

Document downloaded from:

<http://hdl.handle.net/10251/155006>

This paper must be cited as:

Mendes De Jesus, C.; Sanabria-Codesal, E. (2018). Realization of graphs by fold Gauss maps. *Topology and its Applications*. 234:248-258.
<https://doi.org/10.1016/j.topol.2017.11.016>



The final publication is available at

<https://doi.org/10.1016/j.topol.2017.11.016>

Copyright Elsevier

Additional Information

Realization of graphs by fold Gauss maps

C. Mendes de Jesus¹ *and E. Sanabria-Codesal² †

¹ Departamento de Matemática, Universidade Federal de Viçosa, Brazil

² Departamento de Matemática Aplicada, Universitat Politècnica de València, Spain

e-mails: *cmendes@ufv.br and esanabri@mat.upv.es*

Abstract We define here a special type of bipartite graph, called 2-negative, and prove that any 2-negative graph with total weight equal to zero can be associated with some fold Gauss maps from a closed orientable surface.

AMS Classification: 57R45, 57R65.

Key words: stable Gauss maps; cusps of Gauss maps; graphs; parabolic curve; global classification.

1 Introduction

The Gauss map on a surface generically immersed in Euclidean 3-space is described in [2]. The singularities of a stable Gauss map, in Whitney's sense, being fold curves with isolated cusp points, are called the parabolic set of the surface. Each parabolic curve in this set separates a hyperbolic region from an elliptic region of the surface.

In order to study the global behavior of the Gauss maps it is useful to codify all the information relative to the topological type of the complement of the parabolic set on the surface in the simplest possible way. In [10], the authors introduce the study of graphs with weights associated with stable Gauss maps, where it has been shown that *any weighted bipartite graph can be associated to stable Gauss maps from appropriate closed orientable surfaces.*

In the particular case of the parabolic set of stable Gauss map having no cusp points, which is called a fold Gauss map, they also prove that the number of connected components to the parabolic curve (or equivalently the number of edges of the associated graph) is even. A natural question at this point is whether there is a special type of graph that can be associated to stable fold Gauss maps.

Our main objective here is to study the particular case of graphs with a total weight equal to zero. In Section 3, we introduce the definition of the 2-negative graph and in Section 5 we show that *a connected graph with V vertices, E edges and total weight equal to zero is a graph corresponding to a fold Gauss map of a closed orientable surface, with genus $E - V + 1$, if and only if it is a 2-negative graph.*

*Work partially supported by CNPq (200921/2012) and FAPEMIG (APQ-00715-14)

†Work partially supported by DGICYT grant n° MTM2015-64013-P

In order to prove this result we shall use an inductive constructive process starting from simple basic graphs of well known examples and then apply suitable codimension one transitions (see [3]) and surgeries (see [10]), that we describe in Section 4. The codimension one transitions of their singularities are determined through the study of the families of height functions associated to generic 1-parameter families of embeddings. The surgery of immersions on a closed surface consists in joining two elliptic regions by one hyperbolic region. By means of suitable combinations of surgeries and codimension one transitions, we can produce a required immersions associated to Gauss maps. A useful factor in this process is the existence of basic graphs with a total weight of zero (Figure 6) and one process of suitable manipulation of the immersions of a closed orientable surface in Euclidean 3-space that generates the fold Gauss maps associated to these graphs. These basic graphs correspond to fold Gauss maps from torus whose two parabolic curves separate a hyperbolic region from an elliptic disc.

2 Graphs of stable Gauss maps

Let M and P be smooth closed orientable surfaces and $f, g : M \rightarrow P$ smooth maps between them. The f and g maps are **\mathcal{A} -equivalent** (or **equivalent**) if there are orientation-preserving diffeomorphisms, l and k , such that $g \circ l = k \circ f$. A map $f : M \rightarrow N$ is said to be **stable** if all maps sufficiently close to f , in the Whitney C^∞ -topology (see [4]), are equivalent to f .

A point of the source surface M is a **regular** point of f if the map f is a local diffeomorphism around that point and **singular** otherwise. We denote by Σf the **singular set** of a map f and its image $f(\Sigma f)$ is the **branch set** of f .

