Strong transitivity properties for operators

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Abstract

Given a Furstenberg family $\mathscr{F}$ of subsets of $\mathbb{N}$, an operator T on a topological vector space X is called $\mathscr{F}$-transitive provided for each non-empty open subsets U, V of X the set $\{n \in \mathbb{Z}_+ : T^n(U) \cap V \neq \emptyset\}$ belongs to $\mathscr{F}$. We classify the topologically transitive operators with a hierarchy of $\mathscr{F}$-transitive subclasses by considering families $\mathscr{F}$ that are determined by various notions of largeness and density in $\mathbb{Z}_+$.

1 Introduction

Throughout this paper X denotes a topological space and $\mathcal{U}(X)$ the set of non-empty open subsets of X. When X is a topological vector space, $\mathcal{L}(X)$ stands for the set of operators (i..e, linear and continuous self-maps) on X. An operator $T \in \mathcal{L}(X)$ is called *hypercyclic* if there exists a vector $x \in X$ such that for each V in $\mathcal{U}(X)$ the time return set

$$N_T(x, V) = N(x, V) := \{n \geq 0 : T^n x \in V\}$$

is non-empty, or equivalently (since X has no isolated points) an infinite set. When X is an F-space (that is, a complete and metrizable topological vector space), we know thanks to Birkhoff's transitivity theorem that T is hypercyclic if and only if it is *topologically transitive*, that is, provided

$$N_T(U, V) = N(U, V) := \{n \geq 0 : T^n(U) \cap V \neq \emptyset\}$$

is infinite for every $U, V \in \mathcal{U}(X)$.

Since 2004, several refined notions of hypercyclicity based on the properties of time return sets $N(x, V)$ have been investigated: frequent hypercyclicity [3, 2], $\mathcal{U}$-frequent hypercyclicity [21, 9], reiterative hypercyclicity

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[7]. More recently a general notion called $\mathcal{A}$-hypercyclicity, which generalizes the abovementioned notions of hypercyclicity, has been used to investigate the different types of hypercyclic operators, see [7, 9].

Our aim here is to investigate refined notions of topological transitivity based on properties satisfied by the return sets $N(U, V)$. Some of these are already well-known, such as the topological notions of *mixing*, *weak-mixing*, and *ergodicity*, say. Recall that a continuous self-map T on X is called *mixing* provided $N(U, V)$ is cofinite for each $U, V \in \mathcal{U}(X)$. Also, T is called *weakly mixing* whenever $T \times T$ is topologically transitive on $X \times X$, and this occurs precisely when the return set $N(U, V)$ is *thick* (i.e. contains arbitrarily long intervals) for each $U, V \in \mathcal{U}(X)$ [19]. Finally, T is *topologically ergodic* provided $N(U, V)$ is syndetic (i.e. has bounded gaps) for each $U, V \in \mathcal{U}(X)$. It is known that topologically ergodic operators are weakly mixing [14]. The above mentioned notions may be stated through the concept of a (Furstenberg) family. The symbols $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the sets of integers and of positive integers, respectively.

Definition 1.1. We say that a non-empty collection $\mathscr{F}$ of subsets of $\mathbb{Z}_+$ is a *family* provided that each set $A \in \mathscr{F}$ is infinite and that $\mathscr{F}$ is hereditarily upward (i.e. for any $A \in \mathscr{F}$, if $B \supset A$ then $B \in \mathscr{F}$). The *dual* family $\mathscr{F}^*$ of $\mathscr{F}$ is defined as the collection of subsets A of $\mathbb{Z}_+$ such that $A \cap B \neq \emptyset$ for every $B \in \mathscr{F}$.

Some standard families are the following: The family $\mathcal{I}$ of infinite sets, whose dual family $\mathcal{I}^*$ coincides with the family of cofinite sets. The family $\mathcal{T}$ of thick sets, whose dual family is $\mathcal{S} = \mathcal{T}^*$, the family of syndetic sets. For a topologically transitive map T a distinguished family is

$$\mathcal{N}_T := \{A \subset \mathbb{Z}_+ : N_T(U, V) \subseteq A \text{ for some } U, V \in \mathcal{U}(X)\}.$$

From now on the symbol $\mathscr{F}$ will always denote a family.

Definition 1.2. We say that a continuous map T on X is *$\mathscr{F}$-transitive* (or an $\mathscr{F}$-map, for short) provided $\mathcal{N}_T \subset \mathscr{F}$, that is, provided $N(U, V) \in \mathscr{F}$ for each $U, V \in \mathcal{U}(X)$. If in addition X is a topological vector space and $T \in \mathcal{L}(X)$ we call T an *$\mathscr{F}$-transitive operator* (or $\mathscr{F}$-operator for short).

Hence the $\mathcal{I}$-operators are precisely those operators which are topologically transitive, and the $\mathcal{I}^*$-operators and $\mathcal{T}$-operators are precisely those which are mixing and weak mixing, respectively. The $\mathcal{T}^* = \mathcal{S}$-operators, that is, the topologically ergodic operators.

We present here some new classes of topologically transitive operators by considering families $\mathscr{F}$ defined in terms of various notions of density and largeness in $\mathbb{Z}_+$. A hierarchy of fourteen classes (which include the earlier mentioned classes defined by properties of return sets $N(x, V)$) appears in Figure 2 and summarizes our findings. We stress that while trivially any $\mathscr{F}_1$-map is an $\mathscr{F}_2$-map when $\mathscr{F}_1 \subset \mathscr{F}_2$, it is possible that the classes of

$\mathscr{F}_1$-operators and of $\mathscr{F}_2$-operators coincide even if $\mathscr{F}_1$ is strictly contained in $\mathscr{F}_2$ (see e.g., Proposition 5.1).

The paper is organized as follows. In Section 2 we describe some general facts about families $\mathscr{F}$ and their corresponding $\mathscr{F}$-transitive maps and operators. In Theorem 2.4 we provide an extension of the Hypercyclicity Criterion that ensures an operator to be $\mathscr{F}$-transitive. We apply this criterion in Section 3 to characterize $\mathscr{F}$-transitivity among unilateral and bilateral weighted backward shift operators on c_0 and ℓ_p $(1 \leq p < \infty)$ spaces. To illustrate, we establish in Corollary 3.4 that a unilateral backward shift B_w is topologically ergodic precisely when its weight sequence $w = (w_n)_n$ satisfies that each set

$$A_M = \{n : |\prod_{j=1}^{n} w_j| > M\} \quad (M > 0)$$

is syndetic. Section 4 is dedicated to $\mathscr{F}$-operators induced by families $\mathscr{F}$ given by sets of positive or full (lower or upper) asymptotic density or Banach density. In Section 5, we look at $\mathscr{F}$-operators induced by families $\mathscr{F}$ commonly used in Ramsey theory, and we compare the classes that we obtain with the class of reiteratively hypercyclic operators (Subsection 5.1). Some natural questions conclude the paper.

2 $\mathscr{F}$-Transitivity

In this section we introduce a sufficient condition for an operator to be an $\mathscr{F}$-operator, which we call the $\mathscr{F}$-Transitivity Criterion, and it is in the same vein of the Hypercyclicity Criterion. Moreover, we will study the notion of hereditarily $\mathscr{F}$-operator.

We will be interested in the following three special properties a family $\mathscr{F}$ can have: being a *filter*, being *partition-regular*, and being *shift-invariant*. We use the following notation: given two families $\mathscr{F}_1$ and $\mathscr{F}_2$

$$\mathscr{F}_1 \cdot \mathscr{F}_2 := \{A \cap B : A \in \mathscr{F}_1, \quad B \in \mathscr{F}_2\}.$$

Obviously, $\mathscr{F}_1 \subset \mathscr{F}_1 \cdot \mathscr{F}_2$ and $\mathscr{F}_2 \subset \mathscr{F}_1 \cdot \mathscr{F}_2$. A family $\mathscr{F}$ is a *filter* provided it is invariant under finite intersections (i.e., provided $\mathscr{F} \cdot \mathscr{F} \subset \mathscr{F}$). Say, the family $\mathcal{I}^*$ of cofinite sets is a filter while the families $\mathcal{I}$ and $\mathcal{S}$ of infinite sets and of syndetic sets are not.

The second property, that of being partition regular, will be useful for us to identify filters. A family $\mathscr{F}$ on $\mathbb{Z}_+$ is said to be *partition regular* if for every $A \in \mathscr{F}$ and any finite partition $\{A_1, \ldots, A_n\}$ of A, there exists some $i = 1, \ldots, n$ such that $A_i \in \mathscr{F}$. The family $\mathcal{I}$ is an example of partition regular family, while the families $\mathcal{I}^*$, $\mathcal{T}$ and $\mathcal{S}$ are not. Later we will see other examples of partition regular families: the family of piecewise syndetic sets (see Remark 2.5), the family of sets with positive upper (Banach) density (see Section 4), the families of Δ-sets and of $\mathcal{IP}$-sets (see Section 5).

Lemma 2.1. *Given a family $\mathscr{F}$, the following are equivalent:*

(I) $\mathscr{F}$ is partition regular,

(II) $A \cap A' \in \mathscr{F}$ for every $A \in \mathscr{F}$ and $A' \in \mathscr{F}^*$ (i.e., $\mathscr{F} \cdot \mathscr{F}^* \subset \mathscr{F}$),

(III) $\mathscr{F}^*$ is a filter.

Proof. (I) $\Longrightarrow$ (II): Given $A \in \mathscr{F}$ and $A' \in \mathscr{F}^*$ it is clear that $A \cap A' \neq \emptyset$ by definition of dual family. Since $(A \cap A') \cup (A \setminus A') = A$, either $A \cap A' \in \mathscr{F}$ or $A \setminus A' \in \mathscr{F}$ by (I). Since $(A \setminus A') \cap A' = \emptyset$, by definition of dual family we necessarily have $A \cap A' \in \mathscr{F}$.

(II) $\Longrightarrow$ (III): For arbitrary $A', B' \in \mathscr{F}^*$ and $A \in \mathscr{F}$, by applying (II) and the definition of dual family we have $A \cap (A' \cap B') = (A \cap A') \cap B' \neq \emptyset$, which yields that $\mathscr{F}^*$ is a filter.

