

Sullivan provides an alternative equivalent formulation of this property:

Proposition 4.2 (Sullivan)

A normed space $(X, \|\cdot\|)$ is Hahn–Banach smooth at $x^* \in S_{X^*}$ if, and only if, for every $x^\perp \in X^\perp$

$$\|x^* + x^\perp\| = \|x^*\| = 1 \text{ implies } x^\perp = 0. \tag{4.1}$$

Here, $x^* \in X^*$, where X^* is naturally identified with a subspace of X^{***} (see Section 1.4 in Chapter 1 above), and $x^\perp \in X^\perp$, where $X^\perp := \{x^{***} \in X^{***} : \langle x, x^\perp \rangle = 0, \text{ for all } x \in X\}$.

Proof of Proposition 4.2 First, the Hahn–Banach smooth property at x^* clearly implies (4.1). This is so due to the fact that $x^*+x^\perp$ is a (norm-preserving) extension of x^* to X^{**}. Since $Jx^* \in X^{***}$ (denoted x^* again) is certainly a norm-preserving extension of x^*, the uniqueness concludes that $x^\perp = 0$.
Conversely, if (4.1) holds, and x^{***} is a norm-preserving extension of $x^* \in S_{X^*}$ to X^{**}, then $x^{***}-x^* \in X^\perp$, where x^* is again identified with the element $Jx^* \in X^{***}$. It follows that $x^{***} = Jx^*$. Thus, X is Hahn–Banach smooth at x^*. ■

Sullivan [Su77] proved that Banach spaces $(X, \|\cdot\|)$ for which the topologies w^* and $\|\cdot\|$ coincide on the unit sphere S_{X^*} of X^* (a property that we call **w^*-$\|\cdot\|$-Kadets–Klee**) are Hahn–Banach smooth ([Su77, Theorem 6]). This is a particular case of a result that turns out to be an equivalent characterization of the HBS property in terms of the coincidence of weak topologies. It is attributed to G. Godefroy [Go81]. We prefer to formulate the result in a local way. We have been unable to find the proof of this result in the literature, so we provide here an argument. Let us mention here that Sullivan proves his result (that property w^*-$\|\cdot\|$-Kadets–Klee implies HBS) by using the delicate principle of local reflexivity. Our proof of the stronger Godefroy's result does not rely on this principle, and uses only elementary Banach space theory.

Proposition 4.3 (Godefroy)

Let X be a Banach space. Then, $x^* \in S_{X^*}$ has a unique norm-preserving extension to X^{**} if, and only if, the topologies w and w^* coincide at x^*.

Proof: First assume that $x^* \in S_{X^*}$ has a unique extension to the whole bidual X^{**}, and take a net $\{x_i^*\} \subset B_{X^*}$ that $w(X^*, X)$-converges to x^*. Considering the canonical embedding from a Banach space to its bidual, we know that $\{x_i^*\} \subset B_{X^*} \subset B_{X^{***}}$. The last set is a $w(X^{***}, X^{**})$-compact subset of X^{***}. This means that for every subnet $\{x_{i_j}^*\}$ there exist a point $x^{***} \in B^{***}$ and a subnet $\{x_{i_{j_k}}^*\}$ that $w(X^{***}, X^{**})$-converges to x^{***}. Observe that x^{***} is a norm-preserving extension of x^*. By the assumption, there is only one such an extension. Since the subnet $\{x_i^*\}$ was arbitrary, we conclude easily that $x_\alpha \xrightarrow{w} x^*$.

Assume now that the two topologies w and w^* coincide on S_{X^*} at $x^* \in S_{X^*}$. Let $x^\perp \in X^\perp$ such that $\|x^* + x^\perp\| = 1$. Find a net $\{x_i^*\}$ in S_{X^*} that $w(X^{***}, X^{**})$-converges to $x^*+x^\perp$. Thus, given $x \in X$ we have $\langle x, x_i^*\rangle \to \langle x, x^*+x^\perp\rangle = \langle x, x^*\rangle$, so $\{x_i^*\}$ is $w(X^*, X)$-convergent to x^*. By the assumption, $\{x_i^*\}$ $w(X^*, X^{**})$-converges to $x^* + x^\perp$. This proves that $\langle x^{**}, x^\perp\rangle = 0$ for each $x^{**} \in X^{**}$, hence $x^\perp = 0$. It is enough to apply Proposition 4.2. ■

The following characterization of reflexivity in terms of the Hahn–Banach smoothness will be improved later on, when speaking about nicely smooth spaces (Proposition 4.10 below). We formulate it here because it helps to emphasize (see Remark 4.5 below) that the Hahn–Banach smooth property is an isometric but in general not isomorphic one.

Proposition 4.4

Let $(X, \|\cdot\|)$ be a Banach space. Then X is reflexive if, and only if, every equivalent norm on X is Hahn–Banach smooth.

Proof: Of course, Proposition 4.3 implies that if X is reflexive, then every equivalent norm on X has property HBS. Assume now that X is not reflexive and yet every equivalent norm on X has property HBS. Let $x^{**} \in X^{**} \setminus X$. The hyperplane $H := \ker x^{**}$ is a (proper, closed) norming subspace of X^* (see, e.g., [FHHMZ11]). Let $|||\cdot|||$ be the equivalent norm defined by H. Thus, $B_{(X^*,|||\cdot|||^*)} = \overline{B_{(H,|||\cdot|||^*)}}^{w^*}$. Find a net $\{x_i^*\}$ in $S_{(H,|||\cdot|||^*)}$ that w^*-converges to an element $x^* \in S_{(X^*,|||\cdot|||^*)}$, $x^* \notin H$. By the assumption, it also $w(X^*, X^{**})$-converges to x^*. This is impossible, since $\langle x^{**}, x_i^*\rangle = 0$ for all i, while $\langle x^{**}, x^*\rangle \neq 0$. ■

Remark 4.5

It follows from Proposition 4.4 that the HBS property is an isometric, in general non-isomorphic property. Indeed, it was proved by Sullivan that every separable Banach space has an equivalent norm with the HBS property. If the space is non-reflexive, then it has another equivalent norm without this property.

Along this work we will also talk about a weaker version of the HBS property, introduced by Smith and Sullivan in [SmSu77], which is to the property wU as HBS is to the property U.

Definition 4.6

A normed space X is said to be ***weak Hahn–Banach Smooth*** *(**wHBS** for short) if X has property wU in its bidual X^{**}. That is, every $x^* \in NA(X)$ has a unique norm-preserving extension to X^{**}.*

Of course, by using exactly the same arguments than in Proposition 4.2, we get that X is weak Hahn–Banach Smooth if and only if, in X^{***},

$$\|x^* + x^\perp\| = \|x^*\| = 1 \text{ implies } x^\perp = 0 \quad \text{for every } x^* \in \mathrm{NA}(X)\backslash\{0\} \tag{4.2}$$

Let us formulate the characterizations for HBS and wHBS in a single result, a consequence of Proposition 4.3:

Corollary 4.7

Let X be a Banach space. Then:

(1) X is HBS if, and only if, X^* has property w^*-w-Kadets–Klee, i.e., if the topologies w^* and w coincide on S_{X^*}.

(2) X is wHBS if, and only if, the topologies w^* and w coincide on $S_{X^*} \cap NA(X)$.

The condition w^*-w-KK on a Banach space plays an important role in renorming theory. A landmark is the following result of M. Raja.

Theorem (M. Raja, [Ra99])

Let X be a Banach space such that X^* has the w^*-w-KK property. Then X^* admits an equivalent dual LUR norm.

Proving this result is one of the main goals in Chapter 7 (see Theorem 7.13).

Finally, we end this section proving that wHBS (and so, HBS) spaces belong to two important classes within the renorming theory: Nicely smooth and Asplund spaces.

Definition 4.8

The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be ***nicely smooth*** *if X^* does not contain a proper 1-norming subspace.*

Nicely smoothness is a property introduced by G. Godefroy in [Go81]. This time we explicitly speak about the nicely smoothness property *of the norm*, since the arguments above regarding the HBS property and Proposition 4.10 below ensure that nicely smoothness is an isometric property. Indeed, we may repeat the same argument used in the HBS case: Every separable space X has an equivalent HBS norm —hence nicely smooth, see Proposition 4.9 below. However, if the space is nonreflexive, then Proposition 4.10 below shows that there exists an equivalent not-nicely smooth norm on X.

Proposition 4.9

Every wHBS Banach space is nicely smooth.

Proof: By contradiction, assume that there exists $H \subset X^*$ a closed proper 1-norming subspace. Thus $X^* \backslash H \neq \emptyset$, and by the Bishop-Phelps Theorem, we can find $z^* \in S_{X^*} \backslash H$ and $z \in S_X$ such that $\langle z^*, z\rangle = 1$. Now, take $z^{**} \in H^{\perp} \subset X^{**}$ with $\|z^{**}\| = 1$ and $\langle z^{**}, z^*\rangle = 1$. Define the functional $f : X \oplus H^{\perp} \subset X^{**} \longrightarrow \mathbb{R}$

by

$$f(x + h^\perp) := \langle z^*, x\rangle \quad \text{for every } x \in X,\ h^\perp \in H^\perp,$$

which obviously is an extension of z^* to $X \oplus H^\perp$, and notice that $\|f\| = 1$ since

$$|\langle z^*, x\rangle| \leq \|z^*\|\|x\| = \|x\| \leq \|x + h^\perp\| \quad \text{for every } x \in X,\ h^\perp \in H^\perp,$$

and $\langle z^*, z\rangle = 1$. Now, by the Hahn–Banach Theorem we can find an extension of f to the whole space X^{**}, we call it $z^* + z^\perp$, preserving the norm. Then, we get

$$\|z^* + z^\perp\| = \|z^*\| = 1,$$

but since $\langle z^* + z^\perp, z^{**}\rangle = f(z^{**}) = 0$, we have that $\langle z^\perp, z^{**}\rangle = -\langle z^{**}, z^*\rangle \neq 0$, proving that $z^\perp \neq 0$, a contradiction with equation (4.2) after Definition 4.6. ■

The following result has a proof very similar to the one provided for Proposition 4.4, so we shall omit it. The result clearly extends that proposition.

Proposition 4.10

A Banach space $(X, \|\cdot\|)$ is reflexive if, and only if, every equivalent norm is nicely smooth.

Definition 4.11

A Banach space X is said to be an ***Asplund space*** *if every separable subspace has separable dual.*

This class of spaces (originally named **strong differentiability spaces**) was introduced by Asplund in 1968, taking its original definition as a the thesis of a theorem that he proved:

Theorem 4.12 (Asplund, [Asp68])

Let X be a Banach space such that every separable subspace has a separable dual. Then every real, continuous and convex function defined in X is Fréchet differentiable at points of a G_δ dense subset.

In December 1975, Namioka and Phelps proved the reciprocal to Asplund's result [NaPh75]. Earlier in the same year, Stegall proved in [Ste75] that the hypothesis of Asplund's Theorem is equivalent to a property formulated in terms of measure theory, known as **Radon–Nikodým property** (RNP). Thanks to the successive contributions of these and other authors (namely van Dulst, Huff and Morris) other characterizations were found throughout the last century ([DuNa84], [HuMo75]). The Asplund spaces are a stable class under standard Banach space operations, in whose characterization converge arguments that a priori seemed not so related. For the interested reader, we enunciate (without getting into the definitions) the following theorem, which brings together the fruitful work of great mathematicians of the last century. The proof for many of this implications can be found in [FHHMZ11].

Theorem 4.13 ([FHHMZ11, Theorem 11.8])

Let X be a Banach space. The following statements are equivalent:
(i) *X is an Asplund space.*
(ii) *X^* is w^*-dentable.*
(iii) *X^* is dentable.*
(iv) *X^* is w^*-fragmentable.*
(v) *X^* has RNP.*
(vi) *Every non-empty $M \subset X^*$ which is convex and w^*-compact is the w^*-closed convex hull of its strongly w^*-exposed points.*
(vii) *Every real, continuous and convex function defined on X is Fréchet differentiable at points of a G_δ dense subset.*

Here, we include also some of the good permanence properties that Asplund spaces enjoy.

Theorem 4.14 ([Fa97, Theorem 1.1.2])

(i) *Let X is an Asplund space, Y a Banach space and $T : Y \longrightarrow X$ a linear continuous mapping with T^*X^* is dense in Y^*. Then, Y is Asplund. In particular, every subspace of an Asplund space is also Asplund.*

(ii) *Let X be an Asplund space and $T : X \longrightarrow Y$ a linear continuous and surjective operator. Then Y is Asplund. In particular, quotients of an Asplund space are Asplund.*

(iii) *Let X be a Banach space, and a subspace $Y \subset X$ such that both Y and X/Y are Asplund. Then, X is Asplund.*

(iv) *Let Γ be a nonempty set and for every $\gamma \in \Gamma$, $(X_\gamma, \|\cdot\|_\gamma)$ is an Asplund space. Then, the spaces $\left(\sum_{\gamma\in\Gamma} X_\gamma\right)_{c_0}$ and $\left(\sum_{\gamma\in\Gamma} X_\gamma\right)_{l_p}$, with $1 < p < +\infty$, are Asplund.*

(v) *If X is an Asplund space and (Ω, Σ, μ) is a finite measure space, then $L_p(\Omega, \Sigma, \mu, V)$, with $1 < p < +\infty$, is an Asplund space.*

Now, we prove that wHBS (and so, HBS) spaces belong to this class. The proof we give takes advantage of Proposition 4.9. As mentioned in [BaBa01], if being nicely smooth were inherited by subspaces, then every nicely smooth space would be Asplund. Unfortunately, this is not true (in [JiMo97], Jiménez Sevilla and

Moreno found non-Asplund nicely smooth spaces). However, wHBS is a stronger property than nicely smoothness which is inherited by subspaces. This allows for proving that spaces in this last class are Asplund by using a separable reduction argument.

Proposition 4.15

Every wHBS Banach space is an Asplund space.

Proof: First, note that as wHBS is equivalent to the weak topologies coincidence on $S_{X^*} \cap \mathrm{NA}(X)$ (see Corollary 4.7), the wHBS property of the norm is inherited by subspaces.
By contradiction, let Y be a separable subspace of X such that Y^* is not separable. Let $\{y_n\}_{n=1}^{\infty}$ be a sequence dense in S_Y. By the Hahn–Banach Theorem, for every $n \in \mathbb{N}$ there exists $y_n^* \in S_{Y^*}$ such that $\langle y_n^*, y_n \rangle = 1$. Define $H := \mathrm{span}(\{y_n^*\}_{n=1}^{\infty})$. Since Y^* is not separable, we get $Y \neq H$. Observe that $\|y\| = \sup\{\langle y, h \rangle : h \in B_H\}$. This shows that H is a proper 1-norming subspace, but this is impossible, since by Proposition 4.9 a wHBS space is nicely smooth. ■

4.2 Very Smoothness

Definition 4.16

*Let X be a normed space. It is said that the norm $\|\cdot\|$ is **very smooth** (for short, **VS**) if the bidual norm $\|\cdot\|^{**}$ on X^{**} is Gâteaux differentiable at every point $x \in X \backslash \{0\}$.*

As we know, the duality map $\partial\|\cdot\| : X \longrightarrow 2^{X^*}$ is always $\|\cdot\|$-w^*-upper semicontinuous. To say that the norm is Gâteaux differentiable is equivalent to the assertion that the duality mapping $\partial\|\cdot\|$ is univaluated (so, in particular, is $\|\cdot\|$-w^*-continuous). If the norm is Fréchet differentiable, then the subdifferential of the norm is (univaluated and) $\|\cdot\|$-$\|\cdot\|$-continuous. Diestel defined the property of being very smooth precisely by asking for the $\|\cdot\|$-w-continuity of the subdifferential mapping. Let us prove first that both Definition 4.16 and the aforementioned one in [Di75] are equivalent.

Proposition 4.17

Let X be a normed space. Then, X if very smooth if and only if the duality map $\partial\|\cdot\| : X \longrightarrow 2^{X^*}$ is $\|\cdot\|$-w-continuous.

Proof: Every point of $X\backslash\{0\}$ is a smooth point of X^{**} if and only if the duality map of the bidual space $\partial\|\cdot\|^{**} : X^{**} \longrightarrow 2^{X^{***}}$ is $\|\cdot\|$-w^*-continuous at these points. Then, as $\partial\|\cdot\|$ can be thought as the restriction of $\partial\|\cdot\|^{**}$ to X, and having that the w^* topology of X^{***} induces the w topology in X^*, then the $\|\cdot\|$-w^*-continuity of $\partial\|\cdot\|^{**} : X^{**} \longrightarrow 2^{X^{***}}$ for every $x \in X\backslash\{0\}$ is equivalent to the $\|\cdot\|$-w-continuity of $\partial\|\cdot\| : X \longrightarrow 2^{X^*}$ at these points. ■

We will work with canonical embeddings in higher dual spaces. The map $J_1 : X^* \longrightarrow X^{***}$ denotes the canonical embedding for the dual space in the third one, $J_2 : X^{**} \longrightarrow X^{(4)}$ is the embedding of the bidual into the fourth one, etc. Observe that $J_1 x^* \in X^{***}$ is a norm-preserving extensión to the bidual of $x^* \in X^*$. In this way, the natural embeddings and its adjoints can be used in combination with the following Dixmier's result to find different norm-preserving extensions. We provide here the proof of this Dixmier's result, as it has been impossible to find it in the accessible literature.

Theorem 4.18 (Dixmier)

Let X be a normed space, and $x^{**} \in X^{**}$. Then

$$dist(x^{**}, X) \leq \|J^{**}x^{**} - J_2 x^{**}\|.$$

Proof: It is well known that $X^{***} = X^* \oplus X^{\perp}$. As $J^{**}x^{**}, J_2 x^{**} \in X^{(4)}$, then

$$\|J^{**}x^{**} - J_2 x^{**}\| = \sup\langle J^{**}x^{**} - J_2 x^{**}, B_{X^{***}}\rangle \geq \sup\langle J^{**}x^{**} - J_2 x^{**}, B_{X^{\perp}}\rangle.$$

Now observe that, for $x^{\perp} \in X^{\perp}$, if we denote $\Pi : X^{***} \longrightarrow X^*$ the canonical projection to X^* (the restriction), which moreover satisfy $\Pi = J^*$, we have

$$\langle J^{**}x^{**} - J_2 x^{**}, x^{\perp}\rangle = \langle x^{**}, J^* x^{\perp}\rangle - \langle J_2 x^{**}, x^{\perp}\rangle = \langle x^{**}, \Pi x^{\perp}\rangle - \langle J_2 x^{**}, x^{\perp}\rangle =$$

$$= \langle x^{**}, 0\rangle - \langle x^{**}, x^{\perp}\rangle = -\langle x^{**}, x^{\perp}\rangle.$$

Thus, $\sup|\langle J^{**}x^{**} - J_2 x^{**}, B_{X^{\perp}}\rangle| = \sup|\langle x^{**}, B_{X^{\perp}}\rangle|$, and by Lemma 3.5, the last term is equal to $\operatorname{dist}(x^{**}, X)$. So, $\|J^{**}x^{**} - J_2 x^{**}\| \geq \operatorname{dist}(x^{**}, X)$, as we wanted to prove. ■

Proposition 4.19

Let X be a normed space. If X^* is very smooth or HBS, then X is reflexive.

Proof: Assume that X is not reflexive. By James' theorem there must exist $x_0^* \in S_{X^*}$ and $x_0^{**} \in S_{X^{**}}\backslash S_X$ such that $\langle x_0^*, x_0^{**}\rangle = 1$. Hence, by the previous Dixmier's theorem 4.18, $0 < \operatorname{dist}(x_0^{**}, X) \leq \|J^{**}x_0^{**} - J_2 x_0^{**}\|$.

Now, if X^* was very smooth then $x^* \in S_{X^*} \subset S_{X^{***}}$ would be an element that attains its norm on two distinct elements $J^{**}x^{**}$ and J_2x^{**} of $X^{(4)}$, which would be a contradiction.
For the HBS case, the argument is even simpler. $J^{**}x^{**}$ and J_2x^{**} would be two norm-preserving extensions of x_0^{**}, which contradicts the HBS property of X^*. ■

Besides being very smoothness an intermediate property between the Fréchet and Gâteaux differentiabilities, in the context of this work it plays another role, as it is the combination of two of the aforementioned properties, the Gâteaux smoothness and the wHBS property. This was already observed by Smith and Sullivan in [SmSu77].

Proposition 4.20

Let X be a Banach space. The norm $\|\cdot\|$ is very smooth if and only if is simultaneously Gâteaux differentiable and wHBS.

Proof: It follows immediately from the definition of these properties. ■

4.3 Very Rotundness

A consequence of the Šmulyan's lemma is a certain duality between differentiability and convexity properties on the dual and convexity and differentiability on the space, respectively —although this works only in one direction, namely from the dual to the space. To be precise, (i) if the norm $\|\cdot\|^*$ on the dual space X^* is strictly convex, then $\|\cdot\|$ is Gâteaux differentiable; (ii) if $\|\cdot\|^*$ is Gâteaux differentiable, then $\|\cdot\|$ is strictly convex; (iii) if $\|\cdot\|^*$ is LUR, then $\|\cdot\|$ is Fréchet differentiable; (iv) if $\|\cdot\|^*$ is Fréchet differentiable, then $\|\cdot\|$ is LUR (and the space is reflexive). In general, nothing of the above can be reversed. The next geometric property was introduced by Sullivan as a dual property to the very smoothness.

Definition 4.21 (Sullivan)

Let X be a normed space. It is said that X is ***very rotund*** *(or just* ***VR****) if no $x^* \in S_{X^*}$ is simultaneously a norming element of $x^{**} \in S_{X^{**}}$ and $x \in S_X$, where $J(x) \neq x^{**}$. Equivalently, this happens if the dual norm $\|\cdot\|^*$ in X^* is Gâteaux smooth at every point of S_{X^*} that attains its norm.*

Of course, the nomenclature of Sullivan does suggest that very rotundity should imply rotundity. We shall prove this for the sake of completeness.