The concept of stability for a Gauss maps of a surface immersed in \mathbb{R}^3 is slightly different from the general case of maps between surfaces in the sense that it must depend on perturbations of the immersion rather than on those of the map itself. Given an immersion $f : M \rightarrow \mathbb{R}^3$ of a closed orientable surface M in \mathbb{R}^3 , let $\mathcal{N}_f : M \rightarrow S^2$ be its Gauss map. This map \mathcal{N}_f is said to be **stable** if there exists a neighborhood \mathcal{U}_f of f in the space $\mathcal{I}(M, \mathbb{R}^3)$ of immersions of M into \mathbb{R}^3 such that for all $g \in \mathcal{U}_f$, the Gauss map associated to g \mathcal{N}_g is \mathcal{A} -equivalent to \mathcal{N}_f . It can be seen that this condition is equivalent to stating that the family of height functions associated to f :

$$\begin{aligned} \lambda(f) & : M \times S^2 \longrightarrow \mathbb{R} \\ (x, v) & \longmapsto \langle f(x), v \rangle = f_v(x) \end{aligned}$$

is structurally stable ([2], [11]).

Then we have that two Gauss maps are \mathcal{A} -equivalent if and only if their corresponding height functions (generating families) are \mathcal{R}^+ -equivalent ([1]). We observe that in the particular case of surfaces, the stable germs of Gauss maps correspond, geometrically, to the following situations:

- **Regular points of \mathcal{N}_f** : elliptic or hyperbolic points of M , i.e. points where the height function in the normal direction has a stable singularity of Morse type (A_1).

- **Singular points of \mathcal{N}_f** : parabolic points of M , i.e. points where the height function in the normal direction has a non stable singularity. In this case we may have: **fold point** of \mathcal{N}_f , corresponding to an A_2 singularity of the height function

in the normal direction or **cuspidal point** of \mathcal{N}_f , when the height function in the normal direction has a singularity of type A_3 .

So the singular set of \mathcal{N}_f ($\Sigma\mathcal{N}_f$) is the parabolic set of M associated to the immersion f . By Whitney's theorem (see [4]), the singular set of any stable smooth map between two closed orientable surfaces consists of curves of fold points, possibly containing isolated cuspidal points. Then, the image of $\Sigma\mathcal{N}_f$, called the branch set of \mathcal{N}_f , consists of a collection of closed curves immersed in S^2 with possible isolated cusps and self-intersections (double points) corresponding to parabolic points with parallel normals of the same orientation.

This branch set is oriented as follows: as we transverse a branch curve following the orientation, nearby points on our left have two more inverse images than those on our right. The non-singular set (which is immersed in the surface S^2 by the map \mathcal{N}_f) consists of a finite number of **regions**.

Given the orientations of the surfaces M and S^2 , a region is positive if the map preserves orientation and it is negative otherwise. We denote by M^+ (resp. M^-) the union of all the positive (resp. negative) regions including their boundaries. Clearly, M^+ and M^- meet in their common boundary, the singular set of \mathcal{N}_f , i.e. any singular curve of $\Sigma\mathcal{N}_f$ lies on the border of a component of M^+ and a component of M^- . Let us denote by E the number of connected components of $\Sigma\mathcal{N}_f$, by V^+ (resp. V^-) the number of connected components of M^+ (resp. M^-), by W^+ (resp. W^-) the total genus of M^+ (resp. M^-) and by $\chi(M^+)$ (resp. $\chi(M^-)$) the Euler characteristic of M^+ (resp. M^-).

The singular sets of two equivalent maps are equivalent in the sense that there is a diffeomorphism carrying one singular set onto the other and similarly for the branch sets. Thus any diffeomorphism invariant of singular sets or of branch sets will automatically be a topological invariant of the map. Clearly, both the number of connected components of the singular set and the topological types of the regions are topological invariants.

This information was coded in a weighted graph in [10], where the pair $(M, \Sigma\mathcal{N}_f)$ may be reconstructed (up to diffeomorphism) (see [7, 8, 9]).

In the weighted graph defined by a stable Gauss map \mathcal{N}_f the *edges* correspond to the path-components of the parabolic set of M and its *vertices* to the different regions of the surface with non vanishing Gaussian curvature. A *weight* is defined as the genus of the region that it represents and is attached to each vertex.

A vertex has a *positive* (or *negative*) label depending on whether the region that it represents has positive (or negative) Gaussian curvature.

We must remember that a graph is *bipartite* if its vertices can be divided into two disjoint sets (labeled positive and negative in our case) such that every edge connects vertices with opposite labels. Since M is orientable, each point of the parabolic set is in the frontier of a positive and a negative region, and consequently the corresponding graph is bipartite.