(III) $\Longrightarrow$ (I): We will just show that, given $A \in \mathscr{F}$ and $A_1, A_2 \subset \mathbb{Z}_+$ such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \emptyset$, then either $A_1 \in \mathscr{F}$ or $A_2 \in \mathscr{F}$. The general case can be deduced by an inductive process. Since $\mathscr{F} = \mathscr{F}^{**}$, we need to show that $A_i \cap A' \neq \emptyset$ for every $A' \in \mathscr{F}^*$, for $i = 1$ or $i = 2$. Suppose that there exist $A', B' \in \mathscr{F}^*$ with $A_1 \cap A' = \emptyset$ and $A_2 \cap B' = \emptyset$. Since $\mathscr{F}^*$ is a filter, then $C' := A' \cap B' \in \mathscr{F}^*$. Thus,

$$\emptyset \neq A \cap C' \subset (A_1 \cap A') \cup (A_2 \cap B') = \emptyset,$$

which is a contradiction. □

Notice that $(\mathscr{F}^*)^* = \mathscr{F}$ for any family $\mathscr{F}$: the inclusion $\mathscr{F} \subset (\mathscr{F}^*)^*$ is immediate. Conversely, if $A \in (\mathscr{F}^*)^*$, then $\mathbb{Z}_+ \setminus A \notin \mathscr{F}^*$ by the definition of a dual family. This means that there exists $B \in \mathscr{F}$ such that $B \cap (\mathbb{Z}_+ \setminus A) = \emptyset$. That is, $B \subset A$, which gives $A \in \mathscr{F}$.

Thus any family is a dual family, and Lemma 2.1 also gives that a family $\mathscr{F}$ is a filter if and only if $\mathscr{F}^* \cdot \mathscr{F} \subset \mathscr{F}^*$ and if and only if $\mathscr{F}^*$ is partition regular. Another consequence of Lemma 2.1 is that any family $\mathscr{F}$ that is both a filter and partition regular (called an *ultrafilter*) must satisfy $\mathscr{F} = \mathscr{F}^*$.

Finally, our third property: A family $\mathscr{F}$ on $\mathbb{Z}_+$ is said to be *shift$_-$-invariant* provided for every $i \in \mathbb{Z}_+$ and each $A \in \mathscr{F}$, we have $(A - i) \cap \mathbb{Z}_+ \in \mathscr{F}$. We say that $\mathscr{F}$ is called *shift$_+$-invariant* if for every $i \in \mathbb{Z}_+$ and each $A \in \mathscr{F}$, we have $A + i \in \mathscr{F}$. When $\mathscr{F}$ is both, shift$_-$-invariant and shift$_+$-invariant, we simply call it *shift invariant*. For instance, the families of infinite sets, cofinite sets, thick sets and syndetic sets are shift invariant.

We may gain shift invariance by reducing a family. Given a family $\mathscr{F}$, we define

$$\widetilde{\mathscr{F}}_+ = \{A \subset \mathbb{Z}_+ \ : \ \forall N \in \mathbb{Z}_+ \ \exists B \in \mathscr{F} \text{ such that } A \supset B + [0, N]\},$$

$$\widetilde{\mathscr{F}}_- = \{A \subset \mathbb{Z}_+ \ : \ \forall N \in \mathbb{Z}_+ \ \exists B \in \mathscr{F} \text{ such that } A \supset (B + [-N, 0]) \cap \mathbb{Z}_+\},$$

$$\widetilde{\mathscr{F}} = \{A \subset \mathbb{Z}_+ \ : \ \forall N \in \mathbb{Z}_+ \ \exists B \in \mathscr{F} \text{ such that } A \supset (B + [-N, N]) \cap \mathbb{Z}_+\}.$$

So for any family $\mathscr{F}$ we have the inclusions $\widetilde{\mathscr{F}} \subset \widetilde{\mathscr{F}}_+ \subset \mathscr{F}$ and $\widetilde{\mathscr{F}} \subset \widetilde{\mathscr{F}}_- \subset \mathscr{F}$, and that $\widetilde{\mathscr{F}}_-$ is shift$_+$-invariant, $\widetilde{\mathscr{F}}_+$ is shift$_-$-invariant, and $\widetilde{\mathscr{F}}$ is shift invariant.

Lemma 2.2. *If $\mathscr{F}$ is a filter on $\mathbb{Z}_+$, so is $\widetilde{\mathscr{F}}$. Moreover, for any family $\mathscr{F}$ satisfying $\widetilde{\mathscr{F}} \cdot \widetilde{\mathscr{F}} \subset \mathscr{F}$ the subfamily $\widetilde{\mathscr{F}}$ is a filter.*

Proof. Let $A_1, A_2 \in \widetilde{\mathscr{F}}$. We have to show that $A_1 \cap A_2 \in \widetilde{\mathscr{F}}$. Given $N \in \mathbb{N}$, there are $B_1(N), B_2(N) \in \mathscr{F}$ such that $(B_1(N) + [-2N, 2N]) \cap \mathbb{Z}_+ \subset A_1$ and $(B_2(N) + [-2N, 2N]) \cap \mathbb{Z}_+ \subset A_2$. For $i = 1, 2$ we define

$$\bar{A}_i(N) := \bigcup_{J \geq N} (B_i(J) + [-J, J]) \cap \mathbb{Z}_+.$$

Clearly $\bar{A}_1(N), \bar{A}_2(N) \in \widetilde{\mathscr{F}}$ for each $N \in \mathbb{N}$. By hypothesis, $B(N) := \bar{A}_1(N) \cap \bar{A}_2(N) \in \mathscr{F}$, $N \in \mathbb{N}$. To prove that $A_1 \cap A_2 \in \widetilde{\mathscr{F}}$ we just need to show that $(B(N) + [-N, N]) \cap \mathbb{Z}_+ \subset A_1 \cap A_2$ for every $N \in \mathbb{N}$. Indeed, given $N \in \mathbb{N}$ and $m \in (B(N) + [-N, N]) \cap \mathbb{Z}_+$, we write $m = k(N) + l(N)$ with $k(N) \in B(N)$ and $l(N) \in [-N, N]$. By definition of $B(N)$ we have

$$k(N) = k_1(J_1) + l_1(J_1) = k_2(J_2) + l_2(J_2)$$
$$\text{for some } k_i(J_i) \in B_i(J_i), \ \ l_i(J_i) \in [-J_i, J_i], \ J_i \geq N, \ i = 1, 2.$$

Thus

$$m = k_1(J_1) + l_1(J_1) + l(N) \in (B_1(J_1) + [-2J_1, 2J_1]) \cap \mathbb{Z}_+ \subset A_1,$$

and, analogously, $m \in A_2$, which yields the result. □

The rest of the section is dedicated to $\mathscr{F}$-maps and $\mathscr{F}$-operators. Every $\widetilde{\mathscr{F}}$-map is an $\mathscr{F}$-map, since $\widetilde{\mathscr{F}} \subset \mathscr{F}$. The next lemma gives conditions for the converse, and is used in Proposition 3.1.

Lemma 2.3. *Let $\mathscr{F}$ be a family on $\mathbb{Z}_+$ and let T be a $\mathscr{F}$-map. The following are equivalent.*

(i) T is weakly mixing,

(ii) T is an $\widetilde{\mathscr{F}}$-map.

Proof. (i) implies (ii): Given $N \in \mathbb{N}$ and $U, V \in \mathfrak{U}(X)$, since T is weakly mixing, by Furstenberg result we know that $\mathcal{N}_T$ is a filter, so there are $U', V' \in \mathfrak{U}(X)$ such that

$$N(U', V') \subset N(T^{-m}(U), V) \cap N(U, T^{-m}(V)),$$

for $m = 0, \dots, N$. By $\mathscr{F}$-transitivity we have $N(U', V') \in \mathscr{F}$. We then conclude that $(N(U', V') + [-N, N]) \cap \mathbb{Z}_+ \subset N(U, V)$, and T is $\widetilde{\mathscr{F}}$-transitive.

(ii) implies (i): If T is an $\widetilde{\mathscr{F}}$-map, since every element of $\widetilde{\mathscr{F}}$ is thick, we have that $\mathcal{N}_T$ consists of thick sets and, as we already recalled in the introduction, this means that T is weakly mixing. □

To state the $\mathscr{F}$-Transitivity Criterion, we recall the notion of *limit along a family* $\mathscr{F}$: Given a sequence $\{x_n\}_n$ in X and $x \in X$, we say that

$$\mathscr{F} - \lim_n x_n = x, \text{ or that } x_n \overset{\mathscr{F}}{\to} x,$$

provided $\{n \in \mathbb{Z}_+ : x_n \in U\} \in \mathscr{F}$ for each neighbourhood U of x.

Theorem 2.4. ($\mathscr{F}$-Transitivity Criterion) *Let T be an operator on a topological vector space X and let $\mathscr{F}$ be a family on $\mathbb{Z}_+$ such that $\widetilde{\mathscr{F}}$ is a filter. Suppose there exist D_1, D_2 dense sets in X, and (possibly discontinuous) mappings $S_n : D_2 \to X$, $n \in \mathbb{N}$ satisfying*

(a) $\mathscr{F}$-$\lim_n T^n(x) = 0$ *for every* $x \in D_1$

(b) $\mathscr{F}$-$\lim_n (S_n(y), T^n S_n(y)) = (0, y)$ *for every* $y \in D_2$.

Then T is an $\widetilde{\mathscr{F}}$-operator.

Proof. Let $U, V \in \mathcal{U}(X)$. We fix $U', V' \in \mathcal{U}(X)$ and a 0-neighbourhood W such that $U' + W \subset U$ and $V' + W \subset V$. Given $N \in \mathbb{N}$, pick $x \in D_1 \cap T^{-N}U'$ and $y \in D_2 \cap T^{-N}V'$. By continuity of T we easily get

$$\widetilde{\mathscr{F}_+} - \lim_n T^n x = 0,$$

which yields $N(T^{-N}U', W) \in \widetilde{\mathscr{F}_+}$. That is, there is $A \in \mathscr{F}$ such that $A + [0, 2N] \subset N(T^{-N}U', W)$. Therefore,

$$(A + [-N, N]) \cap \mathbb{Z}_+ \subset (N(T^{-N}U', W) - N) \cap \mathbb{Z}_+ \subset N(U', W),$$

and, since N was arbitrary, we have that $N(U', W) \in \widetilde{\mathscr{F}}$.

Also, we find a 0-neighbourhood $W' \subset W$ with $T^m(W') \subset W$ and $y + W' \subset T^{-N}V'$, $m = 0, \dots, 2N$. There is $A \in \mathscr{F}$ such that $S_n y \in W'$ and $T^n S_n(y) \in y + W'$ for all $n \in A$. Thus,

$$\big(T^{(n-m)}(T^m S_n(y)), T^m S_n(y)\big) \in (y + W', T^m(W')) \subset (T^{-N}V', W),$$

for $m = 0, \dots, 2N$ and for every $n \in A$. In particular, $(A + [-N, N]) \cap \mathbb{Z}_+ \subset N(W, V')$. Since N was arbitrary, we obtain that $N(W, V') \in \widetilde{\mathscr{F}}$. Therefore,

$$N(U, V) \supset N(U' + W, V' + W) \supset N(U', W) \cap N(W, V') \in \widetilde{\mathscr{F}} \cdot \widetilde{\mathscr{F}} \subset \widetilde{\mathscr{F}},$$

that is, T is an $\widetilde{\mathscr{F}}$-operator. □

Remark 2.5. 1. By Lemma 2.2 the assumption that $\widetilde{\mathscr{F}}$ be a filter is trivially satisfied in the case that $\mathscr{F}$ is a filter, but Theorem 2.4 applies beyond this case. For instance, the family $\mathscr{F} = \mathcal{S}$ of syndetic sets is not a filter, and $\widetilde{\mathcal{S}} = \mathcal{TS}$ is the family of *thickly syndetic sets*, which is a filter. So every operator that satisfies the $\mathcal{S}$-Transitivity Criterion is a $\mathcal{TS}$-operator.