Proposition 4.22

Let X be a normed linear space. If X is very rotund, then it is rotund.

Proof: Suppose X is not rotund. Then there exists two distinct points $x_0, y_0 \in S_X$ such that $[x, y] \subset S_X$. For every $x^* \in \partial\|\cdot\|(\frac{x_0+y_0}{2})$, we know that

$$1 = x^*\left(\frac{x_0+y_0}{2}\right) = \frac{1}{2}x^*(x_0) + \frac{1}{2}x^*(y_0) \leq \frac{1}{2}\|x^*\|\cdot\|x_0\| + \frac{1}{2}\|x^*\|.\|y_0\| \leq \frac{1}{2} + \frac{1}{2} = 1.$$

This means that $x^*(x_0) = x^*(y_0) = 1$. Then, considering $y_0 \in S_X$ as an element of X^{**}, we have that x^* is a norming element simultaneously for $x_0 \in X$ and $y_0 \in X^{**}$. This is a contradiction with the fact that X is very rotund. ■

The next result shows the relation between the very rotundity and the smoothness properties. Oberve that (2) in fact is an improvement of the usual statement that the Gâteaux differentiability of the dual norm $\|\cdot\|^*$ implies the strict convexity of the norm $\|\cdot\|$.

Proposition 4.23

Let X be a normed space. Then:
(1) *If X^* is very rotund, then X is very smooth.*
(2) *If X^* is smooth, then X is very rotund.*

Proof: (1) Suppose that $\|\cdot\|^{**}$ is smooth at every point of $S_{X^{**}}$ that attains its norm. In particular $\|\cdot\|^{**}$ must be smooth at every point of S_X because every $x \in S_X \subset S_{X^{**}}$ attains its norm on the w^*-compact B_{X^*}, so $\|\cdot\|$ is very smooth.
(2) If $\|\cdot\|^*$ is smooth, then, in particular, it is smooth at every poins of S_{X^*} that attains its norm, so X is very rotund. ■

Remark 4.24

Šmulyan's lemma is usually applied to conclude that the Gâteaux smoothness of the dual norm on X^* implies the strict convexity of the original norm on X. The previous proposition shows that, indeed, something stronger holds: From the Gâteaux smoothness of the dual norm on X^* we get the very rotundness of the original norm on X. ®

Furthermore, very rotundness can also be translated into a duality map argument. The following result can be considered somehow a dual version of Proposition 4.17.

Proposition 4.25

Let X be a normed space. Then, X is very rotund if and only if the pre-duality map $\partial^{-1}\|\cdot\| : NA(X)\backslash\{0\} \longrightarrow 2^X$ is $\|\cdot\|^*$-w-continuous.

Proof: Assume that X is very rotund, i.e. the dual norm $\|\cdot\|^*$ on X^* is Gâteaux differentiable in $\mathrm{NA}(X)\backslash\{0\}$. This is equivalent to the duality map $\partial\|\cdot\|^* : X^* \longrightarrow X^{**}$ being $\|\cdot\|^*$-w^*-continuous in the set $\mathrm{NA}(X)\backslash\{0\}$. This means in particular that for every $x^* \in \mathrm{NA}(X)\backslash\{0\}$, we get that $\partial\|\cdot\|^*(x^*)$ is a unique element, and it should belong to X, by the definition of a norm-attaining element. Thus, since the w^* topology in X^{**} induces the w topology of X, we know that $\partial\|\cdot\|^*$ restricted to $\mathrm{NA}(X)\backslash\{0\}$ must be $\|\cdot\|^*$-w-continuous, and this last map is exactly the pre-duality map. ■

Indeed, the concept of very rotundness is more interesting than simply being the dual property of very smoothness, since this is closely related to the WLUR property.

Definition 4.26

*Let X be a Banach space. We say that its norm (or the space itself) is **weakly locally uniformly rotund** (or just **WLUR**) if for every net $\{x_i\}_i \subset S_X$ and $x \in S_X$ such that $\left\|\dfrac{x+x_i}{2}\right\| \to 1$, then $x_i \xrightarrow{w} x$.*

Remark 4.27

The property of weak local uniform rotundity is usually stated in terms of sequences. Due to the fact that the topology involved is the weak topology —a non-metrizable topology in the infinite-dimensional case— it is not completely clear "a priori" that both formulations are equivalent. Maybe it is worth to present the details: We Claim that the three following statements are equivalent for the norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ and a point $x_0 \in S_X$:
(i) Given a w-neighbourhood U of x_0, there exists $\delta > 0$ such that $x \in S_X$ and $\|x_0 + x\| > 2(1-\delta)$ imply $x \in U$.
(ii) $\|\cdot\|$ is WLUR at x_0.
(iii) Given a sequence $\{x_n\}$ in S_X such that $\|x_0 + x_n\| \to 2$, then $\{x_n\}$ is w-convergent to x_0.

Obviously, (i)⇒(ii) and (ii)⇒(iii) are trivial. If (iii) holds and (i) fails, there exists a w-neighbourhood U of x_0 such that, for every $\delta > 0$ we can find x_δ in S_X such that $\|x_0 + x_\delta\| > 2(1-\delta)$ and yet $x_\delta \notin U$. For $n \in \mathbb{N}$ and $\delta := 1/n$, find $z_n := x_{1/n}$ as above. We get a sequence $\{z_n\}$ in S_X such that $\|z_n + x_0\| \to 2$ and $\{z_n\}$ does not w-converge to x_0, a contradiction. The Claim is proved. ®

Sullivan, by using the Principle of Local Reflexivity, gave a proof of Proposition 4.28 below in [Su77]. We provide here a proof of this fact that does not require the aforementioned Principle.

Proposition 4.28

Let X be a Banach space such that $\|\cdot\|$ is WLUR. Then, $\|\cdot\|$ is very rotund.

Proof: Let $\|\cdot\|$ be WLUR. Let $x \in S_X$ and $x^{**} \in S_{X^{**}}$, and assume that for some $x^* \in S_{X^*}$ we have $\langle x^{**}, x^*\rangle = \langle x, x^*\rangle = 1.$. Find a net $\{x_i\}_i$ in S_X that $w(X^{**}, X^*)$-converges to x^{**}. Then $\dfrac{x + x_i}{2}$ is $w(X^{**}, X^*)$-convergent to $\dfrac{x + x^{**}}{2}$, hence $\|(x + x_i)/2\| \to 1$. Applying the WLUR condition, we get $x_i \xrightarrow{w} x$, so $x^{**} = Jx$. This shows that $\|\cdot\|$ is very rotund. ■

In [ZhZh2000], Zhang and Zhang proved that, adding some extra properties, the very rotundness of the norm implies that it is WLUR. Moltó, Orihuela, Troyanski, and Valdivia proved that if a Banach space admits a WLUR then it has an equivalent LUR norm ([MOTV99]). In Chapter 5 we shall make a slightly improvement of a theorem in [MOTV09] that also provides a relation between very rotund and LUR renorming.

4.4 A remark on HBS and wHBS norms

Corollary 4.7 provides a nice way of checking if a given norm $\|\cdot\|$ in X is wHBS or HBS. In addition, it hints at the fact that a good way to build a norm at X with any of these properties (if possible) would be to find a norm in the dual with good coincidence properties of topologies —essentialy the weak, the weak*, and the norm— on the unit sphere. However, only with what is stated in the aforementioned corollary, this procedure would require checking the extra step that the norm built in X^* is indeed a dual norm, i.e., that induces a norm in the predual space X. Along this section, we prove that this extra step can be avoided, because building a norm on the dual space with certain properties on the coincidence of topologies automatically turns it into a dual norm (i.e., it is w^*-lower semicontinuous). Indeed, G. A. Alexandrov proved in [Al99] that letting τ be a topology defined on X which is weaker than the topology defined by the norm $\|\cdot\|$, *if τ and the norm topology agree on the unit sphere, then the norm is τ-lower semicontinuous*. In the same spirit, we present below an extension of this argument to the notion of τ_1-τ_2-Kadets-Klee norm (or just τ_1-τ_2-KK) (meaning that both topologies coincide in the unit sphere of X) for vector topologies $\tau_1 \subset \tau_2 \subset \|\cdot\|$ where the norm is τ_2-lower semicontinuous (if $\tau_1 = w$ and $\tau_2 = \|\cdot\|$, we just say that the Banach space X has *Kadets-Klee property*). Moreover, we stress that the results we give are applicable in a more general scenario, since we will work with $\|\cdot\|$-dense sets and we are not requiring the metrizability of the two considered topologies τ_1 and τ_2. The results here are new.

Proposition 4.29

Let $\tau_1 \subset \tau_2 \subset \|\cdot\|$ two vector topologies defined on the Banach space X, and its norm $\|\cdot\|$ being a τ_1-τ_2-Kadets-Klee norm which is τ_2-lower semicontinuous. Then, it is also τ_1-lower semicontinuous.

Proof: Assume by contradiction that the norm is not τ_1-lower semicontinuous and find a net $\{x_\alpha\}_\alpha \subset B_X$ that τ_1-converges to some $x \in X$ with $x \notin B_X$. For each α consider the continuous function

$$f_\alpha(t) := \|x + t(x_\alpha - x)\|,$$

that satisfies $f_\alpha(1) = 1$ and $\lim_{t\to\infty} f_\alpha(t) = +\infty$. Therefore, we can choose $t_\alpha > 1$ such that $f_\alpha(t_\alpha) = \|x\| > 1$. Set $z_\alpha := x + t_\alpha(x_\alpha - x)$. Observe that the net $\{t_\alpha\}_\alpha$ is bounded. Indeed,

$$|t_\alpha| = \frac{\|t_\alpha(x_\alpha - x)\|}{\|x_\alpha - x\|} \leq \frac{\|x\| - \|x_\alpha\|}{\|z_\alpha\| + \|x\|} \leq \frac{2\|x\|}{\|x\| - 1}.$$

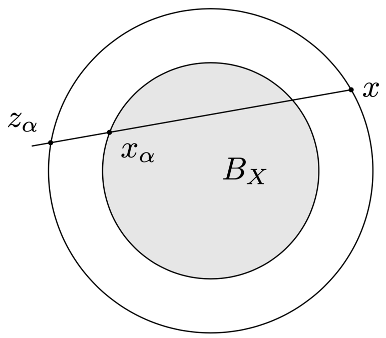

Figure 4.1: Sketch of the construction in Proposition 4.29

So, it is straightforward that the net $\{z_\alpha\}_\alpha$ τ_1-converges to x. Now, the net and the point lie on the sphere $\|x\|S_X$ and, using the τ_1-τ_2-Kadets-Klee property of the norm, we deduce that the net $\{z_\alpha\}_\alpha$ also τ_2-converges to x or, equivalently, that $\frac{1}{t_\alpha}(z_\alpha - x) = x_\alpha - x$ τ_2-converges to 0. This implies that $\{x_\alpha\}_\alpha$ converges to x. However, this is a contradiction, since the τ_2-lower semicontinuity of $\|\cdot\|$ means that B_X is a τ_2-closed set, so the net cannot τ_2-converge to a point that is outside. ■

Now, we are interested in the following weakening of the Kadets-Klee property.

Definition 4.30

Let X be a Banach space, $A \subset X$ be a cone, and let $\tau_1 \subset \tau_2 \subset \|\cdot\|$ be two *vector topologies on X. An equivalent norm $|||\cdot|||$ on X is called* **τ_1-τ_2-Kadets-Klee with respect to A** *when both topologies coincide when restricted to the intersection $A \cap S_{(X,|||\cdot|||)}$*

Notice that, using this notation, statement (ii) in Corollary 4.7 can be rewritten in the following way: *A norm $\|\cdot\|$ on X is wHBS if and only if $\|\cdot\|^*$ has the w^*-w-Kadets-Klee property with respect to NA(X).*

Let us present several generalizations of the previous proposition adapted to these new settings. The celebrated Bishop–Phelps theorem allows us to approximate in norm by elements in $\mathrm{NA}(X)$. Thanks to this we may extend Proposition 4.29 above:

Proposition 4.31

Let $\|\cdot\|$ be a τ_1-τ_2-Kadets-Klee with respect to NA(X) norm on X^* which is τ_2-lower semicontinuous. Then, it is also τ_1-lower semicontinuous.

Proof: Assume by contradiction that the norm is not τ_1-lower semicontinuous. Again, find a net $\{y^*_\alpha\}_\alpha \subset B_{X^*}$ that τ_1-converges to $y^* \notin B_{X^*}$. By the Bishop–Phelps Theorem, we can find $x^* \in \mathrm{NA}(X)$ such that

$$\|x^*\| = \|y^*\|, \text{ and } \|x^* - y^*\| \leq \frac{\|y^*\| - 1}{3}.$$

Now, set $x^*_\alpha := y^*_\alpha + (x^* - y^*)$. Therefore, we have a net

$$\{x_\alpha\}_\alpha \subset \Big(1 + \frac{\|y^*\| - 1}{3}\Big) B_{X^*}$$

that τ_1-converges to $x^* \in \mathrm{NA}(X)$, with $\|x^*\| > (1 + \frac{\|y^*\|-1}{3})$. As in the proof of the previous proposition, for each α choose $t_\alpha > 1$ such that $f_\alpha(t_\alpha) = \|x^*\|$. Set $z^*_\alpha := x^* + t_\alpha(x^*_\alpha - x^*)$ and observe again that the net $\{t_\alpha\}_\alpha$ is bounded. Now, a final trick: for every α and for every $n \in \mathbb{N}$ take, again by Bishop–Phelps Theorem, $w^*_{\alpha,n} \in \mathrm{NA}(X)$ with $\|w^*_{\alpha,n}\| = \|x^*\|$ and such that $\|w^*_{\alpha,n} - z^*_\alpha\| \leq 1/n$ for every α. Then, the net $\{w^*_{\alpha,n}\}$ clearly τ_1-converges to x^* (the set of indices is directed by the lexicographic partial order, i.e., $(\alpha, n) \leq (\beta, m))$ if and only if $\alpha \leq \beta$ and $n \leq m$). Now, as the net and the point lie on the sphere $\|x^*\|S_{X^*}$ and, using that the norm is τ_1-τ_2-Kadets-Klee with respect to $\mathrm{NA}(X)$, we deduce that the net $\{w^*_{\alpha,n}\}_{\alpha,n}$ τ_2-converges to x^*, or equivalently, that $\{x^*_\alpha - x^*\}$ τ_2-converges to 0. Indeed, let W be a neighbourhood of 0. Take U a τ_2-neighbourhood of zero such that $U + U \subset W$ and $n_0 \in \mathbb{N}$ such that $B_{X^*} \subset n_0 U$. Then, take (α_1, n_1) such that for every $(\alpha, n) \geq (\alpha_1, n_1)$, $w^*_{\alpha,n} - x^* \in U$. Then

$$t_\alpha(x^*_\alpha - x^*) = z^*_\alpha - x^* = (w^*_{\alpha,n} - x^*) + (z^*_\alpha - w^*_{\alpha,n}) \in U + (1/n)B_{X^*} \subset W$$

for every $(\alpha, n) \geq (\alpha_1, \max(n_0, n_1))$. The conclusion follows from the fact that $\{t_\alpha\}_\alpha$ is bounded below by 1. This implies that $\{x^*_\alpha\}_\alpha$ τ_2-converges to x^*, which again is a contradiction by the τ_2-lower semicontinuity of the norm. ■

Applying this result with $\tau_1 = w^*$ and $\tau_2 = w$ allow us to state, as we claimed previously, that *any w^*-w-Kadets-Klee norm with respect to NA(X) in X^* is indeed a dual norm*. Finally, we just remark that the previous argument is still valid in more general settings, since the dual space and the set NA(X) do not play any rôle, apart from the norm density provided by the Bishop–Phelps Theorem. Changing this to a suitable hypothesis allows us to prove:

Proposition 4.32

Let $\|\cdot\|$ be a norm on X that is τ_1-τ_2-Kadets-Klee with respect to a cone $A \subset X$. If $\|\cdot\|$ is τ_2-lower semicontinuous and the cone A satisfies $\overline{A \cap B_X}^{\|\cdot\|} = B_X$, then $\|\cdot\|$ is also τ_1-lower semicontinuous.

Proof: Follow the argument in the proof of Proposition 4.31, using condition $\overline{A \cap B_X}^{\|\cdot\|} = B_X$ instead of the Bishop–Phelps Theorem. ■

Chapter 5

A slight improvement of a renorming result of Moltó, Orihuela, Troyanski and Valdivia

5.1 Introduction

This short chapter focuses on a renorming resul of Moltó, Orihuela, Troyanski, and Valdivia [MOTV09, Proposition 4.4]. We arrived at this because, while checking the fundamental renorming results appeared recently, we discovered that one of the geometric concepts introduced in earlier chapters can be used advantageously to slightly improve the aforementioned result, that we reproduce here (for the non-defined concepts see below):

Theorem 5.1 (Moltó, Orihuela, Troyanski, Valdivia, [MOTV09])

Let $(X, \|\cdot\|)$ be a Banach space with a Fréchet differentiable norm. Then, the duality mapping $\partial\|\cdot\|$ is σ-slicely continuous. If, in addition, $\|\cdot\|^*$ is Gâteaux differentiable, then $\partial\|\cdot\|$ is co-σ-continuous, and hence X is LUR renormable.

We shall not prove this result here. Instead, we suggest the reader to check the proof in the given reference. Our purpose is very humble: We notice here that the renorming result can be slightly improved by using the available techniques and an observation about the concept of very rotundness, that was introduced in Definition 4.21.

Our result (Theorem 5.5) substitutes the requirement of Gâteaux smoothness of the dual norm by the less demanding of the very rotundness of the original norm. This may appear as only a formal improvement. However, let us mention some results related to this (and to a Banach space $(X, \|\cdot\|)$):

1. If $\|\cdot\|^*$ is Gâteaux differentiable, then $\|\cdot\|$ is very rotund (see Proposition 4.23 above). Thus, our Theorem 5.5 extends Theorem 5.1 above.

2. Every LUR norm is, certainly, WLUR.

3. Every WLUR norm is very rotund (see Proposition 4.28 or [Su77, Corollary to Lemma 7]).

4. For any set Γ, the space $c_0(\Gamma)$ admits an equivalent norm which is Fréchet differentiable, LUR, and which is the limit (uniform on bounded sets) of C^∞-smooth norms. In particular, $c_0(\Gamma)$ admits a C^∞-smooth norm (see, e.g., [DGZ93, Theorem V.I.5]).

Thus, for an uncountable set Γ, the space $c_0(\Gamma)$ has a Fréchet differentiable norm which is very rotund. However, the dual space $\ell_1(\Gamma)$ (for such a set Γ) does not admit any Gâteaux differentiable equivalent norm. This means that we may apply, as it is, our result, but not Theorem 5.1 above. Of course. the LUR renorming of any $c_0(\Gamma)$ is well known after a result of J. Rainwater [Rainw69].

5.2 Results

Definition 5.2 (Moltó, Orihuela, Troyanski, Valdivia [MOTV09])

Let (X,d) and (Y,p) be metric spaces. A map $\Phi : X \longrightarrow Y$ is said to be **co-σ-continuous** *if for every $\varepsilon > 0$ we can write*

$$X = \bigcup_{n\in\mathbb{N}} X_{n,\varepsilon}$$

and for every $x \in X_{n,\varepsilon}$ we can find $\delta_n(x) > 0$ such that if $y \in X_{n,\varepsilon}$ and $p(\Phi x, \Phi y) < \delta_n(x)$ then $d(x,y) < \varepsilon$.

Definition 5.3 (Moltó, Orihuela, Troyanski, Valdivia [MOTV09])

Let A be a subset of the topological space X, (Y,d) a metric space. We say that the map $\Phi : A \subset X \longrightarrow Y$ is **σ-slicely continuous** *on A if for every $\varepsilon > 0$ we can write*

$$A = \bigcup_{n\in\mathbb{N}} A_{n,\varepsilon}$$

in such a way that for every $x \in A_{n,\varepsilon}$ there exists an open half space $H \subset X$ containing x with $\operatorname{diam}\Phi(H \cap A_{n,\varepsilon}) < \varepsilon$.

Proposition 5.4 (Moltó, Orihuela, Troyanski, Valdivia [MOTV09])

Let (X, d) be a metric space and $(Y, \|\cdot\|)$ a normed space. A mapping $\Phi : A \subset X \longrightarrow Y$ is co-σ-continuous if, and only if, for every $x \in A$ there exist a separable subset $Z_x \subset X$ such that if $\lim_{n\to\infty} \Phi x_n = \Phi x$ then

$$x \in \overline{\operatorname{span}}^{\|\cdot\|} \bigcup \{Z_{x_n} : \ n \in \mathbb{N}\}.$$

Theorem 5.5

Let X be a Banach space such that its norm is simultaneously Fréchet differentiable and very rotund. Then X admits a LUR renorming.