Notation: We denote this graph by $\mathcal{G}_W(V, E)$, where W is the total weight and V, E are the number of vertices and edges, respectively.

In Figure 1 we illustrate two stable Gauss map of the torus with their corresponding graphs: \mathcal{N}_f has the bipartite graph $\mathcal{G}_0(2, 2)$ and \mathcal{N}_h has the tree $\mathcal{G}_1(2, 1)$. The branch set of \mathcal{N}_f has two curves with 4 cuspidal points each one with alternate signs, nevertheless the branch set of \mathcal{N}_h has one curve with 6 cuspidal points (see [2]).

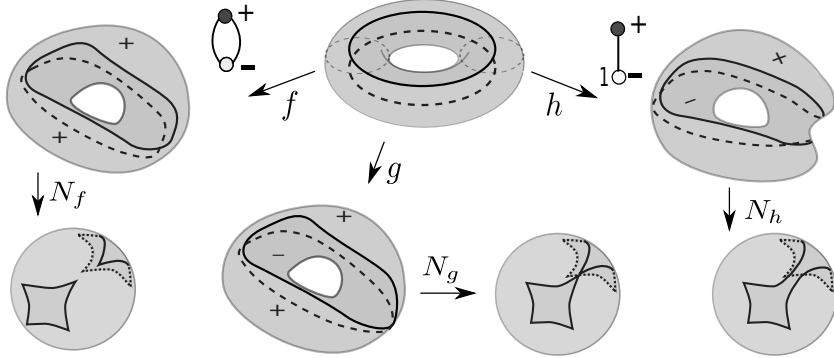


Figure 1: Example of beaks transition (B-transition).

Definition 2.1. A cusp point $x \in \Sigma\mathcal{N}$ is called *positive* (resp. *negative*) if its local mapping degree, in a neighborhood U_x of x is $+1$ (resp. -1) with respect to the given orientations. Let us denote by C^+ (resp. C^-) the number of positive cusp points (resp. negative cusp points) and by C the total number of cusps.

Definition 2.2. A **Gauss fold map** is a Gauss stable map with zero cusp points.

Definition 2.3. Let $\mathcal{G}_W(V, E)$ be a graph associated to a stable Gauss map $\mathcal{N} : M \rightarrow S^2$, where M is a closed orientable surface in \mathbb{R}^3 . By using the above notations we define by:

$$\theta_C = C^+ - C^-, \quad \theta_V = V^+ - V^-, \quad \theta_W = W^+ - W^-.$$

Definition 2.4. We said that a graph $\mathcal{G}_W(V, E)$ can be realized by a Gauss map of a surface M if and only if there exists an immersion $f : M \rightarrow \mathbb{R}^3$ whose Gauss map \mathcal{N}_f has $\mathcal{G}_W(V, E)$ as its associated graph.

The following results from stable Gauss maps were given in [10]:

Theorem 2.5 ([10]). Any weighted graph $\mathcal{G}_W(V, E)$ can be realized by a Gauss stable map of a closed orientable surface M in \mathbb{R}^3 , with $g(M) = 1 - V + E + W$. Any graph can be realized by a Gauss stable map of an embedded sphere if and only if it is a tree with weight zero at each vertex.

By applying the degree formulae ([5]): $d = 1 - g(M)$, where $g(M)$ is the genus of M , we have:

Proposition 2.6. ([10]) If $\mathcal{G}_W(V, E)$ is a graph of a stable Gauss map then

$$\theta_V - \theta_W = 1 - g(M) - \frac{\theta_C}{2}.$$

Consequently, $\theta_C = 4(V^- - W^-) - 2E$.

An immediate consequence of Proposition 2.6 is the following corollary:

Corollary 2.7. If $\mathcal{G}_W(V, E)$ is a graph of a fold Gauss map then $\theta_C = 0$ and

$$E = 2(V^- - W^-).$$

3 Special bipartite graphs: 2-negative

We pursue the necessary and sufficient conditions so that a graph can be associated with a fold Gauss map of some closed orientable surface in \mathbb{R}^3 . The Corollary 2.7 gives us a necessary condition for general graphs. We analyze in this section some special types of bipartite graphs, in order to find a sufficient condition at least for weight zero graphs.

Definition 3.1. A closed walk of a graph consists of a sequence of vertices and its corresponding adjacent edges starting and ending at the same vertex. A **cycle** is defined as a closed walk without repetitions of vertices. A graph without cycles is called a **tree**.