In contrast, if we consider the family of *piecewise syndetic sets* $\mathcal{PS} = \mathcal{TS}^* = \mathcal{T} \cdot \mathcal{S}$ (i.e., A is piecewise syndetic if, and only if, it is the intersection of a thick set with a syndetic set), then $\widetilde{\mathcal{PS}} = \mathcal{T}$, and $\emptyset \in \mathcal{T} \cdot \mathcal{T}$. Thus the hypotheses of Theorem 2.4 are not satisfied. Actually, it is not hard to construct an operator T such that conditions (a) and (b) in Theorem 2.4 are satisfied for $\mathscr{F} = \mathcal{PS}$, with T not even transitive.

2. Another remarkable case is provided by, given a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$, considering the filter

$$\mathscr{F} := \{A \subset \mathbb{N} : \exists j \in \mathbb{N} \text{ with } A \supset \{n_k : k \geq j\}\}.$$

In this case Theorem 2.4 turns out to coincide with the classical Hypercyclicity Criterion. Moreover, since the Hypercyclicity Criterion characterizes the weakly mixing operators on separable F-spaces [8], we have that every weakly mixing operator T on a separable F-space X supports a strictly increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that T is an $\mathscr{F}$-operator, where

$$\mathscr{F} := \{A \subset \mathbb{N} : \forall N \in \mathbb{N} \ \exists j \in \mathbb{N} \text{ with } A \supset \{n_k : k \geq j\} + [-N, N]\}.$$

3. We note that for an $\widetilde{\mathscr{F}}$-operator T with $\widetilde{\mathscr{F}}$ a filter it is not true in general that T must satisfy the $\mathcal{G}$-Transitivity Criterion for some filter $\mathcal{G} \subset \widetilde{\mathscr{F}}$: just consider the family $\mathscr{F} = \mathcal{I}^*$ of cofinite sets and the fact that there exist mixing operators not satisfying Kitai's Criterion [12, Theorem 2.5].

4. Recall that for the case $\mathscr{F} = \mathcal{I}$, Furstenberg [10, Proposition II.3] showed that once $T \oplus T$ is an $\mathcal{I}$-map on X^2, every direct sum $\oplus_{j=1}^r T$ on X^r is an $\mathcal{I}$-map too ($r \in \mathbb{N}$). The assumptions of the $\mathscr{F}$-Transitivity Criterion on an operator T clearly ensure that (any direct sum $\oplus_{j=1}^r T$ will satisfy the $\mathscr{F}$-Transitivity Criterion on the space X^r and thus that) $\oplus_{j=1}^r T$ is an $\widetilde{\mathscr{F}}$-operator on X^r, for every $r \in \mathbb{N}$.

We next introduce the concept of a hereditarily $\mathscr{F}$-operator, and we establish links with that of an $\mathscr{F}$-operator.

Definition 2.6. We say that a continuous map T is a *hereditarily $\mathscr{F}$-map* if $N(U,V) \cap A \in \mathscr{F}$ for every $U, V \in \mathcal{U}(X)$ and every $A \in \mathscr{F}$ (that is, $\mathcal{N}_T \cdot \mathscr{F} \subset \mathscr{F}$). In addition, if X is a topological vector space and $T \in \mathcal{L}(X)$, we say that T is a *hereditarily $\mathscr{F}$-operator*.

Clearly, hereditarily $\mathscr{F}$-maps are $\mathscr{F}$-maps. Moreover, they are automatically $\mathscr{F}^*$-maps since $\mathcal{N}_T \cdot \mathscr{F} \subset \mathscr{F} \not\ni \emptyset$. Also, for a filter $\mathscr{F}$ the concepts of $\mathscr{F}$-map and hereditarily $\mathscr{F}$-map are equivalent. More generally, we have:

Proposition 2.7. *Let T be a continuous map on a complete separable metric space X without isolated points.*

(A) Let $\mathscr{F}$ be a partition regular family. Then the following are equivalent:

(1) T is an $\mathscr{F}^*$-map;

(2) T is a hereditarily $\mathscr{F}^*$-map;

(3) T is a hereditarily $\mathscr{F}$-map;

(4) $hcA := \{x \in X : \overline{\{T^n x : n \in A\}} = X\}$ is a dense (G_δ) set in X for any $A \in \mathscr{F}$.

(B) Let $\mathscr{F}$ be a filter. Then the following are equivalent:

(i) T is an $\mathscr{F}$-map;

(ii) T is a hereditarily $\mathscr{F}$-map;

(iii) T is a hereditarily $\mathscr{F}^*$-map;

(iv) $hcA := \{x \in X : \overline{\{T^n x : n \in A\}} = X\}$ is a dense (G_δ) set in X for any $A \in \mathscr{F}^*$.

Proof. We will just show (A) since (B) follows by taking duals and Lemma 2.1. Indeed, condition (1) is equivalent to (2) because $\mathscr{F}^*$ is a filter. The fact that (1) implies (3) is a consequence of Lemma 2.1 too, while the converse was already noticed before for general families. Finally the equivalence between (1) and (4) can be shown in a similar way as Birkhoff's transitivity theorem [15]. □

Note that when considering the family $\mathscr{F} = \mathfrak{I}$ of infinite sets in Proposition 2.7 (A) we obtain the known equivalences for mixing maps.

Remark 2.8. By the same argument for an operator T on a separable topological vector space X, the first three equivalences of statements (A) and (B) still hold. We also point out that as with the hypercyclic case we have the following comparison principle for $\mathscr{F}$-maps and transference principle for $\mathscr{F}$-operators, see [15, Chapter 12].

1. ($\mathscr{F}$-Comparison Principle) Any quasifactor of an $\mathscr{F}$-map is an $\mathscr{F}$-map. Indeed, let $T : X \to X$ be an $\mathscr{F}$-map and let $S : Y \to Y$ and $\phi : X \to Y$ be maps so that $\phi \circ T = S \circ \phi$, where ϕ has dense range. Then for any non-empty open subsets U and V of Y we have $N_S(U, V) = N_T(\phi^{-1}(U), \phi^{-1}(V)) \in \mathscr{F}$.

2. (Transference Principle) Let $\mathscr{F}$ be a family and let T be an operator on a topological vector space X so that each operator S on an F-space that is quasi-conjugate to T via an operator (that is, it supports a dense range operator $J : X \to Y$ with $JT = SJ$) is an $\mathscr{F}$-map. Then T is an $\mathscr{F}$-map.

3 $\mathscr{F}$-transitive weighted shift operators

Each bounded bilateral weight sequence $w = (w_k)_{k\in\mathbb{Z}}$, induces a *bilateral weighted backward shift* operator B_w on $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$ $(1 \leq p < \infty)$ given by $B_w e_k := w_k e_{k-1}$, where $(e_k)_{k\in\mathbb{Z}}$ denotes the canonical basis of X.

Similarly, each bounded sequence $w = (w_n)_{n\in\mathbb{N}}$ induces a *unilateral weighted backward shift* operator B_w on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$ $(1 \leq p < \infty)$, given by $B_w e_n := w_n e_{n-1}, n \geq 1$ and $B_w e_0 := 0$, where $(e_n)_{n\in\mathbb{Z}_+}$ denotes the canonical basis of X.

Our characterization of $\mathscr{F}$-transitive weighted backward shifts will rely on the properties of the sets $A_{M,j}$ and $\bar{A}_{M,j}$ defined as

$$A_{M,j} := \Big\{ n \in \mathbb{N} : \prod_{i=j+1}^{j+n} |w_i| > M \Big\}$$

$$\bar{A}_{M,j} := \Big\{ n \in \mathbb{N} : \frac{1}{\prod_{i=j-n+1}^{j} |w_i|} > M \Big\},$$

where $M > 0$ and $j \in \mathbb{Z}$. In the case $j = 0$, we just write $A_M, \bar{A}_M$ instead of $A_{M,0}, \bar{A}_{M,0}$ respectively. We note that Salas' [20] characterization of hypercyclic (i.e., transitive) bilateral weighted shifts on the above sequence spaces may be formulated as

$$B_w \text{ is hypercyclic } \Leftrightarrow \ \forall M > 0 \ \ \forall N \in \mathbb{N} \quad \bigcap_{j=-N}^{N} (A_{M,j} \cap \bar{A}_{M,j}) \neq \emptyset.$$

In other words, since $A_{M',j} \subset A_{M,j}$ and $\bar{A}_{M',j} \subset \bar{A}_{M,j}$ whenever $M' > M > 0$, the collection of subsets $\{A_{M,j}, \bar{A}_{M,j}\}_{M>0, j\in\mathbb{Z}}$ should form a filter subbase for the hypercyclicity of B_w. In that case, we denote by $\mathcal{A}_w$ the generated filter. Therefore, for the characterization of weighted shifts B_w that are $\mathscr{F}$-operators for a certain family $\mathscr{F}$ we need to assume that $\mathcal{A}_w$ is a filter.

When B_w is hypercyclic (i.e., when $\mathcal{A}_w$ is a filter), we can describe a filter base of $\mathcal{A}_w$, which will be very useful in the characterization of weighted shifts that are $\mathscr{F}$-operators, and it is given by the collection of sets

$$\{A_{M,j} \cap \bar{A}_{M,j} \ : \ M > 0 \text{ and } j \in \mathbb{N}\}.$$

Actually, this is a consequence of the observation that, if $M_1, M_2 > 0$ and $j_1, j_2, j_3 \in \mathbb{Z}$ with $j_3 > \max\{|j_1|, |j_2|\}$, then there is $M_3 > 0$ such that

$$A_{M_3,j_3} \subset A_{M_1,j_1} \cap A_{M_2,j_2} \quad \text{and} \quad \bar{A}_{M_3,j_3} \subset \bar{A}_{M_1,j_1} \cap \bar{A}_{M_2,j_2}.$$

Indeed, let $M := \sup_{i\in\mathbb{Z}} |w_i|$. We fix $M_3 > K(M_1 + M_2)(1 + M)^{2j_3}$, where

$$K := 1 + \max_{-j_3 \leq m_1 \leq m_2 \leq j_3} \prod_{i=m_1}^{m_2} |w_i|^{-1}.$$

If $n \in A_{M_3,j_3}$ then

$$\prod_{i=j_1+1}^{j_1+n} |w_i| = \left(\prod_{i=j_3+1}^{j_3+n} |w_i|\right) \frac{\prod_{i=j_1+1}^{j_3} |w_i|}{\prod_{i=j_1+n+1}^{j_3+n} |w_i|} > M_3 \frac{\prod_{i=j_1+1}^{j_3} |w_i|}{M^{j_3-j_1-1}} > M_1.$$

That is, $n \in A_{M_1,j_1}$. The same argument shows $n \in A_{M_2,j_2}$. Analogously, we also have $\bar{A}_{M_3,j_3} \subset \bar{A}_{M_1,j_1} \cap \bar{A}_{M_2,j_2}$.