Proof: Recall that we use the notation $F^x := \partial\|\cdot\|(x)$. Fix $\varepsilon > 0$. Let $x \in S_X$. Since the norm is Fréchet differentiable at x we have, according to the Šmulyan test, that there exists $\delta_x > 0$ such that $\|F^x - f\| < \varepsilon$ whenever $f \in S_{X^*}$ and $f(x) > 1 - \delta_x$. Set $S_n := \{x \in S_X : \ \delta_x > \frac{1}{n}\}$. Observe that $S_X = \bigcup_{n\in\mathbb{N}} S_n$. Let us prove that for every $x \in S_n$ we can find a slice S of S_n that contains x and such that the diameter of $\partial\|\cdot\|(S)$ is less than 2ε. We Claim that we can take $S := S(S_n, F^x, 1/n)$. Indeed, if $y \in S$, then we have $\delta_y > 1/n$ and $\langle y, F^x\rangle > 1/n$. Thus, $\langle y, F^x\rangle > 1 - \delta_y$, hence $\|F^x - F^y\| < \varepsilon$. This shows the Claim. Thus, $\partial\|\cdot\|$ is σ-slicely continuous.
Now, assume that $\|\cdot\|$ is a very rotund norm. Then, by Proposition 4.25, we have that the preduality map $\partial^{-1}\|\cdot\| : X^* \longrightarrow X$ is $\|\cdot\|$-w-continuous. In particular, if we take a sequence $\{x_n\}_{n=1}^{\infty}$ such that $\|F^x - F^{x_n}\| \xrightarrow{n\to\infty} 0$, then $x \in \overline{\operatorname{conv}}^{\|\cdot\|}(\{x_n\}_{n=1}^{\infty}\})$. Now, applying Proposition 5.4, we conclude that $\partial\|\cdot\|$ is co-σ-continuous.
Finally, by [MOTV09, Theorem 3.1] we have that this is equivalent to having an equivalent LUR renorming. ■

Chapter 6

Projectional Resolutions of the Identity

For decades, the concept of the PRI has been a key tool and a great contribution to the study of nonseparable spaces. That is why there have been numerous articles in the field trying to obtain this type of construction in different kinds of spaces. One of the highlights is the paper of D. Amir and J. Lindestrauss ([AmLin68]), where they deduced some fundamental properties on WCG spaces using PRI's. Tacon was also one of the pioneers who saw the potential of the tools used by Amir and Lindestrauss and applied them to prove the existence of PRI for the dual of a very smooth space and some other generalizations ([Tac70]). However, in those years projectional resolutions of the identity were obtained through a long and costly process (for instance, the whole paper of Tacon is devoted to this end). Thus, finding sufficient conditions for the existence of a PRI was a worthwhile contribution to the field. For this reason we will invest some effort in studying projectional generators, a powerful tool introduced by J. Orihuela and M. Valdivia in [OrVa90], cleverly identifying some common pattern in previous constructions. Thanks to this concept, ensuring the existence of a PRI is much simpler, since finding a projectional generator guarantees that the construction of a PRI can be carried out in a natural inductive way. To illustrate its usefulness, we include at the end of Section 6.1 an alternative proof of Tacon's result in [Tac70], in a much more compact way (Theorem 6.12). Section 6.2 is fully devoted to prove the existence of a PRI in the dual of every Asplund space by using projectional generators, following [Fa97] instead of the original construction in [FaGo88].

6.1 A General Method for constructing a PRI

Recall that having $A \subset X$, $\operatorname{dens}(A)$ is the smallest cardinal of the form $\operatorname{card}(D)$, where $D \subset A$ and $A \subset \overline{D}$.

Definition 6.1

Let X be a nonseparable Banach space, and let μ be the first ordinal with $\operatorname{card}(\mu) = \operatorname{dens}(X)$. A ***projectional resolution of the identity*** *(or* ***PRI*** *for short) on X is a family $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ of linear projections on X such that $P_\omega \equiv 0$, P_μ is the identity mapping, and for all $\omega < \alpha \leq \mu$ the following hold:*
(i) $\|P_\alpha\| = 1$,
(ii) $\operatorname{dens}(P_\alpha X) \leq \operatorname{card}(\alpha)$,
(iii) *if* $\omega \leq \beta \leq \alpha$, *then* $P_\alpha \circ P_\beta = P_\beta \circ P_\alpha = P_\beta$, *and*
(iv) $\bigcup_{\beta<\alpha} P_{\beta+1}X$ *is* $\|\cdot\|$*-dense in* $P_\alpha X$.

We will start by proving some properties granted by the existence of a PRI. These will be key for getting subsequent results.

Proposition 6.2 ([Fa97, Proposition 6.2.1])

Let X be a Banach space with a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$. Then, for any $x \in X$, the mapping $\alpha \mapsto P_\alpha x$ from $[\omega, \mu]$ (endowed with the order topology) into X with the norm topology is continuous.

Proof: Let $\omega < \alpha \leq \mu$ and $\varepsilon > 0$ be given. By (iv) in Definition 6.1, there are $\omega \leq \gamma < \alpha$ and $y \in P_\gamma X$ so that $\|P_\alpha x - y\| < \varepsilon/2$. Then, for all $\gamma \leq \beta \leq \alpha$ we have

$$\|P_\beta x - u\| = \|P_\beta(P_\alpha x - y)\| \leq \|P_\alpha x - y\| < \frac{\varepsilon}{2}.$$

So

$$\|P_\alpha x - P_\beta x\| \leq \|P_\alpha x - y\| + \|y - P_\beta x\| < 2\frac{\varepsilon}{2} = \varepsilon,$$

and the proof is over. ■

Proposition 6.3 ([DGZ93, Lemma 1.2, VI.1], [FHHMZ11, Proposition 13.14])

Let X be a Banach space with a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$. Then:

(i) *For every $x \in X$, $\{\|(P_{\alpha+1} - P_\alpha)x\| : \omega \leq \alpha < \mu\}$ belongs to $c_0(\Gamma)$, letting $\Gamma := [\omega, \mu[$. Thus, sets of the form $\{\alpha \in \Gamma : \|(P_{\alpha+1} - P_\alpha)x\| > \varepsilon\}$ are finite, so as a consequence, all sets $\{\alpha \in \Gamma : (P_{\alpha+1} - P_\alpha)x \neq 0\}$ are countable.*

(ii) *For every $x \in X$, we have $x = \sum_{\omega\leq\alpha<\mu}(P_{\alpha+1} - P_\alpha)(x)$, where only a countable* *number of summands are non-zero (so this sum should be understood as a usual series). In particular* $X = \overline{\bigcup_{\gamma \leq \alpha < \mu} P_\alpha(X)}$.

Proof: By contradiction, assume that the assertion (i) is false. Then, there exist $x_0 \in X$, $\varepsilon_0 > 0$ and $\omega < \alpha_1 < \alpha_2 < \ldots < \mu$ such that for every $i \geq 1$, $\|(P_{\alpha_i+1} - P_{\alpha_i})x_0\| > \varepsilon_0$. If we let $\alpha_0 := \sup\{\alpha_i : i \geq 1\}$, then this leads to the conclusion that the mapping $\alpha \mapsto P_\alpha(x)$ is not continuous at α_0, which is a contradiction with Proposition 6.2. (ii) follows from the fact that, if $\{\alpha \in \Gamma : (P_{\alpha+1} - P_\alpha)x \neq 0\} = \{\alpha_n\}_{n=1}^\infty$ such that $\alpha_1 < \alpha_2 < \ldots$, then

$$\sum_{\omega \leq \alpha < \mu} (P_{\alpha+1} - P_\alpha)(x) = P_{\alpha_1}x + (P_{\alpha_1+1} - P_{\alpha_1})x + (P_{\alpha_2+1} - P_{\alpha_2})x + \ldots,$$

together with Proposition 6.2. ■

Once we have reviewed the good properties of PRIs, we proceed to define the notion of projectional generator, our main tool to obtain a PRI. Notice that the definition we use is not exactly that given by Orihuela and Valdivia in [OrVa90], but a slightly different one, employed by Fabian in its monograph [Fa97].

Definition 6.4

Let X be a Banach space, $W \subset X^*$ a 1-norming subset such that $\overline{W}$ is a linear subspace, and $\Phi : W \longrightarrow 2^X$ an at most countably valued mapping. The pair (W, Φ) is called ***projectional generator*** *on X if for every nonempty set $B \subset W$ such that $\overline{B}$ is linear, the following condition is satisfied:*

$$\Phi(B)^\perp \cap \overline{B}^{w^*} = \{0\}.$$

From here, the results are aimed to prove the construction of a PRI from a projectional generator. The following lemmata is the basic setup that will be applied on a transfinite inductive process in Proposition 6.8 and the main proof during Theorem 6.9.

Recall that a **linear norm 1 projection** on a Banach space X is a linear mapping $P : X \longrightarrow X$ satisfying $\|P\| = 1$ and $P \circ P = P$.

Lemma 6.5 ([Fa97, Lemma 6.1.1])

Let X be a Banach space and $P : X \longrightarrow X$ a linear norm 1 projection. Put $E = PX$ and $F = P^*X^*$, then

(i) $\|x\| = \sup\langle x, F \cap B_{X^*}\rangle$ *for every* $x \in E$ *and*
(ii) $E^\perp \cap F = \{0\}$.
Conversely, assume that there exist two sets $A \subset X$, $B \subset X^*$ *such that* $\overline{A}, \overline{B}$ *are linear subspaces such that*
(i') $\|a\| = \sup\langle a, B \cap B_{X^*}\rangle$ *for every* $a \in A$ *and*
(ii') $A^\perp \cap \overline{B}^{w^*} = \{0\}$.
Then, there exists a linear norm 1 projection $P : X \longrightarrow X$ *such that* $PX = \overline{A}$, $P^{-1}(0) = B_\perp$ *and* $P^*X^* = \overline{B}^{w^*}$.

Proof: If $x \in PX$, then

$$\|x\| = \|Px\| = \sup\langle Px, B_{X^*}\rangle = \sup\langle x, P^*X^* \cap B_{X^*}\rangle = \sup\langle x, F \cap B_{X^*}\rangle.$$

If $\xi \in E^\perp \cap F$, then for every $x \in X$

$$\langle x, \xi\rangle = \langle x, P^*\xi\rangle = \langle Px, \xi\rangle = 0,$$

so $\xi = 0$.

For the converse, assume A, B are such in the second part of the statement. If $x \in \overline{A} \cap B_\perp$, then using (i'),

$$\|x\| = \sup\langle x, B \cap B_{X^*}\rangle = 0.$$

Notice that (i') also ensures us that $\overline{A} + B_\perp$ is closed, since for every $a \in A$ and every $u \in B_\perp$,

$$\|a\| = \sup\langle a, B \cap B_{X^*}\rangle = \langle a + u, B \cap B_{X^*}\rangle \leq \|a + u\|.$$

By contradicion, if we assume that $\overline{A} + B_\perp \neq X$, then we can find $0 \neq \xi \in X^*$ which vanish on $\overline{A} + B_\perp$. Thus, we have that $\xi \in (\overline{A} + B_\perp)^\perp = A^\perp \cap \overline{B}^{w^*}$,and this last set has only the element zero, by (ii'). So $\xi = 0$, which is a contradiction. This proves that $X = \overline{A} \oplus B_\perp$. This decomposition will be used to define the projection $P : X \longrightarrow X$ by

$$P(a + u) = a, \qquad \text{for } a \in \overline{A}, u \in B_\perp.$$

Thus, P is a linear projection with $\|P\| = 1$, $PX = \overline{A}$ and $P^{-1}(0) = B_\perp$. Only lacks to prove that $P^*X^* = \overline{B}^{w^*}$. We do it by double inclusion. First, if $\xi \in B$ then for all $x \in X$ we have

$$\langle x, P^*\xi\rangle = \langle Px, \xi\rangle = \langle x, \xi\rangle,$$

because $x - Px \in B_\perp$, so $\xi \in P^*X^*$. Hence, $B \subset P^*X^*$, and taking w^*-closures, we get $\overline{B}^{w^*} \subset P^*X^*$. Finally, by contradiction, assume that there is $\xi \in P^*X^* \setminus \overline{B}^{w^*}$.

By the separation theorem, we can find a $x \in X$ such that $\langle x, \xi \rangle \neq 0 = \sup\langle x, B\rangle$ (recall that $\overline{B}$ is a linear subspace). It follow that $x \in B_\perp$, and so $Px = 0$. By thus we obtain a contradiction, since

$$0 \neq \langle x, \xi \rangle = \langle Px, \xi \rangle.$$

■

It is useful to know that in the second part of Lemma 6.5, the condition (ii') can be replaced by a weaker one. This will be mentioned in Remark 6.10, which is a key step for some results ahead.

Lemma 6.6 ([Fa97, Lemma 6.1.2])

Let $A \subset X$ and $B \subset X^*$ be such that $\overline{A}$ and $\overline{B}$ are linear subspaces, and such that
(i') $\|a\| = \sup\langle a, B \cap B_{X^*}\rangle$ for every $a \in A$, and
(ii") $A^\perp \cap \overline{B \cap B_{X^*}}^{w^*} = \{0\}$.
Then,
(ii") $A^\perp \cap \overline{B}^{w^*} = \{0\}$.

Proof: Define

$$Y := \bigcup_{n=1}^{\infty} \overline{nB \cap B_{X^*}}^{w^*}.$$

Notice that, since $\overline{B}$ is linear, Y must be a linear subspace too. If we show that Y is w^*-closed, then the proof is done, since from this we would have $Y = \overline{B}^{w^*}$ and (ii') will immediately follow from this fact. Take $E := \overline{A}$ and let $Q : X^* \longrightarrow E^*$ be the restriction mapping, that is,

$$\langle e, Qx^* \rangle = \langle e, x^* \rangle, \quad \text{for every } x^* \in X^*,\ e \in E.$$

Observe that Q is injective on Y, since if we take $y \in Y$ such that $Qy = 0$, then $y \in A^\perp$, and also $(1/n)y \in \overline{B \cap B_{X^*}^{w^*}}$ for some $n \in \mathbb{N}$, so applying (ii"), we deduce $y = 0$.
Now, we shall check $Q(\overline{B \cap B_{X^*}}^{w^*}) = B_{E^*}$. The "$\subset$" inclusion always holds. By contradiction, if the reverse one is not true, using that $Q(\overline{B \cap B_{X^*}}^{w^*})$ is w^*-closedf and convex, the Separation Theorem provides us with two elements $e \in E$ and $e^* \in B_{E^*}$ such that

$$\sup\langle e, Q(\overline{B \cap B_{X^*}}^{w^*})\rangle < \langle e, e^* \rangle.$$

Hence, by (i'), we get $\|e\| < \langle e, e^* \rangle$, which is a contradiction. Finally, we claim that

$$Y \cap B_{X^*} = \overline{B \cap B_{X^*}}^{w^*}.$$

If this is true, then the Krein–Šmulyan Theorem guarantees that Y is a w^*-closed set, as we wanted to prove. In order to prove the claim, we check that

$$Q(Y \cap B_{X^*}) \subset Q(B_{X^*}) = B_{E^*} = Q(\overline{B \cap B_{X^*}}^{w^*}).$$

Since we proved that Y is injective on Y, we get $Y \cap B_{X^*} \subset \overline{B \cap B_{X^*}}^{w^*}$. The reverse inclusion is trivial. ■

Lemma 6.7 ([Fa97, Lemma 6.1.3])

Let X be a Banach space and $Y \subset X$ a subspace. Let $W \subset X^*$ with $\overline{W}$ is linear and let $\Phi : W \subset X^* \longrightarrow 2^X$ and $\Psi : X \longrightarrow 2^W$ be two, at most countable valued mappings. Let $\aleph$ an infinite cardinal number and let $A_0 \subset X$, $B_0 \subset W$ be two subsets with $card(A_0), card(B_0) \leq \aleph$. Then, there exist sets $A_0 \subset A \subset X$ and $B_0 \subset B \subset W$ such that

1. *$\overline{A}$, $\overline{B}$ are linear subspaces;*
2. *$\overline{A} \cap Y = \overline{A \cap Y}$;*
3. *$card(A), card(B) \leq \aleph$.*
4. *$\Phi(B) \subset A$ and $\Psi(A) \subset B$.*

Proof: The proof will consist on a classical gluing argument due to Mazur. Consider $f : X \longrightarrow Y$ be a mapping assigning to each $x \in X$ a point $f(x) \in Y \subset X$ such that $\|x - f(x)\| \leq 2 \operatorname{dist}(x, Y)$. By induction, we shall construct sequences of sets $A_0 \subset A_1 \subset A_2 \subset \cdots \subset X$ and $B_0 \subset B_1 \subset B_2 \subset \cdots \subset W$ as follows. Suppose that for some fixed $n \in \mathbb{N}$ we have already constructed the sets A_i and B_i for $i = 0, 1, ..., n-1$. Then, we define the new sets

$$A_n := \left\{ \sum_{i=1}^{m} x_i v_i : v_i \in A_{n-1} \cup f(A_{n-1}) \cup \Phi(B_{n-1}),\ r_i \in \mathbb{Q},\ m \in \mathbb{N} \right\};$$

$$B_n := \left\{ \sum_{i=1}^{m} r_i x_i^* : v_i \in B_{n-1} \cup \Psi(A_{n-1}),\ r_i \in \mathbb{Q},\ m \in \mathbb{N} \right\}.$$

With this, put $A := \bigcup_{n=1}^{\infty} A_n$ and $B := \bigcup_{n=1}^{\infty} B_n$. Considering two points $a_1, a_2 \in A$, we have that $a_1, a_2 \in A_n$ for some $n \in \mathbb{N}$, and notice that by the construction, $a_1, a_2 \in A_{n+1} \subset A$. Also, if $a \in A$ and $\lambda \in \mathbb{R}$, there must be an $n \in \mathbb{N}$ such that $\lambda a \in \lambda A_n$, and by the density of $\mathbb{Q}$ on $\mathbb{R}$, we have that $\lambda A_n \subset \overline{A_{n+1}} \subset A$. Both facts together show that $\overline{A}$ is linear, and an analogous argument applies for $\overline{B}$. Now, we prove $\overline{A} \cap Y = \overline{A \cap Y}$ by double inclusion. The $\overline{A} \cap Y \supset \overline{A \cap Y}$ always hold. For the reverse inclusion, take any $y \in \overline{A} \cap Y$. So, there is a sequence $\{a_i\}_{i=1}^{\infty} \subset A$ converging to y. For each $i \in \mathbb{N}$ we can find $n_i \in \mathbb{N}$ such that $a_i \in A_{n_i}$. Then

$$\begin{aligned}\|f(a_i) - y\| &\leq \|f(a_i) - a_i\| + \|a_i - y\| \\ &\leq 2 \operatorname{dist}(a_i, Y) + \|a_i - y\| \leq 3\|a_i - y\| \xrightarrow{i\to\infty} 0.\end{aligned}$$

Since $f(a_i) \in f(A_{n_i}) \subset A_{n_i+1} \subset A$, it follows that $y \in \overline{A \cap Y}$. Thus, $\overline{A} \cap Y \subset \overline{A \cap Y}$, so we have the desired equality. The last two properties follow from the construction of the sets A and B. ■

The previous lemma is complemented by this result, which basically is obtained applying repeatedly the previous lemma in a transfinite induction process.

Proposition 6.8 ([Fa97, Proposition 6.1.4])

Let X, Y, W, Φ and Ψ be as in the previous Lemma 6.7. Assume that $\text{dens}(Y) > \aleph_0$ *and let μ be the first ordinal with $\text{card}(\mu) = \text{dens}(Y)$. Then, there exist families $\{A_\alpha : \omega < \alpha \leq \mu\}$ and $\{B_\alpha : \omega < \alpha \leq \mu\}$ of subsets in X and W, respectively, such that $Y \subset \overline{A_\mu}$, and for each $\omega < \alpha \leq \mu$ the following holds:*

1. $\overline{A_\alpha}$, $\overline{B_\alpha}$ *are linear subspaces;*
2. $\overline{A_\alpha} \cap Y = \overline{A_\alpha \cap Y}$,
3. $\text{card}(A_\alpha), \text{card}(B_\alpha) \leq \text{card}(\alpha)$,
4. $\Phi(B_\alpha) \subset A_\alpha$ *and* $\Psi(A_\alpha) \subset B_\alpha$,
5. *if* $\omega < \beta \leq \alpha$, *then* $A_\beta \subset A_\alpha$, $B_\beta \subset B_\alpha$, *and*
6. $A_\alpha = \bigcup_{\beta<\alpha} A_{\beta+1}$, $B_\alpha = \bigcup_{\beta<\alpha} B_{\beta+1}$.

Proof: Let f be defined as in the proof of Lemma 6.7. Let $\{y_\alpha : \omega \leq \alpha < \mu\}$ be a dense subset in Y. We will construct the family by using transfinite induction on α. As the initial case, apply Lemma 6.7 to $A_0 = \{y_\omega\}$, $B_0 = \emptyset$, $\aleph = \aleph_0$, and thus we obtain sets $A_0 \subset A \subset X$, $B_0 \subset B \subset W$ as in the statement. We denote $A_{\omega+1} := A$, $B_{\beta+1} := B$.
Now, let $\omega + 1 < \gamma \leq \mu$ be fixed and assume that for every $\omega < \alpha < \gamma$ the sets $A_\alpha \subset X$ and $B_\alpha \subset W$ are already constructed satisfying the properties listed in the previous lemma, as well as the extra condition that

$$\text{if } \omega < \beta < \alpha, \text{ then } f(A_\beta) \subset A_\alpha.$$

In the case in which γ is a limit ordinal, then just take $A_\gamma := \bigcup_{\alpha<\gamma} A_\alpha$ and $B_\gamma := \bigcup_{\alpha<\gamma} B_\alpha$. If γ is a successor ordinal, then we use Lemma 6.7 for the sets $A_0 = A_{\gamma-1} \cup \{y_{\gamma-1}\} \cup f(A_{\gamma-1})$, and $\aleph = \text{card}(\gamma)$ to obtain the sets A_γ and B_γ. In both cases, it is easy to verify that they share all the properties listed above, perhaps except the identity $\overline{A_\gamma} \cap Y = \overline{A_\alpha \cap Y}$ in the case when γ is a limit ordinal. Allow us to show that this also holds.
Consider a sequence $\{a_i\}_{i=1}^\infty \subset A_\gamma$ that converges to some $y \in Y$. Then

$$\|f(a_i) - y\| \leq 3\|a_i - y\| \xrightarrow{i\to\infty} 0$$

and $f(a_i) \in f(A_{\beta_i}) \subset A_{\beta_i+1} \subset A_\gamma$ for some $\beta_i < \gamma$. So $y \in \overline{A_\gamma \cap Y}$ and thus $\overline{A_\gamma} \cap Y = \overline{A_\alpha \cap Y}$. Finally, notice that A_μ contains the set $\{y_\alpha : \omega \leq \alpha < \mu\}$, which is dense in Y, so this means that $Y \subset \overline{A_\mu}$. ■

Finally, we arrive at the result that justifies the inclusion of projectional generators

Theorem 6.9 ([Fa97, Proposition 6.1.7])

A nonseparable Banach space with a projectional generator admits a PRI

Proof: Consider (W, Φ) a projectional generator on X. Put $P_\omega \equiv 0$. Notice that, as W is a 1-norming set, for each $x \in X$ we can find a countable set $\Psi(x) \subset W$ such that

$$\|x\| = \sup\langle x, \Psi(x) \cap B_{X^*}\rangle.$$

This gives us the (at most) countably valued mapping $\Psi : X \longrightarrow 2^X$. Now, applying Proposition 6.8 with $Y = X$, we get the families $\{A_\alpha : \omega < \alpha \leq \mu\}$ and $\{B_\alpha : \omega < \alpha \leq \mu\}$, of sets belonging to X and W, respectively. Now, fix any $\omega < \alpha \leq \mu$. Then, we have, for every $a \in A_\alpha$,

$$\|a\| = \sup\langle a, \Psi(a) \cap B_{X^*}\rangle \leq \sup\langle a, B_\alpha \cap B_{X^*}\rangle \leq \|a\|$$

as $\Psi(A_\alpha) \subset B_\alpha$ and, since $\Phi(B_\alpha) \subset A_\alpha$ and (W, Φ) is a projectional generator, we get

$$A_\alpha^\perp \cap \overline{B_\alpha}^{w^*} \subset \Phi(B_\alpha)^\perp \cap \overline{B_\alpha}^{w^*}.$$

Now, we are able to use Lemma 6.5 to ensure that there exists a linear norm 1 projection $P_\alpha : X \longrightarrow X$ such that $P_\alpha(X) = \overline{A_\alpha}$, $P_\alpha^{-1}(0) = B_{\alpha\perp}$, and $P_\alpha^* X^* = \overline{B_\alpha}^{w^*}$. Performing this for every $\alpha \in]\omega, \mu]$, since families $\{A_\alpha : \omega < \alpha \leq \mu\}$ and $\{B_\alpha : \omega < \alpha \leq \mu\}$ satisfy the conditions listed in the thesis of Proposition 6.8, it follows immediately that $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ is a projectional resolution of the identity on X. ■

To get the PRI's needed along this work, we shall rely on the following fact.