Remark 3.2. A set of **independent cycles** is any set of cycles in a graph such that each cycle contains at least one edge that does not belong to other cycles in this graph. The maximum number of independent cycles in a graph is known as the **first Betti number**. In the particular case of a connected graph $\mathcal{G}_W(V, E)$ the first Betti number is given by $\beta(G) = V - E + 1$.

Definition 3.3. The degree of a vertex v in a graph is the number of edges incident to it. A vertex v is said to be **extremal** if v has degree one.

Definition 3.4. A bipartite graph, labeled positive and negative in its vertices, without extremal vertices or all of which are positive is called a **positive graph** and is called **2-negative** if all its negative vertices have degree two.

Figure 2 shows three examples of 2-negative graphs: (a) tree, (b) bipartite graph with 13 cycles and (c) bipartite graph with 15 independent cycles.

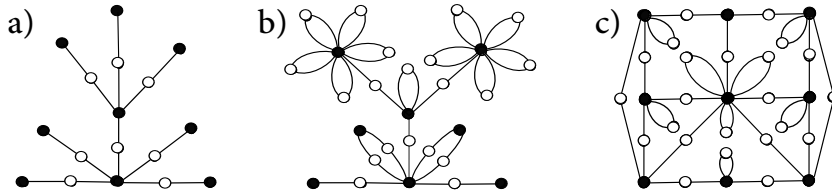


Figure 2: Examples of 2-negative graphs.

Below we analyze some of the properties of this type of graphs:

Lemma 3.5. A bipartite graph is 2-negative if and only if it is positive and $E = 2V^-$.

Proof. All the extremal vertices of a 2-negative bipartite graph are positive by definition. We will prove that any 2-negative bipartite graph satisfies the identity $E = 2V^-$ by induction on the number of negative vertices. If $k = 1$, the 2-negative bipartite graph has two possible configurations: one with two extremal positive vertices and the other with one positive vertex of degree two, and both of them have $E = 2$, then we have proved the base case. We now assume that the assertion is true for any 2-negative bipartite graph with k negative vertices and suppose that \mathcal{G} is a 2-negative bipartite graph with $k + 1$. By removing a negative vertex and the two corresponding edges, we obtain a 2-negative bipartite graph with k vertices

\mathcal{G}' , then $E(\mathcal{G}') = 2k$. We need to add two edges to construct a 2-negative bipartite graph with a $k+1$ negative vertex, then in this case $E = 2k+2$. Inversely, a positive bipartite graph with $E = 2V^-$ is 2-negative because none of its negative vertices has degree 1. \square

We observe that only one of the above conditions, i.e. to be positive or $E = 2V^-$, does not guarantee a 2-negative graph.

Theorem 3.6. *A connected bipartite graph is 2-negative if and only if it is positive and $\theta_V = 1 - \beta$, where β denotes the first Betti number of the graph.*

Proof. By using the definition of Betti number of a connected graph $\mathcal{G}_W(V, E)$ given by $V - E + 1 = \beta$ and applying the Lemma 3.5 we obtain that a connected bipartite graph is 2-negative if and only if it is positive and $\theta_V = 1 - \beta$. \square

4 Tools for constructing fold Gauss maps

In order to analyze the graph associated to fold Gauss maps, we shall use transitions in one-parameter families of height functions and surgeries in the immersions of the corresponding surface M associated to it.

4.1 Lips and Beaks Transitions

The generic transitions in one-parameter families of height functions (defined by generic one-parameter families of embeddings) together with their effects on the corresponding Gauss maps have been described in [3] both in the local and multi-local situation. According to this study, the local codimension 1 phenomena are the following:

1. Morse transitions of the parabolic curve at a non-versal A_3 . This corresponds to lips and beaks transitions in the Gauss map.
2. Birth/annihilation of a pair of cusps of the Gauss map on a smooth parabolic curve (at an A_4 point of the height function). This corresponds to a swallowtail type singularity in the Gauss map.
3. Cone sections at a D_4^\pm point of the height function (flat umbilic). Correspondingly, we have the purse and the pyramid transitions for the Gauss map.

Figure 1 illustrates a Morse transition that alters the cusp number. Here we are interested in those transitions affecting the graphs of the Gauss maps, namely, the beaks and the lips (see description below).