Proposition 3.1. *Let B_w be a bilateral weighted backward shift on $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, $1 \le p < \infty$. Then the following are equivalent:*

(1) B_w is an $\widetilde{\mathscr{F}}$-operator;

(2) B_w is an $\mathscr{F}$-operator;

(3) for every $j \in \mathbb{N}$ and $M > 0$, $A_{M,j} \cap \bar{A}_{M,j} \in \mathscr{F}$;

(4) B_w is hypercyclic, $\mathcal{A}_w \subset \mathscr{F}$, and B_w satisfies the $\mathcal{A}_w$-Criterion.

In addition, if $\widetilde{\mathscr{F}}$ is a filter, then the above conditions are equivalent to

(5) for every $j \in \mathbb{N}$ and $M > 0$ we have $A_{M,j} \in \mathscr{F}$ and $\bar{A}_{M,j} \in \mathscr{F}$.

Proof. Obviously, (1) implies (2). The reverse implication is a consequence of Lemma 2.3 since transitive weighted shifts are weakly mixing. Also, (4) implies (2). To show that (2) implies (3), given $N, j \in \mathbb{N}$ arbitrary, we must find nonempty open sets $U, V \subset X$ such that

$$N(U,V) \subset A_{N,j} \cap \bar{A}_{N,j}. \tag{3.1}$$

Indeed, we fix $R > N$,

$$U := \{x \in X \ : \ |x_j| > \frac{1}{R}\} \cap \{x \in X \ : \ \|x\| < 1\},$$

and we set

$$V = \big\{x \in X : \big\|x - (N+1)e_j\big\| < \frac{1}{R^2}\big\}.$$

If $m \in N(U,V)$ and $x \in U$ is such that $B_w^m x \in V$, then

$$\left|\left(\prod_{i=j+1}^{j+m} w_i\right) x_{j+m} - (N+1)\right| < \frac{1}{R^2} < 1 \quad \text{and} \tag{3.2}$$

$$\left|\left(\prod_{i=l+1}^{l+m} w_i\right) x_{l+m}\right| < \frac{1}{R^2} \quad \text{if } l \ne j. \tag{3.3}$$

Since $x \in U$, we deduce from (3.2) that

$$\prod_{i=j+1}^{j+m} |w_i| > \left(\prod_{i=j+1}^{j+m} |w_i|\right) |x_{j+m}| > N,$$

which implies that $m \in A_{N,j}$.

On the other hand, $B_w^m x \in V$ forces $m > 0$ since U and V do not intersect. Thus, $l := j - m \neq j$, and (3.3) implies

$$\left(\prod_{i=j-m+1}^{j} |w_i|\right) < \left(\prod_{i=j-m+1}^{j} |w_i|\right) R\,|x_j| < \frac{1}{R} < \frac{1}{N},$$

that yields $m \in \bar{A}_{N,j}$. Thus the inclusion (3.1) is satisfied, and property (3) holds.

To prove that (3) implies (4), since B_w is hypercyclic (i.e., $\mathcal{A}_w$ is a filter) and $\mathcal{A}_w \subset \mathscr{F}$ because $\mathscr{F}$ contains a basis of $\mathcal{A}_w$, we just need to show that B_w satisfies the $\mathcal{A}_w$-criterion.

Let D be the set of all finitely supported vectors in X and let S_w be the weighted forward shift defined on D by

$$S_w e_i := \frac{1}{w_{i+1}} e_{i+1}.$$

If we consider $S_n := S_w^n$ then we have $B_w^n S_n x = x$ for every $x \in D$. It suffices to show that

- $\mathcal{A}_w$-$\lim_n B_w^n x = 0$ for every $x \in D$;
- $\mathcal{A}_w$-$\lim_n S_n x = 0$ for every $x \in D$.

For the rest of the proof we assume that $X = \ell^p(\mathbb{Z})$ with $1 \leq p < \infty$. The proof is similar if $X = c_0(\mathbb{Z})$. Let $x \in D$, $\varepsilon > 0$ and $V_\varepsilon := \{x \in \ell^p(\mathbb{Z}) : \|x\| < \varepsilon\}$. First, we show that $\{n \in \mathbb{N} : B_w^n x \in V_\varepsilon\} \in \mathcal{A}_w$. Since $x \in D$, we can write $x = \sum_{j=-m}^{m} x_j e_j$ for some $m \in \mathbb{N}$ and we then have

$$B_w^n x = \sum_{j=-m-n}^{m-n} \left(\prod_{i=j+1}^{j+n} w_i\right) x_{j+n} e_j.$$

Let $M = \|x\|_\infty 2m/\varepsilon$ and $n \in \bigcap_{j=-m}^{m} \bar{A}_{M,j} \in \mathcal{A}_w$. We have

$$\|B_w^n x\|^p = \sum_{j=-m}^{m} \left|\prod_{i=j-n+1}^{j} w_i\right|^p |x_j|^p < \sum_{j=-m}^{m} \left(\frac{\varepsilon}{\|x\|_\infty 2m}\right)^p |x_j|^p < \varepsilon^p,$$

which implies

$$\bigcap_{j=-m}^{m} \bar{A}_{M,j} \subseteq \{n \in \mathbb{N} : B_w^n y \in V_\varepsilon\},$$

thus $\{n \in \mathbb{N} : B_w^n y \in V_\varepsilon\} \in \mathcal{A}_w$. It remains to show that $\{n \in \mathbb{N} : S_n x \in V_\varepsilon\} \in \mathcal{A}_w$. Indeed, we have

$$S_n x = S_w^n x = \sum_{j=-m}^{m} \frac{x_j}{\prod_{i=j+1}^{j+n} w_i} e_{j+n}.$$

Let $M = \|x\|_\infty 2m/\varepsilon$ and $n \in \bigcap_{j=-m}^{m} A_{M,j}$. We then have

$$\|S_n x\|^p = \sum_{j=-m}^{m} \left| \frac{x_j}{\prod_{i=j+1}^{j+n} w_i} \right|^p < \frac{2m\varepsilon^p}{(2m)^p} \leq \varepsilon^p,$$

which implies

$$\bigcap_{j=-m}^{m} A_{M,j} \subseteq \{n \in \mathbb{N} : S_n y \in V_\varepsilon\}.$$

Consequently, $\{n \in \mathbb{N} : S_n y \in V_\varepsilon\} \in \mathcal{A}_w$, and B_w is an $\mathscr{F}$-operator.

Certainly, condition (3) implies (5). If (5) holds, the argument preceding this Proposition yields that, for each $j \in \mathbb{N}$ and for every $M > 0$, the sets $A_{M,j}$ and $\bar{A}_{M,j}$ belong to $\widetilde{\mathscr{F}}$, which gives (3) since $\widetilde{\mathscr{F}}$ is a filter. □

When $\mathscr{F} = \mathcal{I}^*$ is the filter of cofinite sets, we obtain as a consequence the well known characterization of mixing bilateral weighted shifts. On the other hand, the case $\mathscr{F} = \mathcal{S}$ offers again an interesting result.

Corollary 3.2. *Let B_w be a bilateral weighted backward shift on $X = c_0(\mathbb{Z})$ or $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$. Then the following are equivalent:*

(1) B_w is a topologically ergodic operator;

(2) for every $j \in \mathbb{N}$ and $M > 0$, $A_{M,j}$ and $\bar{A}_{M,j}$ are syndetic sets.

The unilateral version of Proposition 3.1 we provide next relies only on the sets $A_{M,j}$. Notice that for a hypercyclic unilateral weighted shift B_w the collection of sets $\{A_{M,j} : M > 0 \text{ and } j \in \mathbb{N}\}$ forms a base of a filter (which we call again $\mathcal{A}_w$) since, as before, if $M_1, M_2 > 0$ and $j_1, j_2, j_3 \in \mathbb{N}$ with $j_3 > \max\{j_1, j_2\}$, then there is $M_3 > 0$ such that

$$A_{M_3,j_3} \subset A_{M_1,j_1} \cap A_{M_2,j_2}.$$

This fact yields a simplification of the corresponding characterization of unilateral weighted shifts that are $\mathscr{F}$-operators, which can be further simplified if $\mathscr{F}$ is a shift$_-$-invariant family. The unilateral version of Proposition 3.1 can be stated as follows.

Proposition 3.3. *Let B_w be an unilateral weighted backward shift on $c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$ $(1 \leq p < \infty)$. The following are equivalent:*

(1) B_w is an $\widetilde{\mathscr{F}}$-operator;

(2) B_w is an $\mathscr{F}$-operator;

(3) for every $j \in \mathbb{N}$ and $M > 0$, the set $A_{M,j} \in \mathscr{F}$;

(4) B_w is hypercyclic, $\mathcal{A}_w \subset \mathscr{F}$, and B_w satisfies the $\mathcal{A}_w$-Criterion.

If in addition $\mathscr{F}$ is shift$_{-}$-invariant, the above conditions are equivalent to

(5) for every $M > 0$ the set $A_M \in \mathscr{F}$.

Proof. We only prove that if $\mathscr{F}$ is shift$_{-}$-invariant then condition (5) implies (3). Let $M > 0$ and $j \in \mathbb{N}$. We fix $M' > M(\sup_{i\in\mathbb{N}} |w_i|)^j$ such that $A_{M'} \subset [j+1, +\infty[$. Given $n \in A_{M'}$, we have

$$\prod_{s=j+1}^{n} |w_s| = \frac{\prod_{s=1}^{n} |w_s|}{\prod_{s=1}^{j} |w_s|} > \frac{M'}{(\sup_{i\in\mathbb{N}} |w_i|)^j} > M.$$

This implies that $A_{M'} - j \subset A_{M,j}$. Since $\mathscr{F}$ is a shift$_{-}$-invariant family, we conclude that $A_{M,j} \in \mathscr{F}$. □

In consequence we have the following characterization of topologically ergodic unilateral backward weighted shifts.