Remark 6.10 ([Fa97, Remark 6.1.8])

Notice that from the above proof and thanks to Lemma 6.6, instead of the requirement

$$\Phi(B)^\perp \cap \overline{B}^{w^*} = \{0\}$$

that figures in the definition of a projectional generator, we can use the weaker condition

$$\Phi(B)^\perp \cap \overline{B \cap B_{X^*}}^{w^*} = \{0\},$$

and the previous PRI construction still valid. ®

If the space of interest is a dual space and supports a projectional generator given by its predual X, we can deduce some more properties of the PRI.

Proposition 6.11 ([Fa97, Proposition 6.1.9])

Suppose that a nonseparable dual space X^* admits a projectional generator defined by X, i.e., W, Φ such that $W \subset X (\subset X^{**})$ and $\overline{W} = X$. Then, X^* admits a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ together with a nondecreasing "long sequence" $\{E_\alpha : \omega \leq \alpha \leq \mu\}$ of subspaces of X such that $E_\omega = \{0\}$ and for all $\omega < \alpha \leq \mu$ the following hold:

(i) $\operatorname{dens} E_\alpha \leq card(\alpha)$,

(ii) $E_\alpha = \overline{\bigcup_{\beta<\alpha} E_{\beta+1}}$,

(iii) *the mapping R_α sending $\xi \in P_\alpha X^*$ to its restriction $\xi_{|E_\alpha}$ maps $P_\alpha X^*$ isometrically onto E_α^*, and*

(iv) *if $\alpha < \mu$, then $(P_{\alpha+1} - P_\alpha)X^*$ is isometric to $(E_{\alpha+1}/E_\alpha)^*$.*

Proof: Put $E_\omega = \{0\}$ and let μ be the first ordinal with $\operatorname{card}(\mu) = \operatorname{dens}(X^*)$. We shall repeat, almost word by word, the proof of Theorem 6.9. Thus, for each $\alpha \in]\omega, \mu]$ we obtain sets $A_\alpha \subset X^*$, $B_\alpha \subset W (\subset X \subset X^{**})$ and projections $P_\alpha : X^* \longrightarrow X^*$ such that $P_\alpha X^* = \overline{A_\alpha}$, $P_\alpha^{-1}(0) = B_\alpha^\perp$, $P_\alpha^* X^{**} = \overline{B_\alpha}^{w^*}$ and $\|a\| = \sup\langle a, B_\alpha \cap B_X\rangle$ for all $a \in A_\alpha$. Then, $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ is a PRI on X^*. Put $E_\alpha = \overline{B_\alpha}$; then (i) and (ii) hold. Now, define the mappings $R_\alpha : P_\alpha X^* \longrightarrow E_\alpha^*$ by

$$R_\alpha \xi := \xi_{|E_\alpha}, \qquad \text{for every } \xi \in P_\alpha X^*.$$

Thus, each R_α is an isometric embedding. Now, take any $\eta \in E_\alpha^*$ and let $\xi \in X^*$ be such that $\xi_{|E_\alpha} = \eta$. Then for all $x \in E_\alpha$ we have $\langle P_\alpha \xi, x\rangle = \langle \eta, x\rangle$, since $P_\alpha \xi - \xi \in B_\alpha^\perp$. Hence $R_\alpha(P_\alpha \xi) = \eta$, which means that R_α is surjective.

As to the proof of (iv), we define a mapping $\Psi : (P_{\alpha+1} - P_\alpha)X^* \longrightarrow (E_{\alpha+1}/E_\alpha)^*$ by

$$\langle \Phi(\xi), [x]\rangle = \langle \xi, x\rangle, \quad \text{for every } \xi \in (P_{\alpha+1} - P_\alpha)X^*,\ [x] \in E_{\alpha+1}/E_\alpha;$$

here $[x]$ means the class $x + E_\alpha$. It is well defined since for $\xi \in (P_{\alpha+1} - P_\alpha)X^*$ and for $x \in E_\alpha$ we have

$$\langle \xi, x\rangle = \langle (P_{\alpha+1} - P_\alpha)\xi, x\rangle = \langle P_{\alpha+1}^* x, \xi\rangle - \langle P_\alpha^* x, \xi\rangle = 0,$$

because $P_\alpha^* X^{**} = \overline{E_\alpha}^{w^*}$. Moreover, for every $\xi \in (P_{\alpha+1} - P_\alpha)X^*$ we get

$$\begin{aligned}\|\Psi(\xi)\| &= \sup\{\langle \Psi(\xi), [x]\rangle : [x] \in E_{\alpha+1}/E_\alpha,\ \|[x]\| < 1\} \\ &= \sup\{\langle \xi, x\rangle : x \in E_{\alpha+1},\ \|x\| < 1\} = \|R_{\alpha+1}\xi\| = \|\xi\|\end{aligned}$$

as $(P_{\alpha+1} - P_\alpha)X^* \subset P_{\alpha+1}X^*$, and we already know that $R_{\alpha+1}$ is an isometry. It remains to prove that Φ is onto. Let $\xi \in (E_{\alpha+1}/E_\alpha)^*$ be given and define $\zeta \in E^*_{\alpha+1}$ by

$$\langle \zeta, x\rangle := \langle \eta, [x]\rangle, \quad \text{for } x \in E_{\alpha+1}.$$

Also, put $\xi = R^{-1}_{\alpha+1}(\zeta)$. Take any $x \in X$. Then $P^*_\alpha x \in P^*_\alpha X^{**} = \overline{B_\alpha}^{w^*}$. Now, let $\{u_\tau\}_\tau \subset B_\alpha$ be a net which is w^*-covergent to $P^*_\alpha x$. We then have

$$\begin{aligned}\langle P_\alpha \xi, x\rangle &= \langle P^*_\alpha x, \xi\rangle = \lim_\tau \langle \xi, u_\tau\rangle \\ &= \lim_\tau \langle \zeta, u_\tau\rangle = \lim_\tau \langle \eta, [u_\tau]\rangle = 0\end{aligned}$$

since $u_\tau \in B_\alpha \subset E_\alpha$, and so $P_\alpha \xi = 0$. Hence, for all $[x] \in E_{\alpha+1}/E_\alpha$ we get

$$\begin{aligned}\langle \Psi((P_{\alpha+1} - P_\alpha)\xi), [x]\rangle &= \langle (P_{\alpha+1} - P_\alpha)\xi, x\rangle \\ &= \langle P_{\alpha+1}\xi, x\rangle = \langle \xi, x\rangle = \langle \zeta, x\rangle \\ &= \langle \eta, [x]\rangle;\end{aligned}$$

that is, $\Psi((P_{\alpha+1} - P_\alpha)\xi) = \eta$, which means that Ψ is surjective. ■

Thus, we end this section with the aforementioned alternative prove to the Tacon's result, proving that the dual of a very smooth space admits a projectional generator (and so, a PRI).

Theorem 6.12

Let X be a Banach space that admits a very smooth norm. Then, there exists a projectional generator on X^* defined by X. In particular, X^* admits a PRI.

Proof: Consider X with $\|\cdot\|$ being a very smooth norm. By Proposition 4.17, the duality map $\partial\|\cdot\|$ is s $\|\cdot\|$-w-continuous map. Take $W := X$ as a subspace of X^{**} and $\Phi : W \longrightarrow X^*$ such that

$$\Phi(x) := \begin{cases} 0, & \text{if } x = 0 \\ \partial\|\cdot\|(x), & \text{otherwise.}\end{cases}$$

Then, the map Φ is univaluated and $\|\cdot\|$-w-continuous in $X\backslash\{0\}$. Take any $\emptyset \neq B \subset X^*$ such that $\overline{B}$ is linear. By Remark 6.10, it is enough to check

$$\Phi(B)^\perp \cap \overline{B \cap B_X}^{w(X^{**},X^*)} = \{0\}. \tag{6.1}$$

Furthermore, we can check that $\Phi(B)^\perp = \Phi(\overline{B})^\perp$, by double inclusion. The first one is trivial, since $\Phi(B) \subset \Phi(\overline{B})$, and thus $\Phi(\overline{B}) \subset \Phi(B)^\perp$. To prove the other one, notice that by the $\|\cdot\|$-w-continuity of Φ, we have

$$\Phi(\overline{B}) \subset \overline{\Phi(B)}^{w(X^{**},X^{*})},$$

and because $\Phi(B)^{\perp}$ is a closed linear subspace,

$$\Phi(B)^{\perp} = \left(\overline{\Phi(B)}^{w(X^{**},X^{*})}\right)^{\perp} \subset \Phi(\overline{B})^{\perp}.$$

Hence, as claimed, $\Phi(B)^{\perp} = \Phi(\overline{B})^{\perp}$. Thus, instead of (6.1), it is the same to check

$$\Phi(\overline{B})^{\perp} \cap \overline{B \cap B_{X^*}}^{w(X^{**},X^{*})} = \{0\}. \tag{6.2}$$

By contradiction, assume that there is

$$0 \neq x^{**} \in \Phi(\overline{B})^{\perp} \cap \overline{B \cap B_X}^{w(X^{**},X^{*})}.$$

Then, we can find a net $\{b_\alpha\}_\alpha \subset B \cap B_X$ that $w(X^{**}, X^*)$-converges to x^{**}. Then, for every $b \in \overline{B}$, we should get

$$\langle \Phi(b), b_\alpha \rangle \to \langle \Phi(b), x^{**} \rangle.$$

Considering $B_{(\overline{B})^*}$ as the unit ball of the dual space of $\overline{B}$, the Bishop–Phelps Theorem ensures us that

$$B_{(\overline{B})^*} \subset \overline{\mathrm{NA}(\overline{B})} = \overline{\{\Phi(b)_{|\overline{B}} : b \in \overline{B}\}}.$$

Combining the last two equation we deduce that the net $\{b_\alpha\}_\alpha$ converges to 0 in the weak topology of $\overline{B}$, and thus, on the weak topology of X. Hence we get that $x^{**} = 0$, a contradiction. ■

Remark 6.13

Of course, the statement of this theorem will become obsolete when we prove that the dual of an Asplund space admits such a projectional generator. However, it remains as an interesting particular case, since our proof guarantees that the projective generator is single-valued and has good continuity properties ($\|\cdot\|$-w-continuity), which is not the case for the dual of an Asplund space.

6.2 Constructing the PRI in the dual of an Asplund space

This whole section is devoted to obtaining a projectional generator in the dual of an Asplund space, and thus, we can ensure the existence of a PRI. That was

a longstanding open problem during almost twenty years. Several articles managed to prove partial results assuming extra hypothesis until Fabian and Godefroy provide the construction in [FaGo88]. The difficulty of the problem lies in the fact that the auxiliary application used to build the PRI (what for us is now a projective generator) is not easy to obtain. Fabian and Godefroy (and also, the approach we are following) used a powerful result from Jayne and Rogers to solve this.
First, we need a lemma about $\ell_\infty(\Gamma)$, which is known in the literature as *Simon's inequality*.

Lemma 6.14 (Simon's inequality)

Let Γ be a non-empty set, $\{g_n\}_{n=1}^\infty \subset \ell_\infty(\Gamma)$ a bounded sequence, and $\Delta \subset \Gamma$ such that whenever $\lambda_1, \lambda_2, ... \geq 0$ and $\lambda_1 + \lambda_2 < +\infty$, then there exists $\gamma \in \Delta$ satisfying $\|\lambda_1 g_1 + \lambda_2 g_2 + \cdots\| = \lambda_1 g_1(\gamma) + \lambda_2 g_2(\gamma) + \cdots$. Then,

$$\sup\{\limsup_{n\to\infty} g_n(\gamma) : \gamma \in \Gamma\} \geq \inf\{\|g\| : g \in conv(\{g_n\}_{n=1}^\infty)\}$$

Proof: First, we denote

$$A := \inf\{\|g\| : g \in \operatorname{conv}(\{g_n\}_{n=1}^\infty)\}, \quad B := \sup\{\|g_n\| : n = 1, 2, ...\}.$$

Then, $0 \leq A \leq B < +\infty$. Take an arbitrary $\delta > 0$, and choose $\lambda \in]0, 1[$ such that

$$A - \delta(1 + \lambda) - \lambda B > (A - 2\delta)(1 - \lambda).$$

By induction, for every $m = 1, 2, ...$ we can choose $h_m \in \operatorname{conv}(\{g_n : n \geq m\})$ such that

$$\left\|\sum_{n=1}^{m} \lambda^{n-1} h_n\right\| < \delta\left(\frac{\lambda}{2}\right)^m + \inf\left\{\left\|\sum_{n=1}^{m-1} \lambda^{n-1} h_k + \lambda^{m-1} h\right\| : h \in \operatorname{conv}(\{g_n : n \geq m\})\right\}.$$

Here, we use the abuse of notation $\sum_{n=1}^{0} \cdots = 0$. Since,

$$\frac{h_m + \lambda h_{m+1}}{1 + \lambda} \in \operatorname{conv}(\{g_n : n \geq m\}) \quad \text{for every } m \in \mathbb{N},$$

thus, we have

$$\left\|\sum_{n=1}^{m} \lambda^{n-1} h_n\right\| < \delta\left(\frac{\lambda}{2}\right)^m + \left\|\sum_{n=1}^{m-1} \lambda^{n-1} h_k + \lambda^{m-1} \frac{h_m + \lambda h_{m+1}}{1 + \lambda}\right\|.$$

With this, now we take $f_0 \equiv 0$, define $f_m := \sum_{n=1}^{m} \lambda^{n-1} h_n$ for every $m = 1, 2, ...$ and $f := \sum_{n=1}^{\infty} \lambda^{n-1} h_n$. This last f is well defined, because $0 < \lambda < 1$ and $\{h_n\}_{n=1}^\infty$

is a bounded sequence. Thus, using this notation and multiplying both sides by $(1+\alpha)$, we reach that, for every $m = 1, 2, ...$

$$\begin{aligned}(1+\lambda)\|f_m\| &< \delta(1+\lambda)\left(\frac{\lambda}{2}\right)^m + \|\lambda f_{m-1} + f_{m+1}\| \\ &\leq \delta(1+\lambda)\left(\frac{\lambda}{2}\right)^m + \lambda\|f_{m-1}\| + \|f_{m+1}\|,\end{aligned}$$

and so,

$$\frac{\|f_{m+1}\| - \|f_m\|}{\lambda^m} > \frac{\|f_m\| - \|f_{m-1}\|}{\lambda^{m-1}} - \frac{\delta(1+\lambda)}{2^m}.$$

Since $\|f_1\| - \|f_0\| = \|f_1\| \geq A$, by the last equation, we deduce that for every m

$$\frac{\|f_m\| - \|f_{m-1}\|}{\lambda^{m-1}} > A - \delta(1+\lambda)(\sum_{n=1}^{\infty}\frac{1}{2^n}) = A - \delta(1+\lambda).$$

Hence, we get

$$\|f\| - \|f_{m-1}\| = \sum_{n=m}^{\infty}(\|f_n\| - \|f_{n-1}\|) > \sum_{n=m}^{\infty}\lambda^{n-1}(A - \delta(1+\lambda)),$$

that is,

$$\|f\| - \|f_{m-1}\| > \frac{\lambda^{m-1}}{1-\lambda}(A - \delta(1+\lambda)).$$

Now, by hypothesis, there must be $\gamma \in \Delta$ such that $f(\gamma) = \|f\|$. Then, for every $m = 1, 2, ...$ we have

$$\begin{aligned}\lambda^{m-1}h_m(\gamma) &= f(\gamma) - f_{m-1}(\gamma) - \sum_{n=m+1}\lambda^{n-1}h_n(\gamma) \\ &\geq \|f\| - \|f_{m-1}\| - \sum_{n=m+1}\lambda^{n-1}B \\ &> \frac{\lambda^{m-1}}{1-\lambda}(A - \delta(1+\lambda)) - \frac{\lambda^m}{1-\lambda}B.\end{aligned}$$

Thus, by the choice of λ, we get that $h_m(\lambda) \geq A - 2\delta$. Now, if we take any $m \in \mathbb{N}$, since h_m belongs to $\text{conv}(\{g_n : n \geq m\})$, there exists $n_0 \geq m$ such that $g_{n_0}(\gamma) \geq A - 2\delta$. Thus, we have

$$\limsup_{n\to\infty} g_n(\gamma) \geq A - 2\delta.$$

Since the $\delta > 0$ was chosen arbitrary, this ends the proof. ■

We mention here the fundamental concept of a *selector* and the Jayne and Rogers result on the existence of a particular selector for the duality mapping in the class of Asplund spaces.

Definition 6.15

Let X and Y two topogical spaces. A mapping $f : X \longrightarrow Y$ is said to be ***Baire class 1*** *if there are continuous mappings $f_n : X \longrightarrow Y$ with $n = 1, 2, ...,$ such that $f_n \to f$ pointwise, i.e., $f_n(x) \xrightarrow{n\to\infty} f(x)$ for every $x \in X$.*

Theorem 6.16 (Jayne–Rogers, [JaRo85])

Let M be a complete metric space, $(X, \|\cdot\|)$ be an Asplund space, and $G : M \longrightarrow 2^{(B_{X^*}, w^*)}$ a usco mapping. Then, there exists a Baire class 1 mapping $g : M \longrightarrow (B_{X^*}, \|\cdot\|)$ such that $g(t) \in G(t)$ for every $t \in M$.

Now, we proof the main theorem of the section.

Theorem 6.17 ([Fa97, Theorem 8.2.1])

Let X be an Asplund space. Then, its dual X^* admits a PG defined by X (i.e., it has the form (W, Φ), with $W \subset X$ and $\overline{W} = X$).

Proof: Consider the duality map $\partial\|\cdot\| : X \longrightarrow 2^{(X^*, w^*)}$, which is usco. According to Theorem 6.16, there is a Baire class 1 mapping $f_0 : (X, \|\cdot\|) \longrightarrow (B_{X^*}, \|\cdot\|)$ such that $f_0(x) \in \partial\|\cdot\|(x)$ for every $x \in X$. Now, let $f_n : X \longrightarrow B_{X^*}$ be $\|\cdot\|$-$\|\cdot\|$-continuous functions such that

$$\|f_n(x) - f_0(x)\| \xrightarrow{n\to\infty} 0 \quad \text{for every } x \in X.$$

Define the countable-valued map $\Phi : X \longrightarrow 2^{B_{X^*}}$ with

$$\Phi(x) := \{f_1(x), f_2(x), ...\} \quad \text{for every } x \in X.$$

We claim that the pair (X, Φ) is a projectional generator. To see this, take any non-empty set $B \subset X$ such that $\overline{B}$ is a linear subspace. By Remark 6.10 we only need to prove that

$$\Phi(B)^\perp \cap \overline{B \cap B_X}^{w^*} = \{0\}.$$

By contradiction, assume that $x^{**} \in \Phi(B)^\perp \cap \overline{B \cap B_X}^{w^*}$ with x^{**}. Then, there must exists $x_0^* \in X^*$ such that $\langle x^{**}, x_0^*\rangle \neq 0$.