Let be M a closed orientable regular surface and \mathcal{N} its corresponding Gauss map. The **lips transition**, that we denote by L , corresponds to a Morse transition of maximum or minimum type in the parabolic curve. It may be done in a region of positive (or negative curvature, respectively) X of M , giving rise to a new region with negative (positive, respectively) curvature Z . Their common boundary is a connected component of the parabolic set whose image through the Gauss map is a closed curve with two cusp points in S^2 . The effect of this on the graph of \mathcal{N} is to add a new edge attached to the positive (negative, respectively) vertex corresponding to the initial region, now renamed X_1 (see Figure 3).

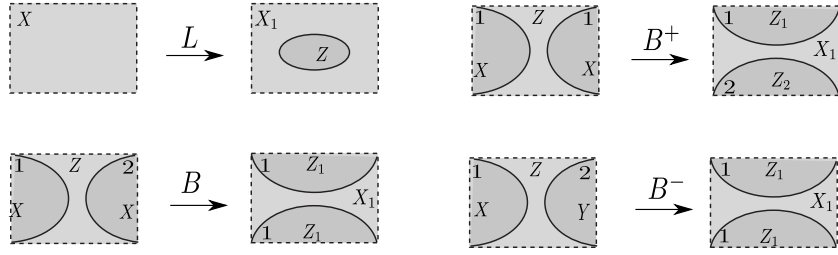


Figure 3: Lips and beaks transition.

Figure 4 below shows lips transitions on an embedding of the sphere with no parabolic points. The initial graph consists of a unique positive vertex, the second graph being made of an edge, corresponding to the newly created parabolic curve, attached to two vertices (one positive and one negative). The branch set of this second maps is a closed curve with two cusps.

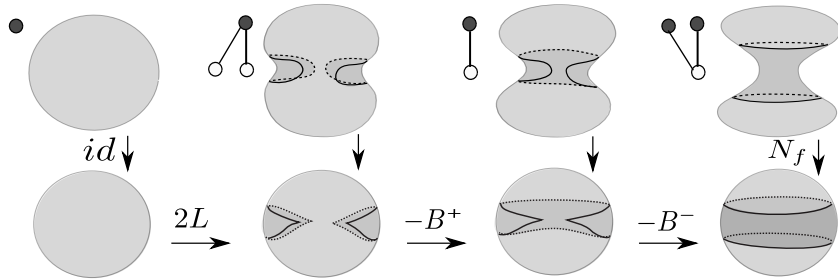


Figure 4: Realization of the graph $\mathcal{G}_0(3, 2)$.

The **beaks transitions** correspond to a Morse transition of saddle type in the parabolic set. Such a transition occurs when we approach two arcs of the parabolic set until they join in a common point *beaks point* and break again giving rise to a new pair of arcs and as a result, a couple of cusp points are introduced in the branch set. This process, in the sense of increasing the cusp points, can be separated into three different cases (see Figure 3): B^+ -transition increases by 1 the number of regular regions, i.e. adds a vertex and an edge on the graph of \mathcal{N} , B^- -transition decreases by 1 the number of regular regions, therefore removes 1 vertex and 1 edge on the graph and B -transition maintains the number of regular regions but increases the number of edges by 1 and also the weight by 1 .

Figure 4 illustrates beaks transitions (B^+ , B^-) in a surface, starting with an elliptical and two hyperbolic regions and ending up with two elliptical regions and a hyperbolic region. It also shows the effect of this transition on the graph.

Remark 4.1. If $\mathcal{G}_0(V, E)$ is a graph of a fold Gauss map with weight zero in all vertices, by using the Corollary 2.7 we know that $E = 2V^-$, or equivalently $\theta_V = 1 - \beta$, with β the first Betti number of $\mathcal{G}_0(V, E)$, and also the graph is positive (by lips transitions).

Then by using the Theorem 3.6 we have that

Theorem 4.2. *Any graph with weight zero at each vertex associated to a fold Gauss map of a closed and orientable surface is 2-negative.*

4.2 Surgeries of fold Gauss maps

We shall now describe the surgery on the immersions, as defined in [10], that alter the singular set of their Gauss maps, and hence their graphs, in suitable ways.

The *Surgery* on one closed surface M consists of joining two elliptic regions of M with an intermediary hyperbolic region. This process is carried out by removing two discs, one in each elliptic region and connecting a hyperbolic tube to their boundaries and clearly this process can be done smoothly.