Corollary 3.4. *Let B_w be an unilateral weighted backward shift on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$, $1 \le p < \infty$, then the following are equivalent:*

(1) B_w is topologically ergodic;

(2) for every $M > 0$ the set A_M is syndetic.

We conclude this section by considering finite products of $\mathscr{F}$-maps.

Proposition 3.5. *Let $T_1, \ldots, T_m$ be continuous maps on X, then*

(1) for $n \ge 1$, T_1^n is an $\mathscr{F}$-map on X if and only if T_1 is an $\mathscr{F}_n$-map where $\mathscr{F}_n := \{A \subset \mathbb{Z}_+ : \frac{1}{n}(A \cap n\mathbb{Z}_+) \in \mathscr{F}\}$. In other words, T_1^n is an $\mathscr{F}$-map on X if and only if for every $U, V \in \mathcal{U}(X)$, $N_{T_1}(U, V) \cap n\mathbb{Z}_+ \in n\mathscr{F}$.

(2) If $\mathscr{F}$ is a filter then $T_1 \times T_2 \times \cdots \times T_m$ is an $\mathscr{F}$-map on X^m if and only if T_l is an $\mathscr{F}$-map on X for every $1 \le l \le m$.

Proof. (1) If $n \ge 1$, then T_1^n is an $\mathscr{F}$-map on X if and only if $N_{T_1^n}(U, V) \in \mathscr{F}$ for every $U, V \in \mathcal{U}(X)$. We remark that $N_{T_1^n}(U, V) = \frac{1}{n}(N_{T_1}(U, V) \cap n\mathbb{Z}_+)$. Therefore, $N_{T_1^n}(U, V) \in \mathscr{F}$ if and only if $N_{T_1}(U, V) \in \mathscr{F}_n$.

(2) Note that $T_1 \times T_2 \times \cdots \times T_m$ is an $\mathscr{F}$-map on X^m if and only if $\bigcap_{l=1}^m N_{T_l}(U_l, V_l) \in \mathscr{F}$, for any $(U_l, V_l)_{l=1}^m \in (\mathcal{U}(X) \times \mathcal{U}(X))^m$. The conclusion follows since $\mathscr{F}$ is a filter. □

Hence by Proposition 3.1 and Proposition 3.5 we have the following corollary.

Corollary 3.6. *Let $\mathscr{F}$ be a filter and B_w be a bilateral weighted backward shift on $X = \ell^p(\mathbb{Z})$ or $c_0(\mathbb{Z})$. Then, for every $m \in \mathbb{N}$, the following are equivalent:*

(1) $B_w \oplus B_w^2 \oplus \ldots \oplus B_w^m$ is an $\mathscr{F}$-operator on X^m;

(2) For every $1 \le l \le m$, $M > 0$ and $j \in \mathbb{Z}$, $A_{M,j} \cap l\mathbb{Z}_+ \in l\mathscr{F}$ and $\bar{A}_{M,j} \cap l\mathbb{Z}_+ \in l\mathscr{F}$.

4 Return sets and densities

The purpose of this section is to analyze which kind of density properties the sets $N(U,V)$ can have for a given hypercyclic operator, and classify the hypercyclic operators accordingly. We first recall the definitions of the asymptotic densities and the Banach densities in $\mathbb{Z}_+$.

Definition 4.1. Let $A \subseteq \mathbb{Z}_+$ be given. The *upper and lower asymptotic density* of A are defined respectively by

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{|A \cap \{1,2,...,n\}|}{n} \text{ and } \underline{d}(A) = \liminf_{n\to\infty} \frac{|A \cap \{1,2,...,n\}|}{n}.$$

The *upper and lower Banach density* of A are defined by

$$\overline{Bd}(A) = \lim_{s\to\infty} \alpha^s/s \quad \text{and} \quad \underline{Bd}(A) = \lim_{s\to\infty} \alpha_s/s,$$

where for each $s \in \mathbb{Z}_+$

$$\alpha^s = \limsup_{k\to\infty} |A \cap [k+1, k+s]| \quad \text{and} \quad \alpha_s = \liminf_{k\to\infty} |A \cap [k+1, k+s]|.$$

In general we have $\underline{Bd}(A) \le \underline{d}(A) \le \overline{d}(A) \le \overline{Bd}(A)$ and

$$\underline{d}(A) + \overline{d}(\mathbb{Z}_+ \setminus A) = 1 \text{ and } \underline{Bd}(A) + \overline{Bd}(\mathbb{Z}_+ \setminus A) = 1. \tag{4.1}$$

We will consider the following families.

$$\overline{\mathcal{D}} = \{A \subseteq \mathbb{Z}_+ : \overline{d}(A) > 0\}, \quad \underline{\mathcal{D}} = \{A \subseteq \mathbb{Z}_+ : \underline{d}(A) > 0\},$$

$$\begin{aligned} \underline{\mathcal{BD}} &= \{A \subseteq \mathbb{Z}_+ : \underline{Bd}(A) > 0\}, \quad \overline{\mathcal{BD}} = \{A \subseteq \mathbb{Z}_+ : \overline{Bd}(A) > 0\}, \\ \overline{\mathcal{D}}_1 &= \{A \subseteq \mathbb{Z}_+ : \overline{d}(A) = 1\}, \quad \underline{\mathcal{D}}_1 = \{A \subseteq \mathbb{Z}_+ : \underline{d}(A) = 1\}, \\ \underline{\mathcal{BD}}_1 &= \{A \subseteq \mathbb{Z}_+ : \underline{Bd}(A) = 1\}, \quad \overline{\mathcal{BD}}_1 = \{A \subseteq \mathbb{Z}_+ : \overline{Bd}(A) = 1\}. \end{aligned}$$

Notice that each of these families is shift invariant, and that $\underline{\mathcal{D}}_1$ and $\underline{\mathcal{BD}}_1$ are filters. Moreover,

1. $\overline{\mathcal{BD}}_1 = \mathcal{T}$, the family of thick sets,
2. $\underline{\mathcal{BD}} = \mathcal{S}$, the family of syndetic sets,
3. $\overline{\mathcal{BD}} \supset \mathcal{PS}$, the family of piecewise syndetic sets,
4. $\underline{\mathcal{BD}}_1 \subset \mathcal{TS}$, the family of thickly syndetic sets,
5. $\underline{\mathcal{BD}}_1 = \overline{\mathcal{BD}}^*$, $\underline{\mathcal{D}}_1 = \overline{\mathcal{D}}^*$, $\overline{\mathcal{D}}_1 = \underline{\mathcal{D}}^*$, and $\overline{\mathcal{BD}}_1 = \underline{\mathcal{BD}}^*$ by (4.1).

In consequence, T is weakly mixing if and only if T is a $\overline{\mathcal{BD}}_1$-map.

Weighted shift operators and Proposition 3.3 help us to provide some counterexamples which allow us to distinguish the different notions of $\mathscr{F}$-operators.

Proposition 4.2. *Let $X = c_0(\mathbb{Z}_+)$, then*

(1) there exists a $\overline{\mathcal{BD}}_1$-operator which is not $\overline{\mathcal{D}}$-operator.

(2) there exists a $\overline{\mathcal{D}}_1$-operator which is not $\underline{\mathcal{D}}$-operator.

(3) there exists a $\underline{\mathcal{D}}_1$-operator which is not $\underline{\mathcal{BD}}$-operator.

Proof. (1) Consider the weight sequence

$$w = (\underbrace{1,\dots,1}_{m_0}, 2, 2^{-1}, \underbrace{1,\dots,1}_{m_1}, 2, 2, 2^{-2}, \underbrace{1,\dots,1}_{m_2}, 2, 2, 2, 2^{-3}, \underbrace{1,\dots,1}_{m_3}, \dots)$$

We first observe that $\sup_n \prod_{i=1}^n w_i$ is infinite, hence B_w is weakly mixing, see Chapter 4 in [15]. In other words B_w is $\overline{\mathcal{BD}}_1$-operator.

On the other hand, by Proposition 3.3, we know that it suffices to show that $\overline{d}(A_1) = 0$ in order to deduce that B_w is not a $\overline{\mathcal{D}}$-operator. In other words, it suffices to show that $\overline{d}\Big(\Big\{n \in \mathbb{N} : \prod_{i=1}^n w_i > 1\Big\}\Big) = 0$ and this holds if (m_k) grows sufficiently rapidly.

(2) Consider the weight

$$w = \big(\underbrace{1,\cdots,1}_{m_0}, \underbrace{2,\cdots,2}_{n_0}, \underbrace{2^{-n_0}, 1,\cdots,1}_{m_1}, \underbrace{2,\cdots,2}_{n_1}, \underbrace{2^{-n_1}, 1,\cdots,1}_{m_2}, \underbrace{2,\cdots,2}_{n_2}, \cdots\big).$$

Thanks to Proposition 3.3, it suffices to find sequences $(m_k)_k, (n_k)_k$ such that

- $\overline{d}\big(\{n : \prod_{i=1}^n w_i = 1\}\big) = 1$
- $\overline{d}(A_M) = \overline{d}\big(\{n : \prod_{i=1}^n w_i > M\}\big) = 1$, for every $M > 0$.

Indeed, if $\overline{d}\big(\{n : \prod_{i=1}^n w_i = 1\}\big) = 1$ then

$$\underline{d}\big(\{n : \prod_{i=1}^n w_i > 1\}\big) = 1 - \overline{d}\big(\{n : \prod_{i=1}^n w_i \le 1\}\big) = 0.$$

Define sequences of intervals in the following way: $\mathcal{A}_k = [10^{2^{2k+1}}, 10^{2^{2k+2}}[$ and $\mathcal{B}_k = [10^{2^{2k+2}}, 10^{2^{2k+3}}[$ for every $k \in \mathbb{Z}_+$.

So $\mathcal{A} = \bigcup_{k\in\mathbb{N}} \mathcal{A}_k$ and $\mathcal{B} = \bigcup_{k\in\mathbb{N}} \mathcal{B}_k$ are disjoint with $\overline{d}(\mathcal{A}) = \overline{d}(\mathcal{B}) = 1$. Hence, setting $m_k = |\mathcal{A}_k|, n_k = |\mathcal{B}_k|$ for every k, we are done.