Now, fix any $0 \neq x_1 \in B \cap B_X$ and put $F_1 := \{x_0^*\}$. Assume that for some $n \geq 0$ we have chosen $x_1, x_2, ..., x_n$ and n finite sets $F_1, F_2, ..., F_n \subset X^*$. Consider the set

$$\bigcup_{i=1}^{n} f_i(\text{span}(\{x_1, ..., x_n\}) \cap nB_X)$$

endowed with the metric given by the dual norm. Thus, we can take H a $1/n$-net of this set. Since every f_i is continuous, the set on the previous equation is compact, and so we can take H to be finite. Writing $F_{n+1} := F_n \cup U$, we can find $x_{n+1} \in B \cap B_X$ such that

$$|\langle x^{**} - x_{n+1}, x^* \rangle| < \frac{1}{n} \quad \text{for each } x^* \in F_{n+1}.$$

Keeping with this inductive process, we have already constructed $\{F_n\}_{n=1}^{\infty}$ and $\{x_n\}_{n=1}^{\infty}$. Then, defining $F := \bigcup_{n=1}^{\infty} F_n$ and $B_0 := \operatorname{span}(\{x_n\}_{n=1}^{\infty})$, by the construction of the sets F_n we have that F must be $\|\cdot\|$-dense in $\Phi(B_0)$ and that

$$\langle x^*, x_n \rangle \to \langle x^*, x^{**} \rangle \quad \text{for each } x^* \in \Phi(B_0) \cup \{x_0^*\}.$$

Notice that $B_0 \subset \operatorname{span}(B) \subset \operatorname{span}(\overline{B}) = \overline{B}$, and as every f_i is $\|\cdot\|$-continuous, $\Phi(\overline{B}) \subset \overline{\Phi(B)}$. This means that $x^{**} \in \Phi(B)^{\perp}$ implies that $x^{**} \in \Phi(B_0)^{\perp}$. Therefore $\langle x^*, x^{**} \rangle = 0$ and so, by the last equation

$$\langle x^*, x_n \rangle \to 0 \quad \text{for every } x^* \in \Phi(B_0),$$

and since the sequence $\{x_n\}_{n=1}\infty$ was bounded, we have this also when $x^* \in \overline{\operatorname{span}}\big(\Phi(B_0)\big)$. Now, we take $Y := \overline{B_0}$, which is a separable linear subspace of the Asplund space X, and thus, it will be also an Asplund space. Finally, we claim that

$$Y^* = \{x^*_{|Y} : x^* \in \overline{\operatorname{span}\big(\Phi(B_0)\big)}\}. \tag{6.3}$$

Notice that then the proof is done, because this will mean $\{x_n\}_{n=1}^{\infty} \xrightarrow{w} 0$ in Y, and so in the whole space X, from where we obtain

$$\langle x^*, x_0 \rangle = \lim_{n\to\infty} \langle x_0^*, x_n \rangle = 0,$$

a contradiction with the assumption $\langle x^*, x_0 \rangle \neq 0$. We prove the claim also by contradiction. If (6.3) is false, then there exists $y_0^{**} \in Y^{**}$ and $y_0^* \in B_{Y^*}$ such that

$$\langle y_0^{**}, y_0^* \rangle > 0 = \langle y_0^{**}, x_Y^* \rangle \quad \text{for every } x^* \in \overline{\Phi(B_0)}.$$

Notice that since Y is separable and Asplund, its own dual Y^* must be separable, and then $(B_{Y^{**}}, w^*)$ is metrizable. Also, Goldstine's Theorem assure us that B_X is always w^*-dense in $(B_{Y^{**}}, w^*)$. Combining both facts, we know we can take a bounded sequence $\{y_n\}_{n=1}^{\infty}$ which w^*-converges to y_0^*. Without loss of generality, we may assume that

$$\langle y_0^*, y_n \rangle > \frac{1}{2} \langle y_0^{**}, y_0^* \rangle \quad \text{for every } n = 1, 2, ...$$

Here, we apply Lemma 6.14 with $\Gamma := B_{Y^*}$, $\Delta := \{x^*_{|Y} : x^* \in \overline{\Phi(B_0)}\}$, and $g_n := y_n$ for every n. By the way Φ has been defined, we know that the hypothesis of the lemma is satisfied, and so, applying it, we get

$$\begin{aligned} 0 &= \sup\{\langle y_0^{**}, x^*_{|Y}\rangle : x^* \in \overline{\Phi(B_0)}\} \\ &= \sup\{\lim_{n\to\infty} \langle y_n, x^*_{|Y}\rangle : x^* \in \overline{\Phi(B_0)}\} \\ &\geq \inf\{\|y\| : y \in \operatorname{conv}(\{y_n\}_{n=1}^{\infty})\} \\ &\geq \inf\{\langle y_0^*, y\rangle : y \in \operatorname{conv}(\{y_n\}_{n=1}^{\infty})\} \\ &> \frac{1}{2}\langle y_0^{**}, y_0^*\rangle > 0, \end{aligned}$$

a contradiction that proves the claim, and finishes the whole proof. ■

6.3 Renorming Techniques using PRI

Along this section we review some of the interest on finding projectional resolutions of the identity for renorming purposes, which is the construction of a LUR norm. Recall that the norm $\|\cdot\|$ in a Banach space X is called **LUR** if for every $x, x_n \in X$ such that $\left\|\frac{x+x_n}{2}\right\| \to 1$, then $x_n \xrightarrow{\|\cdot\|} x$.

The following result was one of the first to identify a class of LUR renormable spaces, and its probably one of the most important results on renorming theory.

Theorem 6.18 (Kadets)

Let X be a separable Banach space. Then, X admits an equivalent LUR norm.

This was one of the great achievements in renorming theory that took place in the early 1960s. Since then, there have been numerous attempts to develop new techniques to deal with nonseparable spaces. One of the most fruitful proposals for the treatment of these spaces was the decomposition through projectional resolutions of the identity. The idea behind this is that a long sequence of projections that grow progressively to the identity allows, informally speaking, to divide the original space into smaller ones, and if for all these small spaces some good property can be deduced, it may also be attributed to the original space. The way to apply this to renorm spaces is to achieve smaller spaces which admit norms with good geometrical properties, and build a new norm on the original space based on combining the previous ones. The "glue" we use for putting together these norms in the proper way is known as the Deville Master Lemma (6.23). Thus, the main goal of this section is to prove that the dual of an Asplund space admits an equivalent LUR norm, from the constructed PRI during the last Section.

We start introducing the *quadratic formula* Q_f of a given convex function f. Despite being such a simple expression, it is a really useful tool, as it provides valuable information about the convexity properties of f. This provide us with an easy way to check the strict convexity or LUR properties of a given norm, and write some proofs in a much more compact way.

Lemma 6.19

Let X be a normed space and f a convex function defined on X. Consider the symmetric function

$$Q_f(x,y) := \frac{f(x)^2+f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2.$$

Then, the following properties hold:

(1) $Q_f \geq 0$;

(2) *if $\{f_n\}_{n=1}^{\infty}$ is a sequence of convex functions such that $\sum_{n=1}^{\infty} f_n^2$ is convergent, then the positive function f defined by $f^2 = \sum_{n=1}^{\infty} f_n^2$ is convex and*

$$Q_f = \sum_{n=1}^{\infty} Q_{f_n}\ ;$$

(3) *given $x, x_n \in X$, the following are equivalent:*

a) $\lim_{n\to\infty} f(x_n) = f(x)$ *and* $\lim_{n\to\infty} f\Big(\frac{x+x_n}{2}\Big) = f(x)$,

b) $\lim_{n\to\infty} Q_f(x,x_n) = 0$.

(4) *The norm $\|\cdot\|$ is strictly convex if and only if, for every $x, y \in X$ such that $Q_{\|\cdot\|}(x,y) = 0$ implies $x = y$.*

Proof: Applying the convexity of f, the next inequalities can be obtained:

$$Q_f(x,y) = \frac{f(x)^2+f(y)^2}{2} - f\Big(\frac{x+y}{2}\Big)^2 \geq \frac{f(x)^2+f(y)^2}{2} - \Big(\frac{f(x)+f(y)}{2}\Big)^2 = \\ = \frac{(f(x)-f(y))^2}{4} \geq 0. \tag{6.4}$$

This proves (1). The b) $\Longrightarrow$ a) of (3) follows directly by writing x_n instead of y in the previous inequalities, while a) $\Longrightarrow$ b) is trivial. Statement (2) can be obtained just by doing the direct computation:

$$Q_f(x,y) = \frac{f(x)^2 + f(y)^2}{2} - f\left(\frac{x+y}{2}\right)^2 =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} f_n(x)^2 + \frac{1}{2}\sum_{n=1}^{\infty} f_n(y)^2 - \sum_{n=1}^{\infty} f_n\left(\frac{x+y}{2}\right)^2 = \sum_{n=1}^{\infty} Q_{f_n}(x,y).$$

Finally, we provide the proof for statement (4). First assume that $\|\cdot\|$ is strictly convex and for $x, y \in X$ we have $Q_{\|\cdot\|}(x,y) = 0$. Then, according to equation (6.4),

$$0 = Q_{\|\cdot\|}(x,y) \geq \frac{(\|x\| - \|y\|)^2}{4} \geq 0,$$

so $\|x\| = \|y\|$. Thus we can suppose $x, y \in S_X$, and with this

$$0 = Q_{\|\cdot\|}(x,y) = 1 - \left\|\frac{x+y}{2}\right\|^2.$$

Hence, $\frac{x+y}{2} \in S_X$, and as x and y belong to S_X too, then by the strict convexity of $\|\cdot\|$ we conclude $x = y$. Conversely, whenever it occurs $x, y, \frac{x+y}{2} \in S_X$, then $Q_{\|\cdot\|}(x,y) = 0$, and by (ii), $x = y$. This means that there cannot be line segments in the unit sphere, proving the strict convexity of $\|\cdot\|$. ■

Notice that by statement (3) of the previous lemma, a norm $\|\cdot\|$ is LUR if and only if, $x_n \xrightarrow{\|\cdot\|} x$ whenever, for every $x, x_n \in S_{(X,\|\cdot\|)}$, $\lim_{n\to\infty} Q_{\|\cdot\|}(x,x_n) = 0$.

Here we prove a first norm-combination result. The existence of a simultaneously Kadets-Klee and rotund norm is guaranteed by the existence of two norms, one with the first of them, the other with the second. This frequently appears in the literature, although most of the time the proof is missing. We provide here a short proof of this fact.

Proposition 6.20

Let X be a normed space that admits a rotund norm $\|\cdot\|_1$ and a Kadets–Klee norm $\|\cdot\|_2$. Then, the norm $|||\cdot|||$ defined by the equation

$$|||\cdot|||^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2$$

is simultaneously rotund and Kadets–Klee.

Proof: First, let us prove that it is rotund. To see this, we recall the function Q_f defined in Lemma 6.19. Given $x, y \in X$ it holds that

$$\begin{aligned} Q_{|||\cdot|||}(x,y) &= \frac{|||x|||^2 + |||y|||^2}{2} - \frac{|||x+y|||^2}{4} \\ &= \frac{\|x\|_1^2 + \|y\|_1^2}{2} - \frac{\|x+y\|_1^2}{4} + \frac{\|x\|_2^2 + \|y\|_2^2}{2} - \frac{\|x+y\|_2^2}{4} \\ &= Q_{\|\cdot\|_1}(x,y) + Q_{\|\cdot\|_2}(x,y). \end{aligned}$$

If $Q_{|||\cdot|||}(x,y) = 0$ is assumed, then both $Q_{\|\cdot\|_1}(x,y)$ and $Q_{\|\cdot\|_2}(x,y)$ must also vanish, as we know that they are non-negative functions by Lemma 6.19(1), so as $Q_{\|\cdot\|_1}(x,y) = 0$ and $\|\cdot\|_1$ is rotund, by Lemma 6.19(4) we know that $x = y$. Then, by applying again Lemma 6.19(4), but in the other direction for $Q_{|||\cdot|||}$, we get that $|||\cdot|||$ is rotund.

Now, to prove that $\|\cdot\|_2$ has Kadets–Klee property, take a net $\{x_i\} \subset S_{|||\cdot|||}$ such that $x_i \xrightarrow{w} x_0 \in S_{|||\cdot|||}$. Considering an arbitrary subnet of $\{x_i\}$ (to which we will refer as $\{x_i\}$ again), we have that

$$1 = |||x_i|||^2 = \|x_i\|_1^2 + \|x_i\|_2^2$$

so both terms $\|x_i\|_1^2$ and $\|x_i\|_2^2$ are less or equal than 1. So we know that there exist a subnet (still called $\{x_i\}$) such that

$$\begin{aligned} \|x_i\|_1^2 &\longrightarrow \varepsilon_0, \\ \|x_i\|_2^2 &\longrightarrow 1 - \varepsilon_0. \end{aligned}$$

for a certain $1 > \varepsilon_0 > 0$. It should be noted that this ε_0 cannot be zero, because by being equivalent norms, there must exist an $M > 0$ such that $\|x_i\|_1 \geq M|||x_i||| = M$. A completely analogous reasoning would show that ε_0 cannot be 1.

Thus, as $x_i \xrightarrow{w} x_0$, notice that we got before that $\|x_i\|_1^2 \leq \varepsilon_0$ and $\|x_i\|_2^2 \leq 1 - \varepsilon_0$, but such that $1 = |||x_0|||^2 = \|x_0\|_1^2 + \|x_0\|_2^2 \leq \varepsilon_0 + (1 - \varepsilon_0) = 1$. So we must have that indeed $\|x_0\|_1^2 = \varepsilon_0$ and $\|x_0\|_2^2 = 1 - \varepsilon_0$.

All we know so far is that

$$\begin{aligned} x_i &\xrightarrow{w} x_0 \quad \text{and} \\ \|x_i\|_2^2 &\longrightarrow \|x_0\|_2^2 \neq 0. \end{aligned}$$

Hence, we deduce that

$$\frac{x_i}{\|x_i\|_2} \xrightarrow{w} \frac{x_0}{\|x_0\|_2},$$

but as the norm $\|\cdot\|_2$ its a Kadets–Klee norm, this means that

$$\frac{x_i}{\|x_i\|_2} \xrightarrow{\|\cdot\|_2} \frac{x_0}{\|x_0\|_2},$$

and thus, since $|||\cdot|||$ is an equivalent norm to $\|\cdot\|_2$, we conclude that $x_i \xrightarrow{|||\cdot|||} x_0$, as we wanted to prove. ■

Remark 6.21

Observe that both parts of the proof are independent. So actually, we were proving that, having

$$|||\cdot|||^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2,$$

where any $\|\cdot\|_i$ for $i = \{1, 2\}$ is rotund (or Kadets-Klee), then $|||\cdot|||$ also has the same property. ®

This kind of arguments that allow for the definition of new norms with certain properties through combining previous ones is the core idea of the important result known as the Deville Master Lemma. It employs a somewhat more elaborate technique, which succeeds in creating a new norm on the original space that inherits the good properties of the family of functions from which it is constructed. Is is very useful and appears in many results on the field, either explicitly or implicitly. We include its proof after the next technical lemma.

Lemma 6.22

Let $c_{n,k}$ be real numbers such that for all $k \geq 1$, $\lim_{n\to\infty} c_{n,k} = 0$ and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of positive numbers. Then, there exist integers k_n, $n = 1, 2, ...$ and N_0 such that

(i) $\lim\limits_{n\to\infty} k_n = \infty$, *and*

(ii) $|c_{n,k_n}| < \alpha_{k_n}$ *for all $n \leq N_0$.*

Proof: We construct the sequence $\{k_n\}_{n=1}^{\infty}$ by induction. Take $k_1 := 1$ and N_0 be such that $\sup\{|c_{n,1}| : n \geq N_0\} < \alpha_1$. If k_n has been already constructed, we take

$$k_{n+1} := \begin{cases} k_n & \text{if } \sup\{|c_{p,k_n+1}| : p \geq n\} \geq \alpha_{k_n+1} \\ k_n + 1 & \text{otherwise} \end{cases}$$

Then, (ii) is obtained by the construction. Now, by contradiction, suppose that (i) is not true. Then, $k_n + 1 = k_0$ for finitely many n, where k_0 is a fixed integer. Then, $\sup\{|c_{p,k_0}| : p \geq n\} \geq \alpha_{k_0}$ for infinite many n, ans this contradicts the fact that $\lim_{n\to\infty} c_{n,k_0} = 0$. ■

Lemma 6.23 (Deville Master Lemma)

Let X be a Banach space, $\{\phi_i\}_{i\in I}$ and $\{\psi_i\}_{i\in I}$ be two families of real valued convex nonnegative function defined on X, which are both uniformly bounded on bounded subsets of X. For $i \in I$ and $k \in \mathbb{N}$, let us denote by

$$\theta_{i,k}(x) := \phi_i^2(x) + \frac{1}{k}\psi_i^2(x),$$
$$\theta_k(x) := \sup_{i\in I}\{\theta_{i,k}(x)\} \quad \textit{and}$$
$$\theta(x) := \|x\|^2 + \sum_{k=1}^{\infty} \frac{\theta_k(x) + \theta_k(-x)}{2^k},$$

where $\|\cdot\|$ is the norm of X. Consider $|||\cdot||| := \mu_B$, i.e., the Minkowski functional of the set

$$B = \{x \in X : \theta(x) \le 1\}.$$

Then, $|||\cdot|||$ is an equivalent norm on X with the following property:
In $x_n, x \in X$ satisfy $\lim_{n\to\infty} Q_{|||\cdot|||}(x, x_n) = 0$, then, there is a sequence $\{i_n\}_{n=1}^{\infty} \in I$ such that
(i) $\lim_{n\to\infty} Q_{\psi_{i_n}}(x, x_n) = 0.$
(ii) $\lim_{n\to\infty} \phi_{i_n}(x) = \lim_{n\to\infty} \phi_{i_n}(x_n) = \lim_{n\to\infty} \phi_{i_n}\Big(\frac{x + x_n}{2}\Big) = \sup_{i\in I}\{\phi_i(x)\}.$

Proof: It is clear that $|||\cdot|||$ is an equivalent norm on X. Let $x_n, x \in X$ satisfy

$$\lim_{n\to\infty} Q_{|||\cdot|||}(x, x_n) = 0. \tag{6.5}$$

If $x = 0$, there is nothing to prove, so we assume $x \neq 0$, by homogeneity, we can also take $|||x||| = 1$. Using equation (6.5), we know that $\lim_{n\to\infty} |||x_n||| = \lim_{n\to\infty} |||(x + x_n)/2||| = 1$. Thus, $\theta(x) = 1$. Since θ is uniformly continuous on bounded subsets of X, we have that $\lim_{n\to\infty} \theta(x_n) = \lim_{n\to\infty} \theta((x + x_n)/2) = 1$, and consequently

$$\lim_{n\to\infty} Q_{\sqrt{\theta}}(x, x_n) = 0. \tag{6.6}$$

Using (ii) of Lemma 6.19 and since for (i) of Lemma 6.19 we know that every, a simple positivity argument shows that

$$\lim_{n\to\infty} Q_{\|\cdot\|}(x, x_n) = 0. \tag{6.7}$$

and also, for every $k \in \mathbb{N}$

$$\lim_{n\to\infty} Q_{\sqrt{\theta_k}}(x, x_n) = 0. \tag{6.8}$$

Let $\{\alpha_n\}_{n=1}^{\infty}$ a sequence of positive real numbers such that $\lim_{n\to\infty} n\alpha_n = 0$. Then, we apply the previous Lemma 6.22 taking $c_{n,k} := Q_{\theta_k}(x, x_n)$, and thus, we obtain a sequence of integers $\{k_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} k_n = \infty$, and

$$Q_{\sqrt{\theta_{k_n}}}(x, x_n) < \alpha_{k_n}, \quad \text{for } n \text{ large enough.} \tag{6.9}$$

By this last equation and the definition of θ_{k_n}, it follows that for each $n \in \mathbb{N}$ there exists $i_n \in I$ such that

$$\frac{\theta_{k_n}(x) + \theta_{k_n}(x_n)}{2} - \theta_{i_n,k_n}\Big(\frac{x + x_n}{2}\Big) < \alpha_{k_n}.$$

Thus, for all $i \in I$ we have

$$\begin{aligned} \alpha_{k_n} > \frac{\theta_{i,k_n}(x) + \theta_{i_n,k_n}(x_n)}{2} - \theta_{i_n,k_n}\Big(\frac{x + x_n}{2}\Big) \geq Q_{\phi_{i_n}}(x, x_n) + \\ + \frac{1}{2}\Big(\phi_i^2(x) - \phi_{i_n}^2(x)\Big) + \frac{1}{k_n} Q_{\psi_{i_n}}(x, x_n) + \frac{1}{2k_n}\Big(\psi_i^2(x) - \psi_{i_n}^2(x)\Big). \end{aligned} \tag{6.10}$$

Now, if we take $i = i_n$, by (i) of Lemma 6.19 we get

$$\alpha_{k_n} \geq Q_{\phi_{i_n}}(x, x_n) \geq 0 \tag{6.11}$$

$$k_n \alpha_{k_n} \geq Q_{\psi_{i_n}}(x, x_n) \geq 0 \tag{6.12}$$

since $\lim\limits_{n\to\infty} k_n \alpha_{k_n} = 0$, equation (6.12) implies (i). Furthermore, equation (6.11) implies that

$$\lim_{n\to\infty} \phi_{i_n}(x) - \phi_{i_n}(x_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} \phi_{i_n}(x) - \phi_{i_n}\Big(\frac{x + x_n}{2}\Big) = 0. \tag{6.13}$$

On the other hand, if we denote by $M := \sup\limits_{i\in I}\{\psi_i^2(x)\}$, then, given $n \in \mathbb{N}$, equation (6.10) yields, for every $i \in I$,

$$\phi_i^2(x) - \phi_{i_n}^2 < \frac{-1}{k_n}\Big(\psi_i^2(x) - \psi_{i_n}^2(x)\Big) + 2\alpha_{k_n} \leq \frac{M}{k_n} + 2\alpha_{k_n}.$$

Thus, for $n \in \mathbb{N}$, we have

$$\phi {i_n}^2(x) \geq \sup_{i\in I}\{\phi_i^2(x)\} - \frac{M}{k_n} - 2\alpha_{k_n}.$$

Hence

$$\liminf_{n\to\infty} \phi_{i_n}(x) \geq \sup_{i\in I}\{\phi_i(x) : i \in I\}. \tag{6.14}$$

Finally, equations (6.13) and (6.14) prove (ii). This concludes the proof of the lemma.

■

Once the preparations are finished, we are able to begin the proof for the renorming results using PRI. First, we will prove that, assuming that every piece $(P_{\alpha+1}-P_\alpha)X$ admits a strictly convex norm, a strictly convex norm can be constructed in X. During the proof, apart from Deville Master Lemma, Proposition 6.3 also plays an important rôle, since it provides a mapping into $c_0(\Gamma)$. This is a key step, because according to the following classical theorem, this space admits an equivalent norm with excellent properties. Recall that a norm $\|\cdot\|$ in the Banach space is a **lattice norm** if for every pair of points $x, y \in X$ such that $|y| \leq |x|$, they satisfy $\|y\| \leq \|x\|$ (where $|x|$ is, as usual, the absolute value of x).