Remark 4.3. *We observe that a lip transition L followed by a beak transition $-B^-$ adds one hyperbolic region and two connected components to the parabolic curve without cusp points, so that this is equivalent to applying surgery. To join two elliptic regions, we can then use surgery (i.e. adding a hyperbolic region between the elliptic regions). To remove this negative region, it is enough to apply a beak transition B^- which joins the singular curves and adds two cusp points, and then using a lips transition $-L$ to remove these cusp points.*

Notation: We denote the surgery as H_1 when it connects two elliptic regions of the same connected surface and as H_0 when it connects two elliptic regions of disjoint connected surfaces.

The surgery adds two connected components but no cusp points to the parabolic curve. This process is illustrated in Figure 5, which also shows the effect of these surgeries on the graphs.

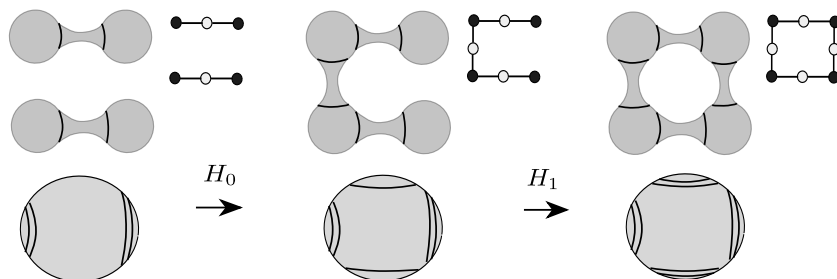


Figure 5: Surgeries H_0 and H_1 of fold Gauss maps.

We will now show how the 2-negative graph $\mathcal{G}_0(2, 2)$ realizes through an embedded torus which has a fold Gauss map (see Figure 6), where the map h indicates the composition of all shown transitions:

- a) We start by making two surgeries (H_0 and H_1) to two ellipsoids. This introduces four new parabolic curves, which are the boundary of the two new hyperbolic regions, and leads to a 2-negative graph $\mathcal{G}_0(4, 4)$ (see Figure 6 (a)).
- b) By applying a beak transition B^- which joins two singular curves and two hyperbolic regions, we add two cusp points in this case.

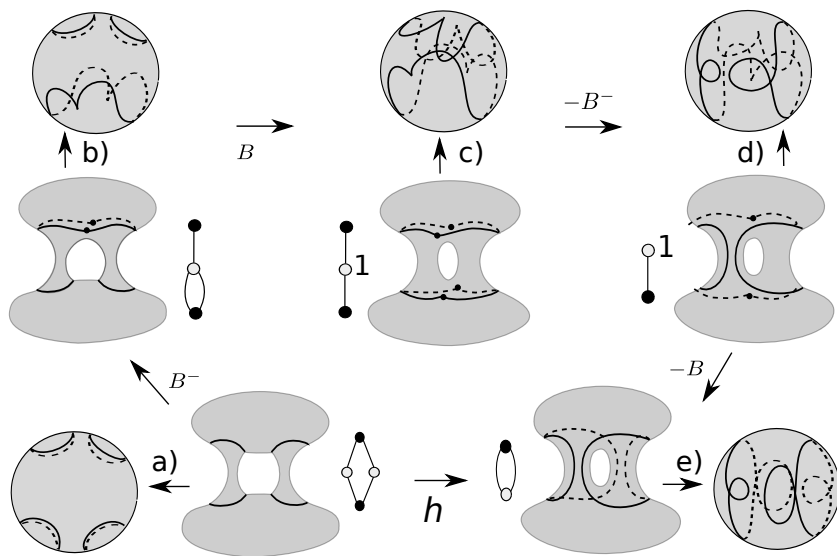


Figure 6: Realization of the graph $\mathcal{G}_0(2, 2)$ by a fold Gauss map of a torus.

- c) Again applying a beak transition B , we add two cusp points, join two singular curves and introduce genus one to the hyperbolic region, and thus obtain a torus with one hyperbolic region and two simple regions (homeomorphic to the disc) with positive curvature.
- d) We approach two cusp points through the hyperbolic region, one of each component of the parabolic curve, and apply a $-B^-$ transition to remove them.
- e) By one $-B$ transition we remove the genus of the hyperbolic region, transforming the positive and negative region into a cylinder.