(3) Let $m_k = 10^{2^k}$ for every $k \in \mathbb{Z}_+$. We consider the weight

$$w = \big(1, 2, 2^{-1}, 1, 1, \underbrace{2,\cdots,2}_{m_0}, 2^{-m_0}, 1, 1, 1, \underbrace{2,\cdots,2}_{m_1}, 2^{-m_1}, 1, 1, 1, 1,$$
$$\underbrace{2,\cdots,2}_{m_2}, 2^{-m_2}, \cdots\big).$$

The set $A_1 = \{n : \prod_{i=1}^n w_i > 1\}$ has arbitrarily large gaps, hence B_w is not an $\underline{\mathcal{BD}}$-operator by Proposition 3.3. On the other hand, we have for every $M > 1$

$$\underline{d}(A_M) = \underline{d}\big(\{n : \prod_{i=1}^n w_i > M\}\big) = 1.$$

Hence, B_w is $\underline{\mathcal{D}}_1$-operator by Proposition 3.3. $\square$

Mixing operators obviously are $\underline{\mathcal{BD}}_1$-operators, but the converse is false, this is the argument of the next result.

Proposition 4.3. *There exists a $\underline{\mathcal{BD}}_1$-operator on $c_0(\mathbb{Z}_+)$ which is not mixing.*

Proof. Consider the weight $w = (w_n)_{n=1}^{\infty}$ defined by

$$w = (1, 2, 2^{-1}, 2, 2, 2^{-2}, \dots, \underbrace{2, \dots, 2}_{n}, 2^{-n}, \dots).$$

The weighted shift B_w is not mixing since $\prod_{i=1}^{n} w_i$ does not tend to infinity as n tends to infinity (see, e.g., Chapter 4 in [15]). It remains to show that $\underline{Bd}(A_M) = 1$ for every $M \geq 1$. Let $M > 1$ and $n \in \mathbb{N}$ such that $2^{n-1} < M \leq 2^n$. If $k > n(n+1)/2$ and $s \geq (n+1) + (n+2) + \dots + 2n = n(3n+1)/2$, then there is $l_s > 1$ such that $(l_s - 1)n((l_s+1)n + 1)/2 \leq s < (l_s)n((l_s+2)n+1)/2$. An easy computation shows that we have $|A_M \cap [k, k+s]| \geq s - l_s(n^2+n) > (l_s^2/2 - l_s - 1)n^2 - l_s n$. Therefore,

$$\alpha_s := \liminf_{k\to\infty} |A_M \cap [k, k+s]| \geq (l_s^2/2 - l_s - 1)n^2 - l_s n,$$

and thus

$$\underline{Bd}(A_M) = \lim_{s\to\infty} \frac{\alpha_s}{s} \geq \lim_{s\to\infty} \frac{(l_s^2/2 - l_s - 1)n^2 - l_s n}{(l_s^2/2 + l_s)n^2 + l_s n} = 1.$$

We conclude by Proposition 3.3. □

Proposition 4.4. *There exists a $\underline{\mathcal{BD}}$-operator on $\ell^1(\mathbb{Z}_+)$ which is not a $\overline{\mathcal{D}}_1$-operator.*

Proof. Let $\mathcal{A}_n = [\underbrace{2, \dots, 2}_{n-times}, 2^{-n}]$, $\mathcal{B}_1 = \mathcal{A}_1$, $\mathcal{B}_n = [\mathcal{B}_{n-1}, \mathcal{A}_n, \mathcal{B}_{n-1}]$, and consider the weight sequence

$$w = (\underbrace{\underbrace{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_1}, \mathcal{A}_3, \underbrace{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_1}}, \mathcal{A}_4, \underbrace{\underbrace{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_1}, \mathcal{A}_3, \underbrace{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_1}}, \dots).$$

Since A_M has bounded gaps for every $M > 0$, we have from Corollary 3.4 that B_w is topologically ergodic, i.e., it is a $\underline{\mathcal{BD}}$-operator.

In view of Proposition 3.3, it now suffices to show that

$$\overline{d}\Big(\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| > 1\}\Big) < 1.$$

We first notice that

$$|\mathcal{B}_n| = 3 \cdot 2^n - n - 3 \quad \text{and} \quad \beta_n := \Big|\Big\{k \leq |\mathcal{B}_n| : \prod_{i=1}^{k} |w_i| = 1\Big\}\Big| = 2^n - 1.$$

Now we observe that $\prod_{i=1}^{k} |w_i| \geq 1$ for all $k \in \mathbb{N}$. Therefore, we have

$$\begin{aligned}
\overline{d}\Big(\Big\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| > 1\Big\}\Big) &= \limsup_{n} \frac{\Big|\Big\{k \in [1,n] : \prod_{i=1}^{k} |w_i| > 1\Big\}\Big|}{n} \\
&= \limsup_{n} \frac{\Big|\Big\{k \leq |\mathcal{B}_n| + n + 1 : \prod_{i=1}^{k} |w_i| > 1\Big\}\Big|}{|\mathcal{B}_n| + n + 1} \\
&= \lim_{n} \frac{|\mathcal{B}_n| - \beta_n + n + 2}{|\mathcal{B}_n| + n + 1} = \lim_{n} \frac{2 \cdot 2^n}{3 \cdot 2^n - 2} = \frac{2}{3} < 1.
\end{aligned}$$

□

Figure 1 below summarizes the results of this section. We remark that:

- by Proposition 4.2 (1), there exists a $\overline{\mathcal{BD}}_1$-operator which is not a $\overline{\mathcal{D}}_1$-operator and a $\overline{\mathcal{BD}}$-operator which is not a $\overline{\mathcal{D}}$-operator;
- by Proposition 4.2 (2), there exists a $\overline{\mathcal{D}}_1$-operator which is not a $\underline{\mathcal{D}}_1$-operator and a $\overline{\mathcal{D}}$-operator which is not a $\underline{\mathcal{D}}$-operator;
- by Proposition 4.2 (3), there exists a $\underline{\mathcal{D}}_1$-operator which is not a $\underline{\mathcal{BD}}_1$ and a $\underline{\mathcal{D}}$-operator which is not a $\underline{\mathcal{BD}}$-operator.

  On the other hand, by Proposition 4.4, there exists a

- $\underline{\mathcal{BD}}$-operator which is not a $\underline{\mathcal{BD}}_1$-operator;
- $\underline{\mathcal{BD}}$-operator which is not a $\overline{\mathcal{D}}_1$-operator;
- $\underline{\mathcal{D}}$-operator which is not a $\underline{\mathcal{D}}_1$-operator;
- $\underline{\mathcal{D}}$-operator which is not a $\overline{\mathcal{D}}_1$-operator;
- $\overline{\mathcal{D}}$-operator which is not a $\overline{\mathcal{D}}_1$-operator.

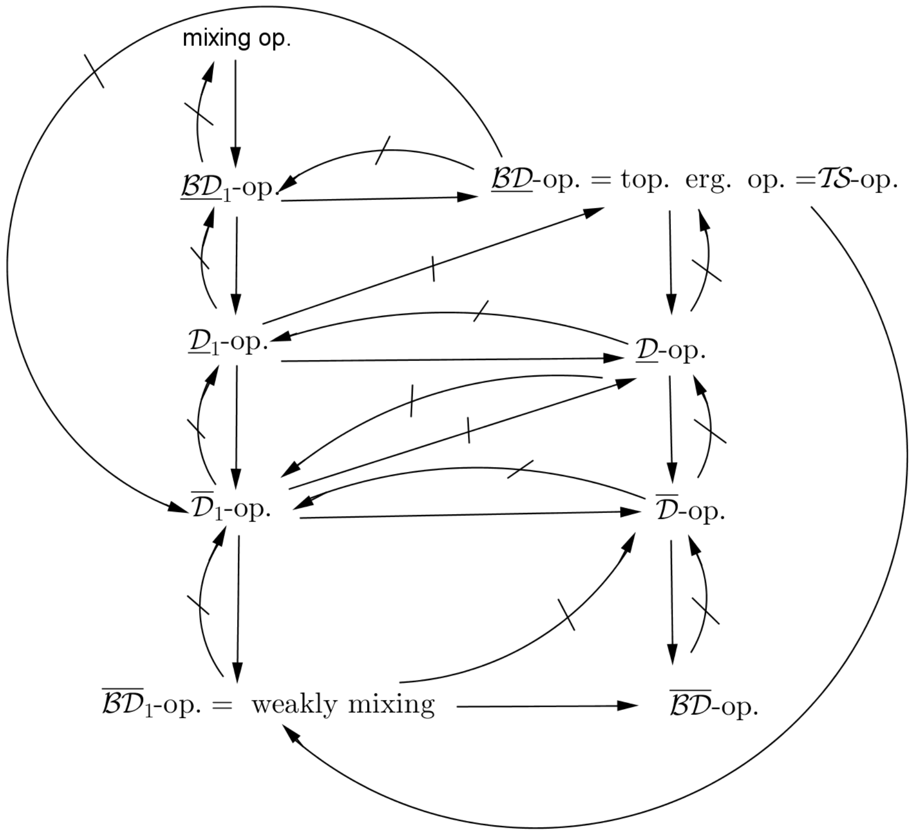

Figure 1: Densities and transitivity properties

5 Some special families

In this section we study new classes of $\mathscr{F}$-transitive operators given by families commonly used in Ramsey Theory. For a rich source on these families see [16]. For instance, we will consider the families of Δ-sets and of $\mathcal{IP}$-sets, as well as their dual families.

$$\Delta = \{A \subseteq \mathbb{Z}_+ : (B - B) \cap \mathbb{Z}_+ \subseteq A, \text{ for some infinite subset } B \text{ of } \mathbb{Z}_+\}$$
$$\mathcal{IP} = \{A \subseteq \mathbb{Z}_+ : \exists (x_n)_n \subseteq \mathbb{N} \text{ with } \sum_{n\in F} x_n \in A, \ \forall F \subset \mathbb{Z}_+ \text{ finite}\}.$$

The families Δ^* and $\mathcal{IP}^*$ are filters since Δ and $\mathcal{IP}$ are partition regular. In addition, we have

$$\mathcal{I}^* \subsetneqq \Delta^* \subsetneqq \mathcal{IP}^* \subsetneqq \mathcal{S}$$
$$\mathcal{I}^* \subsetneqq \mathcal{PS}^* \subsetneqq \mathcal{S}, \tag{5.1}$$

see [6] for details. In linear dynamics, some of the widely studied classes are the mixing and weakly mixing operators. As we already mentioned, an operator T is mixing if and only if it is an $\mathcal{I}^*$-operator and T is weakly mixing if and only if T is a $\mathcal{T}$-operator. We recall that the class of $\mathcal{TS}$-operators coincides with the class of topologically ergodic operators by Lemma 2.3 (see also the exercises in [15, Chapter 2]). Moreover, since $\mathcal{TS} = \mathcal{PS}^*$ and $\mathcal{TS}$ is a filter, we know that $\mathcal{PS}^*$ is partition regular (Lemma 2.1). With the help of Proposition 2.7 applied to $\mathscr{F} = \mathcal{PS}$ we can therefore complete the picture.