Theorem 6.24 (Day, Rainwater)

Let Γ be a nonempty subset. Then, $c_0(\Gamma)$ admits a equivalent LUR norm (Day's norm) which is also a lattice norm.

M. M. Day defined the fundamental norm that carries his name on $c_0(\Gamma)$, proving that it is strictly convex. J. Rainwater proved that, in fact, it is LUR. We aim to use the existence of a certain mapping to transfer the strict convexity from $c_0(\Gamma)$ to X.

Proposition 6.25

Let X be a Banach space that admits a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ such that, for every $\omega \leq \alpha \leq \mu$, the space $(P_{\alpha+1}-P_\alpha)(X)$ admits a rotund norm (we call it $|\cdot|_\alpha$). Then, the whole space X admits a rotund norm.

Proof: First, recall that by (i) of Proposition 6.3, taking $\Gamma := [\omega, \mu[$ and $T_\alpha := P_{\alpha+1} - P_\alpha$, we have that $x \mapsto \{\|T_\alpha x\|\}_{\alpha\in\Gamma}$ maps X into $c_0(\Gamma)$. We can assume without loss of generality that the family $\{T_\alpha\}_{\alpha\in\Gamma}$ is uniformly bounded and, since $|\cdot|_\alpha$ is norm on $T_\alpha(X)$ which is equivalent to the inherited norm ($\|\cdot\|$ restricted to $T_\alpha(X)$), we can also $|\cdot|_\alpha \leq \|\cdot\|$, where $\|\cdot\|$ is the norm of X. Applying Theorem 6.24, denote $|||\cdot|||_{c_0}$ the LUR norm on $c_0(\Gamma)$, and define a equivalent norm $|||\cdot|||$ on X by the equation

$$|||x|||^2 = \|x\|^2 + |||\{|T_\alpha x|_\alpha\}_{\alpha\in\Gamma}|||_{c_0}^2.$$

We will check that is a rotund norm. To this, consider $x, y \in X$ such that $|||x||| = |||y||| = |||(x+y)/2|||$. Thus $Q_{|||\cdot|||}(x, y) = 0$, and by Lemma 6.19 we deduce

$$|||\{|T_\alpha x|_\alpha\}_{\alpha\in\Gamma}|||_{c_0} = |||\{|T_\alpha y|_\alpha\}_{\alpha\in\Gamma}|||_{c_0} = |||\{|T_\alpha((x+y)/2)|_\alpha\}_{\alpha\in\Gamma}|||_{c_0}. \tag{6.15}$$

On the other hand, using the convexity of $|\cdot|_\alpha$ and $|||\cdot|||_{c_0}$ and the fact that $|||\cdot|||_{c_0}$ is a lattice norm, we have

$$\begin{aligned}\left|\left|\left|\left\{\left|T_\alpha\left(\frac{x+y}{2}\right)\right|_\alpha\right\}_{\alpha\in\Gamma}\right|\right|\right|_{c_0} &= \left|\left|\left|\left\{\left|\frac{T_\alpha x + T_\alpha y}{2}\right|_\alpha\right\}_{\alpha\in\Gamma}\right|\right|\right|_{c_0} \\ &\leq \left|\left|\left|\left\{\frac{|T_\alpha x|_\alpha + |T_\alpha y|_\alpha}{2}\right\}_{\alpha\in\Gamma}\right|\right|\right|_{c_0} \\ &\leq \frac{|||\{|T_\alpha x|_\alpha\}_{\alpha\in\Gamma}|||_{c_0} + |||\{|T_\alpha y|_\alpha\}_{\alpha\in\Gamma}|||_{c_0}}{2}.\end{aligned}$$

By equation (6.15), we know that the previous chain of inequalities are indeed equalities, since first and last terms are equal. In particular

$$\frac{|||\{|T_\alpha x|_\alpha\}_{\alpha\in\Gamma}|||_{c_0} + |||\{|T_\alpha y|_\alpha\}_{\alpha\in\Gamma}|||_{c_0}}{2} = \left|\left|\left|\left\{\frac{|T_\alpha x|_\alpha + |T_\alpha y|_\alpha}{2}\right\}_{\alpha\in\Gamma}\right|\right|\right|_{c_0}.$$

From this, equation (6.15) and the rotundity of $|||\cdot|||_{c_0}$, it follows that $|T_\alpha x|_\alpha = |T_\alpha y|_\alpha$ for every $\alpha \in \Gamma$.
For every $\alpha \in \Gamma$ denote

$$a_\alpha := \left|\frac{T_\alpha x + T_\alpha y}{2}\right|_\alpha, \quad \text{and} \quad b_\alpha := \frac{|T_\alpha x|_\alpha + |T_\alpha y|_\alpha}{2}.$$

Since the norm $|||\cdot|||_{c_0}$ is a lattice norm and $a_\alpha \leq b_\alpha$ for every $\alpha \in \Gamma$, for every $\alpha \in \Gamma$, we have

$$2\ |||\{a_\alpha\}_{\alpha\in\Gamma}|||_{c_0} \leq |||\{a_\alpha + b_\alpha\}_{\alpha\in\Gamma}|||_{c_0} \leq 2\ |||\{b_\alpha\}_{\alpha\in\Gamma}|||_{c_0} = |||\{a_\alpha\}_{\alpha\in\Gamma}|||_{c_0}.$$

From the rotundity of $|||\cdot|||_{c_0}$ it follows that $a_\alpha = b_\alpha$ for every $\alpha \in \Gamma$ i.e.,

$$\left|\frac{T_\alpha x + T_\alpha y}{2}\right|_\alpha = \frac{|T_\alpha x|_\alpha + |T_\alpha y|_\alpha}{2}$$

for every $\alpha \in \Gamma$. Since $|T_\alpha x|_\alpha = |T_\alpha y|_\alpha$ for every $\alpha \in \Gamma$, from the rotundity of $|\cdot|_\alpha$, we have $T_\alpha x = T_\alpha y$, for every $\alpha \in \Gamma$. Finally, notice that from (ii) of Proposition 6.3, it can be deduced that $\bigcap\{\mathrm{Ker} T_\alpha : \alpha \in \Gamma\} = \{0\}$.Combining both last facts, we have $x = y$, proving the rotundity of $|||\cdot|||$. ■

Now, we prove the LUR version of the previous result, which is a celebrated theorem of Zizler. However, we are not reviewing the proof given in [Ziz84], but the modern approach from [DGZ93], using the previous proposition as a key ingredient. This lemma show how we take advantage of having a strictly convex norm at the beginning on the proof.

Lemma 6.26

Let X be a Banach space such that $\|\cdot\|$ is rotund. Let $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} Q_{\|\cdot\|}(x, x_n) = 0.$$

If the sequence $\{x_n\}_{n=1}^{\infty}$ is relatively compact, then $x_n \xrightarrow{\|\cdot\|} x$ (and thus, $\|\cdot\|$ is LUR).

Proof: Since $\{x_n\}_{n=1}^{\infty}$ is relatively compact, every subsequence admits a convergent subsequence. We claim that every convergent subsequence of $\{x_n\}_{n=1}^{\infty}$ must converge to x. By contradiction, assume that there is $\{x_{n_k}\}_{k=1}^{\infty}$ a subsequence of $\{x_n\}_{n=1}^{\infty}$ that converges to $y \neq x$. Then, it must satisfy

$$\lim_{n\to\infty} Q_{\|\cdot\|}(x, x_{n_k}) = 0.$$

Thus, from (3) of Lemma 6.19 we deduce

$$\|y\| = \lim_{k\to\infty} \|x_{n_k}\| = \|x\|, \quad \text{and} \quad \left\|\frac{x+y}{2}\right\| = \lim_{k\to\infty} \left\|\frac{x+x_{n_k}}{2}\right\| = \|x\|.$$

But then,

$$Q_{\|\cdot\|}(x, y) = \frac{\|x\|^2}{2} + \frac{\|y\|^2}{2} - \left\|\frac{x+y}{2}\right\|^2 = 0,$$

and using (4) of Lemma 6.19, we would get $x = y$, a contradiction.

■

Finally, we present the aforementioned Zizler's result.

Theorem 6.27 (Zizler)

Let X be a Banach space and $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ a PRI such that, for every $\omega \leq \alpha \leq \mu$, the space $(P_{\alpha+1} - P_\alpha)(X)$ admits a LUR norm (we call it $|\cdot|_\alpha$). Then, the whole space X admits a LUR norm.

Proof: Put $T_\alpha := (P_{\alpha+1} - P_\alpha)$. Since $|\cdot|_\alpha$ is a norm on $T_\alpha(X)$ which is equivalent to the inherited norm ($\|\cdot\|$ restricted to $T_\alpha(X)$), we can assume without loss of generality that $|\cdot|_\alpha \leq \|\cdot\|$, where $\|\cdot\|$ is the norm of X. Furthermore, the previous Proposition 6.25 allow us to assume that $\|\cdot\|$ is a rotund norm. We denote $\Gamma := [\omega, \mu[$ and also $\mathcal{T}_\alpha(x) := \{\sum_{\gamma \in A} T_\alpha(x) : A \subset [\omega, \alpha[, A \text{ finite }\}$. We claim that $P_\alpha(x) \in \overline{\mathcal{T}_\alpha(x)}$. This follows by transfinite induction on α. Now, if A is a finite subset of Γ and $x \in X$ a given element, put

$$\phi_A(x) := \Big(\sum_{\alpha \in A} |T_\alpha x|_\alpha^2\Big)^{1/2};$$

$$\psi_A(x) := \Big(\sum_{A' \subset A} \|x - \sum_{\alpha \in A'} T_\alpha x\|^2\Big)^{1/2}.$$

For $p \in \mathbb{N}$, we denote $I_p := \{A \subset \Gamma : |A| = p\}$ and $\|\cdot\|_p$ the norm $|||\cdot|||$ obtained in the Deville Master Lemma (6.23) for the set $I = I_p$. Finally, let us take $|||\cdot|||$ defined by the equation

$$|||x|||^2 = \|x\|^2 + \sum_{p=1}^{\infty} \beta_p \|x\|_p^2.$$

We claim that $|||\cdot|||$ is an equivalent LUR norm defined on X if $\{\beta_p\}_{p=1}^\infty$ is a suitable sequence of positive numbers. In order to see this, note that for every $p \geq 1$, $\|\cdot\|_p$ is an equivalent norm on X. Thus we can choose the sequence $\{\beta_p\}_{p=1}^\infty$ with $\beta_p > 0$ such that $|||\cdot|||$ is an equivalent norm on X. Also, as $\|\cdot\|$ is rotund, $|||\cdot|||$ has this property too (see Remark 6.21). Therefore, in order to show that $|||\cdot|||$ is LUR, it is enough to show that if $x_n, x \in X$ satisfy

$$\lim_{n\to\infty} Q_{|||\cdot|||}(x, x_n) = 0,$$

then $\{x_n\}_{n=1}^\infty$ is relatively compact in X (and then, apply Lemma 6.26). To prove this, fix $\varepsilon > 0$ and, without loss of generality, assume that $\|x\| = 1$. By the definition of $|||\cdot|||$ and (ii) of Lemma 6.19, we know that $Q_{|||\cdot|||} = Q_{\|\cdot\|} + \sum_{p=1}^\infty \beta_p Q_{\|\cdot\|_p}$ and for (i) of Lemma 6.19 we also know that every Q_f is non-negative, so the previous equation implies in particular

$$\lim_{n\to\infty} Q_{\|\cdot\|_p}(x, x_n) = 0 \quad \text{for every } p \in \mathbb{N}. \tag{6.16}$$

By the claim above, we can choose a finite subset $\tilde{A} \subset \Gamma$ such that for every $\alpha \in \tilde{A}$, we have $T_\alpha x \neq 0$ and

$$\|x - \sum_{\alpha \in \tilde{A}} T_\alpha(x)\| < \varepsilon \tag{6.17}$$

Add to $\tilde{A}$ a finite number of points to form a set A such that

$$\max_{\alpha \notin A}\{|T_\alpha x|_\alpha\} < \min_{\alpha \in A}\{|T_\alpha x|_\alpha\}. \tag{6.18}$$

Put $p := |A|$. Using equation (6.16) and the Deville Master Lemma (6.23) applied with $I = I_p$, we know that there is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X with $|A_n| = |A|$, such that: (i) $\lim_{n\to\infty} Q_{\psi_{A_n}}(x, x_n) = 0$;

(ii) $\lim_{n\to\infty} \sum_{\alpha\in A_n} |T_\alpha((x+x_n)/2)|_\alpha^2 = \lim_{n\to\infty} \sum_{\alpha\in A_n} |T_\alpha x_n|_\alpha^2 = \lim_{n\to\infty} \sum_{\alpha\in A_n} |T_\alpha x|_\alpha^2 = \sum_{\alpha\in A} |T_\alpha x|_\alpha^2$

The last equality of (ii) and equation (6.18) show that there exists $n_0 \in \mathbb{N}$ such that $A_n = A$ for every $n \geq n_0$. Nos, using a standard convexity argument, the two first equalities of (ii) imply that, for every $\alpha \in A$ we have

$$\lim_{n\to\infty} Q_{|\cdot|_\alpha}(T_\alpha x, T_\alpha x_n) = 0$$

Since the norm $|\cdot|_\alpha$ is LUR, we get that for every $\alpha \in A$ we have

$$\lim_{n\to\infty} |T_\alpha x - T_\alpha x_n|_\alpha = 0.$$

Hence, there exists $n_1 \geq n_0$ such that for $n \geq n_1$, we have

$$\| \sum_{\alpha\in\tilde{A}} (T_\alpha(x) - T_\alpha(x_n))\| \leq \varepsilon. \tag{6.19}$$

On the other hand, from (i) and the fact that $A_n = A$ for $n \geq n_0$ and using again Lemma 6.19, it follows that

$$\lim_{n\to\infty} 2\|x - \sum_{\alpha\in\tilde{A}} T_\alpha(x)\|^2 + 2\|x_n - \sum_{\alpha\in\tilde{A}} T_\alpha(x_n)\|^2 - \|x + x_n - \sum_{\alpha\in\tilde{A}} T_\alpha(x + x_n)\|^2.$$

Using again Lemma 6.19, we get

$$\lim_{n\to\infty} \|x_n - \sum_{\alpha\in\tilde{A}} T_\alpha(x_n)\| = \|x - \sum_{\alpha\in\tilde{A}} T_\alpha(x)\|.$$

From this and equation (6.17) we obtain the existence of $n_2 \geq n_1$ such that for $n \geq n_2$, we have:

$$\|x_n - \sum_{\alpha\in\tilde{A}} T_\alpha(x_n)\| < \varepsilon. \tag{6.20}$$

By equations (6.19) and (6.20), we have for every $n \geq n_2$

$$\|x_n - \sum_{\alpha \in \tilde{A}} T_\alpha(x)\| < \varepsilon.$$

Therefore, we can cover the sequence $\{x_n\}_{n=1}^{\infty}$ by finitely many balls of radius 3ε, so $\{x_n\}_{n=1}^{\infty}$ is a relatively compact set in X. Using Lemma 6.26, the proof is done. ■

Notice that this is a powerful tool. For example, in combination with the Kadets Theorem, we obtain the following corollary.

Corollary 6.28

Let X be a Banach space and $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ a PRI such that, for every $\omega \leq \alpha \leq \mu$, the space $(P_{\alpha+1} - P_\alpha)(X)$ is separable. Then, the whole space X admits a LUR norm.

Proof: By Kadets Theorem (6.18), every $(P_{\alpha+1} - P_\alpha)(X)$ admits a LUR norm. Then, apply Theorem 6.27. ■

However, not every PRI enjoy this property. That is a particular case of PRI called *separable PRI*. For example, the argument cannot be applied in a dual of an Asplund space. Indeed, we cannot ensure that, despite having a PRI, every $(P_{\alpha+1} - P_\alpha)(X)$ admits a LUR norm. Then, proving the LUR renorming in the dual of an Asplund space requires an additional argument.

Speaking informally, what we will do is to prove that in a dual of an Asplund space, every piece of the PRI admits a PRI again. In this way, we are allowed to repeatedly obtain smaller and smaller decompositions of the space, until we reach "*sufficiently small*" (separable) spaces, where the Kadets Theorem ensures the LUR norms. Once that is done, we will have to go back, building LUR norms until we reach the original space. The following result of [DGZ93] formalizes this idea, in a more general context.

Theorem 6.29 ([DGZ93, VII.1 Theorem 1.8])

Let $\mathcal{P}$ be a class of Banach space such that every X in $\mathcal{P}$ admits a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ such that every $(P_{\alpha+1} - P_\alpha)(X)$ belongs to $\mathcal{P}$ again. Then, every X in $\mathcal{P}$ admits a LUR norm.

Proof: We proceed by induction on the density character μ of X. First, if $\mu = \omega$ (i.e., X is separable), then the Kadets Theorem (6.18) provides a LUR norm on X.

Let us now assume that the density character of X is μ and that every Banach space Y in $\mathcal{P}$ with $\operatorname{dens}(Y) < \mu$ admits a LUR norm. If $\operatorname{dens}(X) = \mu$ and

$\{P_\alpha : \omega \leq \alpha \leq \mu\}$ is a PRI on X such that $(P_{\alpha+1} - P_\alpha)(X)$ belongs to $\mathcal{P}$ for every $\alpha < \mu$, then by the induction hypothesis every $(P_{\alpha+1} - P_\alpha)(X)$ would admit a LUR norm. Then, we are able to apply Zizler Theorem 6.27, obtaining a LUR norm on X. ■

Now, we finally get the LUR renorming in the dual of an Asplund space by proving that the class of duals of Asplund spaces satisfies the hypotheses on Theorem 6.29.

Theorem 6.30

Let X be an Asplund space. Then, X^* admits a LUR norm.

Proof: Let X be an Asplund space. By Theorem 6.17, we know that X^* admits a projectional generator defined by X, and by Proposition 6.11, we know that X^* admits a PRI $\{P_\alpha : \omega \leq \alpha \leq \mu\}$ such that every $(P_{\alpha+1} - P_\alpha)X^*$ is isometric to a space of the form $(E_{\alpha+1}/E_\alpha)^*$, where $E_{\alpha+1}$ and E_α are subspaces of X (see Proposition 6.11 above). Thus, every $(P_{\alpha+1} - P_\alpha)X^*$ is isometric to a dual of an Asplund space. Then, apply Theorem 6.29 for the property $\mathcal{P}$ "to be the dual of an Asplund space". We thus obtain that X^* has an equivalent LUR norm. ■

Chapter 7

More LUR Renormings Results

According to the title, this chapter aims to finish exposing the LUR renorming results which are necessary for our totally smooth renorming theorem. During Section 7.1 we review Troyanksi's LUR renorming theorem, one of the most important theorems within the area, which apart from being interesting in itself, is (along with Theorem 6.30) a fundamental ingredient for the key M. Raja's dual LUR renorming result, which will be proved by the end of Section 7.2.

7.1 LUR Renorming Characterization

Obtaining an equivalent LUR norm through its construction is not an easy process. For example, we have invested a whole chapter to justify the LUR renorming of X^* when X is Asplund. Precisely because some constructions are very demanding, it is important to know if a strong property can be decomposed into several that are easier to obtain. For this reason, Troyanski's theorem has had a high impact on LUR renorming field, since it offers the possibility to study the existence of LUR renorming from the existence of a strictly convex equivalent norm, and also a Kadets-Klee renorming. Originally, the proof given by Troyanski in [Troy85] was very sophisticated, using probability arguments and *martingales*. An important contribution of M. Raja [Ra99] was an elegant alternative proof using a different and more geometrical approach, based on a method of Lancien [Lan95].

We start by showing the relationship between LUR norms and agreement of topologies.

Proposition 7.1

(1) *Let X be a Banach space such that $\|\cdot\|$ is LUR. Then, the norm has the Kadets-Klee property (i.e., the weak and the norm topologies coincide on the unit*

sphere).

(2) *Let X be a Banach space such that the dual norm $\|\cdot\|^*$ of X^* is LUR. Then, $\|\cdot\|^*$ has w^*-$\|\cdot\|^*$-Kadets-Klee property (i.e., norm, weak and weak star topologies coincide on the unit sphere).*

Proof: To prove (1), take a sequence $\{x_n\}_{n=1}^{\infty} \subset S_X$ such that $x_n \xrightarrow{w} x_0 \in S_X$. By the Hahn–Banach Theorem, there must exists $x_0^* \in S_{X^*}$ such that $\langle x_0^*, x_0 \rangle = 1$. Then, by the weak convergence $\langle x_0^*, x_n \rangle \to \langle x_0^*, x_0 \rangle = 1$ and so

$$2 \geq \|x_0 + x_n\| \geq \langle x_0^*, x_n + x_0 \rangle \to 2.$$

Thus, applying the LUR condition we get $x_n \xrightarrow{\|\cdot\|} x_0$.

Now, for (2) its enough to check that w and $\|\cdot\|^*$ topologies agree on S_{X^*}, since $w^* \subset w \subset \|\cdot\|$. Indeed, by the LUR property, given $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $x^* \in S_{X^*}$ satisfies $\|x_0^* + x^*\| > 2(1-\delta)$, then $\|x_0^* - x^*\| < \varepsilon$. Let $\{x_i^*\}$ be a net in S_{X^*} that w^*-converges to x_0^*. Find, by Riesz's Lemma, $x_0 \in S_X$ such that $\langle x_0, x_0^* \rangle > 1 - \delta$. There exists i_0 such that $\langle x_0, x_i^* \rangle > 1 - \delta$ for $i \geq i_0$. Thus, $\|x_0^* + x_i^*\| \geq \langle x_0, x_0^* + x_i^* \rangle > 2(1 - \delta)$ for $i \geq i_0$, hence $\|x_0^* - x_i^*\| < \varepsilon$ for $i \geq i_0$, and the conclusion follows. ■

Definition 7.2

Given M a non-empty subset of the normed space X, we will call **slice of** M **generated by** $x^* \in X^*$ *to any set of the form $\mathcal{S}(M, x^*, \alpha) := M \cap S(x^*, \alpha)$, where $S(x^*, \alpha) := \{x \in X : x^*(x) > \alpha\}$, for $\alpha \in \mathbb{R}$. With this given notation, $S(x^*, \alpha) = S(X, x^*, \alpha)$. These sets will be called just slices.*

A point x of a convex set C is said to be a ***denting point of*** *C if there are slices of C containing x of arbitrary small diameter. If these slices are given by the elements of some subspace Z of the dual, we say that x is a Z-denting point.*

Lemma 7.3 (Choquet)

Let X be a locally convex space, and $C \subset X$ a w-compact convex subset. For every $x \in \mathrm{Ext}(C)$, the slices of C containing x form a neighborhood base of x in the relative weak topology of C.