Figure 7 show four examples of 2-negative graphs realized by fold Gauss maps of closed orientable surfaces:

- a) We consider the embedded sphere in \mathbb{R}^3 , by applying two transitions as in Figure 4, and obtain a parabolic set with 4 regular curves.
- b) The double torus with 8 singular curves, can be obtained by two H_1 surgeries on the embedding (a).
- c) This closed orientable surface of genus 4, can be obtained by two H_0 surgeries between the embedding of type (b) and Figure 6 (e), adding two new hyperbolic regions and 4 singular curves.
- d) This last embedding can be obtained by two B^- beaks in (c), joining two singular curves and adding two cup points, and two $-L$ lip transitions, eliminating two singular curves and its cusp points (reverse path Figure 4, removing the two hyperbolic regions inserted by the H_0 surgeries).

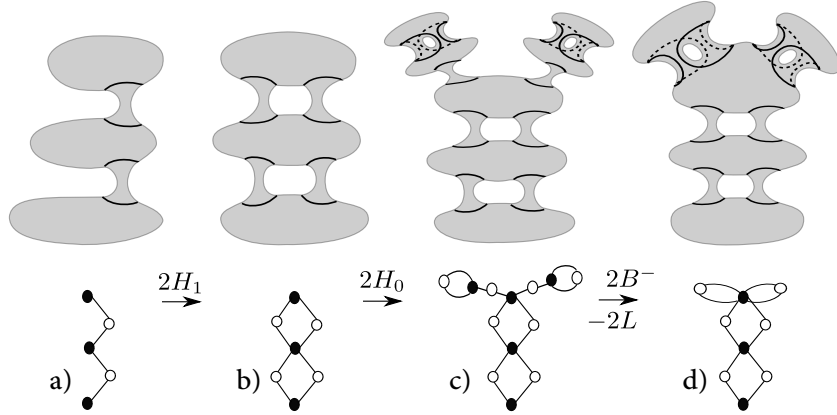


Figure 7: Examples of graphs realized by different Gauss fold maps.

5 Realization of weight zero graph by fold Gauss map

This section contains an analysis of the reverse of Theorem 4.2, i.e. the existence of some immersed surface whose fold Gauss map has a prefixed 2-negative graph of weight zero at each vertex. We will start by considering the particular case of trees.

Theorem 5.1. *Any 2-negative tree with weight zero at each vertex can be realized by a fold Gauss map on an embedded sphere.*

Proof. If the tree has a unique vertex (0 negative vertices), obviously it can be realized by the identity Gauss map associated to the standard embedding of S^2 in \mathbb{R}^3 . When $V = 3$ (2-negative tree with 1 negative vertex), this graph can be realized by using a H_0 surgery between two spheres embedded in \mathbb{R}^3 . We observe that the H_0 surgery between a surface associated to a fold Gauss map and the embedded S^2 associated to the identity Gauss map add two vertices to the original graph (with the corresponding edges), one negative of degree 2 (corresponding to the hyperbolic tube) and one extremal (corresponding to the S^2). So we assume that the assertion is true for any 2-negative tree with weight zero and at least k negative vertices and we assume that \mathcal{A} is a 2-negative tree with weight zero and $k + 1$ negative vertices. By removing one negative vertex joined to an extremal vertex (and the corresponding edges), we obtain a 2-negative tree \mathcal{A}' with k negative vertices and by using the induction hypothesis we know that \mathcal{A}' is realized by a fold Gauss map $N_1 : S^2 \rightarrow S^2$. Then by using the H_0 surgery between the sphere associated to the fold Gauss map N_1 and the S^2 associated to the identity Gauss map we obtain a new embedding $f : S^2 \rightarrow \mathbb{R}^3$ whose fold Gauss map $N : S^2 \rightarrow S^2$ corresponds to the graph \mathcal{A} . \square

Lemma 5.2. *Any 2-negative graph with total weight equal to zero such that its cycles have at least 4 edges, can be realized by a fold Gauss map of a closed orientable surface M , with $g(M) = \beta$, where β is the Betti number of the graph.*

Proof. Given a 2-negative graph $\mathcal{G}_0(V, E)$ with a maximum of c independent cycles, such that its cycles have at least 4 edges, we remove one negative vertex (degree 2) of each of these c cycles and thus we obtain a 2-negative tree \mathcal{A} with $V - c$ vertices. Then, by using Theorem 5.1, \mathcal{A} can be realized by a fold Gauss map $N_1 : S^2 \rightarrow S^2$ and the graph $\mathcal{G}_0(V, E)$ by a fold Gauss map of a closed orientable surface M , obtained by applying c surgeries, of type H_1 , to the elliptical regions corresponding to the positive vertex of \mathcal{A} , which realize the c independent cycles. Then we obtain that $g(M) = c = \beta$. \square