Proposition 5.1. *Let $T \in \mathcal{L}(X)$, where X is a separable F-space. The following are equivalent:*

(1) T is a topologically ergodic operator;

(2) T is a hereditarily $\mathcal{TS}$-operator;

(3) T is a $\mathcal{TS}$-operator;

(4) T is a hereditarily $\mathcal{PS}$-operator;

(5) $hcA := \{x \in X : \overline{\{T^n x : n \in A\}} = X\}$ is a dense (G_δ) set in X for any $A \in \mathcal{PS}$.

We will distinguish different classes of $\mathscr{F}$-operators by means of Proposition 3.3. Given a family $\mathscr{F}$, the following are two standard ways to induce shift-invariant families

$$\mathscr{F}_+ := \bigcup_{k\in\mathbb{Z}} (\mathscr{F} + k)$$

$$\mathscr{F}_\bullet := \bigcap_{k\in\mathbb{Z}} (\mathscr{F} + k),$$

where $\mathscr{F} + k := \{A \subset \mathbb{Z}_+ : \exists B \in \mathscr{F} \text{ with } (B + k) \cap \mathbb{Z}_+ \subset A\}$, $k \in \mathbb{Z}$. We have

$$\widetilde{\mathscr{F}} \subset \mathscr{F}_\bullet \subset \mathscr{F} \subset \mathscr{F}_+.$$

Moreover, for any $A \subseteq \mathbb{Z}_+$ we have

$$A \in (\mathscr{F}^*)_\bullet \text{ if and only if } A \in (\mathscr{F}_+)^*. \tag{5.2}$$

It is well-known that Δ and $\mathcal{IP}$ are not shift invariant, while $\mathcal{PS}$ is. Also, if $\mathscr{F} = \Delta, \mathcal{IP}$ or $\mathcal{PS}$ and $\mathscr{G} = \mathscr{F}$ or $\mathscr{F}_+$ then

$$A \in \mathscr{G}^* \text{ if and only if } \mathbb{Z}_+ \setminus A \notin \mathscr{G}, \tag{5.3}$$

since $\mathscr{G}$ is partition regular.

Proposition 5.2. *Every $\mathscr{F}$-operator is an $\mathscr{F}_\bullet$-operator.*

Proof. Let $U, V \in \mathfrak{U}(X)$ and $k \geq 0$. We have $N(U, T^{-k}V) + k \subset N(U, V)$. Moreover, since X has no isolated points, by transitivity we can find non-empty open sets $U' \subset U$ and $V' \subset V$ such that $N(T^{-k}U', V') \subset [k, +\infty[$. Thus we have

$$N(T^{-k}U', V')) - k \subset N(U, V).$$

We can conclude that every $\mathscr{F}$-operator is an $\mathscr{F}_\bullet$-operator. □

We next compare the notions of mixing operator, Δ^*-operator, $\mathcal{IP}^*$-operator and topologically ergodic operator.

Proposition 5.3. *There exists a topologically ergodic weighted backward shift on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$, $1 \leq p < \infty$, which is not an $\mathcal{IP}^*$-operator.*

Proof. Consider the set

$$B = \Big\{ \sum_{n \in F} 2^{2n} : F \text{ finite set of } \mathbb{N} \Big\}.$$

Clearly $B \in \mathcal{IP}$ and thus $\mathbb{Z}_+ \setminus B \notin \mathcal{IP}^*$ by (5.3). Let (b_n) be the increasing enumeration of B. We define the weight $w = (w_m)_{m=1}^{\infty}$ as follows

$$w = (2, \dots, 2, \underbrace{\frac{1}{2^{b_1 - 1}}}_{w_{b_1}}, 2, \dots, 2, \underbrace{\frac{1}{2^{b_2 - b_1 - 1}}}_{w_{b_2}}, 2, \dots, 2, \underbrace{\frac{1}{2^{b_3 - b_2 - 1}}}_{w_{b_3}}, 2, \dots). \tag{5.4}$$

Now, $A_1 := \{n \geq 1 : \prod_{i=1}^n w_i > 1\} = \mathbb{Z}_+ \setminus B$, hence B_w is not an $\mathcal{IP}^*$-operator by Propositon 3.3. On the other hand, it is easy to see that $B \notin \mathcal{PS}$. Then $(B+i) \notin \mathcal{PS}$ for every $i \geq 0$, since $\mathcal{PS}$ is shift invariant. Hence, by (5.3) the set $\mathbb{Z}_+ \setminus (B+i) \in \mathcal{PS}^*$ for every $i \geq 0$. Now observe that $A_{2^j} := \{n \geq 1 : \prod_{i=1}^n w_i > 2^j\} = \mathbb{Z}_+ \setminus \Big(\bigcup_{i=0}^j B + i \Big) = \bigcap_{i=0}^j (\mathbb{Z}_+ \setminus (B+i)) \in \mathcal{PS}^*$, since $\mathcal{PS}^*$ is a filter. Hence B_w is a $\mathcal{PS}^*$-operator, or equivalently a topologically ergodic operator, by Proposition 3.3. $\square$

Proposition 5.4. *There exists a weighted backward shift operator on $X = c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$, $1 \leq p < \infty$, which is an $\mathcal{IP}^*$-operator but not a Δ^*-operator.*

Proof. Let B be an infinite subset of $\mathbb{N}$ with unbounded gaps and let $(b_n)_n$ be an increasing enumeration of B. So there exists an increasing sequence (n_k) such that

$$b_{n_k+1} - b_{n_k} \to \infty. \tag{5.5}$$

Consider the weight sequence $w = (w_m)_{m=1}^{\infty}$ given by (5.4). As before $\{n \geq 1 : \prod_{i=1}^n w_i > 1\} = \mathbb{Z}_+ \setminus B$, so it would be desirable that $B \in \Delta$ and thus that $\mathbb{Z}_+ \setminus B \notin \Delta^*$ since this would imply that B_w is not a Δ^*-operator.

On the other hand, it can be verified that for every $M > 0$ and $j \in \mathbb{N}$ there exists a finite subset F of $\mathbb{Z}$ such that $A_{M,j} = \mathbb{Z}_+ \setminus (\cup_{i \in F} B + i)$. Hence, in order to conclude that B_w is an $\mathcal{IP}^*$-operator, by Proposition 3.3 and condition (5.3) we need to verify

$$\bigcup_{i \in F} (B + i) \notin \mathcal{IP} \tag{5.6}$$

for any finite subset F of $\mathbb{Z}$. Now, since $\mathcal{IP}$ is partition regular, condition (5.6) is obtained if $B \notin \mathcal{IP}_+$ and this in turn is equivalent to $\mathbb{Z}_+ \setminus B \in$

$\left(\mathcal{IP}^*\right)_\bullet$ by (5.3) and (5.2). Now, an obvious modification in the proof of [6, Theorem 2.11 (1)] ensures the existence of a set $E \in \left(\mathcal{IP}^*\right)_\bullet$ which is not $\bigcup_{n\in\mathbb{Z}_+}\left(\Delta^* + n\right)$-set in $\mathbb{N}$, hence not Δ^*-set. In addition, $\mathbb{Z}_+ \setminus E$ has unbounded gaps. Setting $B = \mathbb{Z}_+ \setminus E$ we are done. □

Evidently, every mixing operator is a Δ^*-operator but the converse is not true.

Proposition 5.5. *There exists a Δ^*-weighted backward shift on $c_0(\mathbb{Z}_+)$ or $\ell^p(\mathbb{Z}_+)$, $1 \le p < \infty$, which is not mixing.*

Proof. Let $B = \{b_i : b_1 = 2, b_{i+1} = b_i + i + 2, i \in \mathbb{N}\}$. Consider the weight sequence $w = (w_m)_{m=1}^\infty$ given by (5.4), so we have

$$w = \left(2, 2^{-1}, 2, 2, 2^{-2}, 2, 2, 2, 2^{-3}, \dots\right).$$

We know that B_w is not mixing since $\prod_{i=1}^n w_i$ does not tend to infinity as n tends to infinity. On the other hand, it can be verified that for every $M > 0$ and $j \in \mathbb{N}$ there exists a finite subset F of $\mathbb{Z}$ such that $A_{M,j} = \mathbb{Z}_+ \setminus (\cup_{i\in F} B + i)$. Hence, in order to conclude that B_w is a Δ^*-operator, by Proposition 3.3 and condition (5.3) we need to verify $\bigcup_{i\in F} B + i \notin \Delta$, for every finite subset F of $\mathbb{Z}$.

So, let F be a finite subset of $\mathbb{Z}$ with $N = \max_{a,b\in F} |a - b|$. Suppose that $\bigcup_{i\in F} B + i$ is a Δ-set. Then, there exists an increasing sequence $(d_m)_m$ such that $\bigcup_{i\in F} B + i = \Delta\big((d_m)_m\big)$, where $\Delta\big((d_m)_m\big)$ denotes the *set of differences of* $(d_m)_m$ defined by $\Delta\big((d_m)_m\big) = \{d_j - d_i : 1 \le i < j\}$. Fix $d_{j_1}, d_{j_2} (j_1 < j_2)$ such that $|d_{j_2} - d_{j_1}| > N$. Then for each $m \in \mathbb{N}$ we have

$$|d_{j_2} - d_{j_1}| = |(d_{j_m} - d_{j_1}) - (d_{j_m} - d_{j_2})|,$$

which means that the distance $|d_{j_2} - d_{j_1}|$ between elements of $\bigcup_{i\in F} B + i$ is attained infinitely many times, which is not the case taking into account the way in which B was defined. We thus conclude that $\bigcup_{i\in F} B + i \notin \Delta$. □

5.1 Connection with $\mathcal{A}$-hypercyclicity

In this subsection we investigate the connection between the classes of hypercyclic operators considered throughout this work and the notion of $\mathcal{A}$-hypercyclicity studied in [7].

Given a family $\mathcal{A}$ on $\mathbb{Z}_+$, an operator $T \in \mathcal{L}(X)$ is called $\mathcal{A}$-*hypercyclic* if there exists $x \in X$ such that $N(x, V) \in \mathcal{A}$ for each V in $\mathcal{U}(X)$. Such a vector x is called an $\mathcal{A}$-*hypercyclic vector* for T.

When $\mathcal{A} = \underline{\mathcal{D}}$, the operator T is called *frequently hypercyclic.* This class was introduced by Bayart and Grivaux in [3], [2]. When $\mathcal{A} = \overline{\mathcal{D}}$, the operator T is called $\mathfrak{U}$-*frequently hypercyclic*; this class was introduced by Shkarin [21]. When $\mathcal{A} = \overline{\mathcal{BD}}$, the operator T is called *reiteratively hypercyclic* [18] (see a detailed study in [7]).