Proof: Take any V neighbourhood of x in the relative weak topology of C. Then, it can be written as a finite intersection of slices of C, that is, $V = V_1' \cap \cdots \cap V_k'$, where V_i' is a slice of C, i.e., $V_i' = V_i \cap C$ where V_i is a slice (a half-space of X).

Then, since x is an extreme point,

$$x \notin \overline{\text{conv}}\left(\bigcup_{i=1}^{k}\left((X\backslash V_i)\cap C\right)\right),$$

which is a convex w-compact set. Then, by the Separation Theorem we can find a half-space $\mathcal{S}$ such that $x \in \mathcal{S}$ and $\mathcal{S} \cap \overline{\text{conv}}\left(\bigcup_{i=1}^{k}\left((X\backslash V_i)\cap C\right)\right) = \emptyset$, so $\mathcal{S} \cap C \subset V$. ■

Now we include the proof of one of the most celebrated theorems on renorming theory. Originally, this result was proven by Troyanski using martingales and probability arguments. Instead, we follow the later proof given by M. Raja in [Ra99], that only uses geometric arguments. The key of this modification lies on the proof of (3) $\implies$ (1), which is a slightly modification of the Deville Master Lemma (Lemma 6.23).

Theorem 7.4 (Troyanski)

Let X be a Banach space. Then, the following are equivalent:
(1) *X admits a LUR norm.*
(2) *X admits both a rotund norm and a Kadets-Klee norm.*
(3) *X admits a norm such that every point of the unit sphere is a denting point of the unit ball.*

Proof: (1) $\implies$ (2) is trivial, since if $\|\cdot\|$ is a LUR norm, then it is in particular a rotund norm, and by (1) of Proposition 7.1, a LUR norm is also Kadets-Klee.
The implication (2) $\implies$ (3) is a result of Lin-Lin-Troyanski. Assume that X admits $\|\cdot\|_1$ a rotund norm and another norm $\|\cdot\|_2$ which is Kadets-Klee. Then, by Proposition 6.20 the norm $\|\cdot\|$ defined by the equation

$$\|\cdot\|^2 = \|\cdot\|_1^2 + \|\cdot\|_2^2$$

has simultaneously both properties. We endow X with this norm. Now, we will proof that every point of S_X is an extreme point of $B_{X^{**}}$. To see this, take any $x \in S_X$ and assume that there are $x_1, x_2 \in B_{X^{**}}$ such that $x = (x_1 + x_2)/2$. Since the norm is Kadets-Klee, for every $\varepsilon > 0$ we can take U a w-open neighbourhood of x such that

$$\text{diam}\,(B_X \cap U) < \varepsilon/2$$

Now, notice that by the Goldstine theorem, $B_{X^{**}} = \overline{B_X}^{w^*}$, and that the w^*-closure must preserve diameter (for instance, because the distance is w^*-lower semicontinuous). Then, we have that

$$\operatorname{diam}(B_{X^{**}} \cap U) < \varepsilon/2$$

Considering U_1 and U_2 two w^*-neighbourhoods of x_1 and x_2 respectively, such that $U_1 + U_2 \subset 2U$. If $y \in B_{X^{**}} \cap U_1$, then

$$\frac{y + x_2}{2} \in B_{X^{**}} \cap U,$$

because of convexity. Thus, we deduce that $\operatorname{diam}(B_{X^{**}} \cap U_1) < \varepsilon$. Since $B_X \cap U_1$ is non-empty, we deduce that x_1 can be approximated uniformly by points of B_X, which is norm complete, so $x_1 \in B_X$, and analogously $x_2 \in B_X$. But, since $x \in \operatorname{Ext}(B_X)$, and the norm is rotund, then $x = x_1 = x_2$.

Thus, we have proved that every extreme point of B_X is an extreme point of the bidual unit ball. Again, since the norm is rotund, every point on the unit ball is extreme, and so, extreme in $B_{X^{**}}$. Finally, we recall that the topological space (X^{**}, w^*) is locally convex space, $B_{X^{**}}$ is a compact convex subset, and the topological dual of (X^{**}, w^*) is X^*, so we are able to apply the Choquet's Lemma 7.3 and conclude that for every extreme point of $B_{X^{**}}$, the slices of $B_{X^{**}}$ given by elements of X^* form a neighbourhood base for the w^* topology of X^*. So, restricting to B_X, we have that the slices of B_X given by elements of X^* form a neighbourhood base for the w topology, and since the norm is Kadets-Klee, the w and $\|\cdot\|$ topologies agree on the unit sphere, so we have a local base for the norm topology.

For the (3) $\implies$ (1) implication, let $\|\cdot\|$ be a norm in X such that every point on the unit sphere is a denting point of the unit ball. For every $\varepsilon > 0$, define the set

$$B_\varepsilon := \{x \in B_X : \text{ for every slice } \mathcal{S}, \text{ if } x \in \mathcal{S} \text{ then } \operatorname{diam}(B_X \cup \mathcal{S}) > \varepsilon\}.$$

The set B_ε is what remains of B_X after removing all its slices of diameter at most ε. Recall that a slice is an open and convex set, so for every $0 < \varepsilon < 2$, B_ε is a closed symmetric convex set, and has nonempty interior. For every $n \in \mathbb{N}$, let μ_n be the Minkowski functional of the set $B_{1/n}$. Note that every set of the form $B_{1/n}$ contains $\frac{1}{2}B_X$, so by the definition of Minkowski functional

$$\mu_n = \mu_{B_{1/n}} \leq \mu_{\frac{1}{2}B_X} = 2\mu_{B_X} = 2\|\cdot\|. \tag{7.1}$$

This last expression implies that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n^2$ converge, so by (2) of Lemma 6.19 we can define the positive and convex function $f(x) := \left(\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n(x)^2\right)^{1/2}$. Thus, we can define the new norm $|||\cdot|||$ by the formula

$$|||x|||^2 := \|x\|^2 + f(x)^2 = \|x\|^2 + \sum_{n=1}^{\infty} \frac{1}{2^n}\mu_n(x)^2$$

Trivially, it is an equivalent norm because by the equation (7.1) , we have that $\|\cdot\|^2 \leq |||\cdot|||^2 \leq 4\|\cdot\|^2$. Now, we only need to check that $|||\cdot|||$ is a LUR norm.

Take any $x \in X$ and let $\{x_k\}_{k=1}^{\infty} \subset X$ such that $|||x_k||| = |||x|||$ and $\lim_{k\to\infty} |||x + x_k||| = 2|||x|||$. This means that

$$0 = \lim_{k\to\infty}\left(\frac{1}{2}|||x|||^2 + \frac{1}{2}|||x_k|||^2 - \frac{1}{4}|||x + x_k|||^2\right) =$$

$$= \lim_{k\to\infty}\left(\frac{1}{2}\|x\|^2 + \frac{1}{2}\|x_k\|^2 - \left\|\frac{x + x_k}{2}\right\|^2\right) + \left(\frac{1}{2}f(x)^2 + \frac{1}{2}f(x_k)^2 - f\Big(\frac{x + x_k}{2}\Big)^2\right).$$

So both last brakets must tend to 0 when k goes to infinity, hence by Lemma 6.19 applied to both functions $\|\cdot\|$ and f, we deduce that

$$\lim_{k\to\infty}\|x_k\| = \lim_{k\to\infty}\left\|\frac{x + x_k}{2}\right\| = \|x\|,$$

$$\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty} f_n\Big(\frac{x + x_k}{2}\Big) = f_n(x),$$

for every $n \in \mathbb{N}$. Now, we are ready to prove the convergence of x_k to x.
Notice that we can assume without loss of generality that $\|x_k\| = \|x\| = 1$ (just taking the sequence $\{\frac{x_k}{\|x_k\|}\}_{k=1}^{\infty}$ instead of $\{x_k\}_{k=1}^{\infty}$). By hypothesis, every point on the unit sphere of $\|\cdot\|$ is a denting point, so if we fix $\varepsilon > 0$, there is a slice $\mathcal{S}$ such that

$$x \in \mathcal{S} \quad \text{and} \quad \operatorname{diam}_{\|\cdot\|}(B_X \cap \mathcal{S}) \leq \frac{\varepsilon}{2}$$

Considering $n \in \mathbb{N}$ such that $\varepsilon/2 \leq 1/n \leq \varepsilon$, then $x \notin B_{1/n}$. It means that $\mu_n(x) > 1$, and so for big enough k, we have $\mu_n((x + x_k)/2) > 1$ also, and thus

$$\frac{x + x_k}{2} \in B_X \backslash B_{1/n}.$$

By definition of $B_{1/n}$, there must exist a slice $\mathcal{S}'$ such that

$$\frac{x + x_k}{2} \in \mathcal{S}' \quad \text{and} \quad \operatorname{diam}_{\|\cdot\|}(B_X \cap \mathcal{S}') \leq \frac{\varepsilon}{2}.$$

Notice that $B_X\mathcal{S}'$ is a convex set, so, at least one of x_k or x must be in $B_X \cap \mathcal{S}'$, and since

$$\|x_k - x\| = 2\left\|x_k - \frac{x + x_k}{2}\right\| = 2\left\|x - \frac{x + x_k}{2}\right\|$$

we deduce that for a big enough k, we get $\|x_k - x\| < \varepsilon$. Consequently, x_k converges to x in the norm $\|\cdot\|$, but since $|||\cdot|||$ is an equivalent norm, also converges in $|||\cdot|||$ and the proof is finished. ■

7.2 Dual LUR Renorming

This section is entirely devoted to prove the dual LUR renorming result of M. Raja. To this end, we have selected the necessary results from [Ra99] and [Ra02] so the proof is self-contained. Since the effort was the same, we have decided to include the preparatory results in their most general version.

This section is, along with Chapter 6, the most technical part of this Memoir, so allow us to explain in simple words what is the idea.

We aim to prove that if X is a Banach space whose dual X^* admits a w^*-w-KK dual norm, then X^* has an equivalent dual LUR norm. All we know so far is that, under this conditions, X must be an Asplund space, so by Theorem 6.30, X^* admits a LUR norm. Here, the key step is that the w^*-w-KK condition also provides a family of convex sets (the family of balls centred at the origin) which has good properties with respect to weak topologies (what M. Raja calls convex-$P(w, w^*)$). In combination with LUR, this guarantees an even stronger property (convex-$P(\|\cdot\|^*, X)$) that allows the construction of an equivalent dual LUR norm in X^* by a slightly modification of the Deville Master Lemma.

We star by proving a technical lemma in [Ra99].

Lemma 7.5 ([Ra99, Lemma 2])

Let X be a vector space and $\tau_2 \subset \tau_1$ locally convex topologies on X. The set $\mathbb{S}(\tau_i)$ will denote a sub-basis of τ_i given b y a family of sublinear functions. Fix any point $x \in X$ and let Δ be a family of sets of X with the property that, for every set $V \in \mathcal{S}(\tau_1)$ with $x \in V$, there exist $A \in \Delta$ and $U \in \mathcal{S}(\tau_2)$ such that

$$x \in A \cap U \subset V.$$

Then, for every $V \in \mathcal{S}(\tau_1)$ with $x \in V$ there must exists $A \in \Delta$, $W \in \tau_1$ and $U \in \mathcal{S}(\tau_2)$ such that

$$x \in (A + W) \cap U \subset V.$$

Proof: Assume that we have a fixed $x \in X$ and $V \in \tau_1$ with $x \in V$ as in the statement of the hypothesis. We shall prove that there exist $W_1, V' \in \mathcal{S}(\tau_1)$ with $0 \in W_1$, $x \in V'$ and $W_1 + V' \subset V$. Indeed, as $\mathcal{S}(\tau_1)$ is given by a family of sublinear functions, there must exists a sublinear function f such that $V = \{y \in X : f(x - y) < \varepsilon\}$ for some $\varepsilon_0 > 0$. Then, we just have to take the sets

$W_1 := \{y \in X : f(y) < \varepsilon_0/2\}$ and $V' := \{y \in X : f(x-y) < \varepsilon_0/2\}$. Then if we take $z \in W_1 + V'$, then we have that $z = w_1 + v'$, with $w_1 \in W_1$ and $v' \in V'$. Then

$$f(x-z) = f(x - w_1 - v') \leq f(x - v') + f(w_1) < \varepsilon_0/2 + \varepsilon_0/2 = \varepsilon_0$$

Now, by the property of Δ, we can take $A \in \Delta$ and $U' \in \mathcal{S}(\tau_2)$ such that $x \in A \cap U' \subset V'$.

Analogously, we can also take $W_2, U \in \mathcal{S}(\tau_2)$ with $0 \in W_2$, $x \in U$ and $W_2 + U \subset U'$.

Now, define $W := W_1 \cap (-W_2) \in \tau_1$. All that remains to be seen is that $(A + W) \cap U \subset V$. For this, taking any $y \in (A+W) \cap U$, then there is $a \in A$ such that $y - a \in W \subset (-W_2)$, and so $a = (a-y) + y \in U'$, hence $a \in A \cap U' \subset V'$. Lastly, as $y - a \in W \subset W_1$, we have $y = (y-a) + a \in V$, and the result is proved. ■

We recall here a concept introduced in [Gru84]. Although we do not specifically require this definition, it is an important concept in renorming theory, and appears implicitly during this chapter, so we thought it appropriate to mention it and briefly comment on the connection below in Remark 7.8.

Definition 7.6 ([Gru84])

Let $(X, \mathcal{T})$ a topological space. A family $\mathcal{N}$ of substets of X is called a ***network*** *if for every $x \in X$ and $U \in \mathcal{T}$ with $x \in U$ there exists a subset $N \in \mathcal{N}$ such that $x \in N \subset U$.*

Definition 7.7 ([Ra99])

Let Σ_1 and Σ_2 two families of subsets of a given set X. We say that X has the property $\boldsymbol{P(\Sigma_1, \Sigma_2)}$ *if there is a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of X such that, for every $x \in X$ and every $V \in \Sigma_1$ with $x \in V$, there exist a $n \in \mathbb{N}$ and $U \in \Sigma_2$ such that*

$$x \in A_n \cap U \subset V.$$

If X is a vector space and the sets A_n can be chosen to be convex, then it is said that X has property **convex-$\boldsymbol{P(\Sigma_1, \Sigma_2)}$**.

Remark 7.8

Observe that when $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies defined on X, then property $P(\mathcal{T}_1, \mathcal{T}_2)$ means exactly that the family $\{A_n \cup \mathcal{T}_2\}_{n=1}^{\infty}$ is a network for the topology $\mathcal{T}_1$.

In the particular case when $\mathcal{T}_1$ is the topology defined by the norm $\|\cdot\|$ of X and $\mathcal{T}_2$ is either the weak, topology, weak star topology, or the family of slices defined by a subspace $Z \subset X^*$ will be denoted respectively by $P(\|\cdot\|, w)$, $P(\|\cdot\|, w^*)$ and $P(\|\cdot\|, Z)$.

Also, notice that from the definition it follows that the property P enjoys some kind of *transitivity* in the next sense: if X has property $P(\Sigma_1, \Sigma_2)$ and $P(\Sigma_2, \Sigma_3)$, then X has also property $P(\Sigma_1, \Sigma_3)$. Of course, this also applies to the case of convex-P properties.

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Recall from Chapter 2 that a linear subspace $Z \subset X^*$ is norming whenever $\|x\|_Z := \sup\langle x, Z\rangle$ defines an equivalent norm on X.

Lemma 7.9 ([Ra99, Lemma 3])

Let X be a Banach space and $Z \subset X^*$ a norming linerar subspace. If X has property convex-$P(w, w(X, Z))$, then it has property convex-$P(X^*, Z)$.

Proof: Assume that X has a sequence of convex sets $\{A_n\}_{n=1}^{\infty}$ that satisfy the definition of property $P(w, w(X, Z))$. We will check that the same sequence of convex sets satisfy the definition of property $P(X^*, Z)$. Take a fixed $x \in X$ and $\mathcal{S}$ a slice of X. By hypothesis, we can find $n \in \mathbb{N}$ and $U \in w(X, Z)$ such that $x \in A_n \cap U \subset \mathcal{S}$. Observe that the set $\overline{A_n \backslash \mathcal{S}}^{w(X,Z)}$ cannot contain the point x since U and the convex set $A_n \backslash \mathcal{S}$, so, we can apply the Separation Theorem to find a $\mathcal{S}'$ a slice generated by Z such that $x \in \mathcal{S}'$ and

$$\overline{A_n \backslash \mathcal{S}}^{w(X,Z)} \cap H' = \emptyset,$$

therefore $X \in A_n \cap \mathcal{S}' \subset \mathcal{S}$. ■

This result shows the link between the Z-denting property and convex-$P(\|\cdot\|, Z)$.

Theorem 7.10 ([Ra99])

Let X be a Banach space and $Z \subset X^*$ a norming linear subspace. If X admits a norm such that every point of the unit sphere is Z-denting, then X has convex-$P(\|\cdot\|, Z)$.

Proof: We check that the collection of balls with rational radius centred at 0 is a countable set that satisfies the definition of convex-$P(\|\cdot\|, Z)$. To see this, fix any $x \in X\backslash\{0\}$ (because $x = 0$ is trivial), and take the closed ball $\|x\|B_X$. By the hypothesis, for every $\varepsilon > 0$ there must exists a slice $\mathcal{S}$ given by an element of Z, such that $x \in \mathcal{S}$ and

$$\|x\|B_X \cap \mathcal{S} \subset x + \frac{\varepsilon}{2}B_X.$$

If we fix $\varepsilon > 0$ we can apply Lemma 7.5 and find $\delta > 0$ and $\mathcal{S}$ a slice generated by an element of Z such that $x \in \mathcal{S}$ and

$$(\|x\|B_X + \delta B_X) \cap \mathcal{S} \subset x + \frac{\varepsilon}{2}B_X.$$

Now, we can find a rational number $r > 0$ such that $\|x\| \le r \le \|x\| + \delta$. This implies

$$\|x\|B_X \subset rB_X \subset (\|x\| + \delta)B_X = \|x\|B_X + \delta B_X,$$

hence, $x \in rB_X \cap \mathcal{S}$ and $\operatorname{diam}(rB_X \cap \mathcal{S}) < \varepsilon$. ■

Theorem 7.11 ([Ra99])

Let X be a Banach space and $Z \subset X^*$ a norming linear subspace. If X admits a LUR norm and has property convex-$P(w, w(X,Z))$, then X has property convex-$P(\|\cdot\|, Z)$.

Proof: Observe that if X has a LUR norm, then by Theorem 7.4 we know that X admits a norm such that every point on the unit sphere is denting, so applying now Theorem 7.10 for the weak topology we get that X has convex-$P(\|\cdot\|, X^*)$. Now, by hypothesis, X has property convex-$P(w, w(X,Z))$ and in particular, by Lemma 7.9 it has convex-$P(X^*, Z)$. Now, having convex-$P(\|\cdot\|, X^*)$ and convex-$P(X^*, Z)$, we apply the transitivity property observed in the last paragraph of Remark 7.8, concluding that X has also property convex-$P(\|\cdot\|, Z)$. ■

Theorem 7.12 ([Ra99])

Let X be a Banach space and $Z \subset X^*$ a norming linear subspace. If X has convex-$P(\|\cdot\|, Z)$, then X admits an equivalent norm $|||\cdot|||$ which is simultaneously LUR and $w(X,Z)$-lower semicontinuous.

Proof: First of all, we can assume without loss of generality that the norm $\|\cdot\|$ is already $w(X,Z)$-lower semicontinuous. Suppose that there exist a countable family of convex sets $\{A_n\}_{n=1}^{\infty}$ that fulfill the definition of $P(\|\cdot\|, Z)$. As in the proof of Theorem 7.5, by Lemma 7.5 it is ensured that the countable family of convex sets

$$\{A_n + rB_X : n \in \mathbb{N}, r > 0, r \in \mathbb{Q}\}$$

can also be used to satisfy the definition of convex-$P(\|\cdot\|, Z)$, and so we can assume that the sets A_n are norm open.

For every $n \in \mathbb{N}$ take a point $a_n \in A_n$, take the set $B_n := \overline{A_n - a_n}^{w(X,Z)}$, and then, define the function

$$f_n(x) := \mu_{B_n}(x - a_n) \quad \text{for every } x \in X.$$

where μ_{B_n} is the Minkowski functional of B_n. Clearly, every function f_n is convex, Lipschitz and $w(X,Z)$-lower semicontinuous. Now, for every $m \in \mathbb{N}$ take the sets

$$A_{n,m} := \{x \in \overline{A_n}^{w(X,Z)} : \text{ for every slice } \mathcal{S} \text{ generated by } Z \text{ with } x \in \mathcal{S}, \operatorname{diam}(A_n \cap \mathcal{S}) > 1/m\}.$$

The set A_n is A_n but removing all the slices generated by Z that have diameter at most $1/m$. So, every $A_{n,m}$ is empty or a $w(X,Z)$-closed convex set. Now, for each $p \in \mathbb{N}$, consider the sets

$$A_{n,m,p} = \overline{A_{n,m} + (1/p)B_X}^{w(X,Z)}.$$

We claim that, when $A_{n,m}$ is non-empty, then $A_{n,m} = \bigcap_{p=1}^{\infty} A_{n,m,p}$. Indeed, if $x \notin A_{n,m}$, then, as $A_{n,m}$ is $w(X,Z)$-closed convex set, by the Separation Theorem we can find a slice $\mathcal{S}$ generated by Z such that $x \in \mathcal{S}$ and $A_{n,m} \cap \mathcal{S} = \emptyset$. Furthermore, we can assume that the distance between $A_{n,m}$ and $\mathcal{S}$ is positive. Thus, there must exists $p \in \mathbb{N}$ such that

$$(A_{n,m} + (1/p)B_X) \cap \mathcal{S} = \emptyset.$$

Then, we have that $x \notin A_{n,m,p}$.