Theorem 5.3. *Any 2-negative graph with total weight equal to zero can be realized by a fold Gauss map of a closed orientable surface M , with $g(M) = \beta$, where β is the Betti number of the graph.*

Proof. Given a 2-negative graph $\mathcal{G}_0(V, E)$, we assume that this graph has r cycles with only two edges. By removing a negative vertex (degree 2) of each of these r cycles, we obtain a 4-bipartite 2-negative graph $\mathcal{G}'_0(V - r, E - 2r)$. By using Lemma 5.2, this graph can be realized by a fold Gauss map $N_1 : M_1 \rightarrow S^2$ from the closed orientable surface M_1 where $g(M_1) = 1 - V + E - r$. By applying r surgeries, of type H_0 , between the surface M_1 and the torus associated to the graph $\mathcal{G}_0(2, 2)$ (see Figure 6 (c)), we obtain a new immersion $f_2 : M_2 \rightarrow \mathbb{R}^3$ associated to a fold Gauss map $N_2 : M_2 \rightarrow S^2$, where M has genus $1 - V + E$ and the associated graph is $\mathcal{G}_0(V + c, E + 2c)$ (see Figure 7 (c)). Finally, to obtain the immersion $f : M \rightarrow \mathbb{R}^3$ associated to a fold Gauss maps $N : M \rightarrow S^2$ associated to $\mathcal{G}_0(V, E)$, we can remove the r hyperbolic regions (introduced to make the r surgeries) with B^- beak transitions which join two singular curves and $-L$ transitions which remove the hyperbolic region and the singular curve with 2 cusp point in it (reverse path Figure 4). \square

Finally, by combining the Theorems 4.2 and 5.3, we obtain:

Theorem 5.4. *$\mathcal{G}_0(V, E)$ is a graph corresponding to a fold Gauss map of a closed orientable surface if and only if $\mathcal{G}_0(V, E)$ is a 2-negative graph.*

References

- [1] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of differentiable maps. Vol. I. The classification of critical points, caustics and wave fronts, *Translated from the Russian by Ian Porteous and Mark Reynolds*. *Monographs in Mathematics*, 82. Birkhuser Boston, Inc., Boston, MA, 1985.
- [2] T. Banchoff, T. Gaffney and C. McCrory, *Cusps of Gauss Mappings*, *Pitman Books Limited*, London, (1982). Web version with D. Dreibelbis www.math.brown.edu/~dan/cgm/index.html
- [3] J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: height functions, Gauss maps and duals. *Real and Complex Singularities*, ed. W. L. Marar, *Pitman Research Notes in Mathematics*, 333:148–178, 1995.
- [4] M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Springer Verlag, Berlin (1976).

- [5] V. Guillemin and A. Pollack, *Differential Topology*, Prentice-Hall, New Jersey (1974).
- [6] D. Hacon , C. Mendes de Jesus and M. C. Romero Fuster, Graphs of stable maps from closed orientable surfaces to the 2-sphere. *J. Sing.* 2: 67–80, 2010. doi:10.5427/jsing.2010.2e.
- [7] D. Hacon , C. Mendes de Jesus and M. C. Romero Fuster, Stable maps from surfaces to the plane with prescribed branching data *Topology and Its Appl.* 154 (1): 166–175, 2007. doi:http://dx.doi.org/10.1016/j.topol.2006.04.005.
- [8] D. Hacon , C. Mendes de Jesus and M. C. Romero Fuster, Topological invariants of stable maps from a surface to the plane from a global viewpoint. *Real and Complex Singularities*. Informa UK Limited, 2003. doi:10.1201/9780203912089.ch10.
- [9] C. Mendes de Jesus, Graphs of stable maps between closed orientable surfaces, *Comp. Appl. Math.*, 2016. DOI :10.1007/s40314-016-0317-9.
- [10] C. Mendes de Jesus, S. M. de Moraes and M. C. Romero Fuster, Stable Gauss maps on surfaces from a global viewpoint, *Bulletin of the Brazilian Mathematical Society*, 42: 87-103(2011). DOI: 10.1007/s00574-011-0005-8
- [11] M.C. Romero Fuster, Sphere stratifications and the Gauss map. *Proceedings of the Royal Society of Edinburgh*, 95A:115–136, 1983.