The frequently hypercyclic operators constitute by far the most extensively studied class of operators amongst the three classes mentioned above. Clearly any frequently hypercyclic operator is an $\mathfrak{U}$-frequently hypercyclic operator, which in turn is reiteratively hypercyclic. The hierarchy between frequently hypercyclic and $\mathfrak{U}$-frequently hypercyclic operators as well as a full characterization for weighted shift operators have been established by Bayart and Ruzsa [5]. A complementary study of this kind, taking into account reiterative hypercyclicity can be found in [7].

In particular, we already know that there exists a mixing weighted shift which is not reiteratively hypercyclic as shown in [7]. On the other hand, there exists a frequently hypercyclic (hence reiteratively hypercyclic) operator which is not mixing, see [1]. Reiteratively hypercyclic operators are topologically ergodic [7, 13]. One can therefore wonder whether any reiteratively hypercyclic operator is a Δ^*-operator or an $\mathcal{IP}^*$-operator.

Proposition 5.6. *Let $T \in \mathcal{L}(X)$ be a reiteratively hypercyclic operator. Then*

$$N(U,V) \in \bigcap_{t \in N(U,V)} \Big(\Delta^* + t\Big),$$

for every U, V non-empty open sets in X.

Proof. Let $U, V \in \mathcal{U}(X)$ and $n \in N(U,V)$. The set $U_n = U \cap T^{-n}V$ is a non-empty open set. Since T is reiteratively hypercyclic, there exists $x \in X$ such that $\overline{Bd}\,(N(x,U_n)) > 0$.

Let $s_1, s_2 \in N(x, U_n)$. We have

$$T^{s_1 - s_2 + n}(T^{s_2}x) = T^n(T^{s_1}x) \in V.$$

In other words,

$$N(x,U_n) - N(x,U_n) + n \subseteq N(U,V). \tag{5.7}$$

The desired result then follows from Theorem 3.18 in [11], which implies that $A - A \in \Delta^*$ whenever $A \in \overline{\mathcal{BD}}$. □

The family Δ^* is not shift invariant ($2\mathbb{N} := \{2n : n \in \mathbb{N}\} \in \Delta^*$ while $2\mathbb{N} + 1 \notin \Delta^*$). Hence, we cannot deduce from Proposition 5.6 that every reiteratively hypercyclic operator is a Δ^*-operator. In fact, we are not able to answer in general the following question: is any reiteratively hypercyclic operator either a Δ^*-operator or an $\mathcal{IP}^*$-operator? However we can show that the answer is yes if we consider bilateral or unilateral weighted shifts.

Proposition 5.7. *If B_w is reiteratively hypercyclic on $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$, or $X = c_0(\mathbb{Z})$, then B_w is an Δ^*-operator.*

In order to prove Proposition 5.7, we first state two lemmas. The first one directly follows from Proposition 5.6.

Lemma 5.8. *Let U, V non-empty open sets in X such that $U \cap V \neq \emptyset$, if T is reiteratively hypercyclic on X then $N(U, V) \in \Delta^*$.*

Let $X = \ell^p(\mathbb{Z})$, $1 \leq p < \infty$, or $c_0(\mathbb{Z})$. The second lemma will rely on the non-empty open sets $U_{R,j}$ defined for every $R > 1$ and every $j \in \mathbb{Z}$ by

$$U_{R,j} = \{U \in \mathcal{U}(X) : |x_j| > \frac{1}{R}, \forall x \in U\}.$$

In particular, we remark that if $MR > 1$ then $B((M+1)e_j; \frac{1}{MR}) \in U_{R,j}$, where $B(y; \epsilon)$ stands for the open ball centered at y with radius ϵ.

Lemma 5.9. *Let $M > 0$, $j \in \mathbb{Z}$ and $R > 1$ such that $MR > 1$. Suppose there exists $U \in U_{R,j}$ such that for any non-empty open subset $\widetilde{U}$ of U it holds $N(\widetilde{U}, B((M+1)e_j; \frac{1}{MR})) \in \Delta^*$. Then $A_{M,j} \in \Delta^*$ and $\bar{A}_{M,j} \in \Delta^*$.*

Proof. Let $(z(m))_m$ be a dense set in X such that

$$z(m) = (z(m)_1, \dots, z(m)_m, 0 \dots)$$

and $U_m = B(z(m); 1/m)$. Let $U \in U_{R,j}$ such that for any non-empty open subset $\widetilde{U}$ of U we have $N(\widetilde{U}, B((M+1)e_j; \frac{1}{MR})) \in \Delta^*$. Then there exists m such that $U_m \subset U$ and hence $N(U_m, B((M+1)e_j; \frac{1}{MR})) \in \Delta^*$. Pick $r \in N(U_m, B((M+1)e_j; \frac{1}{MR}))$ with $r > m$ and $x \in U_m$ such that $B_w^r x \in B((M+1)e_j; \frac{1}{MR})$.

Then, we have

$$\left| \left(\prod_{i=j+1}^{j+r} w_i \right) x_{j+r} - (M+1) \right| < \frac{1}{MR} \tag{5.8}$$

and for every $t \neq j$

$$\left| \left(\prod_{i=t+1}^{t+r} w_i \right) x_{t+r} \right| < \frac{1}{MR}. \tag{5.9}$$

By (5.8) we get,

$$\left| \prod_{i=1}^{r} w_{i+j} \right| > \left| \prod_{i=1}^{r} w_{i+j} x_{r+j} \right| > M,$$

where the first inequality follows since $r > m$. We conclude that $N(U_m, B((M+1)e_j; \frac{1}{MR})) \setminus \{1 \dots m\} \subseteq A_{M,j}$ and thus $A_{M,j} \in \Delta^*$.

On the other hand, by (5.9), we get $\prod_{i=j-r+1}^{j} |w_i x_j| < \frac{1}{MR}$, hence

$$\prod_{i=j-r+1}^{j} |w_i| \frac{1}{R} < \prod_{i=j-r+1}^{j} |w_i x_j| < \frac{1}{MR}.$$

We deduce that $\prod_{i=j-r+1}^{j} |w_i| < \frac{1}{M}$ and thus $\bar{A}_{M,j} \in \Delta^*$.

□

Proof of Proposition 5.7

Suppose B_w is not a Δ^*-operator on X, then by Proposition 3.1, there exists $M > 0$ and $j \in \mathbb{Z}$ such that $A_{M,j} \notin \Delta^*$ or $\bar{A}_{M,j} \notin \Delta^*$. Let $R > 1$ such that $MR > 1$. By Lemma 5.9, it follows that

$$\forall U \in U_{R,j} \quad \exists \widetilde{U} \subseteq U : N(\widetilde{U}, B((M+1)e_j; \frac{1}{MR})) \notin \Delta^*.$$

Since $B((M+1)e_j; \frac{1}{MR}) \in U_{R,j}$, we can consider $U = B((M+1)e_j; \frac{1}{MR})$ and there thus exists $\widetilde{U} \subseteq U$ such that $N(\widetilde{U}, U) \notin \Delta^*$, which by Lemma 5.8, is not possible if B_w is reiteratively hypercyclic. This concludes the proof of Proposition 5.7.

Analogously, we have the following result for unilateral weighted shifts.

Proposition 5.10. *If B_w is reiteratively hypercyclic on $X = \ell^p(\mathbb{Z}_+)$, $1 \leq p < \infty$, or on $X = c_0(\mathbb{Z}_+)$, then B_w is a Δ^*-operator.*

Proposition 5.11. *There exists a reiteratively hypercyclic operator on $c_0(\mathbb{Z}_+)$ which is not a $\overline{\mathcal{D}}_1$-operator.*

Proof. Let B_w be the weighted shift on $c_0(\mathbb{Z}_+)$ given by

$$w_k = \begin{cases} 2 & \text{if } k \in S \\ \prod_{\nu=1}^{k-1} w_\nu^{-1} & \text{if } k \in (S+1)\backslash S \\ 1 & \text{otherwise.} \end{cases}$$

where $S := \bigcup_{j,l\geq 1}]l10^j - j, l10^j + j[$. It is shown in [7, Theorem 17] that B_w is reiteratively hypercyclic and that

$$\overline{d}(\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| \geq 2^j\}) \to 0.$$

In particular, we deduce that there exists $j \geq 1$ such that $\overline{d}(\{k \in \mathbb{N} : \prod_{i=1}^{k} |w_i| \geq 2^j\}) < 1$ and in view of Proposition 3.3, we can conclude that B_w is not a $\overline{\mathcal{D}}_1$-operator. □

Figure 2 summarizes what we know after this work.

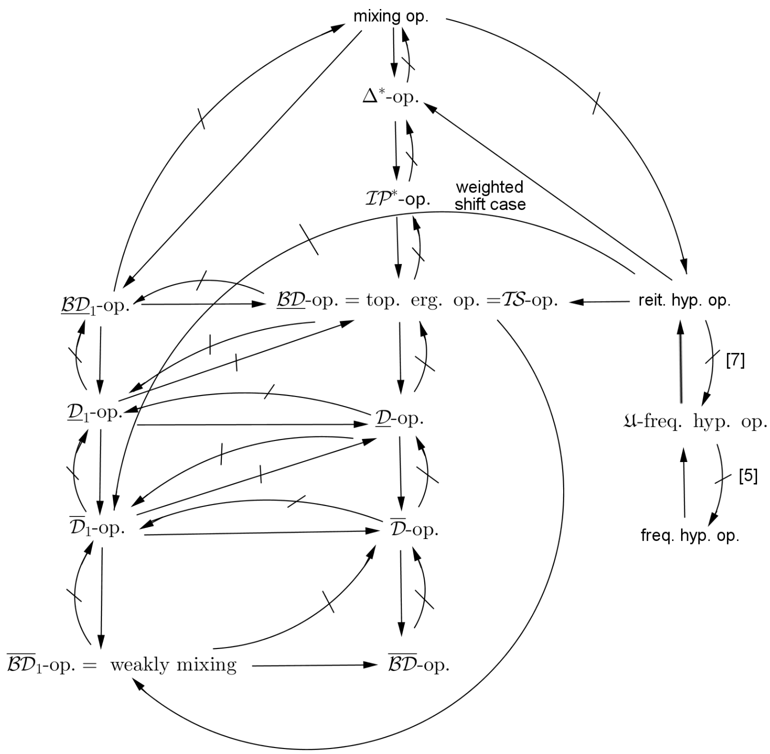

Figure 2: Known relations

We recall the following questions that remain open.

Question 5.12. *Does there exist a $\underline{\mathcal{D}}$-operator which is not a $\overline{\mathcal{BD}_1}$-operator? In other words, does there exist $T \in \mathcal{L}(X)$ being a $\underline{\mathcal{D}}$-operator but not weakly mixing?*

Note that if it were the case, then such operator T must not be weighted shift.

Question 5.13. *Is any reiteratively hypercyclic operator an Δ^*-operator or an $\mathcal{IP}^*$-operator?*

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