Now, we proceed with the construction of the norm. For every $n, m \in \mathbb{N}$, if $A_{n,m} = \emptyset$, then for every $p \in \mathbb{N}$ we take the function $f_{n,m,p}(x) := 0$ for every $x \in X$, while if $A_{n,m} \neq \emptyset$, then fix a point $a_{n,m} \in A_{n,m}$ and take

$$f_{n,m,p}(x) := \mu_{A_{n,m,p}}(x - a_{n,m})$$

which is convex, Lipschitz and $w(X,Z)$-lower semicontinuous. Thus, we can define a symmetric convex function F by the formula

$$F(x)^2 = \|x\|^2 + \sum_{n=1}^{\infty} \alpha_n f_n(x)^2 + \sum_{n,m,p=1}^{\infty} \beta_{n,m,p} f_{n,m,p}(x)^2 +$$
$$+ \sum_{n=1}^{\infty} \alpha_n f_n(-x)^2 + \sum_{n,m,p=1}^{\infty} \beta_{n,m,p} f_{n,m,p}(-x)^2$$

where every α_n and $\beta_{n,m,p}$ are positive coefficients taken in such a way that the series converges uniformly on bounded subsets of X, so that F is uniformly continuous on bounded sets and the absolutely convex set $B := \{x \in X : F(x) \leq 1\}$ contains 0 as an interior point. Note that this can be done by rewriting the series according to a single sub-index $k \in \mathbb{N}$, and for each non-zero function f_k, take its coefficient as $1/2^k$ and divide by the supremum of f_k on the unit ball. Thus, the uniform convergence comes from the Weierstrass M-test.

Now, since a series of $w(X,Z)$-lower semicontinuous functions is also $w(X,Z)$-lower semicontinuous, then the set B must be $w(X,Z)$-closed.
With this, we know that defining

$$|||\cdot||| := \mu_B,$$

this is an equivalent $w(X,Z)$-lower semicontinuous norm defined on X. Now, we only need to prove that it is also a LUR norm.

First, we need to prove that for every sequence $\{x_i\}_{i=1}^{\infty} \subset B$ such that $|||x_i||| \xrightarrow{i\to\infty} 1$, then $\lim_{\imath\to\infty} F(x_i) = 1$. Note that the function F is $\|\cdot\|$-continuous, so we can deduce that

$$\{x \in X : F(x) < 1\} \subset \{x \in X : |||x||| < 1\},$$

because the first one is $\|\cdot\|$-open set contained in B, and the second set is the interior of B. By considering the sequence $x_i' := x_i/|||x_i|||$, we already have that $F(x_i') = 1$. Thus, as $\lim_{i\to\infty} \|x_i' - x_k\| = 0$ and F is uniformly continuous on bounded sets, we deduce that

$$\lim_{i\to\infty} F(x_i) = 1.$$

Now, we are ready to prove that $|||\cdot|||$ is a LUR norm. To see this, take $x \in X$ and $\{x_i\}_{i=1}^{\infty} \subset X$ such that $|||x_i||| = |||x||| = 1$ and $\lim_{i\to\infty} |||x + x_i|||/2$. Thus, by the precious claim, we know that $F(x_k) = F(x) = 1$ and $\lim_{i\to\infty} F((x + x_k)/2) = 1$. Then,

$$\lim_{i\to\infty} \left(\frac{F(x^2) + f(x_i)^2}{2} - F\Big(\frac{x + x_i}{2}\Big)^2 \right) = 0.$$

Here, we are in condition to apply Lemma 6.19, and deduce that for every fixed $n, m, p \in \mathbb{N}$,

$$\lim_{i\to\infty} f_n(x_i) = \lim_{i\to\infty} f_n\Big(\frac{x + x_i}{2}\Big) = f_n(x),$$
$$\lim_{i\to\infty} f_{n,m,p}(x_i) = \lim_{i\to\infty} f_{n,m,p}\Big(\frac{x + x_i}{2}\Big) = f_{n,m,p}(x).$$

Now, fix $0 < \varepsilon < 1/2$, and then, there must be an $n \in \mathbb{N}$ and $\mathcal{S}$ a slice generated by Z such that $x \in A_n\mathcal{S}$ and

$$\operatorname{diam}(\overline{A_n}^{w(X,Z)} \cap \mathcal{S}) \leq \varepsilon/3,$$

because $\mathcal{S}$ is $w(X,Z)$ open and $\|\cdot\|$ is $w(X,Z)$-lower semicontinuous. Since the sets A_n have been taken to be $\|\cdot\|$-open, then $f_x(x) < 1$, and then, $f_n(x_i) < 1$ for a big enough $i \in \mathbb{N}$, and thus $x_i \in \overline{A_n}^{w(X,Z)}$. Analogously, for a big enough $i \in \mathbb{N}$, we have $(x + x_i)/2 \in \overline{A_n}^{w(X,Z)}$.

Now, take $m \in \mathbb{N}$ such that $2/\varepsilon < m < 3/\varepsilon$, so $x \notin A_{n,m}$. If $A_{n,m} = \emptyset$, then trivially $(x + x_i)/2 \notin A_{n,m}$. In the case that $A_{n,m} \neq \emptyset$, by the previous construction, there must be a $p \in \mathbb{N}$ such that $f_{n,m,p}(x) > 1$, so taking a big enough $i \in \mathbb{N}$, $f_{n,m,p}((x + x_i)/2) > 1$ and thus we get again that $(x + x_i)/2 \notin A_{n,m}$.

Therefore, we found that for a $i \in \mathbb{N}$ big enough,

$$\frac{x + x_i}{2} \in \overline{A_n}^{w(X,Z)} \setminus A_{n,m}.$$

By the definition of $A_{n,m}$, there must exists a $\mathcal{S}'$ slice generated by Z such that $(x + x_i)/2 \in \overline{A_n}^{w(X,Z)} \cap \mathcal{S}'$, and diam $(\overline{A_n}^{w(X,Z)} \cap \mathcal{S}') \leq \varepsilon/2$. But note that either x or x_i must be in $\mathcal{S}'$, and so, in $\overline{A_n}^{w(X,Z)} \cap \mathcal{S}'$. Finally, since

$$\|x_i - x\| = 2\left\|x_k - \frac{x + x_i}{2}\right\| = 2\left\|x - \frac{x + x_i}{2}\right\|,$$

we deduce that $\|x_i - x\| \leq \varepsilon$. Thus, $|||\cdot|||$ is LUR. ■

Observe that just combining Theorems 7.11 and 7.12, we get that *if a Banach space X with $Z \subset X^*$ a norming linear subspace has property convex-$P(w, w(X,Z))$ and admits a LUR norm, then X admits an equivalent norm $|||\cdot|||$ which is simultaneously LUR and $w(X,Z)$-lower semicontinuous.*

Finally, we prove here the key result of M. Raja of dual LUR renorming. We have included some steps that had been omitted in the original proof. We also appeal concepts from previous chapters to make it self-contained.

Theorem 7.13 (M. Raja, [Ra99])

Let X be a Banach space such that X^* has w^*-w-KK property. Then, X^* admits an equivalent dual LUR norm.

Proof:
Assume that the dual norm $\|\cdot\|$ on X^* has w-w^*-KK property. We prove first that then X^* should have property convex-$P(w, w^*)$, using a similar argument to the proof of Theorem 7.10. Take any point $x^* \in X^*$ and any a w-open set V

containing x^*. Without loss of generality, we can assume that $x^* \neq 0$, and take the ball $\|x^*\|B_{X^*}$, so $\|x^*\|B_{X^*} \cap V$ is a relative w-open neighbourhood of x^*. Since weak topologies agree on spheres, there must exist a w^*-open set U such that

$$\|x^*\|B_{X^*} \cap U \subset \|x^*\|B_{X^*} \cap V \subset V.$$

Now, by Lemma 7.5 we know that in this situation, there exists $\delta > 0$ such that

$$(\|x^*\|B_{X^*} + \delta B_{X^*}) \cap U \subset V.$$

Then, if we take a rational r such that $\|x^*\| \leq r \leq \|x^*\| + \delta$, we get

$$rB_{X^*} \cap U \subset V,$$

so the countable family of dual unit balls centered at 0 and with rational radius satisfy the definition of convex-$P(w, w^*)$ property.

On the other hand, during Section 4.4 we proved that a w-w^*-KK norm on X^* is already a dual norm. Now, by Corollary 4.7, we know that the norm on X is HBS, and by Proposition 4.15 X must be an Asplund space, so X^* admits a LUR norm.

Finally, combining Theorems 7.11 and 7.12, we obtain that X^* admits a LUR norm which is w^*-lower semicontinuous, i.e., a dual LUR norm. ■

According to the results we developed during Section 4.4, we know that previous theorem admits a stronger formulation, since every w^*-w-KK norm is indeed a dual norm. We strongly believe that M. Raja already knows this detail. However, as it is not explicit in the references, we have considered to emphasize this fact in our Theorem 8.1.

Also, in Chapter 4 we refer the M. Raja result as a landmark. Let us point why: Troyanki's LUR characterization says that having a LUR norm is equivalent to having both, a strictly convex and a Kadets-Klee norm. Of course, none of this hypotheses can be dropped, since there are Banach spaces having just one of these properties (see [DGZ93] or [FHHMZ11]). So, for having a dual LUR renorming, it might seem that the expected analogous assumptions would be the existence of both, a strictly convex dual norm and also a norm such that in its unit sphere, norm, weak and weak star topologies agree.
However, the previous result shows that (surprisingly) the coincidence on weak and weak star topologies is enough.

Chapter 8

Sullivan's amd Oja–Viil–Werner theorems are true in any Banach space

Once reviewed the necessary LUR renorming results, and having given a complete proof of M. Raja's Theorem, we will retrieve Chapter 4 right where we left it, recalling and putting together the pieces that have been presented during the work. We shall give the proof of our improvement of the Oja–Viil–Werner Theorem, showing that the result holds, even in a stronger form, without the assumption of being a WCG Banach space (this is the main result in [CoGuiMon20]). We shall complete the chapter offering a perspective of the relationship between the studied properties.

8.1 A —even more than— totally smooth renorming

Let us recall some of the definitions and results from previous chapters.

The norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ is said to be **strictly convex** (or **rotund**), if for $x, y \in S_X$ such that $\|x + y\| = 2$ we have $x = y$. The norm is said to be **locally uniformly convex** (or **locally uniformly rotund**) (**LUR**, for short), if $x \in S_X$, $x_n \in S_X$ for $n \in \mathbb{N}$, and $\|x + x_n\| \to 2$ implies $\|x_n - x\| \to 0$. The norm $\|\cdot\|$ in a dual Banach space is said to have **property** w^***-LUR** if $x_n^* \to x_0^*$ in the w^*-topology as soon as x_0^*, $x_n^* \in S_{X^*}$ for $n \in \mathbb{N}$, and $\|x_n^* + x_0^*\| \to 2$.

Chapter 3 paid attention to the problem of uniqueness of norm-preserving extensions (also called **Hahn–Banach extensions**) of any continuous functional defined on a closed subspace Y of a Banach space X to the whole of X (one of the main contributions to this subject was due to R. R. Phelps in [Ph60], as we mentioned). We recall that the norm $\|\cdot\|$ of a Banach space $(X, \|\cdot\|)$ has the so-called **Hahn–Banach smooth property** (**HBS**, for short) if every $x^* \in X^*$ has a unique norm-preserving extension to X^{**} (Definition 4.1). Godefroy's Proposition 4.3 (a result also in [HW93]) says that this property is equivalent to the coincidence of the topologies w and w^* on the unit sphere S_{X^*} of the dual space X^* (a property that we called ***w*-w*-KK**, for short). A stronger property, called **total smoothness** (**TS** for short) is that for *any* closed subspace Y of X, any $y^* \in Y^*$ has a unique Hahn–Banach extension to X^{**} ([LiWo2010]). This is equivalent, by the A. E. Taylor and S. R. Foguel result (Theorem 3.14) [Tay39], [Fo58], to the HBS property plus the rotundness of the dual norm $\|\cdot\|^*$. As we mentioned in the Introduction to Chapter 3, in [Su77] it was proved that a *separable* space whose norm has the HBS property has a TS renorming, and in [OVW19] this result was extended to the class of *weakly compactly generated* spaces (i.e., spaces having a weakly compact linearly dense subset).

Here we just point out that the renorming result holds, even in a stronger form, *without any restriction on the space*. This observation is based on M. Raja's Theorem 7.13 above. Let us point out that *our result applies to classes of spaces strictly larger than the class of WCG Banach spaces having a HBS renorming* (see Remark 8.2 below).

We believe that putting together those results as in Theorem 8.1 below may help to clarify the connections between the different properties mentioned above.

Theorem 8.1

Let $(X, \|\cdot\|)$ be a Banach space. Then, the following statements are equivalent:
(i) *X has an equivalent norm with property HBS.*
(ii) *X^* has an equivalent w^*-w-Kadets-Klee norm.*
(iii) *X has an equivalent norm whose dual norm is LUR.*
(iv) *X has an equivalent norm with property TS.*

Proof As noticed during Section 4.4, we know that every w^*-w-Kadets-Klee norm on X^* is already a dual norm. Then, (i)$\Leftrightarrow$(ii) follows from Corollary 4.7.
A dual LUR norm has w^*-w-KK property (see (2) of Proposition 7.1). Also, by Raja's dual LUR renorming theorem (7.13) we know that X^* has an equivalent dual LUR norm if and only if X admits a norm whose dual has w^*-w-Kadets-Klee property. Again by Section 4.4, we get (ii)$\Leftrightarrow$(iii).
(iii)$\Rightarrow$(iv) follows from Taylor–Foguel's Theorem 3.14 and Proposition 7.1.
(iv)$\Rightarrow$(i) is obvious. ■

Remark 8.2

1. There are Banach spaces $(X, \|\cdot\|)$ such that $\|\cdot\|$ has property HBS while they are not WCG Banach spaces. This is important for clarifying that our result (Theorem 8.1) is a real extension of the result of Oja–Viil–Werner. The example is $X := C(K)$, where K is a non-Eberlein compact space such that $K^{(\omega_1)} = \emptyset$. It is proved in [DGZ93, Theorem VII.4.7] that for such a compact space K, the space $C(K)$ has an equivalent dual LUR norm (so the norm in X has property HBS). Since K is not Eberlein, the space $C(K)$ is not WCG. As a particular example, let $K := [0, \omega_1]$, an interval of ordinal numbers, where ω_1 is the first uncountable ordinal. The space K is not an Eberlein compact, since it is not angelic.

2. Observe that, in particular, the TS norm defined in (iii) above on every Banach space with a HBS norm is Fréchet differentiable.

3. Banach spaces that satisfy one (and then all) of the conditions (i) to (iv) in Theorem 8.1 have been characterized in other different ways. Let us mention here that, for example, Theorem 1.4 in [FOR19] provides a few of them, in terms of (a) the existence of a dual norm in X^* such that (S_{X^*}, w^*) is a Moore space, or (b) the existence of an equivalent dual norm such that (S_{X^*}, w^*) is symmetrized by a symmetric ρ such that every point $x^* \in S_{X^*}$ has w^*-neighborhoods of arbitrary small ρ-diameter, or (c) the existence of a dual equivalent norm such that (S_{X^*}, w^*) is metrizable, or even (d) that (B_{X^*}, w^*) is a descriptive compact space (for details, see the op. cit. and the reference list there). ®

Remark 8.3

Let us mention here (only with a hint for the proofs) that, for a Banach space $(X, \|\cdot\|)$ whose norm $\|\cdot\|$ has property HBS,

1. The norm, restricted to any closed subspace of X, has property HBS too, a consequence of the w^*-lower semicontinuity of the dual norm.

2. X is Asplund (Proposition 4.15 above).

3. X is **nicely smooth** (i.e., there is no proper 1-norming subspace in X^*) (Proposition 4.9), and that, in fact, every James boundary is strong (see, e.g., [FHHMZ11, Paragraph 3.11.8.3]).

4. If $(X, \|\cdot\|) = (C(K), \|\cdot\|_\infty)$, where K is a compact topological space, then K is finite. This follows from the fact that the set of extreme points of $B_{C(K)^*}$ is $\{\pm\delta_k : k \in K\}$, that all extreme points are distributed between two closed hyperplanes, the Krein–Milman theorem, and the consequent reflexivity of the space $C(K)$. This observation depends strongly on the fact that the norm on $C(K)$ is the supremum norm. A space $C(K)$, for K an infinite compact space, may admit an equivalent norm $\|\cdot\|$ whose dual is LUR (and so $\|\cdot\|$ has property HBS): Just take K an infinite countable compact space; it is metrizable and scattered (see, e.g., [FHHMZ11, Lemma 14.21]), hence $C(K)$ is Asplund (see, e.g., [FHHMZ11, Theorem 14.25]). Thus, $C(K)^*$ is separable, and the conclusion follows from a classical result of Kadets (see, e.g., [FMZ06, Section 2]).

5. There exists a LUR renorming of X. This follows from the aforementioned Raja's result and a result of R. Haydon in [Hay08]. Note that it is an open problem (see, e.g., [ST10, Problem 1] and [GMZ16, Problem 102]) whether a space X has a LUR renorming as soon as it has a norm whose dual norm has property w^*-LUR. ®

8.2 The huge gap between wHBS and HBS

Much of the work developed until now has focused on the HBS property. Through the Memoir we have tried to improve (or at least maintain) conclusions of certain results but asking for weaker hypotheses. In this spirit, we achieve the extension of the Moltó–Orihuela–Troyanski–Valdivia result on Chapter 5, the results on differentiation of Chapter 2, and Theorem 8.1 above, among others. Looking at Theorem 8.1, we may ask a natural question: Whether there is much difference between being wHBS and HBS, or if there can be an analogous weaker version of Theorem 5.5 for the wHBS property.

In fact, the parallelism that already exists between these properties can reinforce this idea. For example, by the local version of Godefroy's Proposition 4.3, HBS and wHBS have an equivalent formulation by coincidence of weak topologies, w^*-w-KK and w^*-w-KK with respect to $\mathrm{NA}(X)$, respectively. Furthermore, by Taylor–Foguel Theorem, *$\|\cdot\|$ is totally smooth if and only if is simultaneously HBS and its dual norm is strictly convex*, and for the wHBS case, Theorem 3.22 together with Proposition 4.20 show that a similar scheme holds, where the totally smoothness is replaced by the very smooth property, and Gâteaux differentiability plays the same rôle that dual strict convexity for the HBS case. Using compact notation, we have very similar schemes

$$\begin{aligned} \text{TS } &= \text{R}^* + \text{HBS} \\ \text{VS } &= \text{G } + \text{wHBS} \end{aligned}$$

However, Talagrand proved that there are Fréchet differentiable spaces (something much stronger than wHBS) which its dual does not admit any strictly convex dual norm, in particular, there are wHBS spaces that cannot be renormed for being HBS ([DGZ93, VII.5]).
On the other hand, if X admits a wHBS norm, then is an Asplund space, and by the techniques reviewed in Chapter 6, we know that its dual space X^* admits an equivalent LUR norm 6.30. Thus, duals of wHBS spaces have norms with good convexity properties, despite not dual ones.

Conclusions

Some comments in the form of a conclusion

Along this Memoir we achieved several goals: First, we were acquainted with many techniques in Banach space theory (differentiability, convexity, extension of linear functionals, duality, decompositions of nonseparable spaces, geometric properties of the closed unit ball, renormings, weak and weak-star topologies, etc.). We think this has been a systematic training in this part of the theory. Some of the most sophisticated techniques are presented in full detail, because we think that mathematical research and mastering its procedures is learned also by understanding, reproving and rewriting the work done by others.

Second, we made a thoughtful study of the sources (books and papers). All those in the reference list have been consulted, and some of them studied in detail.

Third, and as several mathematicians we know like to point it out, every mathematical piece of research must say something new and must be motivated by some problem. In our case, we made several contributions, as described in the introduction. We do not claim that they will have a profound impact in the theory. However, we may honestly say that they are new, and that they are not a straightforward product of previous results. The proof of this is that our contribution in the case of the HBS and TS renorming went through a regular referee's process —two referees, at least, were involved— and got a very positive evaluation. We plan to submit the others to a publication process.

Fourth, along the study of the sources we made the effort to really assimilate the results —sometimes by producing completely new proofs— and test the scope of applications —providing a whole picture of examples and counterexamples, remarks and comments. By the way, this was how we found improvements and extensions of previously known results.

Last, and not least, we had the possibility to participate in real-life research discussions with our tutors and colleagues. This is maybe the right moment to thank the extraordinary help and encouragement we got from Antonio José Guirao and Vicente Montesinos. It is impossible to reflect here how much we learned from them and how important part of their time they devoted to us and to this work.

Future work

Renorming theory is a huge field. Along the Memoir we touched some of its achievements. We think that our work on the property wHBS must be completed. LUR renorming of spaces with the RNP is a fundamental open problem that we shall like to investigate. We think that Zizler's LUR renorming result by using projectional resolutions of the identity can still be investigated to get in a more friendly way not only his original result, but also transfer results in the direction of the important Moltó–Orihuela–Troyanski and Valvivia Lecture Notes. Due to the fact that we have now a quite good knowledge of the use of norming subsets in getting differentiability results, we think we may apply this to some Bishop–Phelps properties of operators between Banach spaces, and maybe also to the active field of Lipschitz–free spaces. The reason is that it is known that, under some mild requirements, the Lipschitz free space on a pointed metric space has a predual, and in many instances a 1-norming or just a norming subspace of its dual can be identified.

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